Magnitude homology and iterated enrichment

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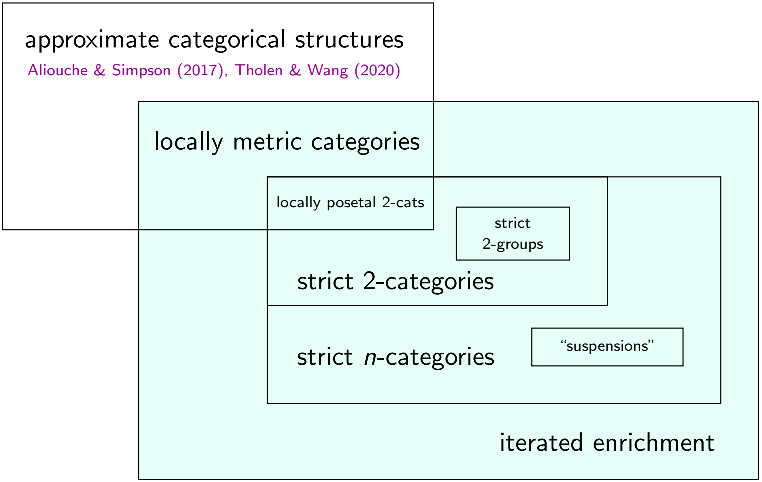
CT20$\to$21
Genova

Why think about iterated enrichment?



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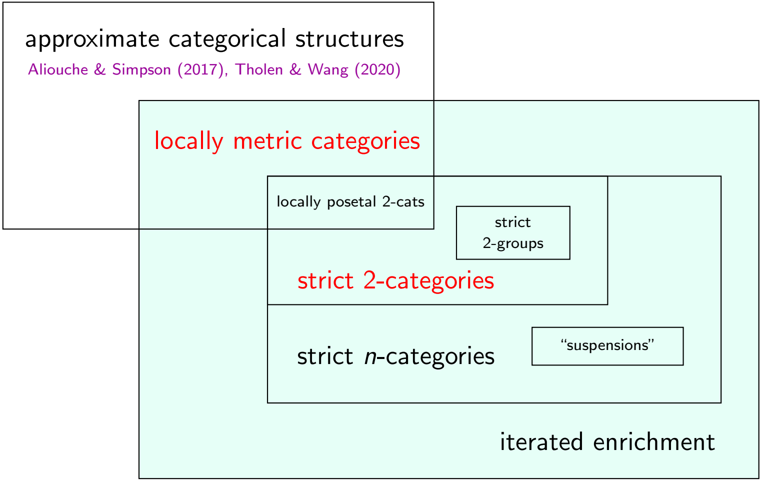

Plan

I. Magnitude homology

II. Iterating magnitude homology

III. The magnitude homology of a locally metric category

Part I

Magnitude homology

Size and magnitude

notion of 'size' for objects in $\mathcal{V}$	$\longrightarrow$	$\boxed{\textbf{magnitude}}$	$\longrightarrow$	notion of 'size' for $\mathcal{V}$-categories

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Examples (Leinster, Willerton, Meckes, Carbery, Gimperlein, ...)

Finite categories: magnitude is a generalized Euler characteristic

Finite metric spaces: magnitude is "effective number of points"

Compact metric spaces: magnitude knows volume, surface area, Euler characteristic...

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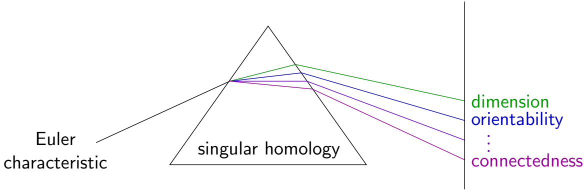

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$$\boxed{\text{notion of 'size' for objects in } \mathcal{V} \longrightarrow \boxed{\textbf{magnitude}} \longrightarrow \text{notion of 'size' for } \mathcal{V}\text{-categories}}$$

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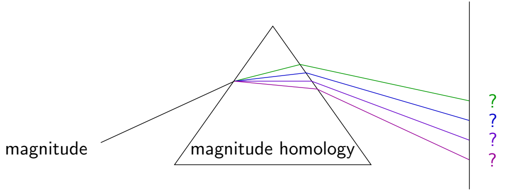

Categorifying size

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{magnitude}\\ \mathrm{Mag}_{\#} : \mathcal{V}\mathbf{Cat} \to R\end{array}} \arrow[r, dashed] & \boxed{\begin{array}{c}\text{‘magnitude functor’}\\ \mathcal{V}\mathbf{Cat} \to \mathbb{A}^{\mathbb{N}}\end{array}} \\
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Suppose R is a ring and

- $\mathcal{V}$ is a semicartesian monoidal category with a size homomorphism

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- $\Sigma : \mathcal{V} \to \mathbb{A}$ is a strong symmetric monoidal functor such that

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Then we say Σ is a **size functor** categorifying $\#$.

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$$\mathcal{V}\mathbf{Cat} \xrightarrow{MB^{\Sigma}} [\triangle^{\mathsf{op}}, \mathbb{A}] \xrightarrow{C} \mathrm{Ch}(\mathbb{A}) \xrightarrow{H_{\bullet}} \mathbb{A}^{\mathbb{N}}$$

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Definition (Leinster & Shulman, 2017)

The **magnitude nerve** of a $\mathcal{V}$-category $\mathbf{X}$ is given for $n \in \mathbb{N}$ by

$$MB_n^{\Sigma}(\mathbf{X}) = \bigoplus_{x_0,\ldots,x_n \in \mathbf{X}} \Sigma\mathbf{X}(x_0, x_1) \otimes \cdots \otimes \Sigma\mathbf{X}(x_{n-1}, x_n)$$

with face maps δ^i induced by composition in $\mathbf{X}$ and terminal maps in $\mathcal{V}$.

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E.g. If $\Sigma = \mathbb{Z} \cdot \text{-} : \mathbf{Set} \to \mathbf{Ab}$, for $\mathbf{X} \in \mathbf{Cat}$, $MB_n^{\Sigma}(\mathbf{X}) = \mathbb{Z} \cdot \{x_0 \to x_1 \to \cdots \to x_n \text{ in } \mathbf{X}\}$.

Magnitude homology

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The **magnitude complex** of $\mathbf{X}$ has $MC_n^{\Sigma}(\mathbf{X}) = MB_n^{\Sigma}(\mathbf{X})$, with boundary maps

$$\partial_n = \sum_{i=0}^{n} (-1)^i \delta^i : MC_n^{\Sigma}(\mathbf{X}) \to MC_{n-1}^{\Sigma}(\mathbf{X}).$$

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E.g. For $\mathbf{X} \in \mathbf{Cat}$, $MH_\bullet(\mathbf{X})$ is the homology of the classifying space $B\mathbf{X}$.

Magnitude homology categorifies magnitude

Theorem (Leinster & Shulman, 2017)

Under finiteness conditions, MH^{Σ} categorifies magnitude for $\mathcal{V}$-categories:

$$\chi(MH^{\Sigma}(\mathbf{X})) := \sum_{i \geqslant 0} (-1)^i \mathrm{rk}(MH_i^{\Sigma}(\mathbf{X})) = \mathrm{Mag}_{\#}(\mathbf{X}).$$

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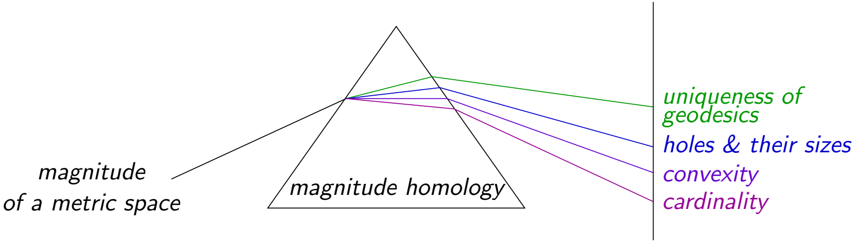
Theorem (Leinster & Shulman, 2017; Kaneta & Yoshinaga, 2018)



Part II

Iterating magnitude homology

Iterating magnitude

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For bicategories, see Tanaka (2014)

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The magnitude nerve as a size functor

Proposition

The magnitude nerve defines a strong symmetric monoidal functor

$$MB^{\Sigma} : (\mathcal{V}\mathbf{Cat}, \otimes_{\mathcal{V}}) \to ([\triangle^{\mathrm{op}}, \mathbb{A}], \otimes_{pw}).$$

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Corollary (Künneth Theorem)

Suppose $\mathbb{A} = \mathbf{Mod}_R$ *for* R *a P.I.D., and let* $\mathbf{X} \in \mathcal{V}\mathbf{Cat}$ *be such that* $MC^{\Sigma}_{\bullet}(\mathbf{X})$ *is flat. Then given any* $\mathcal{V}$*-category* $\mathbf{Y}$, *there is for each* $n \in \mathbb{N}$ *a natural short exact sequence*

$$0 \to \bigoplus_k MH^{\Sigma}_k(\mathbf{X}) \otimes_R MH^{\Sigma}_{n-k}(\mathbf{Y}) \to MH^{\Sigma}_n(\mathbf{X} \otimes_{\mathcal{V}} \mathbf{Y})$$
$$\to \bigoplus_k \mathrm{Tor}\left(MH^{\Sigma}_k(\mathbf{X}), MH^{\Sigma}_{n-k-1}(\mathbf{Y})\right) \to 0.$$

The sequence splits, but the splitting is not natural.

Iterating magnitude homology

$$\begin{tikzcd}
2\mathcal{V}\mathbf{Cat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rr, bend right, "MB^2"'] & {[\triangle^{\mathrm{op}} \times \triangle^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "\mathrm{diag}"] & {[\triangle^{\mathrm{op}}, \mathbb{A}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbb{A}) \arrow[r, "H_\bullet"] & \mathbb{A}^{\mathbb{N}}
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Definition

The **(iterated) magnitude nerve** of a $\mathcal{V}\mathbf{Cat}$-category $\mathbf{X}$ is

$$MB^2(\mathbf{X}) = \mathrm{diag}\left(MB^{MB^{\Sigma}}(\mathbf{X})\right).$$

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When $\mathcal{V} = \mathbf{Set}$, repeated iteration lets us define $MH^n_\bullet : n\mathbf{Cat} \to \mathbf{Ab}^{\mathbb{N}}$.

The classifying space of a 2-category $\mathbf{X}$

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The Segal approach

take the classifying space
of each hom-category in $\mathbf{X}$

$$\begin{tikzcd}
\mathbf{2Cat} \arrow[d, "B_*"] \arrow[ddd, bend left, "B"] \\
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Call $B\mathbf{X}$ the **classifying space** of $\mathbf{X}$.

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The Duskin or Street approach

Define a simplicial set $\triangle\mathbf{X}$ by

$[n] = (0 \to 1 \to \cdots \to n)$

$$\triangle\mathbf{X}_n = \mathbf{BiCat}_{\mathrm{NLax}}([n], \mathbf{X}).$$

bicategories and
normal lax 2-functors

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\mathbf{2Cat} \arrow[d, "B_*"] \arrow[ddd, bend left=60, "B"] \\
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bicategories and *normal* lax 2-functors

Call the topological space $|\triangle\mathbf{X}|$ the **classifying space** of $\mathbf{X}$.

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Theorem (Bullejos & Cegarra, 2003) There's a natural equivalence $B\mathbf{X} \simeq |\triangle\mathbf{X}|$.

MH^2 categorifies iterated magnitude

Lemma

For any 2-category $\mathbf{X}$, $MH^2(\mathbf{X})$ *is the homology of the classifying space of* $\mathbf{X}$.

Proof is via the description of $B\mathbf{X}$ in Bullejos & Cegarra (2003). $\square$

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Theorem
For any finite enough 2-category $\mathbf{X}$ *we have* $\chi(MH^2(\mathbf{X})) = \mathrm{Mag}_{\mathrm{Mag}}(\mathbf{X})$.

Proof Tanaka (2014) showed $\chi(|\triangle\mathbf{X}|) = \mathrm{Mag}_{\mathrm{Mag}}(\mathbf{X})$. Combine with the lemma. $\square$

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Theorem
For any finite enough locally metric category $\mathbf{X}$ we have $\chi(MH^2(\mathbf{X})) = \mathrm{Mag}_{\mathrm{Mag}}(\mathbf{X})$.

Proof uses facts about spectral sequences, plus linear algebra. □

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Part III

The magnitude homology of a locally metric category

The magnitude nerve of a locally metric category

$$\begin{tikzcd}
\mathbf{MetCat} \arrow[r, "MB^{MB^{\Sigma}}"] \arrow[rr, bend right, red, "MB^2"'] \arrow[rrrr, bend left, "MH^2"] & {[\triangle^{\mathsf{op}} \times \triangle^{\mathsf{op}}, \mathbf{Ab}^{\mathbb{R}_+}]} \arrow[r, "\mathsf{diag}"] & {[\triangle^{\mathsf{op}}, \mathbf{Ab}^{\mathbb{R}_+}]} \arrow[r, "C"] & \mathrm{Ch}(\mathbf{Ab}^{\mathbb{R}_+}) \arrow[r, "H_\bullet"] & \mathbf{Ab}^{\mathbb{R}_+ \times \mathbb{N}}
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$$

For a **Met**-category $\mathbf{X}$ the magnitude nerve is given in degrees $n \in \mathbb{N}$ and $\ell \in \mathbb{R}_+$ by

$$
\mathbb{Z} \cdot \left\{
\begin{tikzcd}
x_0 \arrow[r, bend left, "f_{00}"] \arrow[r, "f_{01}"] \arrow[r, bend right, "f_{0n}"'] & x_1 \arrow[r, bend left, "f_{10}"] \arrow[r, "f_{11}"] \arrow[r, bend right, "f_{1n}"'] & x_2 \cdots x_{n-1} \arrow[r, bend left, "f_{n-1,0}"] \arrow[r, "f_{n-1,1}"] \arrow[r, bend right, "f_{n-1,n}"'] & x_n
\end{tikzcd}
\;\middle|\; \sum_{p=0}^{j-1} \sum_{q=0}^{k-1} d(f_{pq}, f_{p,q+1}) = \ell \right\}.
$$

Gaps in a locally metric category

A **gap of width ℓ** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\updownarrow \ell>0"] & y \end{tikzcd}$$

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$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \arrow[r, "h"] & z \end{tikzcd}$$

A pair is reducible if

- we can tighten it by composing with another arrow
  - if (f, g) can't be tightened, call it tight

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$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, dashed] \arrow[r, phantom, "\ell"] & y \end{tikzcd}$$

- or we can bridge it with an arrow strictly between f and g
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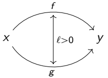
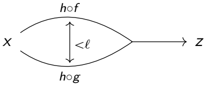
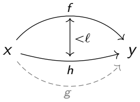
A **gap of width ℓ** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows:







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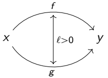

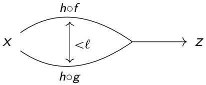

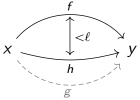

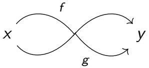

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- we can tighten it by composing with another arrow
  - if (f, g) can't be tightened, call it tight
- or we can bridge it with an arrow strictly between f and g
  - if (f, g) can't be bridged, call it adjacent
- or we can split it into two strictly smaller pairs
  - if (f, g) can't be split, call it simple.

Gaps in a locally metric category

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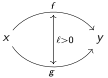

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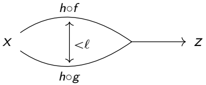

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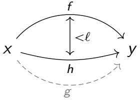

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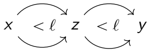

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A **gap of width ℓ** in $\mathbf{X}$ is an equivalence class of irreducible pairs of arrows:
$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell>0"] & y \end{tikzcd}$$

A pair is reducible if

$$\begin{tikzcd} x \arrow[r, bend left, no head, "h\circ f"] \arrow[r, bend right, no head, "h\circ g"'] \arrow[r, phantom, "<\ell"] & \cdot \arrow[r] & z \end{tikzcd}$$

- we can tighten it by composing with another arrow
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$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, "h"'] \arrow[r, bend right=60, dashed, "g"'] \arrow[r, phantom, bend left=25, "<\ell"] & y \end{tikzcd}$$

- or we can bridge it with an arrow strictly between f and g
  - if (f, g) can't be bridged, call it adjacent

$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "<\ell"] & z \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "<\ell"] & y \end{tikzcd}$$

- or we can split it into two strictly smaller pairs
  - if (f, g) can't be split, call it simple.

A gap is a class of simple, tight, adjacent pairs under the equivalence relation gen'd by

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, bend left, no head, "h\circ f"] \arrow[r, bend right, no head, "h\circ g"'] \arrow[r, phantom, "\ell"] & \bullet \arrow[r] & z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} w \arrow[r, no head] & \bullet \arrow[r, bend left, "f\circ k"] \arrow[r, bend right, "g\circ k"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd} \sim \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\ell"] & y \end{tikzcd}$$

The magnitude homology of a locally metric category

Theorem

Let $\mathbf{X}$ *be a locally metric category in which all the hom-spaces are separated.*

In real grading 0*, the magnitude homology of* $\mathbf{X}$ *is the homology of its underlying ordinary category* $\underline{\mathbf{X}}$*:*

$$MH^0_\bullet(\mathbf{X}) \cong H_\bullet(\underline{\mathbf{X}}).$$

In real gradings $\ell > 0$*, the first three magnitude homology groups are given by*

$$MH^\ell_k(\mathbf{X}) \cong \begin{cases} 0 & k = 0, 1 \\ \mathbb{Z} \cdot \{\textit{gaps of width } \ell \textit{ in } \mathbf{X}\} & k = 2. \end{cases}$$

Questions

more speculative

1. What do other choices of $\Sigma : \mathbb{R}_+ \to \mathbb{A}$ give us? For instance, those related to persistent homology—see Otter (2018).
2. Can the iterated magnitude nerve be adapted to give a sensible nerve for approximate categorical structures?
3. If so, can homological tools be used to distinguish locally metric categories among approximate categorical structures?
4. If so, might such tools help detect categorical structure in real-world systems?

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