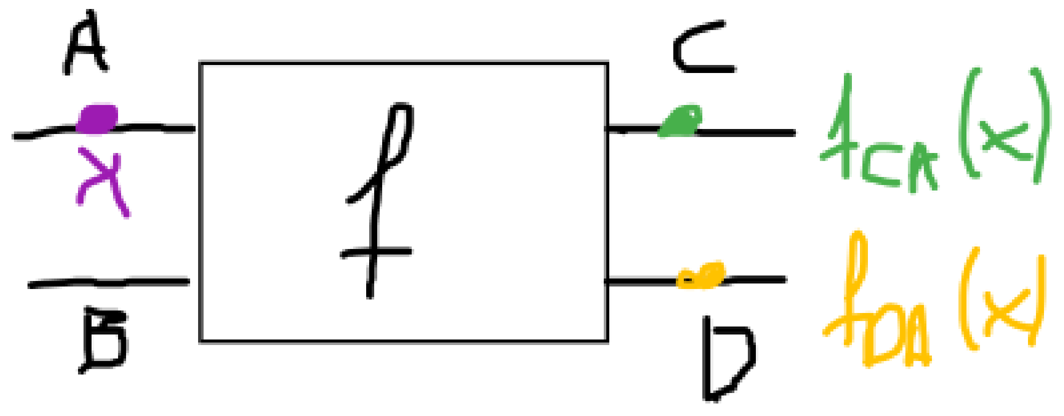
Feedback loops without coproducts

- Haghverdi E. (2000) "Unique decomposition categories, Geometry of Interaction and combinatory logic" Math. Struct. in Comp. Science.
- Hoshino N. (2012) "A representation theorem for unique decomposition categories" Electronic Notes in Theoretical Computer Science.

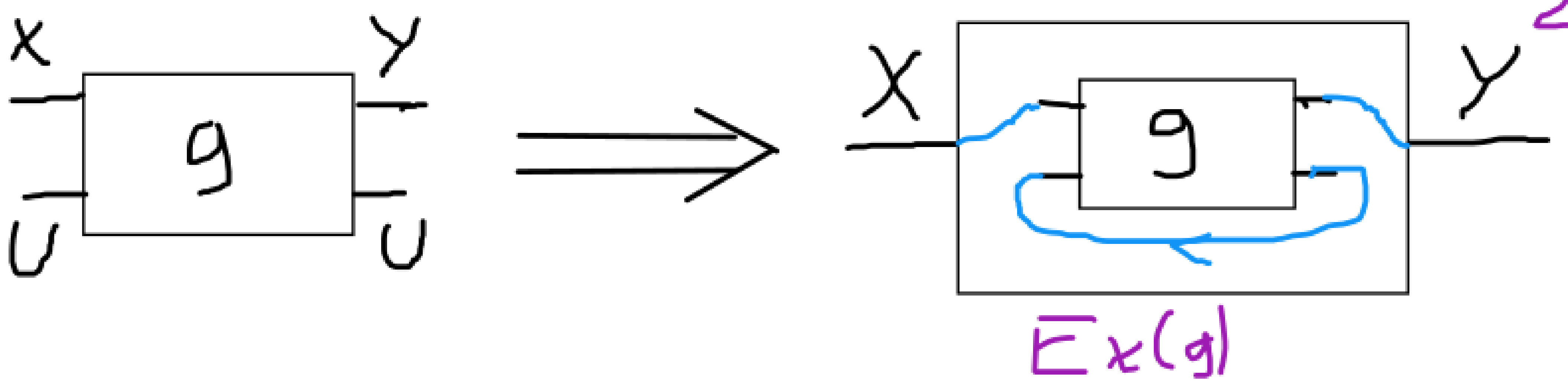
Motivation

Consider a category of "processes". In it, every morphism $f : A \otimes B \longrightarrow C \otimes D$ has a matrix decomposition.

$$f = \begin{pmatrix} f_{CA} & f_{CB} \\ f_{DA} & f_{DB} \end{pmatrix}$$



In this situation, it is reasonable to add feedback loops to morphisms:



$\Sigma \{ g_{yx} , g_{yu} g_{ux} , \dots \}$

$$Ex(g) = g_{yx} + \sum_{n=0}^{\infty} g_{yu} (g_{uu})^n g_{ux}$$

Under certain conditions, this operation on morphisms is called a categorical trace.

- Let's look into concrete examples:

SubStoch sets and substochastic maps

$f : \{a, a'\} \to \{b, b'\}$

$$f = \begin{array}{cc} & \begin{array}{cc} a & a' \end{array} \\ \begin{array}{c} b \\ b' \end{array} & \begin{pmatrix} 0.1 & 0.5 \\ 0.7 & 0.4 \end{pmatrix} \\ & \begin{array}{cc} \le 1 & \le 1 \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccccc} \overbrace{\quad\quad A \quad\quad} & \otimes & \overbrace{\quad\quad B \quad\quad} \end{array} \\ \begin{array}{c} C \\ \otimes \\ D \end{array} & \left(\begin{array}{cc|cc} 0.2 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ \hline 0.6 & 0 & 0.1 & 0.4 \\ 0.1 & 0.2 & 0.4 & 0.2 \end{array}\right) \end{array}$$

$f + g$

partial

$A \longrightarrow A \oplus A$

$\begin{pmatrix} id \\ id \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

PInj sets and partial injections

$f : X \longrightarrow Y$ $\quad X' \subseteq Y$

$X' = \{x \in X \mid f(x) \text{ def}\}$

$f : A \overset{\uplus}{\otimes} B \longrightarrow C \otimes D$

$$f = \begin{pmatrix} f_{CA} & f_{CB} \\ f_{DA} & f_{DB} \end{pmatrix}$$

$(f+g)(x) = \begin{cases} f(x) \\ g(x) \\ \text{undef} \end{cases}$

partial

$A \otimes A \longrightarrow A$

$\langle id, id \rangle$

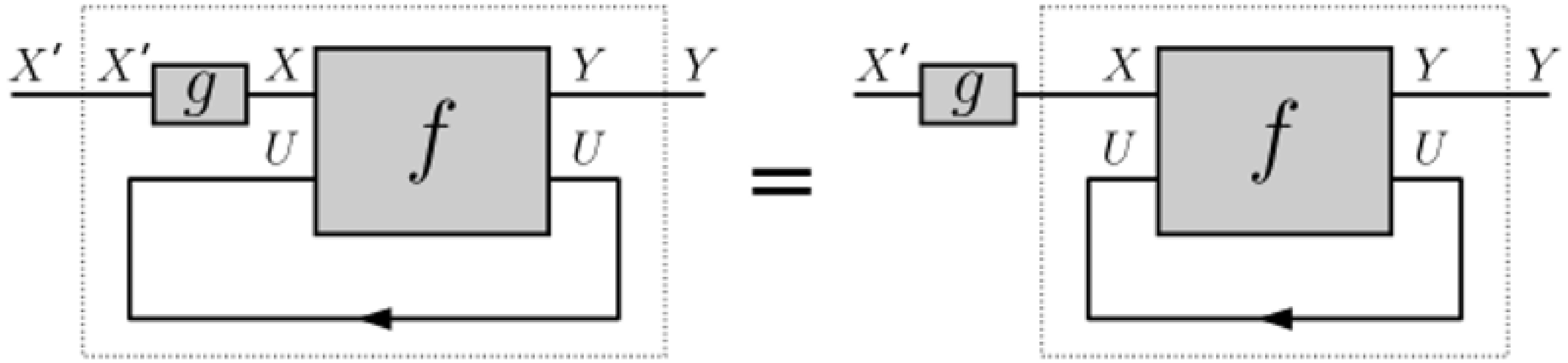
- Essentially, a Unique Decomposition Category (UDC) is a symmetric monoidal category such that it has

1. a (unique) matrix decomposition of morphisms,
2. a (partial) addition on morphisms, such that composition distributes over it,
3. some mild conditions on the symmetric monoidal structure.

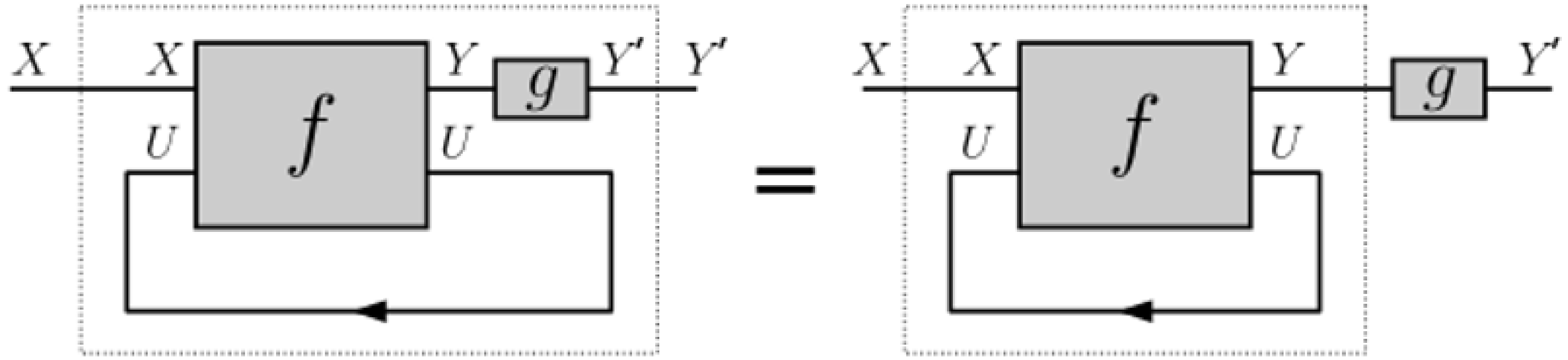
<u>Proposition</u>: In a UDC, if the execution formula is always defined, it is a categorical trace.

<u>Axioms of traced monoidal categories</u>

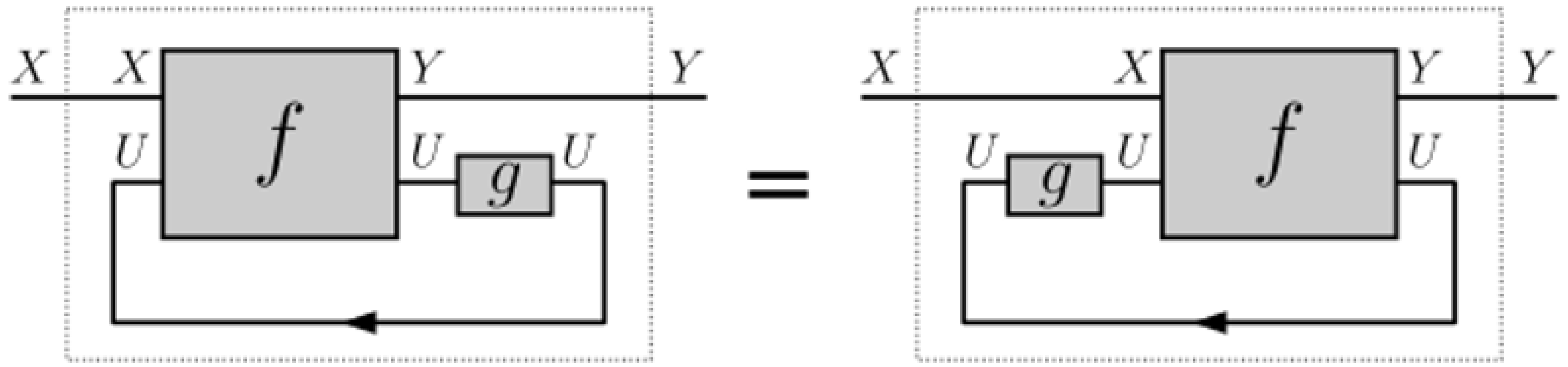
Naturality on X



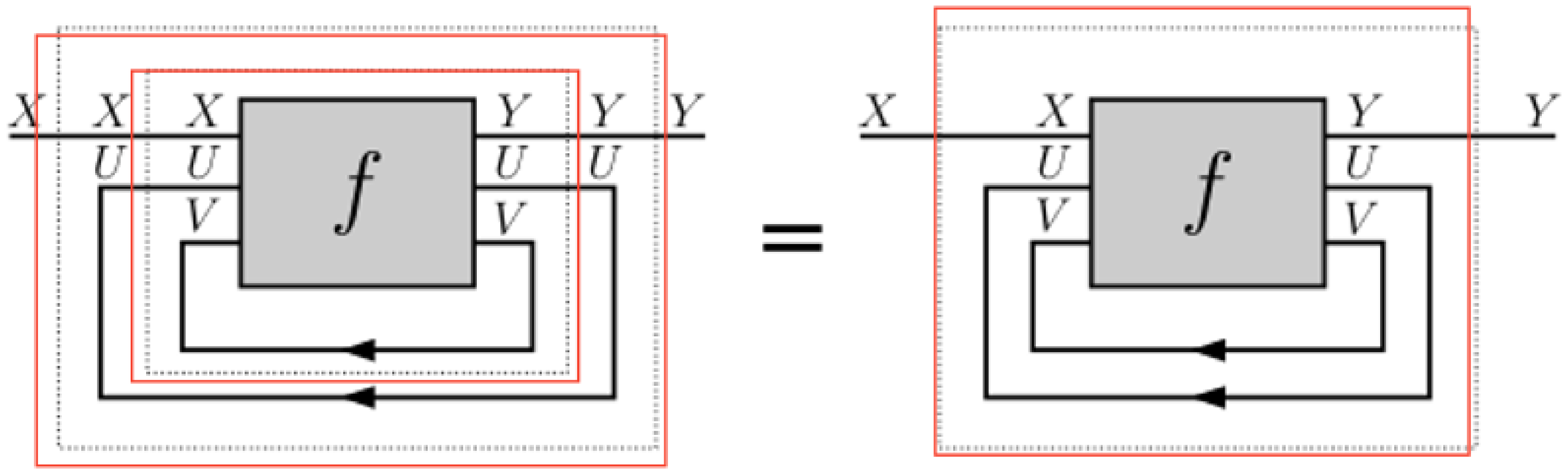
Naturality on Y



Dinaturality



Vanishing II



Axioms of traced monoidal categories

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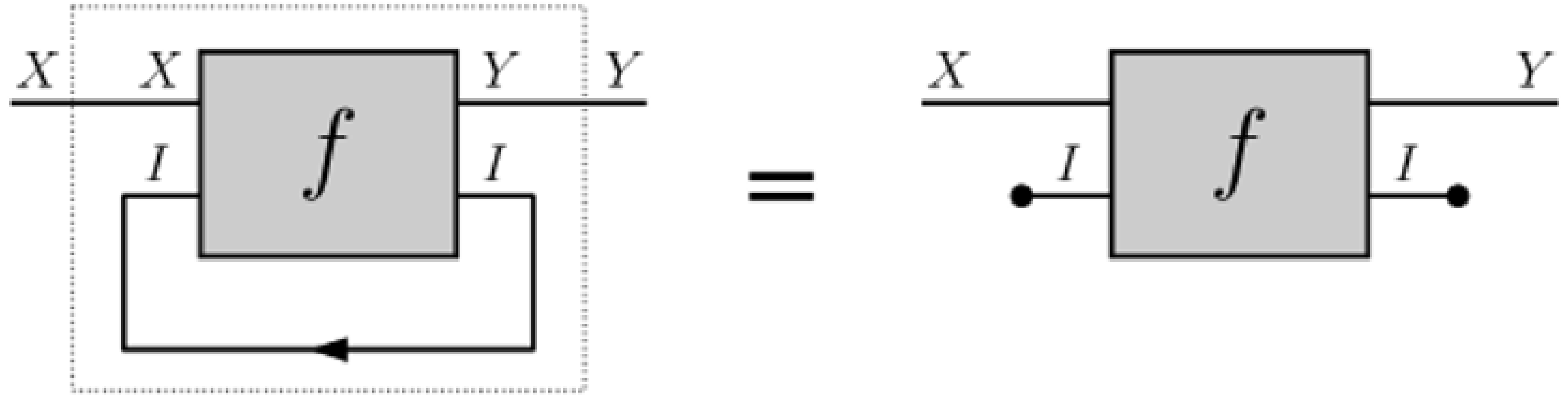
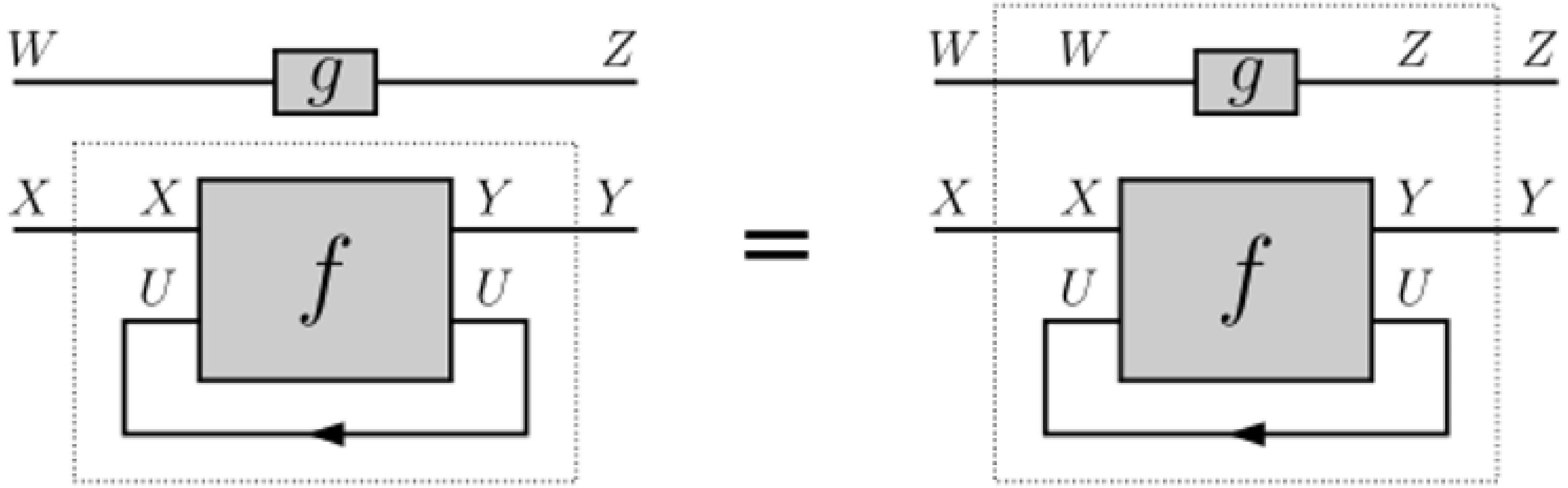
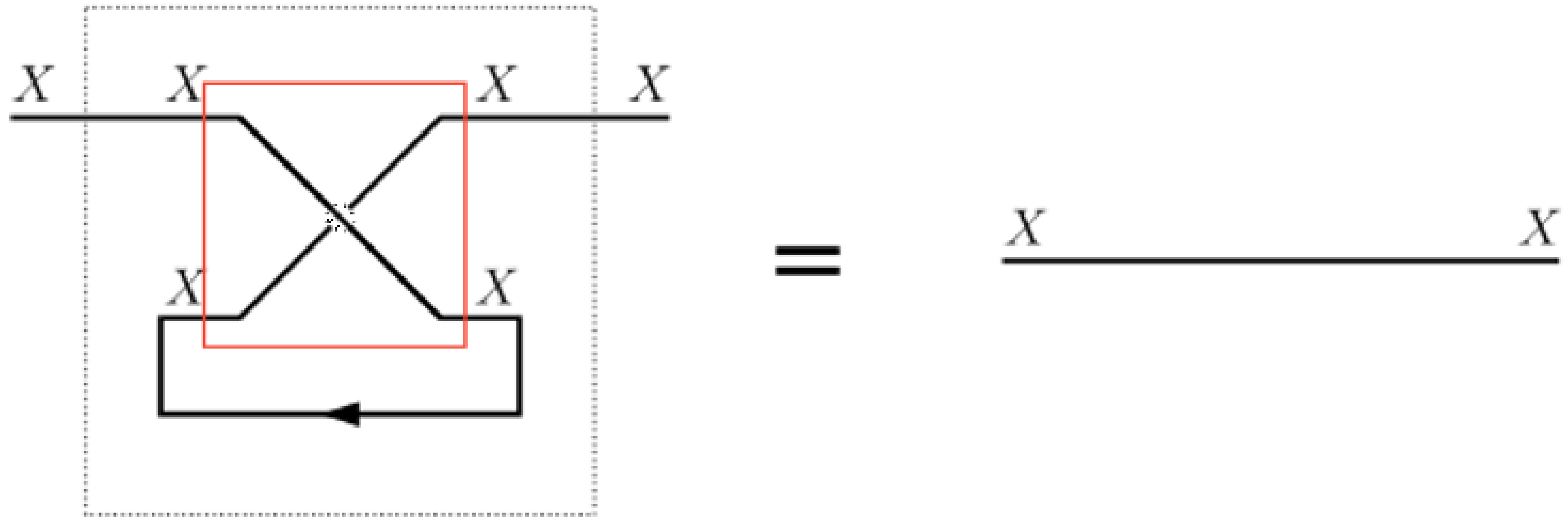


Superposing



Yanking



Defining partial addition abstractly (through enrichment)

Definition 3.1 A *Σ-monoid* is a non-empty set X with a partial map $\Sigma : X^* \rightharpoonup X$ subject to the following axioms:

- If I is a singleton $\{n\}$, then $\Sigma\{x_i\}_{i\in I} \simeq x_n$.
- If $\{I_j\}_{j\in J}$ is a countable partition of a countable subset $I \subset \mathbb{N}$, then for every countable family $\{x_i\}_{i\in I}$ on X, we have $\Sigma\{x_i\}_{i\in I} \simeq \Sigma\left\{\Sigma\{x_i\}_{i\in I_j}\right\}_{j\in J}$.

- Every Σ-monoid has a zero element. $0 = \sum \emptyset$

- We say that a family $\{f_i\}_I$ is summable iff $\sum_{i\in I}\{f_i\}$ is defined.

  In $\mathbf{PInj}$, a family is summable iff all domains and codomains are disjoint.

  In $\mathbf{SubStoch}$, a family is summable iff the standard matrix addition would return a substochastic map.

Defining partial addition abstractly (through enrichment)

We define a category $\mathbf{M}$ of Σ-monoids: objects are Σ-monoids, and a morphism $f : X \to Y$ is a map $f : X \to Y$ such that for each summable countable family $\{x_i\}_{i\in I}$ on X, the summation $\sum_{i\in I} f x_i$ is defined to be $f(\sum_{i\in I} x_i)$.

- Any UDC $\mathcal{C}$ is, by definition, an M-enriched category. This means that

  hom-sets $\mathcal{C}(A,B)$ are objects in M with

  composition $\mathcal{C}(B,C) \otimes \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$ and $f\circ(g+h) = fg + fh$

  identities $I \xrightarrow{f} \mathcal{C}(A,A)$

  $(f\circ -)(g+h) = (f\cdot -)g + (f\cdot -)h$

  $\mathcal{C}(A,B) \to \mathcal{C}(A,C)$

  given by morphisms in M.

- We need a symmetric monoidal structure on M so that the above morphisms may be defined.

$I = \{\cdot\} = \{0\}$ $f(\cdot) = id = 0$

$I = \{0, \cdot\}$

Defining partial addition abstractly (through enrichment)

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Definition 4.1 A *unique decomposition category* is a symmetric monoidal **M**-category such that for all $i \in I$, there are morphisms called *quasi projections* $\rho_i : \bigotimes_{i\in I} X_i \to X_i$ and *quasi injections* $\iota_i : X_i \to \bigotimes_{i\in I} X_i$ subject to the following axioms:

$$\rho_i \circ \iota_j = \begin{cases} \mathrm{id}_{X_i} & (i = j) \\ 0_{X_j, X_i} & (\text{otherwise}), \end{cases} \qquad \sum_{i\in I} \iota_i \circ \rho_i \simeq \mathrm{id}_{\bigotimes_{i\in I} X_i}.$$

- Matricial representation is defined to be

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\sum \{\iota_1 a p_1, \iota_2 b p_1, \iota_1 c p_2, \iota_2 d p_2\}}.$$

- Define each component of $f : \bigotimes_{i\in I} X_i \longrightarrow \bigotimes_{j\in J} Y_j$ as $f_{ji} = p_j f \iota_i$.

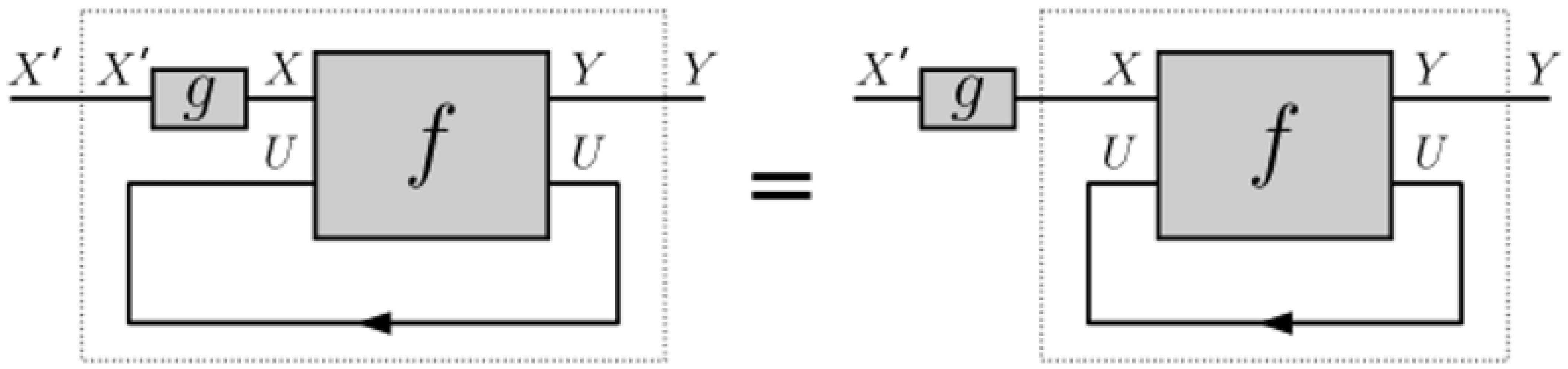
$$f = \mathrm{id} \cdot f \cdot \mathrm{id} = \left(\Sigma \iota_i p_i\right) f \left(\Sigma_j \iota_j p_j\right) = \underline{\Sigma_{i,j} \iota_i (p_i f \iota_j) p_j}$$

<u>Proposition</u>: If the execution formula is defined for all morphisms, then the category is traced.

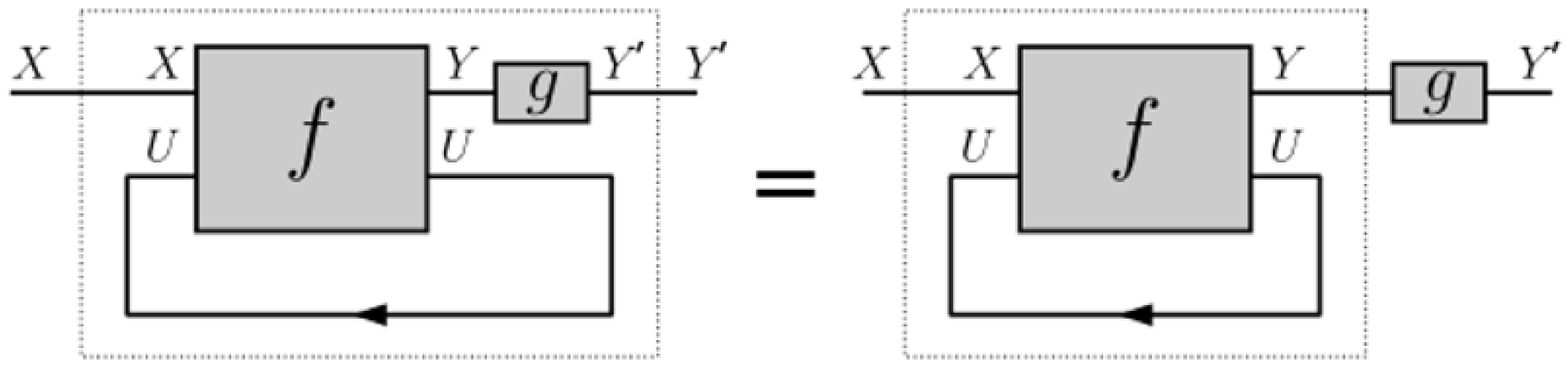
$$\mathrm{Tr}(g) = g_{YX} + \sum_{n=0}^{\infty} g_{YU} (g_{UU})^n g_{UX}$$ (this is almost always true, but there's a catch!)

Axioms of traced monoidal categories

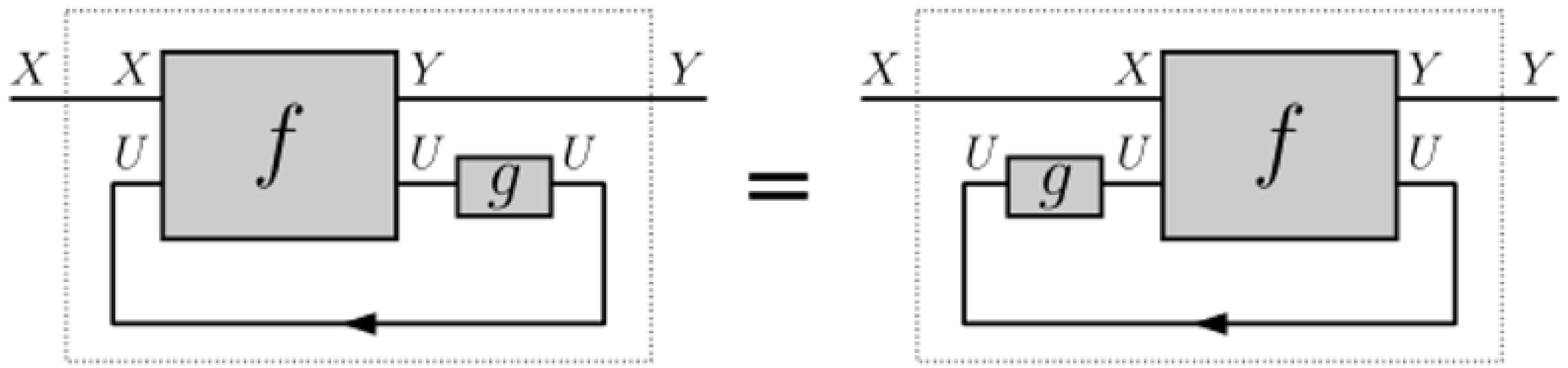
Naturality on X



Naturality on Y



Dinaturality



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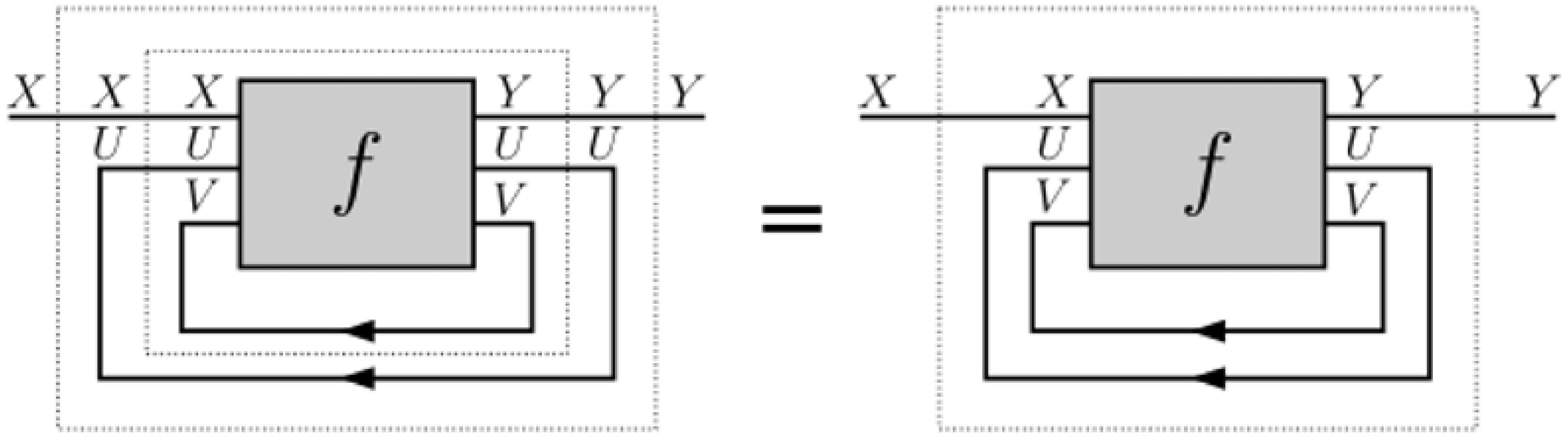
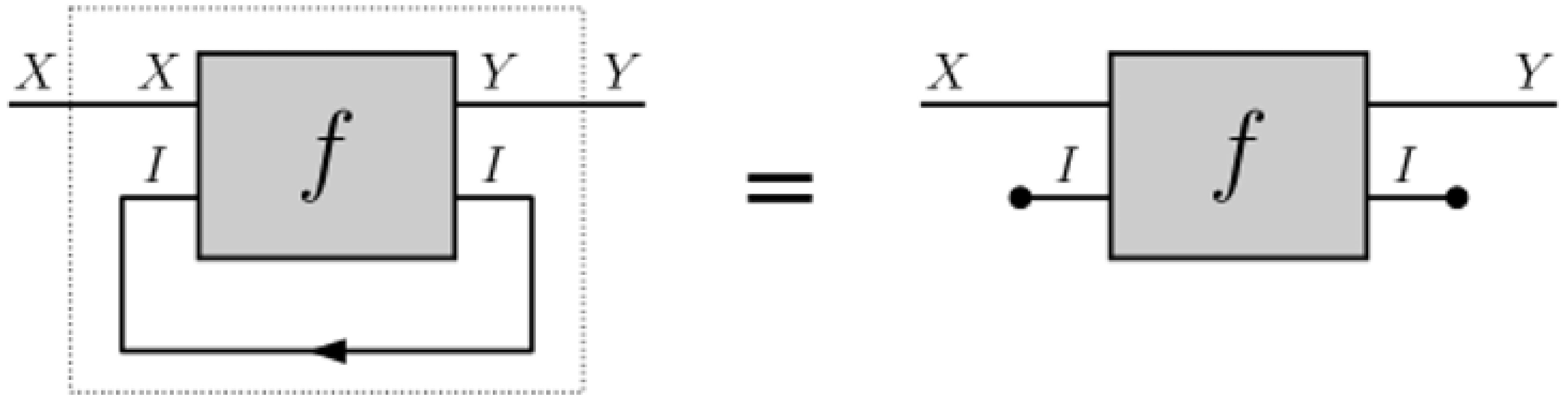


Axioms of traced monoidal categories

Vanishing I



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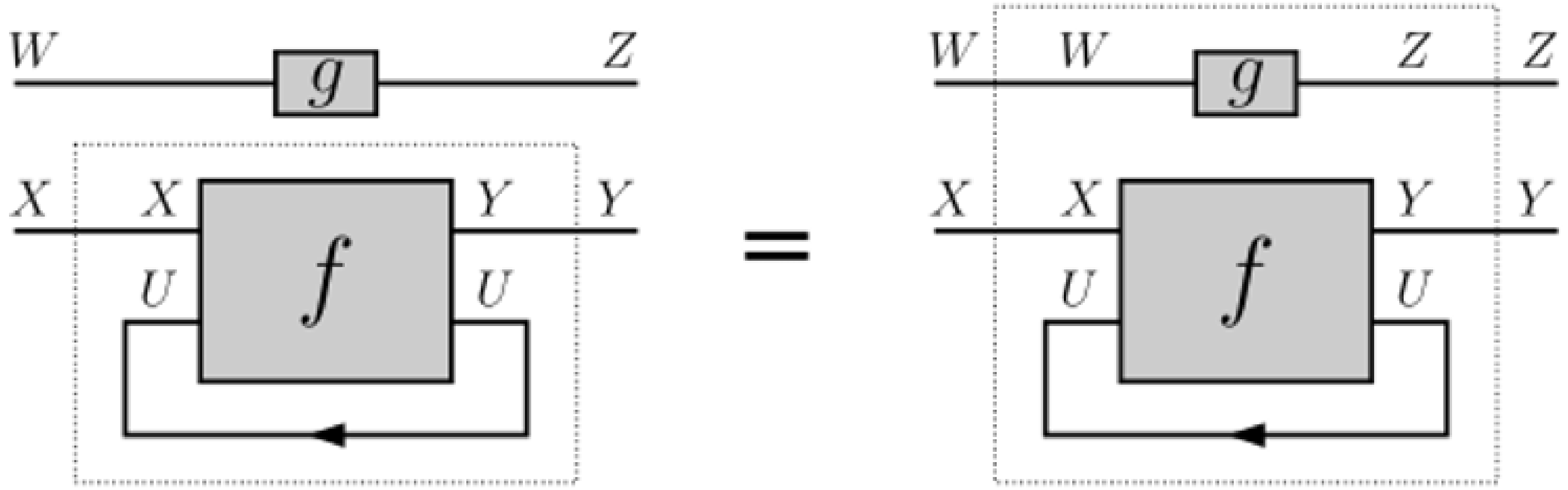

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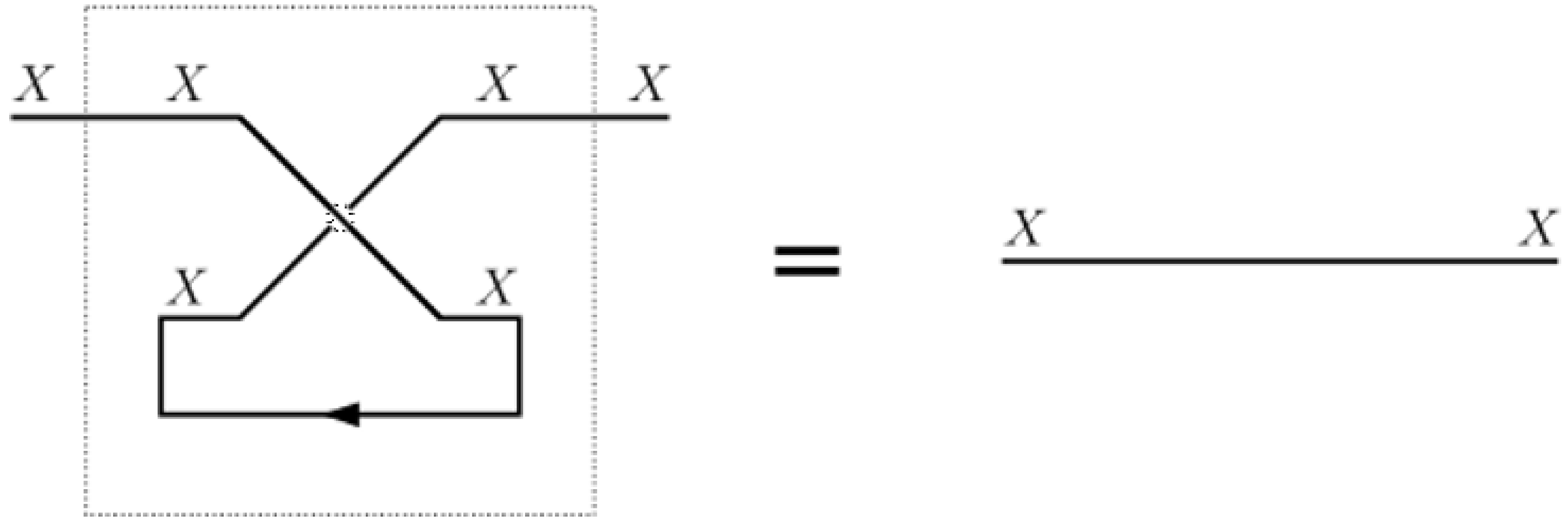

Definition 4.2 A *strong unique decomposition category* $\mathcal{C}$ is a symmetric monoidal **M**-category $\mathcal{C}$ such that

- The identity on the unit I is equal to $0_{\mathrm{I},\mathrm{I}}$.
- $\mathrm{id}_X \otimes 0_{Y,Y} + 0_{X,X} \otimes \mathrm{id}_Y$ is defined to be $\mathrm{id}_{X\otimes Y}$.

$$\rho_{X,Y} := X \otimes Y \xrightarrow{\mathrm{id}_X \otimes 0_{Y,\mathrm{I}}} X \otimes \mathrm{I} \xrightarrow{\cong} X$$

$$\iota_{X,Y} := X \xrightarrow{\cong} X \otimes \mathrm{I} \xrightarrow{\mathrm{id}_X \otimes 0_{\mathrm{I},Y}} X \otimes Y$$

$$f \otimes g = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$$

Proposition: If the execution formula is defined for all morphisms, then the category is traced.

$$Tr(g) = g_{YX} + \sum_{n=0}^{\infty} g_{YU} (g_{UU})^n g_{UX}$$

Related work

- Manes and Arbib (1986) defined partially additive categories (PAC). Essentially, these are UDCs with:

    1. countable coproducts and
    2. Σ-monoids satisfying the limit axiom.

  Proposition: the execution formula is always defined in PACs.

- Hoshino (2012) proves that every strong UDC is partially traced.