

TERNARY STRUCTURES AND TERNARY CATEGORIES ?

(could do n-ary but focus on n=3)

(very much work in progress)

[references]

SOME PHILOSOPHICAL AND NOTATIONAL REMARKS

- strongly binary flavour of modern mathematics
 - e.g. $\left\{ \begin{array}{l} \text{binary algebraic structures (groups, rings, modules...)} \\ \text{categories (maps, relations, homomorphisms...)} \\ \text{logic (truth tables, ...)} \end{array} \right.$
- linear notation ubiquity: $a+b, ab, (x, y, z), \dots$
 - probable influence of European languages (vs e.g. Kanji)
 - constrained by printing technology until recently

CUBIC MATRICES

[R. Kerner (2008) - Ternary and Non-Associative Structures]

[R. Bai, H. Liu, M. Zhang (2014) - 3-Lie Algebras realised by Cubic Matrices]

[M. Ladra, V.A. Roizal (2016) - Algebras of Cubic Matrices]

indices: $i, j, k, m, n \dots \in \{0, 1, \dots, N\} \subset \mathbb{N}$ field $(\mathbb{F}, \cdot, +)$

cubic matrix: $[a_{ijk}]$, $a_{ijk} \in \mathbb{F}$

generalisations of matrix multiplication:

binary: $\sum_{n=0}^N a_{ijn} b_{in-k}, \dots$

ternary: $\sum_{n=0}^N a_{inn} b_{ijn} c_{nkk}, \sum_{n,m,k} a_{inn} b_{nje} c_{mek}, \dots$

most of the existing literature assumes: $[a_{ijk}] \sim a \in V \otimes V \otimes V$ for V a \mathbb{F} -vector space

issues: covariant and contravariant indices cannot be contracted (traced) consistently so that the ternary products are internal operations

$[a_{ijk}] \stackrel{?}{\sim} a \in V \otimes V^* \otimes V, V \otimes V^* \otimes V^*, \dots$

If (square) matrices are the basis expression of linear maps, what are the "linear objects" associated with cubic matrices?

3 - LIE ALGEBRAS

[J.A. Azcárraga, J.M. Izquierdo (2010) - n-ary Algebras: a review with applications]

$(\mathfrak{g}, [\dots])$ \mathfrak{g} \mathbb{F} -vector space, $[\dots]: \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$

$$\forall x, y, a, b, c \in \mathfrak{g} : [x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$$

e.g. cross product in (\mathbb{R}^3, δ)

$$a, b, c \in \mathbb{R}^3 \quad a \times b = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

Integration problem: \exists "ternary Lie group" G : " $T_e G \cong \mathfrak{g}$ "?

OPEN PROBLEM!

$$3\text{-associativity: } (abc)de = a(bcd)e = ab(cde)$$

GENERALISING CATEGORIES

[V.V. Topentcharov (1988) - n-ary Algebraic Structures generalising Categories] (in French)

[N.A. Baas (2015) - On Higher Structures]

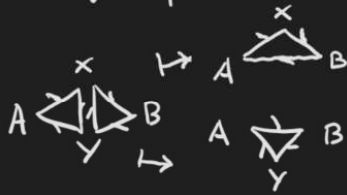
- n-categories: $\begin{matrix} A & \rightarrow & B \\ & \downarrow & \\ x & \rightarrow & y \end{matrix}$
- poly categories: $(A, B) \rightarrow (x, y)$
- these are "fundamentally binary"

Can we define categories with "fundamentally ternary" morphisms?

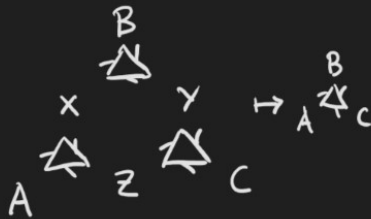
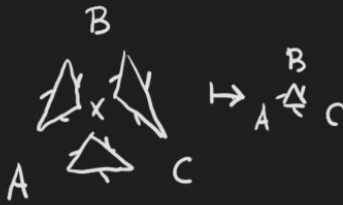


Simplex Analogy

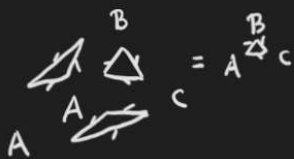
class of objects: $\{A, B, C, X, \dots\}$ (0-simplex)
 binary morphism: $A \xrightarrow{f} B$ (1-simplex)
 ternary morphism: $A \xrightarrow{\quad} B \xrightarrow{\quad} C$ (2-simplex)
 binary compositions:



ternary compositions:



identities:



ASSOCIATIVITY?

Δ and ∇ are (2-) associative $A \overset{A}{\underset{A}{\Delta}} B$ act as 1_B
 Δ and ∇ are not 3-associative (far from it)
 only "not quite" trisassociative:
 $a, b, c \in X$
 $a^b c \in X$
 $a(bc) = (a^b)c$

TERNARY GROUP(OIDS)

Relations

sets: $A, B, C, X, \dots \in \text{SET}$
 binary relation: $R \subset A \times B$
 ternary relation: $T \subset A \times B \times C$
 $(a, b, c) \in T \Leftrightarrow a \overset{b}{\underset{c}{\Delta}} \quad \text{or} \quad a \overset{c}{\underset{b}{\nabla}}$
 binary compositions:

$$T_1 \Delta T_2 = \{(a, x, b) \mid \exists y \in Y: a \overset{x}{\underset{y}{\Delta}} b, y \overset{z}{\underset{b}{\nabla}} c\}$$

$$T_1 \nabla T_2 = \{(a, y, b) \mid \exists x \in X: a \overset{x}{\underset{y}{\Delta}} b, y \overset{z}{\underset{c}{\nabla}} b\}$$

ternary compositions:

$$T_1 \Delta T_2 \overset{c}{\underset{c}{\Delta}} T_3 = \{(a, b, c) \mid \exists x \in X: a \overset{b}{\underset{x}{\Delta}} z, x \overset{b}{\underset{c}{\Delta}} c, a \overset{x}{\underset{c}{\Delta}} c\}$$

$$T_1 \Delta T_2 \overset{c}{\underset{c}{\nabla}} T_3 = \{(a, b, c) \mid \exists x, y, z: a \overset{x}{\underset{y}{\Delta}} b, x \overset{b}{\underset{z}{\Delta}} z, z \overset{b}{\underset{c}{\Delta}} c\}$$

identities:

"partial diagonals"

$$A \overset{A}{\underset{A}{\Delta}} B = \{(a, a, b) \mid a \in A, b \in B\}$$



= T



TRIALITY

