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Isbell conjugacy

$X : \mathbb{A}^{op} \to \mathbf{Set}$
$\rightsquigarrow X' : \mathbb{A}^{op} \to \mathbf{Set}$
$\qquad a \mapsto \mathrm{Hom}(\mathbb{A}(-,a), X)$
$\rightsquigarrow X'', X''', \ldots$

$X : \mathbb{A}^{op} \to \mathbf{Set}$
$\rightsquigarrow X^{\vee} : \mathbb{A} \to \mathbf{Set}$ — Isbell conjugate of X
$\qquad a \mapsto \mathrm{Hom}(X, \mathbb{A}(-,a))$
$\rightsquigarrow X^{\vee\vee} : \mathbb{A}^{op} \to \mathbf{Set}$
$\qquad a \mapsto \mathrm{Hom}(X^{\vee}, \mathbb{A}(a,-))$
$\rightsquigarrow X, X^{\vee}, X^{\vee\vee}, X^{\vee\vee\vee}, \ldots$

$\mathbb{A}$ small

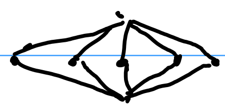
Have functors $[\mathbb{A}^{op}, \mathbf{Set}] \underset{\vee}{\overset{\vee}{\rightleftarrows}} [\mathbb{A}, \mathbf{Set}]^{op}$ ... (*)
forming an adjunction
$$[\mathbb{A}^{op}, \mathbf{Set}](X, Y^{\vee}) \cong [\mathbb{A}, \mathbf{Set}](Y, X^{\vee})$$
$(X : \mathbb{A}^{op} \to \mathbf{Set},\ Y : \mathbb{A} \to \mathbf{Set})$.
In ptc, have $\eta_X : X \to X^{\vee\vee}$, $\eta_Y : Y \to Y^{\vee\vee}$.

A Set-valued functor X is <u>reflexive</u> if $\eta_X : X \to X^{\vee\vee}$ is an iso.

The invariant part of the adjunction (*) consists of (reflexive ftrs $\mathbb{A}^{op} \to \mathbf{Set}$)
$\simeq$ (reflexive ftrs $\mathbb{A} \to \mathbf{Set})^{op} =: R(\mathbb{A})$,
the <u>reflexive completion</u> of $\mathbb{A}$.

E.g. $\mathbb{A}(-,a)^{\vee} = \mathbb{A}(a,-)$ & $\mathbb{A}(a,-)^{\vee} = \mathbb{A}(-,a)$, so $R(\mathbb{A}) \supseteq \mathbb{A}$.

- $R(\emptyset) = 1$
- $R(1) = 1$
- $R($small discrete caty $\mathbb{A}$ with ≥ 2 objects$) \simeq (\mathbb{A}$ with 0 & 1 adjoined$)$



- Group G, nontrivial.
  Then $R(G) = (G$ with init & terminal objs adjoined$)$
  <u>except</u> when $G = C_2$.
  $$R(C_2) \subseteq [C_2^{op}, Set]$$
  consists of:
  - empty C_2-set
  - one-pt C_2-set
  - representable C_2-set
  - the free C_2-set on 2 generators

  (background fact: $C_2 + C_2 \cong C_2 \times C_2$)
- There is a 7-element monoid M s.t. $R(M)$ is large.

Enriched examples:

- Over Ab: for field k as 1-obj Ab-caty,
  $R(k) = $ (reflexive k-modules) $= FinDimVS_k$.
- Over $2 = \{\bot \to \top\}$: for poset A, $R(A) = $ Dedekind-MacNeille completion of A. E.g. $R(\mathbb{Q}) = \mathbb{R} \cup \{\pm\infty\}$.
  In general,
  $$R(A) = \{\text{downwards closed } S \subseteq A : \downarrow\uparrow S = S\}$$
  where
  $$\uparrow S = \{\text{upper bounds of } S\} \quad \text{etc.}$$
- Over $[0,\infty]$: refl completion is related to tight span of metric spaces. (Willerton)

Thm Let $\mathbb{I}$ be a small cat. TFAE:

- $\mathbb{I}$-limits exist in $R(A)$ $\forall$ small cats A
- $\mathbb{I}$ is empty or $\mathbb{I}$-lims are absolute
- $\mathbb{I}$-limits exist in every Cauchy complete caty with a terminal object.

$$\begin{tikzcd} C_2 \arrow[r, shift left, "\Delta"] & C_2 \times C_2 \arrow[l, shift left, "pr_1"] \end{tikzcd}$$

$$\begin{tikzcd} R(C_2) \arrow[r, shift left] & R(C_2 \times C_2) \arrow[l, shift left] \end{tikzcd}$$

$\sim$

Isbell, "Adequate subcats", 1960.

A functor $F : A \to B$ is dense if

$$N_F : B \longrightarrow [A^{op}, Set]$$
$$b \longmapsto B(F-, b)$$

is f&f. A functor $F : A \to B$ is adequate if f&f & dense & codense.

Rough theorem: Let A be a category. Then the Yoneda embedding $A \xrightarrow{J_A} R(A)$ is adequate, and $R(A)$ is the largest caty in which A embeds adequately.

Ignoring smallness, the thm says:

Let A be a caty & $F : A \to B$ an adeq ftr. Then there is an adequate ftr $N(F) : B \to R(A)$ s.t.

$$\begin{tikzcd} B \arrow[rr, dotted, "N(F)"] & & R(A) \\ & A \arrow[ul, "F"] \arrow[ur, "J_A"'] & \end{tikzcd}$$

commutes; moreover, $N(F)$ is unique as such.

Cor $R(R(A)) = R(A)$.

any set!

A functor $X : \mathbb{A}^{op} \to \mathbf{Set}$ is <u>small</u> if it's a small colim of rep'bles.
If X small, $X^\vee$ is well-defined (maybe not small).
X refl $\iff$ X small, $X^\vee$ small, & $\eta_X : X \xrightarrow{\sim} X^{\vee\vee}$.
$\overline{R(\mathbb{A})}$ = (refl ftrs).

Std wise <u>small-adeq</u> ftrs, i.e. adeq F s.t. $\mathbb{B}(F-, b)$ & $\mathbb{B}(b, F-)$ are small $\forall b$.

$[\mathbb{A}^{op} \times \mathbb{A}, \mathbf{Set}]$

$\hat{\mathbb{A}} = [\mathbb{A}^{op}, \mathbf{Set}]$

$\check{\mathbb{A}} = [\mathbb{A}, \mathbf{Set}]^{op}$

$\hat{X}$, $\check{Y}$

$\eta_X : X \longrightarrow X^{\vee\vee}$

$$\begin{aligned} \eta_{X,a} : Xa &\longrightarrow X^{\vee\vee}a = \mathrm{Hom}(X^\vee, \mathbb{A}(a,-)) \\ x &\longmapsto \mathrm{ev}_{x,a} \end{aligned}$$

where

$$\mathrm{ev}_{x,a} : X^\vee \longrightarrow \mathbb{A}(a,-)$$

has b-component

$$\begin{aligned} X^\vee b = \mathrm{Hom}(X, \mathbb{A}(-,b)) &\longrightarrow \mathbb{A}(a,b) \\ \xi &\longmapsto \xi_a(x). \end{aligned}$$