Scalars and Traces in Bicategories

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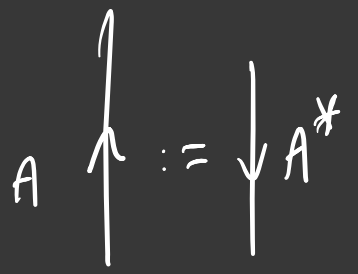
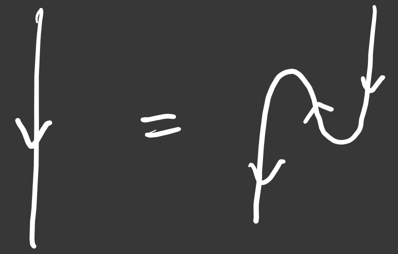
Category Theory and Linear Algebra

	Prof	$\mathbf{FDVect}_{\mathbb{C}}$
Objects	Categories	FD Vector Spaces
Morphisms	$P : B \nrightarrow C \quad P : C^{op} \times B \longrightarrow \mathbf{Set}$	$F : V \longrightarrow W$ OR $F : W^* \otimes V \longrightarrow \mathbb{C}$
Composition	$B \overset{P}{\nrightarrow} C \overset{Q}{\nrightarrow} D$ | $Q \circ P(d, b) = \int^{c \in C} Q(d,c) \times P(c,b)$	$V \xrightarrow{F} W \xrightarrow{G} X$ | $G \circ F(x^* \otimes v) = \sum_{w \in \mathcal{B}_W} G(x^* \otimes w) \cdot F(w^* \otimes v)$
Tensor, unit	$\times, \ *$	$\otimes, \ \mathbb{C}$
Scalars	$\mathbf{Prof}(*, *) \cong \mathbf{Set}$ as monoidal cat.	$\mathbf{FDVect}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ as a monoid
"Inner Product"	$B \overset{P}{\underset{Q}{\rightrightarrows}} C$, $\ \mathrm{Nat}(P, Q)$	$V \overset{F}{\underset{G}{\rightrightarrows}} W$, $\ \langle F, G \rangle$ $\mathrm{Tr}(M_G \cdot \overline{M_F}^T) = \langle F, G \rangle$

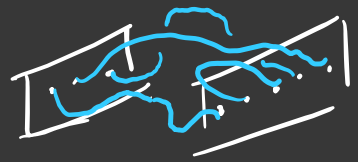
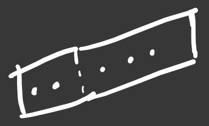
Compact Closed Categories

A sym. monoidal category is compact closed if every object A has a dual A^*.

	$A^* \otimes A \xrightarrow{ev} 1$	$1 \xrightarrow{coev} A \otimes A^*$	+ naturality.
			

- $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ is functorial, so $X \xrightarrow{f} Y$ has transpose $Y^* \xrightarrow{f} X^*$
- $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ is involutive
- $(-)^* : \mathcal{C}^{op} \to \mathcal{C}$ gives closure by $[A, B] := A^* \otimes B$, and coclosure

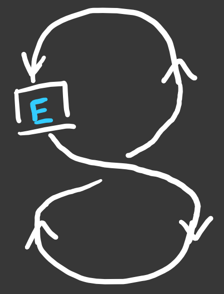
$FDVect_{\mathbb{C}}$ is a compact category, Prof is a compact bicategory

Bicategory	Objects	Homs	2-Homs	Tensor	Duals
V-Prof	V-Categories	$P: Y^{op} \otimes X \to V$	Natural Transformations	$\otimes$	$(-)^{op}$
$\mathrm{Bim}_{\mathbb{C}}$	Algebras/$\mathbb{C}$	${}_A M_B$	Bimodule maps	$\otimes_{\mathbb{C}}$	$(-)^{op}$
DG-$\mathrm{Bim}_{\mathbb{C}}$	— " —	— " —	— " —	— " —	— " —
Span	Sets	$X \leftarrow S \rightarrow Y$	$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, dashed] & Y \\ X & T \arrow[l] \arrow[u] \end{tikzcd}$	$\times$	Id on objects Reflects morphisms
FMKer	$\mathbb{C}$-manifolds	$\begin{array}{c} \mathcal{E}^{\bullet} \\ \downarrow \\ X \times Y \end{array}$ OR nice $D(X) \to D(Y)$	Morphisms in $D(X \times Y)$ OR Nat trans.	$\times$	Reflects morphisms
2-Tang			Cobordisms (embedded in 4D space)		Reflects morphisms

Traces in Compact closed categories

In any comp. closed category an endom. $E : A \to A$ has a trace.

$$1 \xrightarrow{coev} A \otimes A^* \xrightarrow{E \otimes id} A \otimes A^* \xrightarrow{\sim} A^* \otimes A \xrightarrow{ev} 1$$

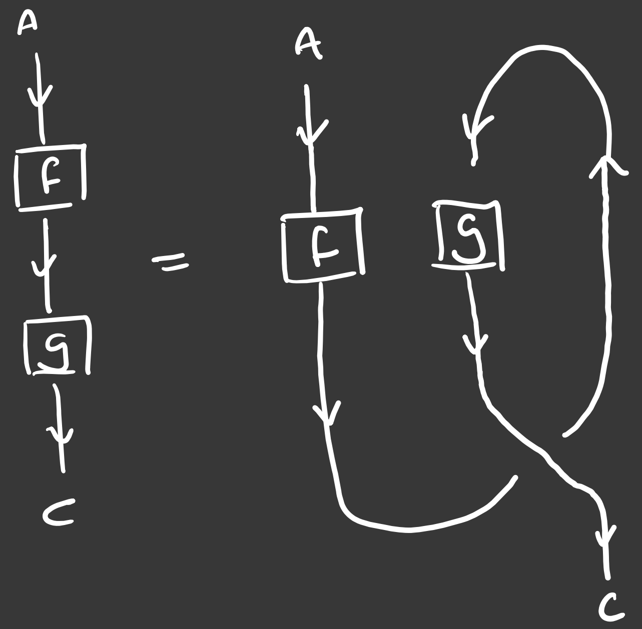


We call this the "round trace", $Tr^{\circlearrowleft}$, and composition can be expressed via the round trace.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\|$$

$$A \xrightarrow{coev} A \otimes B \otimes B^* \xrightarrow{f \otimes g} B \otimes C \otimes B^* \xrightarrow{\sim} B \otimes B^* \otimes C \xrightarrow{ev} C$$



Traces via Inner Products

For an endomorphism $E : A \to A$ in $\mathrm{FDVect}_{\mathbb{C}}$ $\quad \langle id_A, E\rangle = Tr(I_A^T \cdot E) = Tr(E)$

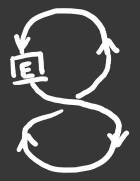
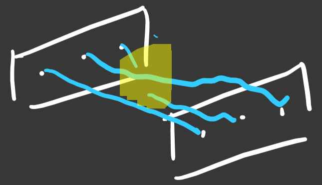
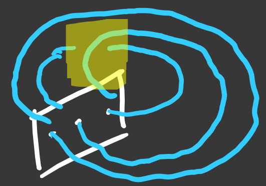
In the Bicategory Prof what happens when we take $2\text{-Hom}(id_A, E)$?

For an endoprofunctor $E : \mathcal{C}^{op} \times \mathcal{C} \to Set$

$$Tr^{\circlearrowleft}(E) = \int^{c \in \mathcal{C}} E(c,c) \in Set$$

$$Tr^{\vee}(E) = 2\text{-Hom}(id, E) = \int_{c \in \mathcal{C}} E(c,c) \in Set$$

Our other examples don't have "inner products" but let's see what happens.

Bicategory	Endomorphism, E	$\mathrm{Tr}^{\circlearrowleft} :=$ 	$\mathrm{Tr}^{\downarrow} := \text{2-Hom}(\mathrm{id}, E)$
V-Prof	$E : \mathcal{C}^{op} \otimes \mathcal{C} \to V$	$\int^{c \in \mathcal{C}} E(c,c)$	$\int_{c \in \mathcal{C}} (E(c,c))_0$
$\mathrm{Bim}_{\mathbb{C}}$	${}_A M_A$	$M / \langle ma - am \rangle$ i.e. coinvariants	$\{ m \in M \mid am = ma \}$ i.e. invariants
DG-$\mathrm{Bim}_{\mathbb{C}}$	${}_{A^\bullet} M^\bullet {}_{A^\bullet}$	$HH_\bullet(A^\bullet, M^\bullet)$	$HH^\bullet(A^\bullet, M^\bullet)$
Span	$X \xleftarrow{f} S \xrightarrow{g} X$	$Gph(f) \cap Gph(g)$	$\{$sections $X \xrightarrow{\sigma} eq(f,g)\}$
FMKer	$\mathcal{E}^\bullet \in D(X \times X)$	$HH_\bullet(X, \mathcal{E}^\bullet)$	$HH^\bullet(X, \mathcal{E}^\bullet)$
2-Tang			?

$\mathrm{Tr}^{\downarrow}$ is forgetting structure.

Should 2-Hom live in $\mathcal{B}(1,1)$?

I.e. should $\mathcal{B}(A,B)$ be a $\mathcal{B}(1,1)$-category?

Is $\mathcal{B}$ always canonically enriched in $\mathcal{B}(1,1)$-cat?

Closure in Bicategories

What do all of our examples have in common?

Monoidal closed

$$\mathrm{Hom}(B \otimes A, C) \cong \mathrm{Hom}(A, [B, C])$$

e.g. for bimodules ${}_A M_B$, ${}_C N_B$

Composition closed

$$2\text{-}\mathrm{Hom}(g \circ f, h) \cong 2\text{-}\mathrm{Hom}(f, g \rhd h)$$

$$M \rhd N = {}_C \mathrm{Hom}(M, N)_A$$

$$\begin{tikzcd}
& C & \\
B \arrow[ur, "g"] \arrow[rr, phantom, "\Rightarrow"] & & A \arrow[ul, "h"'] \arrow[ll, dashed, "g \rhd h"'] \arrow[ll, bend left=60, "f"] \arrow[ll, phantom, bend left=30, "\Uparrow"]
\end{tikzcd}$$

It's a right Kan lift!

Prop If $\mathcal{B}$ compact closed bicategory w/ Kan lifts then $\mathcal{B}$ has Kan extensions

$$\begin{tikzcd}
& A \arrow[dl, "g"'] \arrow[dr, "f"] \arrow[d, phantom, "\Rightarrow"] & \\
B \arrow[rr, "f \lhd g"'] & {} & C
\end{tikzcd}$$

$$f \lhd g \cong (g^* \rhd f^*)^*.$$

Categories of Scalars

$\mathcal{B}(1,1)$ is a nice category to enrich in!

Lemma (R.)

If $\mathcal{B}$ is a monoidal bicategory with Kan lifts then its category of scalars, $\mathcal{B}(1,1)$, is symmetric monoidal closed.

Proof:

Symmetry

Eckmann-Hilton:

$$1 \cong 1 \otimes 1 \xrightarrow{f \otimes g} 1 \otimes 1 \cong 1$$

$$1 \xrightarrow{f} 1 \xrightarrow{g} 1$$

and functoriality of $\otimes$ gives interchange law

Closure

$[f,g] = f \rhd g.$

Scalar Enrichment

Theorem (R.)

Every closed monoidal bicategory $\mathcal{B}$ with Kan lifts is naturally 2-enriched over its category of scalars i.e. $\mathcal{B}$ has "inner products".

Proof

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & {[A,B]} & \\ 1 \arrow[ur, "\bar{f}"] & & 1 \arrow[ul, "\bar{g}"'] \arrow[ll, "\bar{f} \rhd \bar{g}"] \end{tikzcd}$$

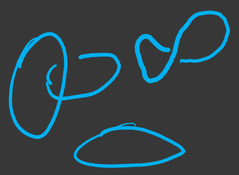
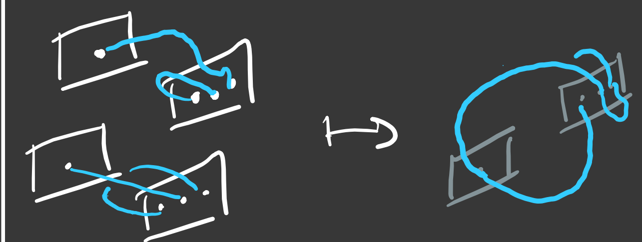
$$\bar{f} \rhd \bar{g} =: \underline{2\text{-Hom}}(f,g)$$

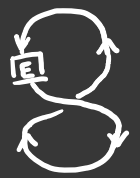
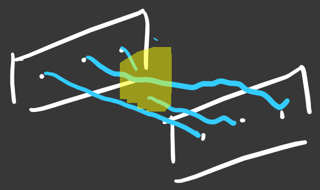
Composition etc. follows from universal properties

If $\mathcal{B}$ compact

$$\begin{tikzcd} & * \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \otimes B^* \arrow[rr] & & A \otimes B^* \end{tikzcd}$$

$$g \lhd f \cong \bar{f} \rhd \bar{g}$$

Bicategory	2-Homs	Category of Scalars	Scalar Enrichment
V-Prof	Natural Transform.	$\{*^{op}\otimes * \to V\} \simeq V$	Natural transformation objects
$\mathrm{Bim}_{\mathbb{C}}$	Bimodule maps	$\{{}_{\mathbb{C}}M_{\mathbb{C}}\} \simeq \mathrm{Vect}_{\mathbb{C}}$	Pointwise vector space structure on bimodule maps
DG-$\mathrm{Bim}_{\mathbb{C}}$	Bimodule maps	$\{{}_{\mathbb{C}}M^{\bullet}_{\mathbb{C}}\} \simeq \text{DG-}\mathrm{Vect}_{\mathbb{C}}$	Pointwise DG-vector space structure on bimodule maps
Span	$\begin{array}{ccc} S & \longrightarrow & Y \\ \downarrow & \searrow & \uparrow \\ X & \longleftarrow & T \end{array}$	$\{ * \leftarrow S \rightarrow * \} \simeq \mathrm{Set}$	Sets of morphisms
FMKer	Derived maps	$D(*) \simeq G\text{-}\mathrm{Vect}_{\mathbb{C}}$	Graded vector space of derived maps
2-Tang	4D-cobordisms	closed 1-manifolds e.g. 	

Bicategory	Endomorphism, E	$\mathrm{Tr}^{\circlearrowleft} :=$ 	$\mathrm{Tr}^{\searrow} := \underline{\text{2-Hom}}(\mathrm{id}, E)$	
V-Prof	$E : \mathcal{C}^{op} \otimes \mathcal{C} \to V$	$\int^{c \in \mathcal{C}} E(c,c)$	$\int_{c \in \mathcal{C}} E(c,c)$	$\in V$
$\mathrm{Bim}_{\mathbb{C}}$	${}_A M_A$	$M / \langle ma - am \rangle$ i.e. coinvariants	$\{ m \in M \mid am = ma \}$ i.e. invariants	$\in \mathrm{Vect}_{\mathbb{C}}$
$\mathrm{DG}-\mathrm{Bim}_{\mathbb{C}}$	${}_{A^\bullet} M^\bullet {}_{A^\bullet}$	$HH_\bullet(R^\bullet, M^\bullet)$	$HH^\bullet(R^\bullet, M^\bullet)$	$\in \mathrm{DG}-\mathrm{Vect}_{\mathbb{C}}$
Span	$X \xleftarrow{f} S \xrightarrow{g} X$	$\mathrm{Gph}(f) \cap \mathrm{Gph}(g)$	$\{ \text{sections } X \xrightarrow{\sigma} \mathrm{eq}(f,g) \}$	$\in \mathrm{Set}$
FMKer	$\mathcal{E}^\bullet \in D(X \times X)$	$HH_\bullet(X, \mathcal{E}^\bullet)$	$HH^\bullet(X, \mathcal{E}^\bullet)$	$\in G-\mathrm{Vect}_{\mathbb{C}}$
2-Tang				$\in$ closed 1-manifolds

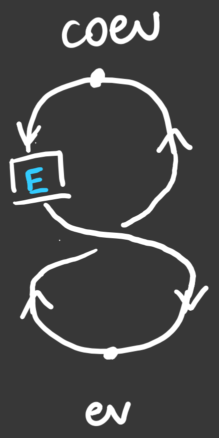
Comparing the two traces

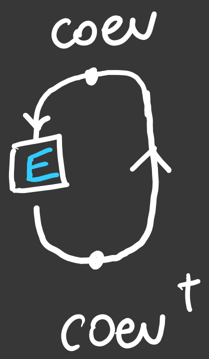
Recall that $\mathrm{Tr}^{\circlearrowleft}(E) := \underbrace{1 \xrightarrow{coev} A\otimes A^* \xrightarrow{E\otimes id} A\otimes A^*}_{\overline{E}} \xrightarrow{\sim} \underbrace{A^*\otimes A \xrightarrow{ev} 1}_{\underline{id}}$

So $\mathrm{Tr}^{\circlearrowleft}(E) \cong \underline{id} \circ \overline{E} \cong \underline{E} \circ \overline{id}$

BUT $\mathrm{Tr}^{\searrow}(E) \cong \overline{id} \triangleright \overline{E} \cong \underline{E} \triangleleft \underline{id}$

In fact if $\overline{id}$ has a right adjoint $\overline{id}^{\dagger}$ then $\mathrm{Tr}^{\searrow}(E) = \overline{id}^{\dagger} \circ \overline{E}$!

$\mathrm{Tr}^{\circlearrowleft}(E) =$ 

$\mathrm{Tr}^{\searrow}(E) =$ 

$$\underline{\text{2-Hom}}(f,g) = \underline{\text{2-Hom}}(id, f \triangleright g) = \mathrm{Tr}^{\searrow}(f \triangleright g) = \mathrm{Tr}^{\searrow}(f^{\dagger} \circ g)$$

$$\langle M, N \rangle = \mathrm{Tr}(\overline{M}^{T} \cdot N)$$

Dimension

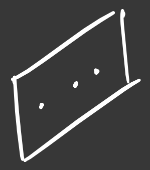
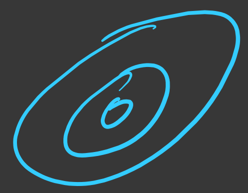
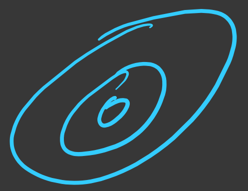
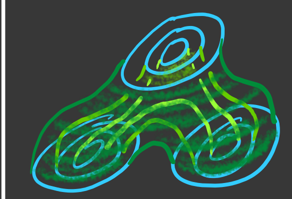
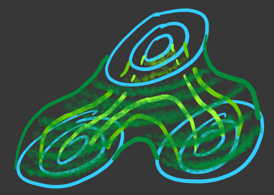
For a vector space V, $\quad \mathrm{Tr}(id_V) = 1 + 1 + \ldots + 1 = \mathrm{Dim}(V)$

For object $A \in \mathcal{B}$ define $\mathrm{Dim}^{\downarrow}(A) = \mathrm{Tr}^{\downarrow}(id_A)$ and $\mathrm{Dim}^{\circlearrowleft}(A) = \mathrm{Tr}^{\circlearrowleft}(id_A)$.

<u>Properties</u>

- $\mathrm{Dim}^{\downarrow}(A) = coev_A \triangleright coev_A$ which gives the codensity monad on $\mathbb{1}$.
  Thus $\mathrm{Dim}^{\downarrow}(A)$ is a monoid object in $\mathcal{B}(\mathbb{1}, \mathbb{1})$.

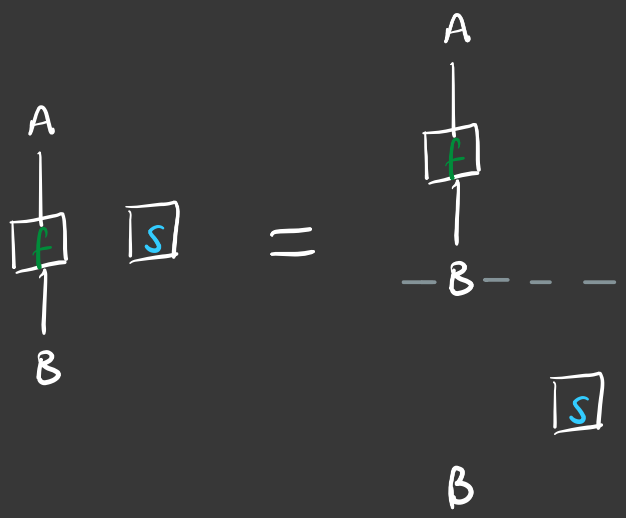
- $\mathrm{Dim}^{\circlearrowleft}(A) = ev_A \circ coev_A$ so
  $\mathrm{Dim}^{\circlearrowleft}(A) \circ \mathrm{Dim}^{\downarrow}(A) = ev_A \circ \cancel{coev_A} \circ (\cancel{coev_A} \triangleright coev_A) \Rightarrow ev_A \circ coev_A = \mathrm{Dim}^{\circlearrowleft}(A)$
  Thus $\mathrm{Dim}^{\circlearrowleft}(A)$ is a $\mathrm{Dim}^{\downarrow}(A)$ module object in $\mathcal{B}(\mathbb{1}, \mathbb{1})$

Bicategory	Object A	$\mathrm{Dim}^{\downarrow}(A)$	$\mathrm{Dim}^{\circlearrowleft}(A)$	Monoid mult.	Monoid action
V-Prof	V-category	$\int_{a\in A} \mathrm{Hom}(a,a)$	$\int^{a\in A} \mathrm{Hom}(a,a)$	—	—
$\mathrm{Bim}_{\mathbb{C}}$	Algebra	$Z(A)$	$A/[A,A]$	Algebra multiplication	Algebra multiplication
DG-$\mathrm{Bim}_{\mathbb{C}}$	DG-Algebra	$HH^{\bullet}(A^{\bullet})$	$HH_{\bullet}(A^{\bullet})$	Cup product	Cap product
Span	Set	$*$	A	trivial	trivial
FMKer	$\mathbb{C}$-manifold	$HH^{\bullet}(A)$	$HH_{\bullet}(A)$	Cup product	Cap product
2-Tang					

Scalar Action

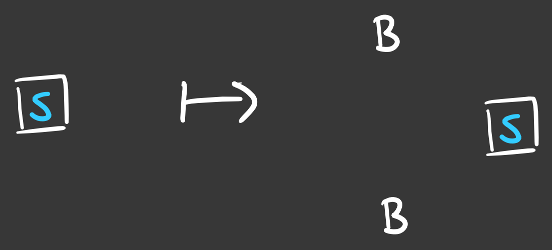
Thm (Kelly, Laplaza '80): In a monoidal category $(\mathcal{C},\otimes,1)$, $\mathcal{C}(1,1)$ is a commutative monoid and every $\mathcal{C}(A,B)$ a $\mathcal{C}(1,1)$-module

Given $s\in\mathcal{C}(1,1)$, $f\in\mathcal{C}(A,B)$ $s\cdot f$ is $A\cong 1\otimes A \xrightarrow{s\otimes f} 1\otimes B\cong B$ but...



we call this map "scalar promotion"

$(-)I_B : \mathcal{C}(1,1)\longrightarrow\mathcal{C}(B,B)$



$\lambda M = \lambda I\cdot M = \begin{pmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{pmatrix}\cdot M$

Scalar Promotion in Bicategories

In a compact closed bicategory with Kan lifts we have two types of scalar promotion.

<u>Diagonal</u>

$$(-)I_A^{\searrow} : \mathcal{B}(1,1) \longrightarrow \mathcal{B}(A,A)$$
$$s \longmapsto s \otimes id_A = \overline{s \circ ev_A}$$

<u>Round</u>

$$(-)I_A^{\circlearrowleft} : \mathcal{B}(1,1) \longrightarrow \mathcal{B}(A,A)$$
$$s \longmapsto \overline{s \triangleright coev_A}$$

$$(-)I_A^{\searrow} \dashv Tr_A^{\searrow} \quad !$$

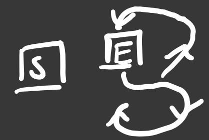
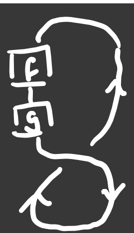
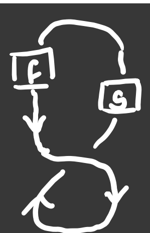
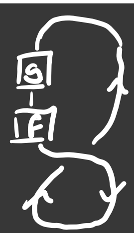
$$Tr_A^{\circlearrowleft} \dashv (-)I_A^{\circlearrowleft} \quad !$$

<u>Lemma (R.):</u> Every scalar-enriched Hom-category $\mathcal{B}(A,B)$ is both powered and copowered

$$\mathcal{B}(1,1)^{op} \otimes \mathcal{B}(A,B) \xrightarrow{(-)I_B^{\searrow}} \mathcal{B}(B,B)^{op} \otimes \mathcal{B}(A,B) \xrightarrow{\triangleright} \mathcal{B}(A,B)$$

$$\mathcal{B}(1,1) \otimes \mathcal{B}(A,B) \xrightarrow{(-)I_B^{\searrow}} \mathcal{B}(B,B) \otimes \mathcal{B}(A,B) \xrightarrow{\circ} \mathcal{B}(A,B)$$

Trace Properties

Vector Spaces	Bicategories
$Tr(sE) = sTr(E)$	$Tr^{\circlearrowleft}(sI^{\searrow} \circ E) = s \circ Tr^{\circlearrowleft}(E)$ } =  $Tr^{\searrow}(sI^{\searrow} \triangleright E) = s \triangleright Tr^{\searrow}(E)$ $Tr^{\searrow}(E \triangleright sI^{\circlearrowleft}) = Tr^{\circlearrowleft}(E) \triangleright s$ } tensor-nom
$Tr(g \circ f) = Tr(f \circ g)$	$Tr^{\circlearrowleft}(g \circ f) =$  $\cong$  $\cong$  $= Tr^{\circlearrowleft}(f \circ g)$
$Tr(g^{-1} \circ f \circ g) = Tr(f)$	$Tr^{\searrow}(g^{\dagger} \circ f \circ g) \Rightarrow Tr^{\searrow}(f)$ $Tr^{\searrow}(g^{-1} \circ f \circ g) \cong Tr^{\searrow}(f)$ } Properties of Homs

Further Work

- More examples!
- Can we reframe FDVect as a bicategory and recover the linear algebra?
- Traced monoidal categories $\underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}}$ compact closed categories (Joyal Street Verity '94)

  Similar result for $\mathrm{Tr}^{\vee}$?

- All the examples so far come from proarrow equipments. Does that structure add anything?
- Characteristic polynomials? (James Cranch)
- Dim in FDVect has image $\mathbb{N} \subseteq \mathbb{R}$, what's the analogue here?

Questions ?

Dimensions for V-Categories

- $V = \bar{\mathbb{R}}_+$ then $\mathrm{Dim}^{\searrow}(\mathcal{C}) = \sup_{x \in X} \mathcal{C}(x,x) = 0 = \inf_{x \in X} \mathcal{C}(x,x) = \mathrm{Dim}^{\circlearrowleft}(\mathcal{C})$

- $V = Ab$ and $\mathcal{C}$ has one object then $\mathcal{C}$ is some ring R

  and $\int_{*} \mathcal{C}(*,*) = Z(R)$, $\int^{*} \mathcal{C}(*,*) = R/[R,R]$

- $V = Set$ then $\int_{c \in \mathcal{C}} \mathcal{C}(c,c) \cong Nat(id_{\mathcal{C}}, id_{\mathcal{C}}) \subseteq Z(\prod_{c \in \mathcal{C}} End(c))$

$$\begin{tikzcd} c \arrow[r, "\varphi_c"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & c \arrow[d, "f"] \\ c \arrow[r, "\varphi_c"'] & c \end{tikzcd}$$

and $\int^{c \in \mathcal{C}} \mathcal{C}(c,c) = \bigcup_{c \in \mathcal{C}} \{c \xrightarrow{\varphi} c\}/\sim$ where $\begin{tikzcd} c \arrow[loop above, "\varphi"] \end{tikzcd} \sim \begin{tikzcd} c' \arrow[loop above, "\varphi'"] \end{tikzcd} \iff \exists \psi : c \to c'$

$$\begin{tikzcd} c \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "="] & c' \arrow[d, "\varphi'"] \\ c \arrow[r, "\psi"'] & c' \end{tikzcd}$$

$\theta \in \mathrm{Dim}^{\searrow}(\mathcal{C})$ and $[\varphi] \in \mathrm{Dim}^{\circlearrowleft}(\mathcal{C})$ then $\theta \cdot \varphi = [\theta \circ \varphi]$.