What are higher-order roles in social systems?

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Joint work in progress with Daniel Cicala, Rachel Hardeman Morrill, Abby Hickok, Nikola Milećević, Nima Motamed and Nina Otter

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Acknowledgements

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The Plan

I. Role and positional analysis for networks

II. Higher-order relations

III. Role analysis for higher-order relations

Part I

Role and positional analysis for networks

The idea

Social positions are collections of actors who are similar in their ties with others.

Social roles are patterns of ties, or compound ties, between actors or positions.

Multirelational networks

Definition

A ***k*-graph** consists of a finite set V and a family $R_1, \ldots, R_k$ of binary relations on V.

We can represent each R_i as a digraph or as a $V \times V$ matrix with Boolean entries.

Elements of V are called **vertices** and pairs $(u, v) \in R_i$ are **edges** of the k-graph.

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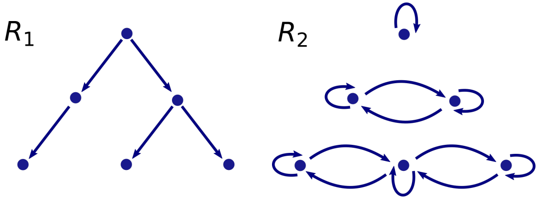
For us, the vertices represent actors in a system and $R_1, \ldots, R_k$ are distinct types of social relation among them. A k-graph is also called a **multirelational network**.

Example

In the 2-graph on the right, perhaps the vertices are employees in a firm, R_1 is the 'boss of' relation and R_2 is the 'collaborator of' relation.



This running example is from Otter & Porter (2020).

Positional analysis on (1-)graphs

Let (V, R) be a 'unirelational network'. (That is, V is a finite set and $R \subseteq V \times V$.)

Definition(ish) The aim of **positional analysis** is to identify a partition $\overline{V}$ of V whose elements are **positions** in the network, and a relation $\overline{R}$ on $\overline{V}$ containing information about R. The pair $(\overline{V}, \overline{R})$ is called a **blockmodel** for (V, R).

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Two types of equivalence relation are commonly used to form blockmodels.

Definition Vertices $u, v \in V$ are **structurally equivalent** if for every $w \in V$

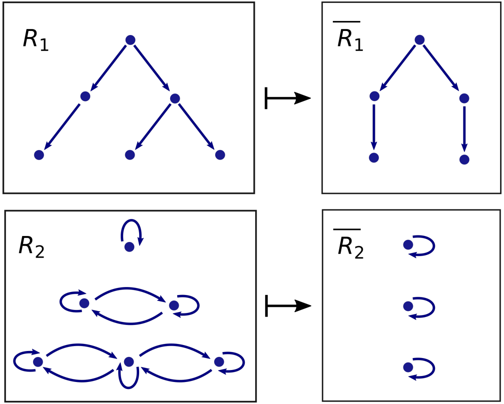
$$[(u, w) \in R \iff (v, w) \in R] \text{ and } [(w, u) \in R \iff (v, u) \in R.]$$

That is, if they have the same ties to all other vertices.

Vertices are **regularly equivalent** if they have "the same ties to *equivalent* vertices".

Positional analysis on (1-)graphs

Partitioning by structural equivalence, the 'boss' and 'collaborator' blockmodels of our firm are



Positional analysis on k-graphs

Definition A **positional reduction** of a k-graph $(V, \{R_i\}_{i=1}^k)$ consists of a k-graph $(\overline{V}, \{\overline{R_i}\}_{i=1}^k)$ and a surjective function $\phi : V \to \overline{V}$ such that, for each i,

$$(u, v) \in R_i \iff (\phi(u), \phi(v)) \in \overline{R}_i.$$

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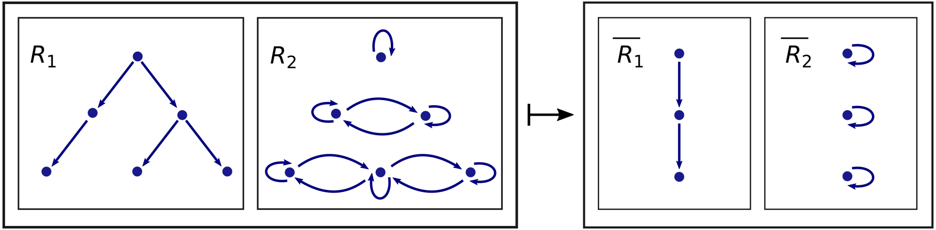
Example Here is a positional reduction of our firm:



Typically, one performs a sequence of several 'nested' positional reductions.

Role analysis on k-graphs

Definition The **semigroup of roles** in a k-graph $(V, \{R_i\}_{i=1}^{k})$ is the semigroup generated by the set $\{R_1, \dots, R_k\}$ under composition of relations. Denote it $\mathrm{Role}(V)$. The elements of $\mathrm{Role}(V)$ are called **roles** or **compound ties**.

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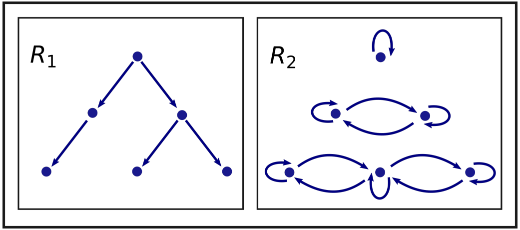
Example Here is the multiplication table for the semigroup of roles in our firm:



	R_1	R_2	R_1R_2	R_1R_1
R_1	R_1R_1	R_1R_2	R_1R_1	$\varnothing$
R_2	R_1R_2	R_2	R_1R_2	R_1R_1
R_1R_2	R_1R_1	R_1R_2	R_1R_1	$\varnothing$
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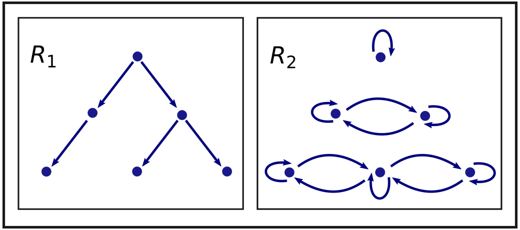

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R_1R_1	$\varnothing$	R_1R_1	$\varnothing$	$\varnothing$

Definition A **role reduction** for $(V, \{R_i\}_{i=1}^k)$ consists of a semigroup S and a surjective semigroup homomorphism $\mathrm{Role}(V) \to S$.

Functoriality of role analysis on graphs

Let $\mathbf{Graph}^k_{\mathbf{Surj}}$ denote the category of k-graphs and positional reductions, and let $\mathbf{SemiGroup}_{\mathbf{Surj}}$ denote the category of semigroups and surjective homomorphisms.

Theorem (Otter & Porter, 2020)

The assignment of the semigroup of roles induces a functor

$$\mathrm{Role} : \mathbf{Graph}^k_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

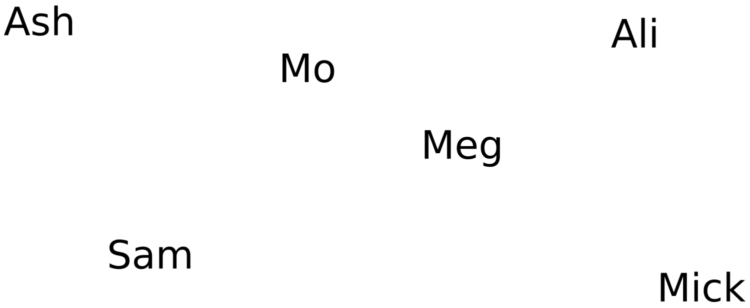
The theorem tells us that every positional reduction induces a canonical role reduction, and these behave well under ‘nesting’.

Part II

Higher-order relations

Graphs are not always enough

You're an ethnographer studying social dynamics among young people in Edinburgh.



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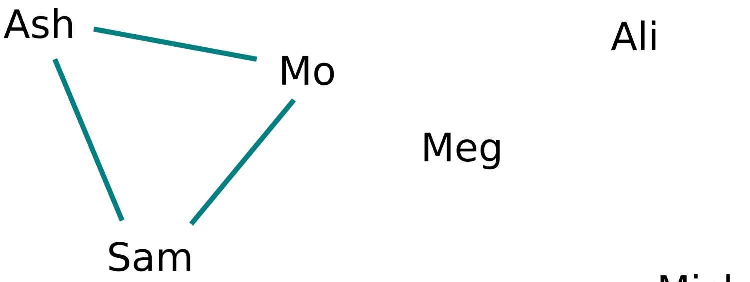

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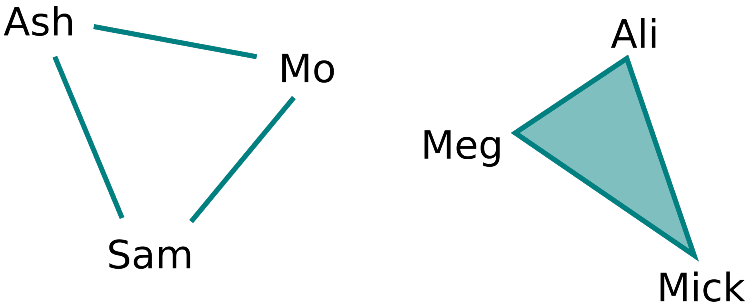
You're an ethnographer studying social dynamics among young people in Edinburgh.

Ash and Sam, Sam and Mo, and Mo and Ash have each been pals at different times.

But Meg, Ali and Mick have been an inseparable trio ever since they all first met.

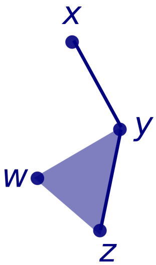


You're dealing with a higher-order relation.

Representing higher-order relations

Definition

A **hypergraph** on a set X is a set K of non-empty subsets of X.
We call the elements of K **hyperedges**.



Representing higher-order relations

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A **hypergraph** on a set X is a set K of non-empty subsets of X.
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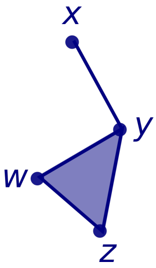
A **simplicial complex** is a hypergraph that's downward-closed: for each $\sigma \in K$ and every $\sigma' \subseteq X$ such that $\sigma' \subseteq \sigma$, we have $\sigma' \in K$.

The hyperedges of a simplicial complex are called **simplices**.

The **dimension** of a simplex σ is $\#\sigma - 1$. If $\dim(\sigma) = n$, it's called an n-simplex.



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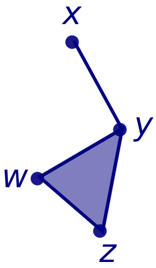

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Simplicial complexes are cheaper to store and have valuable topological structure.

Complexes from relations

Observation (Dowker, 1952)

From any relation $B \subseteq X \times Y$ we can construct two simplicial complexes:

$$K_L(B) = \{(x_0, \dots, x_n) \mid \text{there exists some } y \in Y \text{ with } (x_i, y) \in R \text{ for all } i\}$$

and

$$K_R(B) = \{(y_0, \dots, y_n) \mid \text{there exists some } x \in X \text{ with } (x, y_i) \in R \text{ for all } i\}.$$

Theorem (Dowker, 1952)

The geometric realizations of $K_L(B)$ and $K_R(B)$ have the same homotopy type.

q-analysis

Observation (Atkin, 1970s)

Relations are everywhere! So simplicial complexes are, too.

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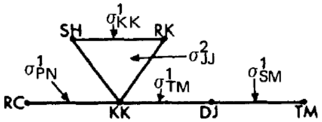
Relations are everywhere! So simplicial complexes are, too.

Atkin developed a method called ***q*-analysis** to study the connectedness of complexes. For a while, it became very popular with social network theorists.

Figure 5. *The prominence of player KK in the Liverpool complex when $\theta \geqslant 4$ at the* q = *1 dimensional level.*

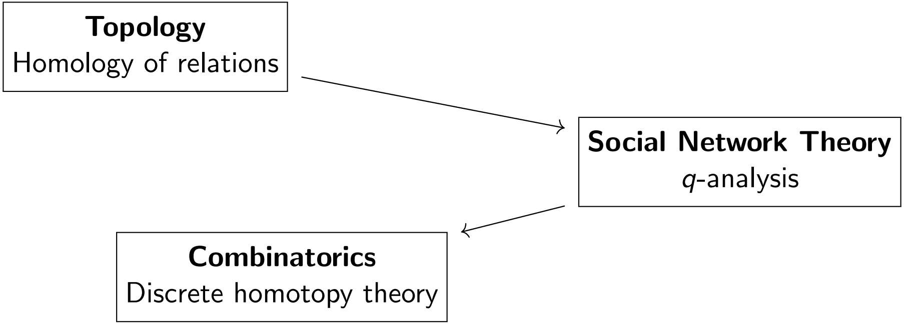


From Gould & Gattrell, A Structural Analysis of a Game: The Liverpool v Manchester United Cup Final of 1977. *Social Networks* 2 (1979).

Discrete homotopy theory

Observation (Barcelo *et al*, 2001)

Atkins' *q*-connected components are the grading-0 part of a **discrete homotopy theory** for complexes, which turns out to have many mathematical applications.



Part III

Role analysis for higher-order relations

Our objective

The Goal A way to 'compose' two hypergraphs on the same set of vertices.

Desirable Properties

1. Given two simplicial complexes, their composite should be a simplicial complex.
2. (a) Composition should be associative.
   (b) Or, if not, it should at least be 'associative for a single relation':

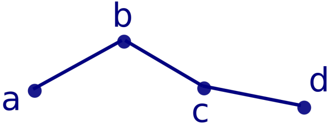
$$K * (K * K) = (K * K) * K.$$

3. The assignment of the 'object of roles' should be functorial with respect to a corresponding 'positional analysis' for higher-order relations.
4. Compound ties should be meaningful for (some) social relations!

Two perspectives

Perspective A When we compose edges in a graph, we are taking ‘steps’ in a ‘path’ through the graph.

R



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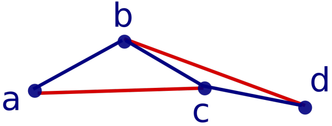

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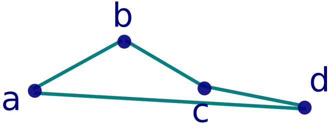

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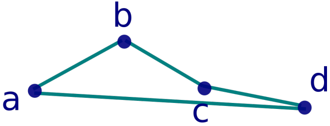

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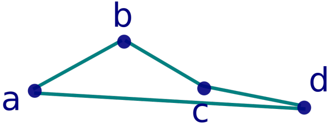

We need a good notion of 'path' for hypergraphs; discrete homotopy theory offers one.

Perspective B Binary relations can be described in several equivalent categorical ways, including as coalgebras for the covariant powerset functor—and that gives us a nice way to understand positional analysis. We should start from there.

Perspective A

Paths

Steps in a q-path

Definition (Atkin 1970s; Barcelo *et al*, 2001)

Let K be a simplicial complex. Fix $q \geqslant 0$. A **q-path** from a simplex σ to a simplex τ is a sequence of simplices

$$\sigma = \sigma_0, \ldots, \sigma_k = \tau$$

such that σ_i and σ_{i+1} share a q-face: they have at least $q+1$ vertices in common.

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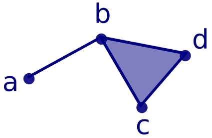

Example

In this complex, $\{a\}, \{a, b\}, \{b, c, d\}, \{c, d\}$ is a 0-path from $\{a\}$ to $\{c, d\}$. But there's no 1-path between these simplices.

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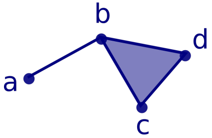

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Rachel's idea

The 'steps' in a q-path are the things we should be 'compounding'.

q-composition

Let K_1 and K_2 be simplicial complexes on the same set of vertices. Fix $q \geqslant 0$.

Definition

A pair of simplices $\sigma_1 \in K_1$ and $\sigma_2 \in K_2$ is **q-composable** if they share at least one face of dimension q. The **set of q-compounds** of σ_1 and σ_2 is

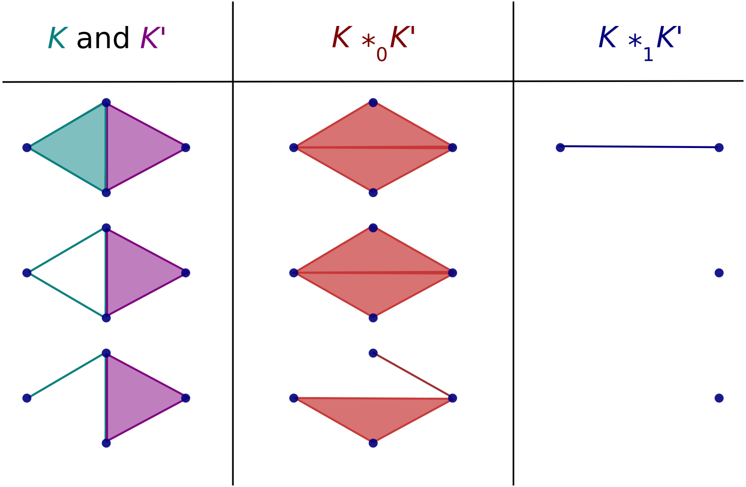
$$\sigma_1 *_q \sigma_2 = \{(\sigma_1 \cup \sigma_2) \backslash \tau \mid \tau \subseteq \sigma_1 \cap \sigma_2 \text{ and } \dim(\tau) = q\}.$$

The **q-composite** of the complexes K_1 and K_2 is the set

$$K_1 *_q K_2 = \bigcup_{(\sigma_1, \sigma_2)} \sigma_1 *_q \sigma_2$$

where the union is over q-composable pairs $(\sigma_1, \sigma_2) \in K_1 \times K_2$.

q-composition in action



Properties of q-composition

Theorem

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Property	$*_q$
1	✓
2(a)	✗
2(b)	✓
3	?
4	?

Perspective B

Coalgebra

From relations to coalgebras

Definition Let $\mathbf{C}$ be a category, and $T : \mathbf{C} \to \mathbf{C}$ a functor. A **coalgebra for T** is an object X in $\mathbf{C}$ and a $\mathbf{C}$-morphism $X \to TX$.

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Example Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the powerset monad. Since $\mathbf{Set}$ is cartesian closed, for every set X there's a bijection $\mathcal{P}(X \times X) \cong \mathbf{Set}(X, \mathcal{P}(X))$. So binary relations on X are the same thing as $\mathcal{P}$-coalgebra structures on X and elements of $\mathbf{Set}_{\mathcal{P}}(X, X)$.

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Nima's insight Regular equivalences are bisimulation equivalences!
So a positional reduction is a quotient in a category of coalgebras.*

* Provided we are careful about the coalgebra morphisms we use.

Positional analysis for coalgebras

Definition

Let T be a functor $\mathbf{C} \to \mathbf{C}$. A **$k$-coalgebra for T** is a pair $\mathbf{V} = (V, \{V \xrightarrow{\rho_i} TV\}_{i=1}^{k})$ where V is an object of $\mathbf{C}$ and the ρ_i are $\mathbf{C}$-morphisms.

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A **positional reduction** of $\mathbf{V}$ consists of a k-coalgebra $(W, \{W \xrightarrow{\sigma_i} TW\}_{i=1}^{k})$ and a split epimorphism $f : V \to W$ in $\mathbf{C}$ which is a coalgebra morphism $\rho_i \to \sigma_i$ for every i.

Let $T\mathbf{Coalg}^{k}_{\mathbf{Surj}}$ denote the category whose objects are k-coalgebras for T and whose morphisms are positional reductions.

Role analysis for coalgebras

Definition

Let $\mathbb{T}$ be a monad on $\mathbf{C}$, and $\mathbf{V}$ a k-coalgebra for $\mathbb{T}$. The **semigroup of $\mathbb{T}$-roles** in $\mathbf{V}$ is the subsemigroup of $\mathbf{C}_{\mathbb{T}}(V, V)$ generated by $\{V \xrightarrow{\rho_i} \mathbb{T}V\}_{i=1}^{k}$. Denote it $\mathrm{Role}_{\mathbb{T}}(\mathbf{V})$.

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Theorem

The assignment of the semigroup of $\mathbb{T}$-roles extends to a functor

$$\mathsf{Role}_{\mathbb{T}} : \mathbb{T}\mathbf{Coalg}^{k}_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

Taking $\mathbb{T}$ to be $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ recovers Otter & Porter's theorem.

Encoding complexes as coalgebras

$$\mathcal{P}(X \times \mathcal{P}(X)) \underset{}{\overset{\cong}{\rightleftarrows}} \mathbf{Set}(X, \mathcal{P}\mathcal{P}(X))$$

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$$\{\text{simplical complexes on } X\} \xleftarrow[K_L]{} \mathcal{P}(X \times \mathcal{P}(X)) \underset{}{\overset{\cong}{\rightleftarrows}} \mathbf{Set}(X, \mathcal{P}\mathcal{P}(X))$$

Encoding complexes as coalgebras

$$\{\text{simplical complexes on } X\} \underset{K_L}{\overset{\text{Belonging}}{\rightleftarrows}} \mathcal{P}(X \times \mathcal{P}(X)) \overset{\cong}{\rightleftarrows} \mathbf{Set}(X, \mathcal{P}\mathcal{P}(X))$$

Definition (The belonging relation)

Let K be a simplicial complex on X. Then $(x, \sigma) \in \text{Belonging}(K)$ if $\sigma \in K$ and $x \in \sigma$.

Encoding complexes as coalgebras

$$\{\text{simplical complexes on } X\} \underset{K_L}{\overset{\text{Belonging}}{\rightleftarrows}} \mathcal{P}(X \times \mathcal{P}(X)) \overset{\cong}{\rightleftarrows} \mathbf{Set}(X, \mathcal{P}\mathcal{P}(X))$$

$$E : \{\text{simplical complexes on } X\} \to \mathbf{Set}(X, \mathcal{P}\mathcal{P}(X))$$

Definition (The belonging relation)

Let K be a simplicial complex on X. Then $(x, \sigma) \in \text{Belonging}(K)$ if $\sigma \in K$ and $x \in \sigma$.

Definition (The coalgebra encoding)

Given a simplicial complex K on X and a point $x \in X$,

$$E(K)(x) = \{\sigma \in K \mid x \in \sigma\} \in \mathcal{P}\mathcal{P}(X).$$

Uh-oh!

Iterated Covariant Powerset is not a Monad[1]

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We don't need a monad

Definition

Let $T : \mathbf{C} \to \mathbf{C}$ be a functor and $\mu : TT \to T$ a natural transformation such that

$$\begin{array}{ccc} TTTT & \xrightarrow{\mu_T} & TT \\ \downarrow T\mu & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

commutes. The **Kleisli semi-category $\mathbf{C}_{(T,\mu)}$** of T has as objects those of X, with morphisms $X \to Y$ given by $\mathbf{C}$-morphisms $X \to TY$. Composition is defined using μ.

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Definition

Let (T, μ) be as above, $\mathbf{V}$ a k-coalgebra for T. The **semigroup of roles** in $\mathbf{V}$ is the subsemigroup of $\mathbf{C}_{(T,\mu)}(V, V)$ generated by $\{V \xrightarrow{\rho_i} TV\}_{i=1}^{k}$. Denote it $\mathrm{Role}_{(T,\mu)}(\mathbf{V})$.

Theorem This extends to a functor $\mathrm{Role}_{(T,\mu)} : T\mathbf{Coalg}^{k}_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}$.

Multiplications on $\mathcal{P}\mathcal{P}$

In 2018 John Baez asked on the n-Category Cafe...

Question. Does there exist an associative multiplication $m\colon P^2P^2 \Rightarrow P^2$? In other words, is there a natural transformation $m\colon P^2P^2 \Rightarrow P^2$ such that

$$P^2P^2P^2 \xRightarrow{mP^2} P^2P^2 \xRightarrow{m} P^2$$

equals

$$P^2P^2P^2 \xRightarrow{P^2m} P^2P^2 \xRightarrow{m} P^2.$$

Greg Egan answered: Yes! There are at least two:

$$\mu_1 : \mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P} \xRightarrow{\mu_{\mathcal{P}\mathcal{P}}} \mathcal{P}\mathcal{P}\mathcal{P} \xRightarrow{\mu_{\mathcal{P}}} \mathcal{P}\mathcal{P} \quad \text{and} \quad \mu_2 : \mathcal{P}\mathcal{P}\mathcal{P}\mathcal{P} \xRightarrow{\mathcal{P}\mathcal{P}\mu} \mathcal{P}\mathcal{P}\mathcal{P} \xRightarrow{\mathcal{P}\mu} \mathcal{P}\mathcal{P}$$

where μ is the multiplication of the monad $\mathcal{P}$.

Associative and functorial role analysis

Theorem

Any associative multiplication μ *on* $\mathcal{P}\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ *gives a role analysis functor*

$$\mathrm{Role}_\mu : \mathcal{P}\mathcal{P}\mathbf{Coalg}^k_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

Associative and functorial role analysis

Theorem

Any associative multiplication μ on $\mathcal{PP} : \mathbf{Set} \to \mathbf{Set}$ gives a role analysis functor

$$\mathrm{Role}_\mu : \mathcal{PP}\mathbf{Coalg}^k_{\mathbf{Surj}} \to \mathbf{SemiGroup}_{\mathbf{Surj}}.$$

So there are at least two functorial ways to assign a semigroup of roles to a multirelational simplicial complex.

What they look like remains to be seen!

Property	$*_q$	$*_{\mu_1}$	$*_{\mu_2}$
1	✓	✗	✗
2(a)	✗	✓	✓
2(b)	✓	✓	✓
3	?	✓	✓
4	?	?	?

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