Hopf Monads + Trace Monads

(joint work with Masahito Hasegawa)

$$\mathbb{T} = (T : \mathbb{X} \longrightarrow \mathbb{X},\ \mu : T^2A \longrightarrow TA,\ \eta : A \longrightarrow TA)$$

monad

functor

nat. trans.

Eilenberg-Moore Category of $\mathbb{T}$-algebras: $\mathbb{X}^{\mathbb{T}}$

objects:

$(A,\ a : TA \longrightarrow A)$

satisfy 2 diagrams.

maps:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Want: Lift structure to EM-cat

$$\begin{tikzcd}
\mathbb{X}^{\pi} & \text{same structure} \arrow[l] \\
\mathbb{X} \arrow[u, bend left, dashed] & \text{structure} \arrow[l]
\end{tikzcd}$$

Want: Lift structure to EM-cat

$$
\begin{tikzcd}
\mathbb{X}^{\mathbb{T}} \arrow[d, "U^{\mathbb{T}}"] \\
\mathbb{X} \arrow[u, dashed, bend left]
\end{tikzcd}
$$

$\mathbb{X}^{\mathbb{T}}$ ← same structure

$\mathbb{X}$ ← structure

Forgetful Functor $U^{\mathbb{T}}$

$U^{\mathbb{T}}(A, a) = A$

$U^{\mathbb{T}}(f) = f$

Lifting structure:

Forgetful Functor preserves structure strictly!

Want: Lift structure to EM-cat

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$\mathbb{X}$ ← structure

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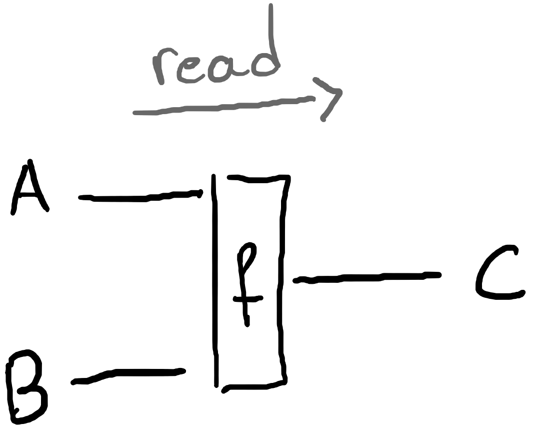
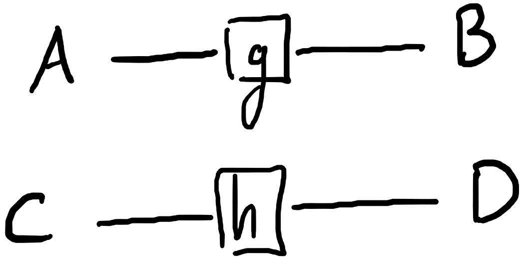
TODAY'S STORY:
Lifting <u>trace</u>.

Graphical Calculus: Sym. Mon. Cat.

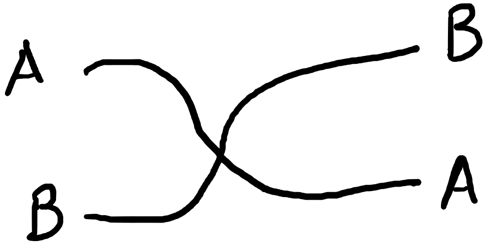
$\mathbb{X}$, $\otimes$, I



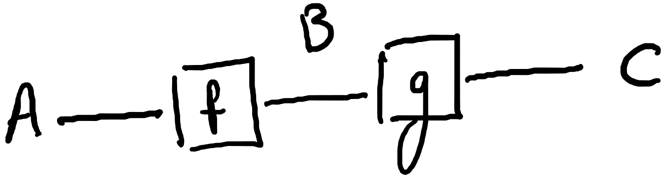
$f : A \otimes B \longrightarrow C$



$g \otimes h$



symmetry



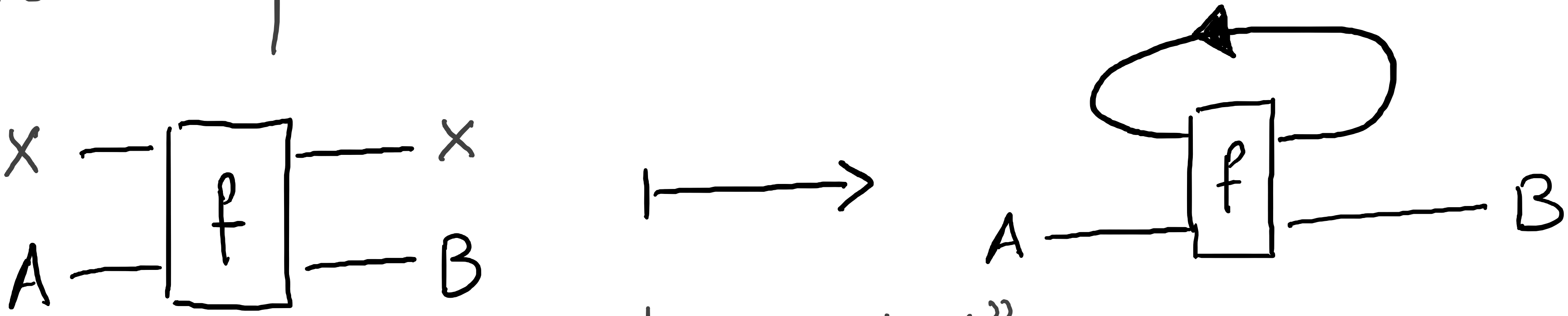
$g \circ f$

Traced Monoidal Categories: (Joyal, Street, Verity)

generalize (partial) trace operation on matrices / feedback.

- Sym. Mon. Cat.

- Trace Operator:

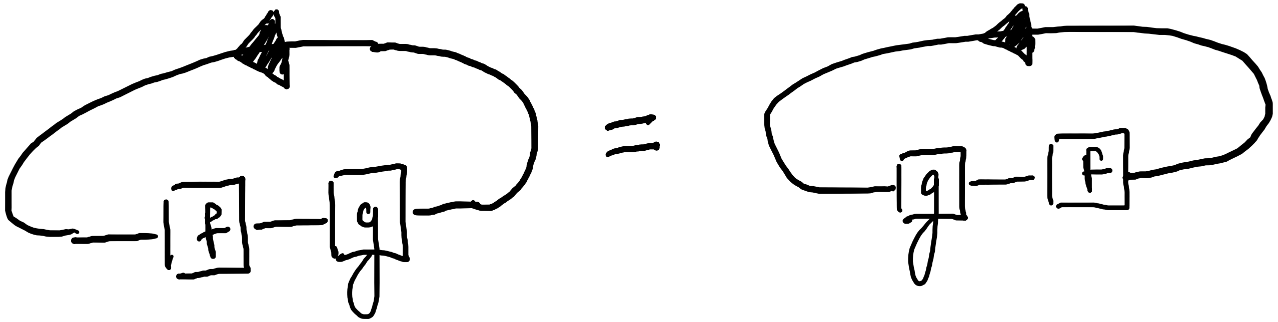


"trace out X"

Some identities of trace:

1. 

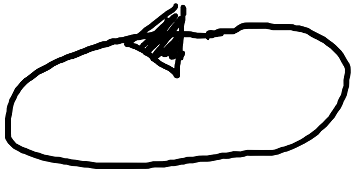
trace of symmetry map is the identity

2. 

Cyclic property

$\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$

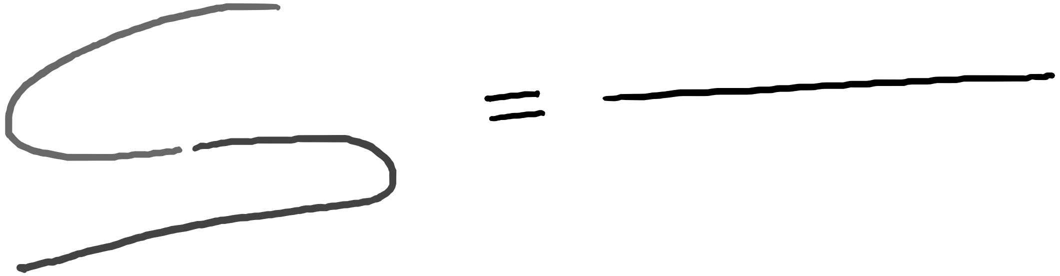
$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

3. 

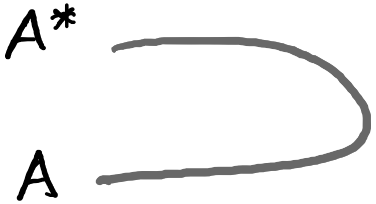
Loops!

Example: Compact Closed Categories

Sym. Mon. Cat.

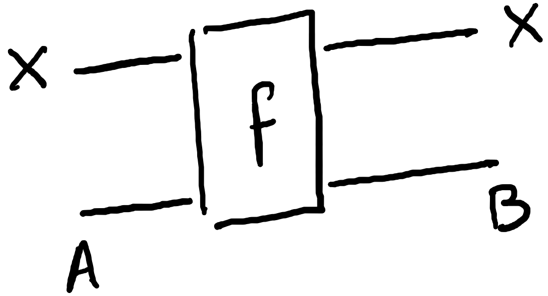


For simplicity: $A^{**} = A$

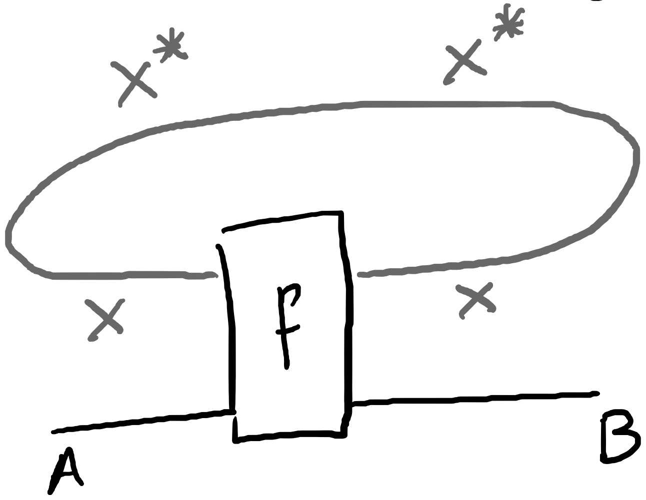


Example: Compact Closed Categories

Trace:



$\longmapsto$



Example: Matrices.

$\mathrm{Mat}(\mathbb{R})$

obj: $n \in \mathbb{N}$

$m \times k$ matrix

map: $n \xrightarrow{A} m$

identity: $n \xrightarrow{I_n} n$ $\quad \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

comp:

$$\begin{tikzcd} n \arrow[r, "A"] \arrow[rr, bend right, "BA"'] & m \arrow[r, "B"] & k \end{tikzcd}$$

$n \otimes m = nm$

$I = 1$

$n^* = n$

$n^2 \xrightarrow{\supset} 1$

$[\underbrace{1\,0\,0 \ldots 0}_{n} \; \underbrace{1\,0\,0 \ldots 0}_{n} \; \ldots \; \underbrace{1\,0 \ldots 0}_{n}]$

$\subset = \supset^{T}$

Trace: Partial Trace of matrices.

$n \xrightarrow{A} n \longmapsto 1 \xrightarrow{Tr(A)} 1$

square matrices

$Tr(A) = \sum A(i,i)$

Not all traced monoidal categories are Compact closed!

Example: Poset $\mathbb{N}$

obj: $n \in \mathbb{N}$

map: $n \xrightarrow{\leq} m$

$n \otimes m = n + m$

$I = 0$

Not com. closed:

$n + n^* \leq 0$

$\Updownarrow$

$n = 0$

Trace:

$$x + n \xrightarrow{\leq} x + m \implies n \xrightarrow{\leq} m$$

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That said, every traced monoidal category induces a compact closed category via INT-construction.

Other examples:

Compact Closed:	Non-compact Closed:
1) $FVEC_K$	1) CPPO, pointed ω-complete partial orders
2) REL, $\otimes = \times$	2) REL, $\otimes = \oplus$
3) A one object compact closed category is an abelian group.	3) Cartesian categories: Conway fix point operators.

Trace Monads: monads which lift trace mon. structure

1) lift sym. mon. structure

2) lift trace operator.

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   symmetric bimonads

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Symmetric Bimonads:

$\mathbb{T} = (T, \mu, \eta)$ on a SMC equipped with:

$$m_2 : T(A \otimes B) \longrightarrow TA \otimes TB \qquad m_I : TI \longrightarrow I$$

satisfying certain equations.

$\boxed{\text{Prop: A monad lifts sym. mon. structure} \iff \text{it is a bimonad.}}$

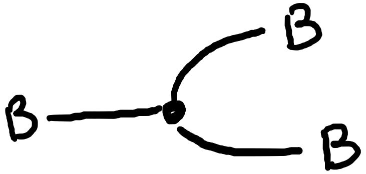
$$(A, a) \otimes (B, b) := \left(A \otimes B,\ T(A \otimes B) \xrightarrow{m_2} TA \otimes TB \xrightarrow{a \otimes b} A \otimes B\right)$$

$(I, m_I) \leftarrow$ unit

$\mathbb{X}^{\mathbb{T}}$ is a SMC.

Example: Bicommutative Bialgebras.

multiplication



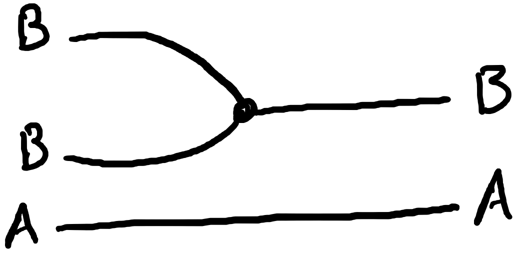
comultipliation

unit

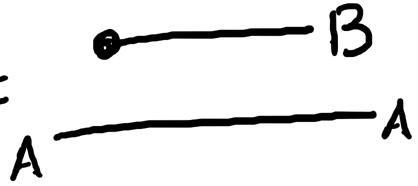
counit.

$TA = B \otimes A$

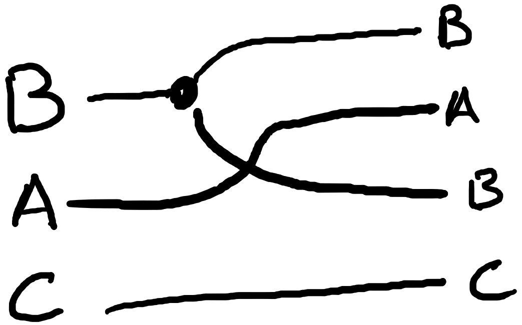
$\mu :=$



$\eta :=$



$m_2 :=$



$m_I :=$

$\mathbb{X}^{\pi} = \mathrm{Mod}(B)$ ← modules over a bialg.

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1) lift sym. mon. structure
   symmetric bimonads

2) lift trace operator.

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$$
\begin{tikzcd}
T(X\otimes A) \arrow[r, "T(f)"] \arrow[d, "m_2"'] & T(X\otimes B) \arrow[d, "m_2"] \\
TX\otimes TA \arrow[d, "X\otimes a"'] & TX\otimes TB \arrow[d, "X\otimes b"] \\
X\otimes A \arrow[r, "f"'] & X\otimes B
\end{tikzcd}
\longmapsto
\begin{tikzcd}
TA \arrow[r, "T(Tr(f))"] \arrow[d, "a"'] & TB \arrow[d] \\
A \arrow[r, "Tr(f)"'] & B
\end{tikzcd}
$$

trace of T-alg maps is a T-alg map.

Prop: A monad lifts trace $\Longleftrightarrow$ it is a trace monad.

Trace monad $\Longrightarrow$ $\mathbb{X}^{\pi}$ is a traced mon. cat.

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We want to give a simple characterization of trace monads without mentioning algebras.

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Trace monad $\Longrightarrow$ $\mathbb{X}^{\pi}$ is a traced mon. cat.

We want to give a simple characterization of trace monads without mentioning T-algebras.

Idea: Let's look at Hopf monads!

- Why Symmetric Hopf monads?

Prop: A monad lifts compact closed structure $\Leftrightarrow$ it is sym. Hopf.

Since every compact closed category is traced, this means sym. Hopf monads on compact closed categories are traced.

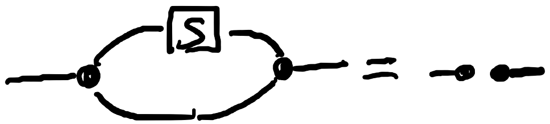
So it natural to ask if this is always the case.

Symmetric Hopf Monad is a sym. bimonad whose fusion operator:

$$h : T(TX \otimes A) \xrightarrow{m_2} T^2X \otimes TA \xrightarrow{\mu \otimes 1} TX \otimes TA$$

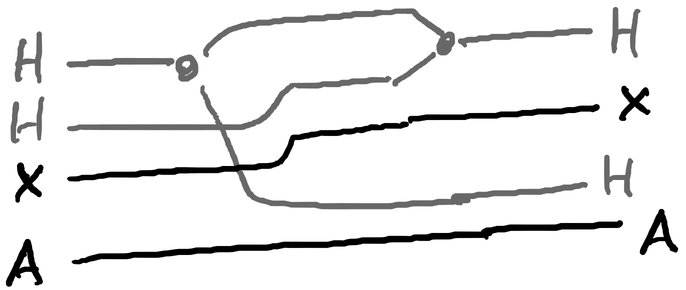
is a natural isomorphism $h^{-1} : TX \otimes TA \longrightarrow T(TX \otimes A)$.

Ex: Bicommutative Hopf Algebras.

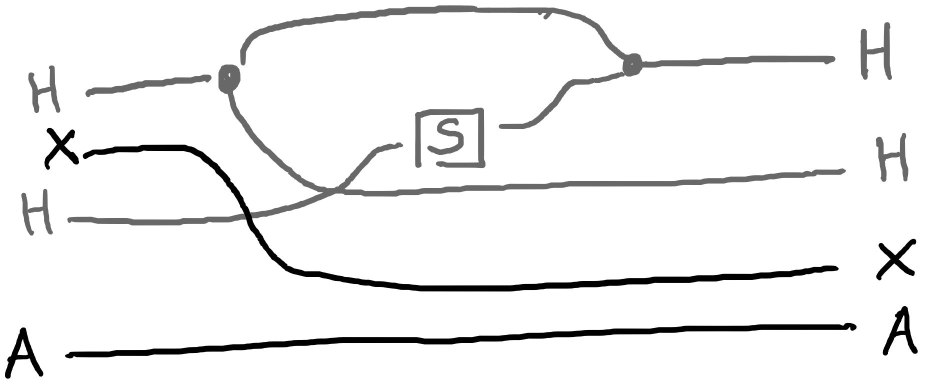


$TA = H \otimes A$

$h :=$



$h^{-1} :=$



$\boxed{\text{Example: Not all Hopf monads come from Hopf algebras!}}$

Ex. Consider the poset $\mathbb{N}$.

$$T(n) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$

$$T^2(n) = T(n)$$

is called an idempotent monad.

Prop: A sym. Hopf monad on a compact closed cat is traced.

Prop: A sym Hopf monad of the form $T(-) = \overset{\text{Hopf}}{H} \otimes -$ on a traced monoidal cat is traced.

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Prop: A sym Hopf monad of the form $T(-) = \overset{\text{Hopf}}{H} \otimes -$ on a traced monoidal cat is traced.

PROBLEMS:

1. Not all traced monoidal categories are compact closed.
2. Not all Hopf monads are of the form $T(-) = H \otimes -$

- Not every trace monad is Hopf!

- Not every Hopf monad is trace!

  But many examples are!

- Not every trace monad is Hopf!

  trace monad is Hopf $\Longleftrightarrow$???

  not clear if trace gives inverse of fusion operator.

- Not every Hopf monad is trace!

  But many examples are!

  Hopf monad is traced $\Longleftrightarrow$ trace coherent

Trace Coherence:

Prop: A sym. Hopf monad is traced if and only if

$f : TX \otimes A \longrightarrow TX \otimes B$ let's trace out TX

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Prop: A sym. Hopf monad is traced if and only if

let's trace out TX

$f: TX \otimes A \longrightarrow TX \otimes B$

1. $$\dfrac{\dfrac{TX \otimes A \xrightarrow{f} TX \otimes B}{A \xrightarrow{Tr(f)} B}}{TA \xrightarrow{T(Tr(f))} TB}$$

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Prop: A sym. Hopf monad is traced if and only if

let's trace out TX

$f : TX \otimes A \longrightarrow TX \otimes B$

① $\dfrac{TX \otimes A \xrightarrow{f} TX \otimes B}{\dfrac{A \xrightarrow{Tr(f)} B}{TA \xrightarrow{T(Tr(f))} TB}}$

② $\dfrac{TX \otimes A \xrightarrow{f} TX \otimes B}{\dfrac{T(TX \otimes A) \xrightarrow{Tf} T(TX \otimes B)}{\dfrac{TX \otimes TA \xrightarrow{h^{-1}} T(TX \otimes A) \xrightarrow{T(f)} T(TX \otimes B) \xrightarrow{h} TX \otimes TB}{TA \xrightarrow{Tr(h \circ T(f) \circ h^{-1})} TB}}}$

Trace Coherence:

Prop: A sym. Hopf monad is traced if and only if

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let's trace out TX

①
$$\begin{array}{c}
TX \otimes A \xrightarrow{\quad f \quad} TX \otimes B \\
\hline
A \xrightarrow{\quad Tr(f) \quad} B \\
\hline
TA \xrightarrow{\quad T(Tr(f)) \quad} TB
\end{array}$$

②
$$\begin{array}{c}
TX \otimes A \xrightarrow{\quad f \quad} TX \otimes B \\
\hline
T(TX \otimes A) \xrightarrow{\quad Tf \quad} T(TX \otimes B) \\
\hline
TX \otimes TA \xrightarrow{h^{-1}} T(TX \otimes A) \xrightarrow{T(f)} T(TX \otimes B) \xrightarrow{h} TX \otimes TB \\
\hline
TA \xrightarrow{\quad Tr(h \circ T(f) \circ h^{-1}) \quad} TB
\end{array}$$

This does not mention T-algebras!

- Not every trace monad is Hopf!

  trace monad is Hopf $\Leftrightarrow$???

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- Not every Hopf monad is trace!

  But many examples are!

  Hopf monad is traced $\Leftrightarrow$ trace coherent

THANK YOU

FOR

LISTENING