Bigraded Path Homology
and the
Magnitude-Path Spectral Sequence

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Osaka University

joint work w/ Richard Hepworth

Topology Seminar
BIMSA, Beijing, April 2024

In this talk

1. The magnitude-path spectral sequence

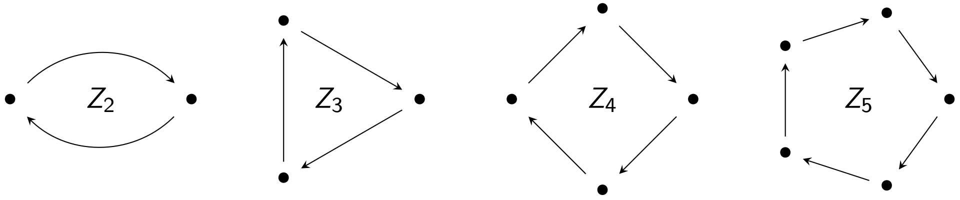
2. Bigraded path homology

3. A little formal homotopy theory

Part I

The magnitude-path spectral sequence

Three perspectives on directed cycles



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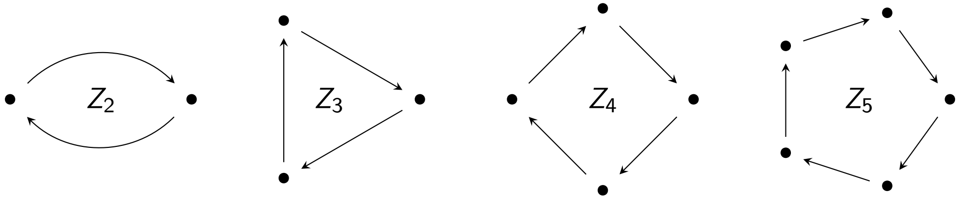

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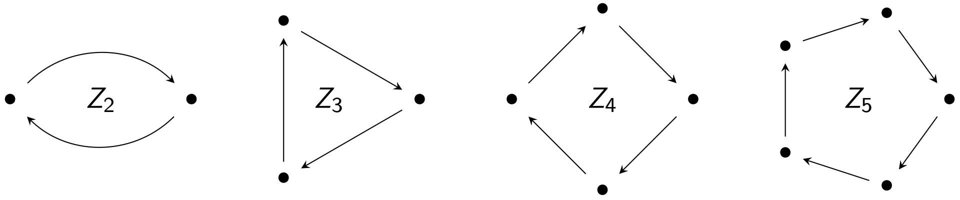

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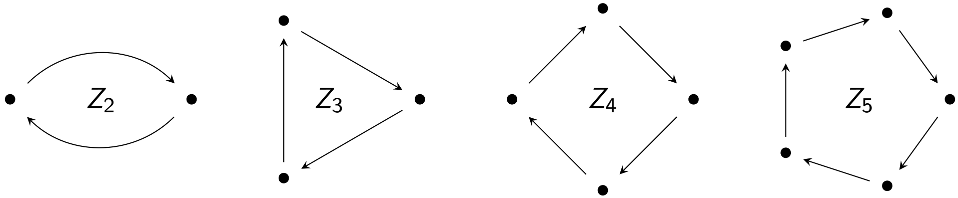

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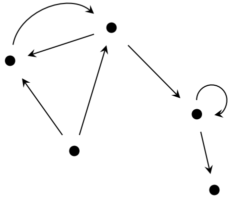
To **GLMY path homology**, Z_2 looks ‘contractible’ and all the rest look ‘circle-like’.

To **reachability homology**, every directed cycle looks ‘contractible’.

The category of directed graphs

Definition A **directed graph** X consists of

- a set of vertices $V(X)$
- a set of edges $E(X) \subseteq V(X) \times V(X)$.



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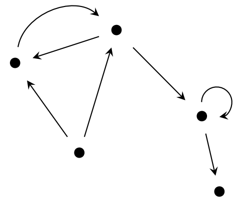

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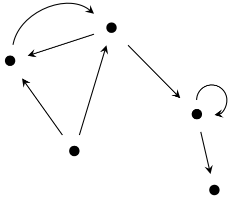

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Definition The **shortest path metric** on X is the distance function

$$d(x, x') = \min\{n \mid \text{there is a path } x = x_0 \to x_1 \to \cdots \to x_n = x' \text{ in } X\}$$

or $d(x, x') = \infty$ if no such path exists.

The reachability complex of a directed graph

Definition (Hepworth & R., 2023) The **reachability complex** of a digraph X is

$$\mathrm{RC}_k(X) = R \cdot \{(x_0, x_1, \ldots, x_k) \mid x_{i-1} \neq x_i \text{ and } d(x_{i-1}, x_i) < \infty \text{ for every } i\}$$

with differential $\partial(x_0, \ldots, x_k) = \sum (-1)^i (x_0, \ldots, \widehat{x_i}, \ldots, x_k)$.

The **reachability homology** of X is $\mathrm{RH}_*(X) = H_*(\mathrm{RC}(X))$.

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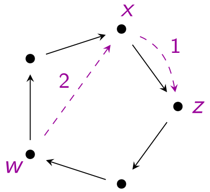

$\mathrm{RC}_*(X)$ can be filtered by the **length** of its generators:

$$F_\ell(\mathrm{RC}_k(X)) = R \cdot \left\{(x_0, x_1, \ldots, x_k) \mid x_{i-1} \neq x_i \text{ for every } i, \text{ and } \sum_{i=1}^{k} d(x_{i-1}, x_i) \leqslant \ell\right\}.$$

Example (w, x, z) is a generator of $F_3(\mathrm{RC}_2(Z_5))$ but not of $F_2(\mathrm{RC}_2(Z_5))$.

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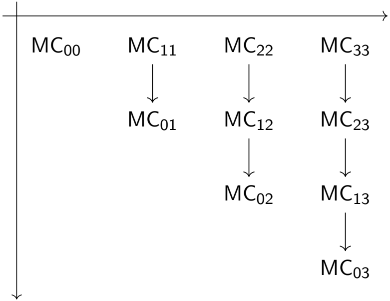
Observation (Hepworth & Willerton, 2015)
$E^0(X)$ is the magnitude chain complex $\mathrm{MC}_{**}(X)$.



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$E^1(X)$ is the magnitude homology $\mathrm{MH}_{**}(X)$.

$$\begin{array}{ccccccc}
\mathrm{MH}_{00} & \longleftarrow & \mathrm{MH}_{11} & \longleftarrow & \mathrm{MH}_{22} & \longleftarrow & \mathrm{MH}_{33} \\
 & & \mathrm{MH}_{01} & \longleftarrow & \mathrm{MH}_{12} & \longleftarrow & \mathrm{MH}_{23} \\
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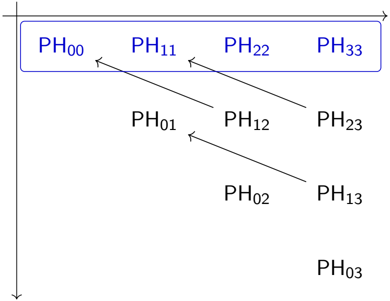
Theorem (Asao, 2022)
$E^2(X)$ contains the GLMY path homology of X.



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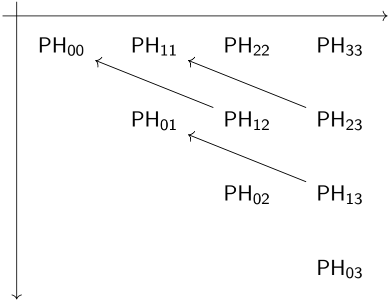

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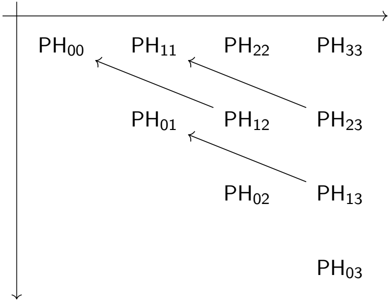

By construction $E^\bullet(X) \Rightarrow \mathrm{RH}_*(X)$ under mild conditions on X.

Functoriality and homotopy invariance of spectral sequences

A chain map $\phi\colon C_* \to D_*$ is **filtered** if $\phi(F_\ell C_*) \subseteq F_\ell D_*$ for every ℓ.

Functoriality A filtered chain map $\phi\colon C_* \to D_*$ induces a **map of spectral sequences**

$$E^\bullet(\phi)\colon E^\bullet(C) \to E^\bullet(D)$$

i.e. for every r a map of bigraded chain complexes $E^r(\phi)\colon E^r(C) \to E^r(D)$.

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Definition Let $\phi, \psi\colon C_* \to D_*$ be filtered chain maps. A chain homotopy $h\colon \phi \to \psi$ is called a **chain r-homotopy** if $h(F_\ell D_*) \subseteq F_{\ell+r} D_*$ for every ℓ.

Homotopy invariance If there exists a chain r-homotopy from ϕ to ψ, then $E^r(\phi)$ and $E^r(\psi)$ are chain homotopic and $E^s(\phi) = E^s(\psi)$ for all $s > r$.

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Moral We can talk about degrees of homotopy equivalence for filtered complexes.

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Degrees of homotopy equivalence

Definition Digraphs X and Y are ***r*-homotopy equivalent**, and write $X \simeq_r Y$, if there exist maps $f : X \rightleftarrows Y : g$ such that

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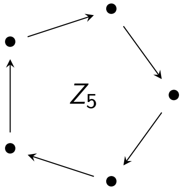
- $f \circ g$ is related to Id_Y by a zig-zag of r-homotopies.



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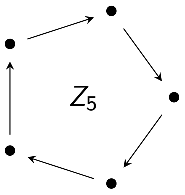

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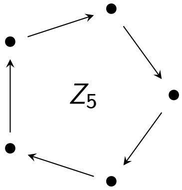

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Lemma If X has diameter D, then $X \simeq_D \bullet$ and $E^r(X)$ is trivial for all $r > D$.

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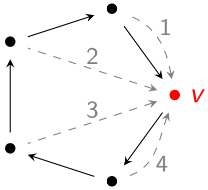

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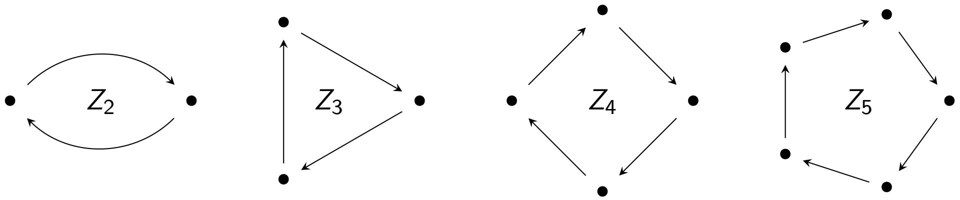
Example Fix $r \in \mathbb{N}$. Then $Z_n \simeq_r \bullet$ for $n \leqslant r+1$, and $Z_m \not\simeq_r Z_n$ for $r+1 \leqslant m < n$.

Example

Directed cycles



Theorem (Hepworth & R., 2024)

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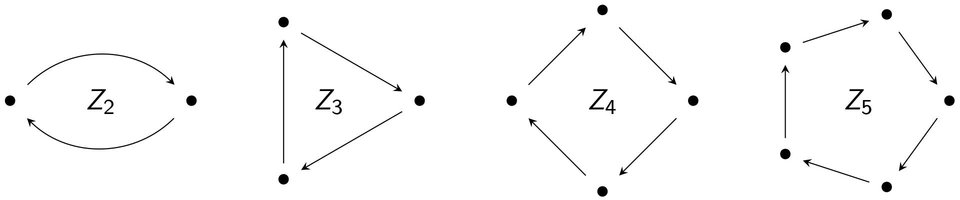

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In particular, bigraded path homology distinguishes the directed m-cycles for all $m \geqslant 2$.

Infinitely many homology theories!

Basic observation

For each $r \geqslant 0$, page $E^r(-)$ of the MPSS is a functor $\mathbf{DiGraph} \to \mathbf{Mod}_R^{\mathbb{N}\times\mathbb{N}}$.

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Corollary	These results hold for magnitude homology & bigraded path homology.

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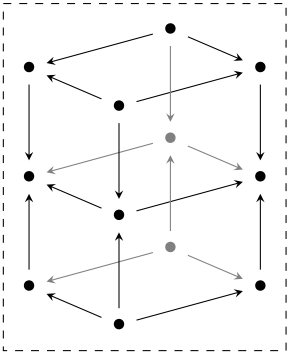
Part II

Bigraded path homology

The Künneth Theorem

The **box product** of X and Y is the digraph $X \square Y$ with

- vertices $(x, y) \in V(X) \times V(Y)$
- an edge $(x, y) \to (x', y')$ if $x = x'$ in X and $y \to y'$ or $x \to x'$ and $y = y'$ in Y.



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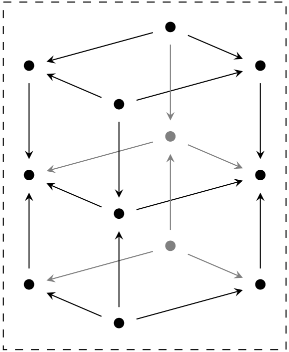
Theorem (Hepworth & R., 2024)

Fix a ground ring R which is a field. Then we have

$$E^r(X \,\square\, Y) \cong E^r(X) \otimes E^r(Y)$$

for every $r \geqslant 1$.



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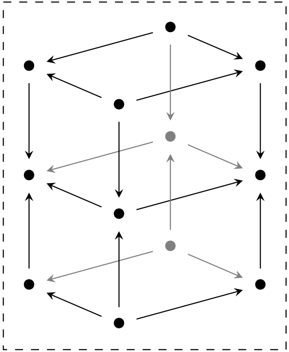


Theorem (Hepworth & R., 2024)

Fix a ground ring R which is a field. Then we have

$$E^r(X \square Y) \cong E^r(X) \otimes E^r(Y)$$

for every $r \geqslant 1$. In particular,

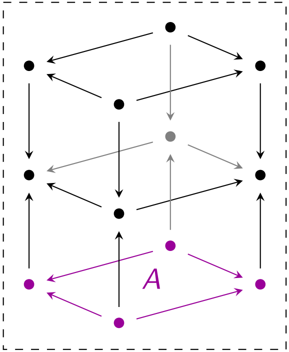
- $\mathrm{MH}_{*,*}(X \square Y) \cong \mathrm{MH}_{*,*}(X) \otimes \mathrm{MH}_{*,*}(Y)$
- $\mathrm{PH}_{*,*}(X \square Y) \cong \mathrm{PH}_{*,*}(X) \otimes \mathrm{PH}_{*,*}(Y)$.

A subtler statement holds when R is a P.I.D.

Cofibrations

Let $A \subseteq X$ be an induced subgraph. The **reach of A** is the induced subgraph rA with

$$V(rA) = \{x \in V(X) \mid \text{there exists a path from } A \text{ to } x\}.$$



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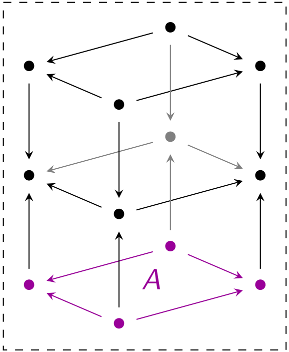
Definition (Carranza *et al*, 2022)

A **cofibration** is an inclusion $A \rightarrowtail X$ such that:

1. There are no edges from $X \backslash A$ to A.
2. There is a function $\pi\colon V(rA) \to V(A)$ satisfying

$$d(a, x) = d(a, \pi(x)) + d(\pi(x), x)$$

for every $a \in V(A)$ and $x \in V(rA)$.



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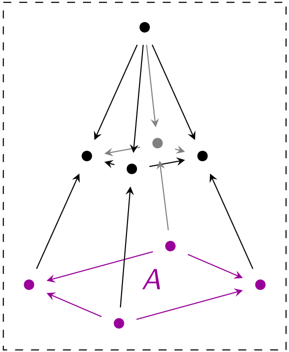

The Excision Theorem

Theorem (Hepworth & R., 2024)

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow{g} & X \cup_A Y \end{array}$$

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Proof sketch. Since a map inducing an isomorphism on E^1 will induce an isomorphism on E^r for all $r \geqslant 1$, it's enough to prove it for $\mathrm{MH}_{*,*}(-)$. For $\mathrm{MH}_{*,*}(-)$ we adapt the proof of the excision theorem in Hepworth & Willerton (2017). □

The Mayer–Vietoris Theorem

Theorem (Hepworth & R., 2024)

Given a pushout of digraphs in which $i\colon A \rightarrowtail X$ is a cofibration, there is a split short exact sequence in magnitude homology

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{g} & X \cup_A Y \end{array}$$

$$0 \to \mathrm{MH}_{*,*}(A) \xrightarrow{(i_*,-f_*)} \mathrm{MH}_{*,*}(X) \oplus \mathrm{MH}_{*,*}(Y) \xrightarrow{g_* \oplus j_*} \mathrm{MH}_{*,*}(X \cup_A Y) \to 0$$

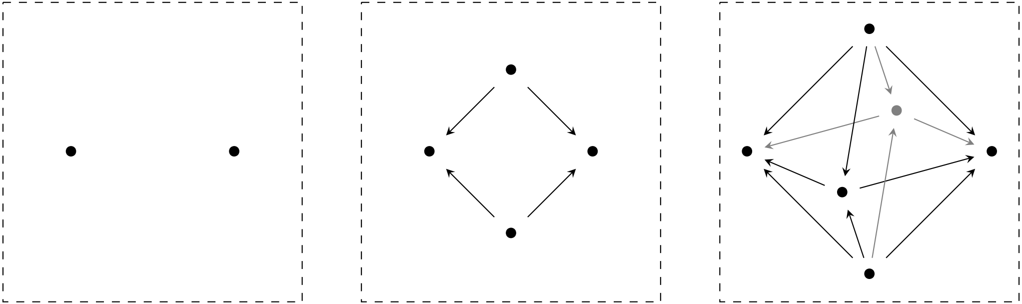
and a long exact sequence in bigraded path homology

$$\cdots \to \mathrm{PH}_{*,*}(A) \xrightarrow{(i_*,-f_*)} \mathrm{PH}_{*,*}(X) \oplus \mathrm{PH}_*(Y) \xrightarrow{g_* \oplus j_*} \mathrm{PH}_{*,*}(X \cup_A Y) \xrightarrow{\partial_*}$$

$$\mathrm{PH}_{*-1,*}(A) \to \mathrm{PH}_{*-1,*}(X) \oplus \mathrm{PH}_*(Y) \to \mathrm{PH}_{*-1,*}(X \cup_A Y) \to \cdots$$

Example

Spheres



Definition For each $n \geqslant 0$, let $\mathbb{S}^n$ be the face poset of the cell-decomposition of the n-sphere into hemispheres. Let $\mathbb{S}^\infty$ be the colimit of $\mathbb{S}^0 \hookrightarrow \mathbb{S}^1 \hookrightarrow \cdots \hookrightarrow \mathbb{S}^n \hookrightarrow \cdots$.

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Spheres

Theorem (Hepworth & R., 2024)

Let $n \geqslant 1$. Then $\mathrm{PH}_{k,\ell}(\mathbb{S}^n) = 0$ for $k \neq \ell$, while

$$\mathrm{PH}_{k,k}(\mathbb{S}^n) \cong \begin{cases} R & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

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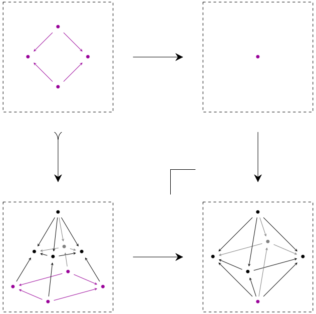

Proof sketch. $\mathbb{S}^n$ is the pushout of the maps $\mathrm{Cone}(\mathbb{S}^{n-1}) \leftarrowtail \mathbb{S}^{n-1} \to \bullet$. Write down the Mayer–Vietoris sequence and use the fact that $\mathrm{Cone}(\mathbb{S}^{n-1}) \simeq_1 \bullet$ to see that $\mathrm{PH}_{k,\ell}(\mathbb{S}^n) \cong \mathrm{PH}_{k-1,\ell}(\mathbb{S}^{n-1})$. Now induct on n. □

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Every page of the MPSS is a finitary functor: it preserves filtered colimits.

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Theorem (Hepworth & R., 2024; Di *et al*, 2023)
Every page of the MPSS is a finitary functor: it preserves filtered colimits.

Corollary Bigraded path homology sees $\mathbb{S}^\infty$ as contractible:

$$\mathrm{PH}_{k,\ell}(\mathbb{S}^\infty) = \begin{cases} R & k = \ell = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $\mathrm{PH}_{*,*}(-)$ is finitary, we have

$$\mathrm{PH}_{k,\ell}(\mathbb{S}^\infty) = \mathrm{PH}_{k,\ell}(\operatorname{colim}_{\mathbb{N}}(\mathbb{S}^n)) \cong \operatorname{colim}_{\mathbb{N}}(\mathrm{PH}_{k,\ell}(\mathbb{S}^n)).$$

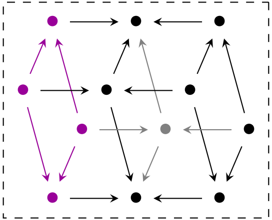
For each n, the map $i_*\colon \mathrm{PH}_{k,\ell}(\mathbb{S}^n) \to \mathrm{PH}_{k,\ell}(\mathbb{S}^{n+1})$ is zero except when $k = \ell = 0$. $\square$

Part III

A little formal homotopy theory

A new cofibration category of directed graphs

A **cofibration category** is 'one half of a model category'. It is a category equipped with two distinguished classes of morphisms:



- weak equivalences $X \xrightarrow{\sim} Y$
- cofibrations $A \rightarrowtail X$

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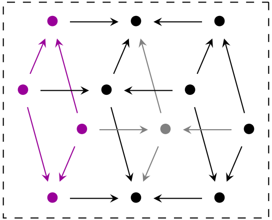

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satisfying several axioms.

Theorem (Hepworth & R., 2024)

DiGraph carries a cofibration category structure in which

- weak equivalences are maps inducing isomorphisms on bigraded path homology;
- cofibrations are those defined in Carranza *et al* (2023).

This structure is strictly finer than that for GLMY homology given by Carranza *et al*.

Proof. Combines all the homological properties of bigraded path homology. □

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Most of the homological properties of PH_{**} and MH_{**} hold for all pages of the MPSS.
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Recall A cofibration is a subgraph inclusion $A \rightarrowtail X$ such that:

1. There are no edges from $X \backslash A$ to A.
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Definition A **short cofibration** is a subgraph inclusion $A \rightarrowtail X$ such that:

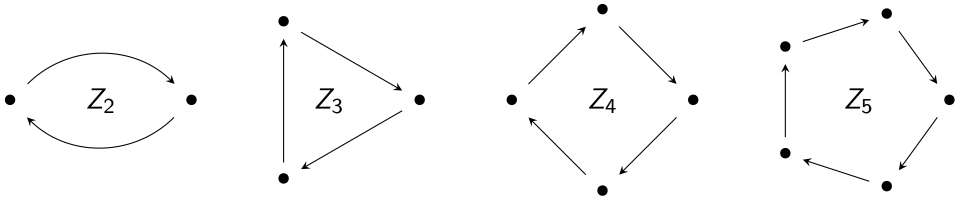
1. There are no edges from $X \backslash A$ to A.
2. There is a map of graphs $\pi \colon rA \to A$ satisfying $d(a, x) = d(a, \pi(x)) + d(\pi(x), x)$ for every $a \in V(A)$ and $x \in V(rA)$.

A conjecture

Conjecture **DiGraph** has a nested family of ∞-many cofibration category structures:

- A weak equivalence at level r is a map $f : X \to Y$ inducing an isomorphism on page E^{r+1} of the magnitude-path spectral sequence.
- A cofibration at any level is a **short cofibration**.

These structures are all distinct: at level m, the cycle Z_m 'becomes contractible'.



Conclusions

- For directed graphs, homotopy equivalence is a matter of degree: for each $r \in \mathbb{N}$ we can define r-homotopy equivalence for digraphs, getting weaker as r grows.
- For each $r \in \mathbb{N}$, page E^r of the magnitude-path spectral sequence is an r-homotopy invariant of directed graphs and has good homological properties.
- Page E^2 is bigraded path homology. It shares the homotopy invariance of GLMY homology, but is strictly finer: it distinguishes directed cycles of different lengths.
- The spectral sequence is a useful tool to prove results about $\mathrm{MH}_{*,*}(-)$, $\mathrm{PH}_{*,*}(-)$ and GLMY homology.
- Eventually, we hope it will cast more light on the homotopy theory of digraphs.

Thank you.

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