Ternary Structures and Ternary Categories ?

(could do n-ary but focus on $n=3$)

(very much work in progress)

[references]

Some Philosophical and Notational Remarks

- strongly binary flavour of modern mathematics

  e.g. $\left\{ \begin{array}{l} \text{binary algebraic structures (groups, rings, modules ...)} \\ \text{categories (maps, relations, homomorphisms ...)} \\ \text{logic (truth tables ...)} \end{array} \right.$

- linear notation ubiquity: $a+b$, ab, (x,y,z), ...
  - probable influence of european languages (vs e.g. Kanji)
  - constrained by printing technology until recently

Cubic Matrices

[R. Kerner (2008) - Ternary and Non-Associative Structures]
[R. Bai, H. Liu, M. Zhang (2014) - 3-Lie Algebras realised by Cubic Matrices]
[M. Ladra, U.A. Rozikov (2016) - Algebras of Cubic Matrices]

indices: $i,j,k,m,n \ldots \in \{0,1,\ldots,N\} \subset \mathbb{N}$ field $(\mathbb{F},\cdot,+)$

cubic matrix: $[a_{ijk}]$, $a_{ijk} \in \mathbb{F}$

generalisations of matrix multiplication:

binary: $\sum_{n=0}^{N} a_{ijn} b_{ink}$, ...

ternary: $\sum_{n=0}^{N} a_{inn} b_{njn} c_{nnk}$, $\sum_{n,m,l}^{N} a_{inm} b_{mjl} c_{mlk}$, ...

most of the existing literature assumes: $[a_{ijk}] \sim a \in V \otimes V \otimes V$ for V a $\mathbb{F}$-vector space

issues: covariant and contravariant indices cannot be contracted (traced) consistently so that the ternary products are internal operations

$$[a_{ijk}] \overset{?}{\sim} a \in V \otimes V^* \otimes V,\ V \otimes V^* \otimes V^*, \ldots$$

If (square) matrices are the basis expression of linear maps, what are the "linear objects" associated with cubic matrices?

3-LIE ALGEBRAS

[J.A. Azcárraga, J.M. Izquierdo (2010) - n-ary Algebras: a review with applications]

$(\mathfrak{g}, [\,,\,,\,])$ $\mathfrak{g}$ $\mathbb{F}$-vector space, $[\,,\,,\,]: \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g} \to \mathfrak{g}$

$\forall X,Y,A,B,C \in \mathfrak{g}$: $[X,Y,[A,B,C]] = [[X,Y,A],B,C] + [A,[X,Y,B],C] + [A,B,[X,Y,C]]$

e.g. cross product in $(\mathbb{R}^4, \delta)$

$\underline{a}, \underline{b}, \underline{c} \in \mathbb{R}^4$ $\quad \underline{a} \overset{\underline{b}}{\times} \underline{c} = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 & \underline{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$

Integration problem: $\exists$ "ternary Lie group" G : "$T_e G \cong \mathfrak{g}$"?

OPEN PROBLEM!

3-associativity: $(abc)de = a(bcd)e = ab(cde)$

GENERALISING CATEGORIES

[V.V. Topentcharov (1988) - n-ary Algebraic Structures generalising Categories] (in French)

[N.A. Baas (2015) - On Higher Structures]

- n-categories : $\begin{array}{ccc} A & \longrightarrow & B \\ & \Downarrow & \\ X & \longrightarrow & Y \end{array}$
- poly categories : $(A,B) \to (X,Y)$
- these are "fundamentally binary"

Can we define categories with "fundamentally ternary" morphisms?

MODELS FOR TERNARY INSPIRATION

Simplex Analogy

class of objects: $\{A, B, C, X \ldots\}$ (0-simplex)

binary morphism: $A \xrightarrow{f} B$ (1-simplex)

ternary morphism: $A \overset{B}{\triangle_t} C$ (2-simplex)

binary compositions:

$A \lhd X \rhd B \mapsto A \overset{X}{\triangle} B$

$A \overset{X}{\underset{Y}{\lhd\rhd}} B \mapsto A \underset{Y}{\triangledown} B$

ternary compositions:

$\overset{B}{\underset{A \quad \triangle \quad C}{\triangle_X \triangle}} \mapsto A \overset{B}{\triangle} C$

$\overset{B}{\triangle}$, $X \quad Y$, $A \triangle Z \triangle C \mapsto A \overset{B}{\triangle} C$

identities:

$A \overset{B}{\triangle} C = A \overset{B}{\triangle} C$

$A \overset{B}{\triangle_T} C$

ASSOCIATIVITY?

$\vartriangle$ and $\triangledown$ are (2-)associative $\;{}_A\overset{A}{1}{}_B$ act as 1_B

$\triangle$ and $\blacktriangle$ are not 3-associative (far from it)

only "not quite" trissociative:

$a,b,c \in X$ $\quad {}_a{}^b{}_c \in X$

$a(bc) = (ab)c$

$\overset{x}{(a\,{}^b c)}\,y = a\,\overset{(b\,c)}{\ }\,y = a\,\overset{x}{(b\,c\,y)}$

TERNARY GROUP(OID)S

Relations

sets: $A, B, C, X \ldots \in SET$

binary relation: $R \subset A \times B$

ternary relation: $T \subset A \times B \times C$

$(a,b,c) \in T \Leftrightarrow a \overset{b}{T} c$

binary compositions:

$T_1 \vartriangle T_2 = \{(a,x,b) \mid \exists y \in Y : a \overset{x}{T_1} y,\ y \overset{x}{T_2} b\}$

$T_1 \triangledown T_2 = \{(a,y,b) \mid \exists x \in X : a \overset{x}{T_1} y,\ y \overset{x}{T_2} b\}$

ternary compositions:

$\underset{T_3}{T_1 \triangle T_2} = \{(a,b,c) \mid \exists x \in X : a \overset{b}{T_1} x,\ x \overset{b}{T_2} c,\ a \overset{x}{T_3} c\}$

$\underset{T_3}{T_1 \blacktriangle T_2} = \{(a,b,c) \mid \exists x,y,z : a \overset{x}{T_1} z,\ x \overset{b}{T_2} y,\ z \overset{y}{T_3} c\}$

identities:

"partial diagonals"

$A \overset{A}{\triangle_{{}_A1_B}} B \qquad {}_A\overset{A}{1}{}_B = \{(a,a,b),\ a \in A, b \in B\}$

$\underset{T}{{}_A\overset{A}{1}{}_B \triangle {}_C\overset{A}{1}{}_A} = T$

$A \lhd B \overset{X}{\triangle} C \rhd D \ldots$

TRIALITY