# The surprising power of free categories

Tom Leinster 1/10/2020

## 1. The free group on 2 elements.

What is it?

- Start with elts $x, y$. Put in whatever you have to, and no more.
- Univ property, v1: $\mathrm{Gp}(F_2, G) \cong \{\text{pair of elements of } G\}$.
- " " v2: for all $G$ & $g, h \in G$, $\exists!\, \varphi : F_2 \to G$ s.t. $\varphi(x) = g$ & $\varphi(y) = h$.

## 2. The free monoidal cat ~~on a monoid~~ containing an internal monoid

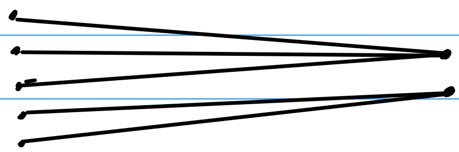
Mon cat: cat $\mathcal{M}$, product $\otimes$ on $\mathcal{M}$, unit obj $I$

Monoid in $\mathcal{M}$: object $A$, maps $\mu : A \otimes A \to A$, $\eta : I \to A$ satisfying assoc & unit axioms.

E.g. $(\mathrm{Set}, \times, 1)$ is a mon cat; a monoid in $(\mathrm{Set}, \times, 1)$ is a monoid.

Build mon cat $\mathbb{D}$ containing monoid $X$ & "free as such":

- $\mathbb{D}$ has obj $X$, hence $X^{\otimes n} = X \otimes \cdots \otimes X$ $\forall n \geq 0$
- $X$ is monoid: $X \otimes X \xrightarrow{m} X \xleftarrow{i} I = X^{\otimes 0}$
- have $n$-fold multip.: $X^{\otimes n} \xrightarrow{m_n} X$
- hence get $X^{\otimes(n_1 + \cdots + n_k)} \xrightarrow{m_{n_1} \otimes \cdots \otimes m_{n_k}} X^{\otimes k}$ $\forall k, n_1, \ldots, n_k \geq 0$



$k = 2$
$n_1 = 3$
$n_2 = 2$

$\mathbb{D} \simeq$ (finite totally ordered sets)

$X^{\otimes n} \longleftrightarrow \{1, \ldots, n\}$

$\otimes \longleftrightarrow \amalg$ or $+$

$$\begin{array}{ccc} X \otimes X & & \{1,2\} \\ \downarrow m & & \downarrow ! \\ X & \longleftrightarrow & \{1\} \\ \uparrow i & & \uparrow ! \\ I & & \emptyset \end{array}$$

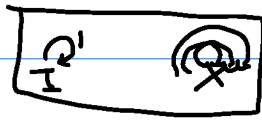
For any mon cat $\mathcal{M}$ & monoid $(A,\mu,\eta)$ in $\mathcal{M}$, $\exists!$ monoidal functor $\mathbb{D}\to\mathcal{M}$ s.t. $(X,m,i)\mapsto(A,\mu,\eta)$.
So
$$\{\text{mon ftrs } \mathbb{D}\to\mathcal{M}\} \simeq \{\text{monoids in } \mathcal{M}\}$$

<u>Summary</u> The free sym mon cat on a comm monoid is the cat. of finite ~~totally ordered~~ sets.

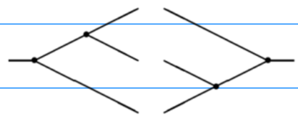
## 3. The free mon cat on an idempotent object

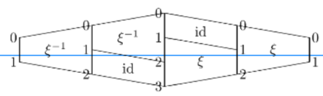
An <u>iden obj</u> in a mon cat $\mathcal{M}$ is an obj $A$ & an iso $\alpha : A\otimes A \xrightarrow{\sim} A$.

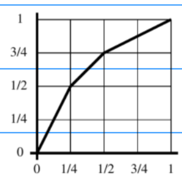
Free mon cat on an iden obj:
- objs $X^{\otimes n}$ $(n\geq 0)$ with $X\cong X^{\otimes n}$ $(n\geq 1)$ $(\& \; \overset{I \,=}{X^{\otimes 0}} \not\cong X)$
- maps gen by $\xi : X\otimes X \xrightarrow{\sim} X$




A nontriv map
$$X \xrightarrow{\xi^{-1}} X\otimes X \xrightarrow{\xi^{-1}\otimes 1} X\otimes X\otimes X \xrightarrow{1\otimes\xi} X\otimes X \xrightarrow{\xi} X$$





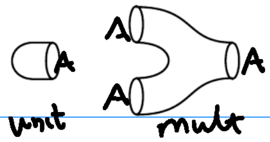



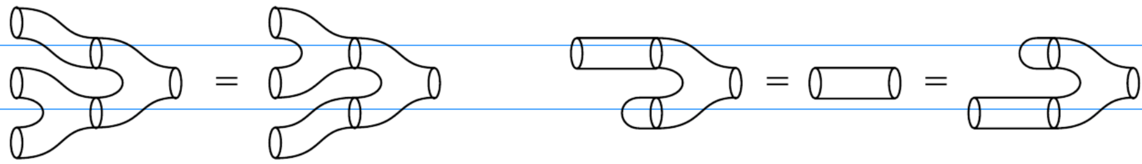

The free mon cat on an iden is eqv to $1 \amalg F$, where $F$ is <u>Thompson's group</u> ($F \subseteq \{$piecewise linear bijections $[0,1]\}$).

## 4. The free sym mon cat on a Frobenius algebra

A <u>Frob alg</u> in a SMC $\mathcal{M}$ is:
- an obj $A$

• a monoid structure:



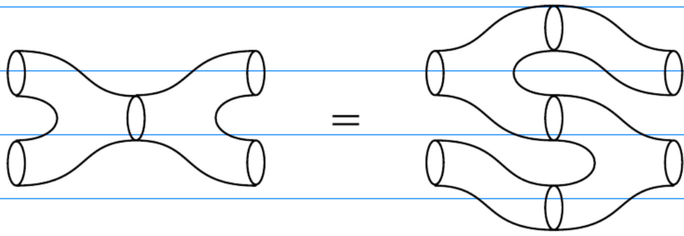
associativity

unit axiom

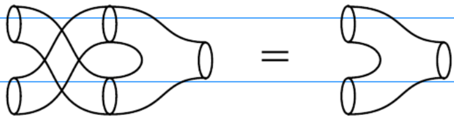
• comonoid structure on $A$ : 

subject to:

• Frob axiom:

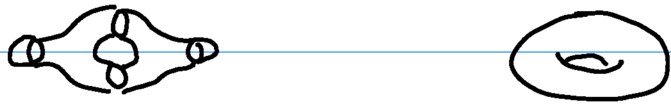


<u>Comm</u> if



$\Leftrightarrow$ cocomm

In the free SMC on a comm Frob alg, objs are $X^{\otimes n}$ $(n \geq 0)$
& e.g.  is a map $X \otimes X \longrightarrow X$ &



is a map $X^{\otimes 0} \longrightarrow X^{\otimes 0}$, i.e. $I \longrightarrow I$.

<u>Defn</u> Let $\Sigma$ & $\Sigma'$ be $(d-1)$-mfds. A <u>cobordism</u> from $\Sigma$ to $\Sigma'$ is a $d$-mfd with boundary $\Sigma \sqcup \Sigma'$.
(E.g.  is a cob. from $\genfrac{}{}{0pt}{}{0}{0}$ to $0$.)

Mfds + Diffeo classes of cobordisms give cat $d\mathrm{Cob}$, monoidal under $\sqcup$.

<u>Thm</u> Free SMC on comm Frob alg is $2\mathrm{Cob}$.

<u>Cor</u> The sym mon ftrs $2\mathrm{Cob} \longrightarrow \mathrm{Vect}$ ("2D TQFTs") correspond to the $\underbrace{\text{Frob algs in Vect}}_{\text{"Frob algs"}}$.