Let $P_{ij} = \{\text{paths from } i \text{ to } j\}$ $i,j \in X$, $G : X \times X \to R$

The Alg Path Problem computes

$$\sum_{p \in P_{ij}} w(p)$$

Does this exist? Seems sort of arbitrary...

Ex) Let $R = ([0,\infty], \min, +)$ zero $= \infty + a = \infty$

This is the shortest path!

$$P_{ad} = \{ \overset{p_1}{\{(a,b),(b,d)\}}, \overset{p_2}{\{(a,b)\}}, \underbrace{\{(a,c),(c,d)\}}_{p_3} \}$$

non-infinite paths

$w(p_1) = .5 + .2 = .7$

$w(p_2) = .9$

$w(p_3) = .1 + .15$

$\min(.7, .9, .25)$

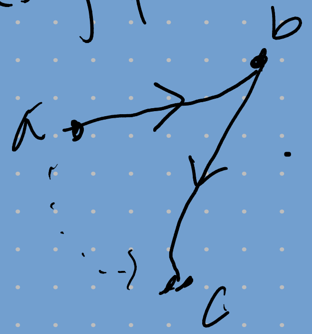
$= .25$

$R = \mathbb{B} = (\{T, F\}, \text{or}, \text{and})$

$T \vee F = T$

$T \wedge F = F$

A $\mathbb{B}$-weighted graph is



$P_{ac} = \{ (a, c), \{a, b) \; (b, c) \}$

F

$T \wedge T$

$= T$

$F \vee T = T$ ... so theyre connected

The algebraic path problem detects connectivity.

There's more

semiring	solutions are...
$[0,\infty], \min, +$	shortest paths
$([0,1], \max, \cdot)$	most likely path in a markov chain
$\mathcal{P}(\Sigma^*), \cup, (\text{concat})$	the language of a NFA
$\mathbb{B}$	Connectivity
$[0,\infty], \max, \min$	tunnel capacities

Where is the CT?

$$G : X \times X \to R \qquad \begin{bmatrix} G(1,1) & & \cdot \\ \cdots & & G(i,j) \\ & & \end{bmatrix}$$

$$G^2(i,j) = \sum_{k \in X} G(i,k)G(k,j)$$

$(i,k),(k,j)$ a path from i to j of length 2. if $R = ([0,\infty], \min, +)$ $G^2(i,j)$ is the shortest path from i to j with length 2. Similarly

$$G^n(i,j) = \sum_{k_1, k_2 \ldots k_n} G(i,k_1) \cdots G(k_n,j)$$

= shortest length n path.

What if you want all the paths?

$$F(G) = \sum_{n \geq 0} G^n \quad \text{with} \quad G^0 = \text{Identity matrix}$$

— its the solution of the alg. path problem.

$$\begin{bmatrix} 0 & & \infty \\ & 0 \quad 0 & \\ \infty & & 0 \end{bmatrix} \text{ in } ([0, \infty_+], \min, +)$$

For ordinary graphs $G = E \overset{s}{\underset{t}{\rightrightarrows}} V$

$$\begin{tikzcd} & & E \times_V E \arrow[dl] \arrow[dr] & & \\ & E \arrow[dl] \arrow[dr] & & E \arrow[dl] \arrow[dr] & \\ V & & V & & V \end{tikzcd}$$

$$E \times_V E = \{(e_1, e_2) \mid t(e_1) = s(e_2)\}$$

$$= \{\text{length 2 paths in } G\}$$

$$E^n = \text{iterated pullback} = \{\text{length } n \text{ paths}\}$$

$$F(G) = \sum_{n \geq 0} G^n \qquad G^0 = V$$

this the free category on G

$$\begin{tikzcd} \mathrm{Grph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Cat} \arrow[l, bend left, "U"] \end{tikzcd}$$

We're doing the same in the R-enriched C.

~~We~~ Let R has a nice $\leq$ s.t.

- $\top$ is max w.r.t. $\leq$
- $\times$ ~~does~~ respects $\leq$ $\quad a \leq b \Rightarrow a \cdot c \leq b \cdot c$
- Arbitrary sums exist
- $\times$ distributes over arb. sums.

Then R is a quantale.
A monoidal closed poset with all joins.

$$\begin{tikzcd} R\text{Graph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & R(a) \arrow[l, bend left, "u"] \end{tikzcd}$$

$\boxed{\text{Realization}}$ The left adjoint $F(G) = \sum_{n \geq 0} G^n$ computes solutions to the alg. path problem.

What's the upshot?

Cospans

ms ⟶ open systems

RGraph we have

Ts given $m : X \times X \to R$,

$f : X \to Y$

$$\begin{tikzcd} X \times X \arrow[rd, "m"] \arrow[dd] & \\ \arrow[r, phantom, "\geq"] & R, \\ Y \times Y \arrow[ru, "n"'] & \end{tikzcd}$$

$f_*(m) \leq n$

$$\boxed{\sum_{(i,j)\in(f\times f)^{-1}(a,b)} m(i,j) \leq n(a,b) \quad \forall a,b \in Y}$$

$$\begin{tikzcd} & m & \\ LX \arrow[ru, "i"] & & LY \arrow[lu, "j"'] \end{tikzcd}$$

where LX, LY are the discrete graphs on X and Y

$$LX \xrightarrow{i} m \xleftarrow{j} LY$$

$LX : X \times X \to R$

$LX(i,j) = 0 \ \forall i,j \in X$

called an open RGraph

$$\begin{tikzcd} & & m +_{LY} n & & \\ & m \arrow[ru, "a"] & & n \arrow[lu, "b"'] & \\ LX \arrow[ru] & & LY \arrow[lu] \arrow[ru] & & LZ \arrow[lu] \end{tikzcd}$$

$m +_{LY} n = a_*(m) + b_*(n)$ ← pointwise sum of matrices

Because left adjoints preserve colimits, ~~you~~ the alg. path problem preserves this gluing*

* pushouts are different in RCat

$m: X \longrightarrow Y, \quad n: Y \longrightarrow Z$

$F(m;n)$ $\sim$ ~~or~~ $\sim$ $F(m);F(n)$

because ; is pushouts comp. formula

$$\boxed{F(m;n) \simeq F(UF(m);UF(n))}$$