GSO talk: Linear scattering theory

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Abstract

Consider a basic scattering experiment: we shoot a beam of electrons toward the origin, allow the electrons to be scatter off of an electric potential, and catch the scattered electrons using particle detectors. Two basic questions arise: 1) If we know the potential beforehand, can we predict the angles and energies of the scattered electrons? 2) If we only know the angles and energies of the scattered electrons, can we recover the potential? In this talk we aim to answer these questions, pose some related problems, and discuss how more enterprising people than myself can use these to make cool things (e.g. MRIs, cloaking devices, a salary).

0 Notation and the Fourier transform

We will use the Japanese bracket notation: $\langle x\rangle = \sqrt{1+|x|^2}$. We write $X \lesssim Y$ as shorthand for $X \leq CY$, where $C > 0$ is a constant. We designate any dependent of C on parameters like so: $X \lesssim_{a,b,c,\dots} Y$ for $X \leq C(a,b,c,\dots)Y$. We employ Landau big-O and little-O asymptotic notation: $f = O(g)$ as $a \to A$ if $|f| \leq C|g|$ for a constant $C > 0$ for a sufficiently close to A, and $f = o(g)$ as $a \to A$ if $(f/g) \to 0$ as $a \to A$.

For $f : \mathbb{R}^d \to \mathbb{C}$, its Fourier transform $\mathcal{F}f = \widehat{f} : \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx.$$

The Fourier transform enjoys many useful properties. The most relevant for us are:

1. $\mathcal{F}$ is a linear operator.

2. $\mathcal{F}$ has an inverse defined by

$$\mathcal{F}^{-1}g(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\, d\xi.$$

3. $\mathcal{F}$ and $\mathcal{F}^{-1}$ intertwine differentiation and polynomial multiplication: $\mathcal{F}(\partial_{x_j} f)(\xi) = i\xi_j \mathcal{F}f(\xi)$, $\mathcal{F}^{-1}(\partial_{\xi_j} g)(x) = -ix_j \mathcal{F}g(x)$.

4. $\mathcal{F}$ and $\mathcal{F}^{-1}$ intertwine pointwise products and convolution: $\mathcal{F}(fg) = \mathcal{F}f * \mathcal{F}g$, $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}f * \mathcal{F}^{-1}g$.

5. $\mathcal{F}(e^{i\omega\cdot x} f)(\xi) = \mathcal{F}(f)(\xi - \omega)$.

Remark 0.1. We will only provide sketches of most proofs in these notes. A gentle introduction to the material is [1, Sections IV.5]. Complete proofs of most assertions can be found in [2, Chapters III - IV], upon which these notes are based.

1 The scattering problem for the linear Schrödinger equation

The linear Schrödinger equation with time-independent potential

$$-i\partial_t \psi - \Delta\psi + Q(x)\psi = 0,\ \psi = \psi(t,x),\ (t,x) \in \mathbb{R}_t \times \mathbb{R}^d_x, \tag{1.1}$$

is a well-known PDE that models the time evolution of a quantum wavefunction in the presence of the potential Q. By *scattering* we mean that we consider solutions to (1.1) of the form $\psi(t,x) = e^{-ik^2 t}u(x)$, modelling a steady beam of electrons with constant energy. (This is justified through the de Broglie relation “frequency $\sim$ energy.”) Substituting this ansatz into the equation, we arrive at the *Helmholtz equation*

$$(-k^2 - \Delta + Q(x))u = 0. \tag{1.2}$$

When $Q(x) \equiv 0$, this equation can be solved using the Fourier transform, yielding *plane wave solutions* of the form $u(x) = e^{ik\omega\cdot x}$, where $\omega \in \mathbb{R}^d$ with $|\omega| = 1$. However, when $Q(x) \not\equiv 0$, we (formally) have a correction term:

$$u(x) = e^{ik\omega\cdot x} + v(x,\omega,k).$$

The function u with this correction $v(x,\omega,k)$ is known as the *distorted plane wave*. Our goal in these notes is to answer, at least for a potential Q with sufficiently rapid decay, the following two questions:

1. *(Forward scattering problem):* Given information about the potential Q, what can we say about the distorted plane wave u?

2. *(Inverse scattering problem):* Given information about the distorted plane wave u, what can we say about the potential Q?

Forward and inverse scattering problems can be formulated with other equations of wave evolution. With different choices of PDE and different conditions imposed on the domain (see Section 5) we can pose problems relevant to a wide variety of applications. Examples:

1. We have stated the simplest version of a scattering problem involving quantum particles. Our understanding of more sophisticated scattering problems can be considered to be one of the primary drivers of much of the advances in physics over the past century.

2. Medical imaging devices, at their core, operate by solving an inverse scattering problem. For example, an MRI places a patient in a strong oscillating magnetic field, which excites hydrogen atoms in the patient's body, resulting in an emission of electromagnetic signal which is measured by a detector. The inverse scattering problem is to use the signal measurements to reconstruct the image of the interior of the patient's body.

3. Sonar and radar technology is a natural application of inverse scattering problems: here the measurement of sound and electromagnetic waves can be used to reconstruct physical objects that interact with them.

4. Geophysical surveying uses inverse scattering problems to detect valuable resources (e.g. oil, water, minerals) deep underground.

5. In materials science, inverse problems arise naturally in nondestructive testing of properties of samples, for instance in electrical impedance tomography (using surface measurements of voltage to measure the interior conductivity of a material). This idea can be flipped on its head to produce interesting metamaterials: for instance, so-called cloaking devices, which are designed to interact with electromagnetic radiation in controlled ways to produce adaptive gradations of refractive index, resulting in the illusion of invisibility.

2 Solution of the Helmholtz equation

From here on we work in $\mathbb{R}^3$ for the sake of concreteness.

We first address the forward scattering problem, in which the potential Q is known and the distorted plane wave v is to be determined. Inserting the distorted plane wave solution $u(x) = e^{ik\omega\cdot x} + v(x,\omega,k)$ into the Helmholtz equation (1.2), we obtain an equation for v:

$$(-k^2 - \Delta + Q)v = -e^{ik\omega\cdot x}Q(x). \tag{2.1}$$

Formally this is an inhomogeneous Helmholtz equation for v:

$$(-k^2 - \Delta + Q)v = f(x). \tag{2.2}$$

How do we solve such equations?

2.1 The Helmholtz equation with no potential

When $Q(x) \equiv 0$, this equation can again be solved by the Fourier transform:

$$(-k^2 + |\xi|^2)\widehat{v}(\xi) = \widehat{f}(\xi), \tag{2.3}$$

and therefore formally

$$v(x) = \mathcal{F}^{-1}\left(\frac{\widehat{f}(\xi)}{|\xi|^2 - k^2}\right)(x) = \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - k^2}\right) * f(x). \tag{2.4}$$

There is an issue here, in that $(|\xi|^2 - k^2)^{-1}$ does not have a well-defined inverse Fourier transform, even as a tempered distribution. However, we can regularize the distribution to make sense of the transform. That is, since

$$\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - (k \pm i\varepsilon)^2}\right)(x) = \frac{1}{4\pi|x|}e^{-(\varepsilon \mp ik)|x|}$$

defines for all $\varepsilon > 0$ a tempered distribution, we can take the distributional limit as $\varepsilon \to 0$ to make sense of $\mathcal{F}^{-1}(|\xi|^2 - k^2)^{-1}$; this is an instance of what is known as a *regularization* of a distribution. We thus obtain

$$E_\pm = \lim_{\varepsilon \to 0} \mathcal{F}^{-1}\left(\frac{1}{|\xi|^2 - (k \pm i\varepsilon)^2}\right) = \frac{1}{4\pi|x|}e^{\pm ik|x|}.$$

These are *fundamental solutions* of the Helmholtz equation, and a solution to (2.2) can now be written in the form

$$u(x) = E_{\pm} f = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{\pm ik|x-y|}}{|x-y|} f(y)\, dy. \tag{2.5}$$

We call $E_+ f$ (resp. $E_- f$) the *outgoing* (resp. *incoming*) solution. One notices that there is no uniqueness of the solution. In fact there are more ways to regularize the distribution, for instance using a principal value integral, and consequently even more fundamental solutions. Of course, as we have set our PDE on an unbounded domain, this is not unexpected. Uniqueness can be recovered once we set boundary conditions at infinity, i.e. a sufficiently strong decay condition. Some minimal decay is required: indeed, $E_+(x) - E_-(x) = \frac{i}{2\pi} \frac{\sin(k|x|)}{|x|}$ solves the homogeneous equation, so there is no uniqueness in the class of functions decaying as $O(|x|^{-1})$.

Definition 2.1. We say that $v(x)$ satisfies the *Sommerfeld conditions for outward radiation* in $\mathbb{R}^3$ if

$$v(x) = O(1/|x|) \text{ and } \frac{\partial}{\partial r} v(x) - ikv(x) = o(1/|x|) \text{ as } |x| \to \infty,$$

where $r = |x|$ and $\frac{\partial}{\partial r} = \frac{x}{|x|} \cdot \frac{\partial}{\partial x}$.

Remark 2.1. The Sommerfeld radiation conditions describe a wave that is radiating energy away from the origin. One can show that the radiation conditions imply that the energy flux through a large sphere centered at the origin over a single period $2\pi/k^2$ is positive. (See [1].) This is considered necessary for physically valid solutions in scattering theory: energy should radiate from sources to infinity, rather than from infinity to sources. The latter scenario is described by the *Sommerfeld conditions for inward radiation*, obtained by replacing the $-$ sign in the outward radiation conditions with a $+$ sign.

Theorem 2.1 (Asymptotic behavior of free solutions)**.** *Suppose f satisfies $|f(x)| \lesssim \langle x \rangle^{-3-\alpha}$, $\alpha > 0$. Then Equation* (2.2) *has a unique solution u satisfying the radiation conditions. It takes the form*

$$\begin{aligned} u(x) = u_+(x) = E_+ f(x) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{|x-y|} f(y)\, dy \\ &= \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} \widehat{f}(k\theta) + O(|x|^{-(1+\alpha')}) \text{ as } |x| \to \infty, \end{aligned}$$

where $\theta = \frac{x}{|x|}$ and $\alpha' = \min(\frac{\alpha}{2} - \varepsilon, 1)$ for arbitrary $\varepsilon > 0$.

Sketch of proof. The function $u = E_+ f$ manifestly solves (2.2). The asymptotic behavior is essentially a stationary phase argument. For $|y| < \frac{|x|}{2}$ we may write

$$|x-y| = |x| - \theta \cdot y + O(|y|^2/|x|).$$

Therefore on this region

$$\frac{e^{ik|x-y|}}{|x-y|} = \frac{e^{ik|x|-ik\theta\cdot y+O(\frac{|y|^2}{|x|})}}{|x|(1+O(\frac{|y|}{|x|}))} = \frac{e^{ik|x|-ik\theta\cdot y+O(\frac{|y|^2}{|x|})}}{|x|}(1+O(|y|/|x|)).$$

Performing the integral over $|y| < \frac{|x|}{2}$ yields the dominant term, with an error. One can show the appropriate decay of this error term, as well as the error coming from the integration over $|y| \geq \frac{|x|}{2}$.

Next, we must verify the radiation conditions. This is done by taking the radial derivative inside the integral. This hits the kernel E_+ in the numerator and denominator through the product rule; we then again perform a careful analysis of the integrals in the regions $|y| < \frac{|x|}{2}$ and $|y| \geq \frac{|x|}{2}$.

Lastly, we verify uniqueness. Suppose v is another solution, and let x_0 be a point, B_R a ball at the origin with $R > |x_0|$. Integrating $v(\Delta+k^2)u - u(\Delta+k^2)v$ over $B_R \setminus \overline{B_\varepsilon(x_0)}$ using Green's theorem, and taking $\varepsilon \to 0$, yields an expression for $-v(x_0)$ which is shown to be $o(1)$ as $R \to \infty$. □

2.2 The Helmholtz equation with potential

We return to Equation (2.2), the Helmholtz equation with potential:

$$(-k^2 - \Delta + Q)v = f(x).$$

To solve this equation, we make the following ansatz: suppose $v = E_+ g$ for some g. Formally substituting into the equation and using the fact that $(-k^2-\Delta)E_+ g = g$, we find that g solves the integral equation

$$g + QE_+ g = (\mathrm{id} + T)g = f(x), \tag{2.6}$$

where $Tg = QE_+ g$. We must then ask the question of existence and uniqueness of solutions to this equation. To do this we first introduce some norms and associated function spaces. For a function f define the weighted L^∞-norm

$$\|f\|_{L^{\infty,r}} = \|\langle x\rangle^r f(x)\|_{L^\infty}$$

and the weighted Sobolev norm

$$\|f\|_{H^{s,r}} = \sum_{|j|\le s} \|\langle x\rangle^r \nabla^j f\|_{L^2}.$$

Definition 2.2. We say that v is an *outgoing solution* of (2.2) if v solves (2.2) and is of the form $v = E_+ g$ for some $g \in L^{\infty,3+\alpha}$, $\alpha > 0$.

Theorem 2.2 (Existence and uniqueness of outgoing solutions)**.** *Let $Q \in L^{\infty,3+\alpha}$. Then for any $f \in L^{\infty,3+\alpha}$, there exists a unique outgoing solution $v \in L^{\infty,1}$.*

Sketch of proof. We will show that T is a compact operator on $L^{\infty,3+\alpha}$. Then $\mathrm{id}+T$ is a compact perturbation of the identity, and the Fredholm alternative holds: the inhomogeneous equation $(\mathrm{id}+T)g = f$ is uniquely solvable in $L^{\infty,\alpha}$ if and only if the homogenous equation $(\mathrm{id}+T)h = 0$ has trivial kernel.

From Theorem 2.1, we find that

$$\langle x\rangle |E_+ g| \lesssim \|g\|_{L^1} \lesssim \|g\|_{L^{\infty,3+\alpha}}$$

and

$$\langle x\rangle |\partial_x E_+ g| \lesssim \|g\|_{L^{\infty,3+\alpha}}.$$

Moreover, $E_+ g$ and $\partial_x E_+ g$ are continuous functions. Define

$$T_N g = Q(x)\eta_N(x) E_+ g,$$

where η_N is a smooth spatial cutoff at scale N. Then our estimates combined with Arzelà-Ascoli show that T_N is a compact operator on $L^{\infty,3+\alpha}$. Moreover we have

$$\begin{aligned}\langle x\rangle^{3+\alpha} |(T-T_N)g| &\lesssim \langle x\rangle^{3+\alpha} \langle x\rangle^{-3-\alpha} (1-\eta_N)(x) |E_+ g| \\ &\lesssim (1-\eta_N)(x)\langle x\rangle^{-1}\langle x\rangle |E_+ g| \\ &\lesssim \langle N\rangle^{-1} \|g\|_{L^{\infty,3+\alpha}}.\end{aligned}$$

Therefore $\|T - T_N\|_{L^{\infty,3+\alpha}} \to 0$ as $N \to \infty$, which proves that T is an operator norm limit of compact operators, and hence a compact operator.

Lastly, we can prove the following result:

Theorem 2.3. *Let $Q \in L^{\infty,1+\varepsilon}$ be real-valued. Suppose $0 < \varepsilon_1 < \frac{\varepsilon}{2}$ and $h \in H^{0,\frac{1}{2}+\varepsilon_1}$ solves the homogeneous equation*

$$h + Q(x)E_+ h = 0.$$

Then $h = 0$.

This verifies the Fredholm alternative hypothesis, and hence our desired conclusion follows. As the proof is rather long, we take this result on faith. We do, however, mention the estimate that is at the heart of its proof:

Theorem 2.4 (Agmon's estimates)**.** *For any* $\varepsilon > 0$*,*

$$\|E_+h\|_{H^{s,-\frac{1}{2}-\varepsilon}} \lesssim_\varepsilon k^{s-1}\|h\|_{H^{0,\frac{1}{2}+\varepsilon}},\ s=0,1,2$$

where the implicit constant is independent of k for $k \geq \varepsilon_0 > 0$*.* □

3 Solution of the forward scattering problem

With Theorem 2.2 in hand, we return to the problem of solving

$$(-k^2 - \Delta + Q)v = -e^{ik\omega\cdot x}Q(x). \tag{3.1}$$

For any given $Q \in L^{\infty,3+\alpha}$ and fixed $k \in \mathbb{R}$, $\omega \in S^2$, we can find the unique outgoing solution of this equation v, namely $v = E_+g$ where g is the unique solution of the integral equation

$$(\mathrm{id} + Q(x)E_+)g = -Q(x)e^{ik\omega\cdot x}.$$

Now letting $k \in \mathbb{R}$ and $\omega \in S^2$ vary, we obtain a function $v = v(x,k\omega,k)$ solving (3.1). Since v is of the form E_+g, Theorem 2.1 tells us that v has the following asymptotic behavior as $|x| \to \infty$:

$$v(x,k\omega,k) = \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|}\widehat{g}(k\theta,k\omega) + O(|x|^{-(1+\alpha')}),$$

where

$$\widehat{g}(k\theta,k\omega) = \int_{\mathbb{R}^3} g(x,k\omega)e^{-ik\theta\cdot x}\,dx.$$

In conclusion, we have shown the following:

Theorem 3.1 (Existence and asymptotics for the distorted plane wave)**.** *Assume* $Q \in L^{\infty,3+\alpha}$*. For any* $\omega \in S^2$*, the distorted plane wave*

$$u(x,\omega,k) = e^{ik\omega\cdot x} + v(x,k\omega,k)$$

exists and has the following form as $|x| \to \infty$*:*

$$u(x,\omega,k) = e^{ik\omega\cdot x} + \frac{e^{ik|x|}}{|x|}a(\theta,\omega,k) + O(|x|^{-(1+\alpha')}).$$

Definition 3.1. The function $a(\theta, \omega, k)$ is called the *scattering amplitude*.

This completely solves the scattering problem: if we know the potential Q, then we can construct the distorted plane wave, and we can describe it completely as $|x| \to \infty$ via the scattering amplitude.

4 Solution of the inverse scattering problem

Lastly, we handle the inverse scattering problem: the reconstruction of the potential $Q(x)$, given information about the scattering amplitude $a(\theta, \omega, k)$.

Remark 4.1. This problem is overdetermined: a depends on $(3-1)+(3-1)+1 = 5$ parameters, while Q depends on just 3 parameters.

Theorem 4.1 (Recovery of potential from scattering amplitude)**.** *Suppose we know $Q \in L^{\infty,3+\alpha}$, and we know the scattering amplitude $a(\theta, \omega, k)$ for all $(\theta, \omega, k) \in S^2 \times S^2 \times \mathbb{R}_+$. Then the value of $Q(x)$ can be determined for all $x \in \mathbb{R}^3$.*

Proof. Substituting $v = E_+ g$ into Equation 3.1, we consider once again its integral form:

$$g + Q(x)E_+ g = (\mathrm{id} + T)g = -e^{ik\omega\cdot x}Q(x).$$

The idea is to take the Fourier tranform of this equation, obtaining:

$$\widehat{g}(\xi, k\omega) + \mathcal{F}[QE_+ g] = -\widehat{Q}(\xi - k\omega), \tag{4.1}$$

and use the fact that $a(\theta, \omega, k) = (4\pi)^{-1}\widehat{g}(k\theta, k\omega)$ to obtain information about $\widehat{Q}$.

We pose the equation for $g \in H^{0,\frac{1}{2}+\alpha}$. Note that if $g \in H^{0,\frac{1}{2}+\alpha}$, then by Agmon's estimates (Theorem 2.4) we have $E_+ g \in H^{0,-\frac{1}{2}-\alpha}$ and hence $Q(x)E_+ g \in H^{0,3-\frac{1}{2}} \subset H^{0,\frac{1}{2}+\alpha}$. Then again by Agmon's estimates, we have

$$\|Tg\|_{H^{0,\frac{1}{2}+\alpha}} = \|QE_+ g\|_{H^{0,\frac{1}{2}+\alpha}} \lesssim k^{-1}\|g\|_{H^{0,\frac{1}{2}+\alpha}}$$

for all $k \geq 1$, with the implicit constant independent of k. Therefore the operator $T : H^{0,\frac{1}{2}+\alpha} \to H^{\frac{1}{2}+\alpha}$ has norm $\|T\| < 1$ for large k. So we conclude that $\mathrm{id} + T$ is invertible for k large, with inverse given by the Neumann series

$$(\mathrm{id} + T)^{-1} = \frac{1}{\mathrm{id} + T} = \sum_{j=0}^{\infty} (-1)^j T^j.$$

Since

$$\|g\|_{H^{0,\frac{1}{2}+\alpha}} \leq \|(\mathrm{id}+T)^{-1}\| \|Q(x)e^{ik\omega\cdot x}\|_{H^{0,\frac{1}{2}+\alpha}} = \|(\mathrm{id}+T)^{-1}\| \|Q(x)\|_{H^{0,\frac{1}{2}+\alpha}},$$

we conclude that $\|g\|_{H^{0,\frac{1}{2}+\alpha}}$ is bounded as $k \to \infty$ uniformly in ω.

Now we return to Equation 4.1. We estimate using Cauchy-Schwarz and Agmon:

$$\begin{aligned}
|\mathcal{F}(QE_+g)| &\leq \int_{\mathbb{R}^3} |Q(x)||E_+g|\,dx \lesssim \int_{\mathbb{R}^3} \langle x\rangle^{-3-\alpha}|E_+g|\,dx \\
&\leq \|\langle x\rangle^{-5/2}\|_{L^2(\mathbb{R}^3)} \|\langle x\rangle^{-\frac{1}{2}-\alpha}E_+g\|_{L^2(\mathbb{R}^3)} \\
&\lesssim \|E_+g\|_{H^{0,-\frac{1}{2}-\alpha}} \lesssim k^{-1}\|g\|_{H^{0,\frac{1}{2}+\alpha}} = O(1/k).
\end{aligned}$$

Therefore

$$\widehat{Q}(\xi - k\omega) = -\widehat{g}(\xi, k\omega) + O(1/k).$$

For a given $\omega \in S^2$, let $\eta \in \mathbb{R}^3$ be orthogonal to ω. Set

$$\theta_k = \frac{k\omega+\eta}{|k\omega+\eta|},\ \xi_k = k\theta_k.$$

Note that

$$|k\omega+\eta|^{-1} = (k^2+|\eta|^2)^{-\frac{1}{2}} = k^{-1}(1+O(|\eta|^2/k^2)).$$

Therefore

$$\xi_k - k\omega = (1+O(|\eta|^2/k^2))(k\omega+\eta) - k\omega \to \eta$$

as $k \to \infty$, and hence

$$\widehat{Q}(\xi_k - k\omega) \to \widehat{Q}(\eta) \text{ as } k \to \infty.$$

Thus we can recover $\widehat{Q}(\eta)$, where η is any vector orthogonal to any $\omega \in S^2$. Hence $\widehat{Q}(\eta)$ can be recovered for arbitrary η, and thus we recover $Q(x)$. □

5 Extensions and conclusion

1. The methods we have described are rather general and work for a larger class of potentials and in all dimensions ≥ 2; we have chosen $d=3$ and $Q \in L^{\infty,3+\alpha}$ purely to simplify some exposition. It is possible to recover similar conclusions for potentials with weaker decay, such as $Q \in L^{\infty,1+\alpha}$.

The "holy grail," so to speak, for these methods would be the Coulomb potential $Q(x) = |x|^{-1}$; sadly some extra work needs to be done to get to this decay class of potentials. One trick that physicists employ is to approximate the Coulomb potential with the Yukawa potential $Q(x) = \frac{e^{-\gamma|x|}}{|x|}$, to which our methods can be applied.

2. We should note that physicists would probably balk at our solution of the forward problem. There are other ways to solve the forward scattering problem: key words here are *partial wave analysis* and *Born approximation*.

3. We have remarked earlier that the inverse scattering problem is overdetermined in dimension 3, and in general for dimensions $d \geq 2$; again the scattering amplitude is determined by $(d-1)+(d-1)+1 = 2d-1$ parameters, while the potential is determined by d parameters. Thus one might believe that it is possible to recover the potential with less information on the scattering amplitude. This turns out to be correct: for example, one can prove inverse scattering results at fixed energy ($a(\theta,\omega,k)$ known for a single value of k), or study backscattering problems ($a(\theta,-\theta,k)$ known).

4. A related problem one can consider is the inverse boundary value problem. Here we consider the Dirichlet problem

$$\begin{cases} (-k^2 - \Delta + Q(x))u = 0 & x \in \Omega, \\ u(x) = h(x) & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with C^∞ boundary. If for k fixed this boundary value problem enjoys existence and uniqueness of solutions, then one can form the *Dirichlet-to-Neumann operator*

$$\Lambda h = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega},$$

where u is the solution to (5.1) with data h. The *inverse boundary value problem* consists of recovering the potential Q from information on the operator Λ. One way in which this type of problem appears (albeit with a different PDE) is in electrical impedance tomography: one can imagine that Ω is a region through which current flows, and which has varying electrical conductivity $\gamma(x)$. Then the voltage u satisfies the equation $\nabla\gamma(x)\cdot\nabla u(x) = 0$ on Ω. The Dirichlet-to-Neumann operator allows us to measure the flux of the current on $\partial\Omega$ if we set the voltage on $\partial\Omega$ to be given by h. Then the inverse boundary value problem is to determine the conductivity using the measurements described by the Dirichlet-to-Neumann operator. Interestingly, one can show that inverse boundary value problems and inverse scattering problems are in some cases equivalent.

5. Another related problem is that of *scattering by obstacles*. In this case we again study the behavior of plane wave solutions to the time-dependent Schrödinger equation satisfying outward radiation conditions, but this time on $\mathbb{R}^d \setminus \Omega$ and with zero potential and satisfying either Dirichlet or Neumann boundary conditions on $\partial\Omega$. The *inverse obstacle problem* is then to determine Ω from the scattering amplitude.

References

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