Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1.

a) We consider the standard Black–Scholes model: a risk-free asset B given by

$$dB_t = rB_t\, dt\,,\ \ B_0 = 1$$

and a risky asset S given by

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t\,,\ \ S_0 > 0 \text{ fixed.}$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. An investor, with initial wealth $X_0 = x > 0$ selects among strategies ν that are *constants* and represent fraction of the wealth invested in the risky asset. The investor seeks to maximise his expected utility at time T for U given by

$$U(x) = \ln x, \qquad x > 0.$$

The optimization problem can be written as

$$u(x) = \sup_{\nu \in \mathbb{R}} \mathbb{E}\Big[U\big(X_T^\nu\big)\big|X_0 = x\Big]\,.$$

(i) Derive the SDE satisfied by the portfolio value process $X = X^\nu$. **[2 marks]**

(ii) Show that there is a risk neutral measure for this model and identify the expression for the "market price of risk", denoting it λ. **[4 marks]**

(iii) The deflator is

$$Y_t = \exp(-r\,t - \tfrac{1}{2}\lambda^2 t - \lambda\, W_t)\,.$$

Let ν be an admissible strategy and X the corresponding wealth process. Show that

$$X_t Y_t = x_0 + \int_0^t X_s Y_s(\nu\sigma - \lambda)\, dW_s\,.$$

[4 marks]

(iv) Use duality theory to identify the optimal wealth random variable $\widehat{X}_T$, the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$ and $\hat{\nu}$. **[8 marks]**

b) Let W be an $\mathbb{R}^d$-valued Wiener process, let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $\gamma = \gamma_t$ be an adapted, bounded process. Let ξ be $\mathcal{F}_T$-measurable s.t. $\mathbb{E}\xi^2 < \infty$. Find an explicit solution to

$$dY_t = \gamma_t\, dt + Z_t\, dW_t \quad t \in [0, T]\,,\ \ Y_T = \xi\,. \tag{1}$$

[7 marks]

2. We consider the standard Black–Scholes model: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. We will consider the optimal investment problem with X_s denoting the portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t\in[0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E}\int_t^T \nu_s^2\, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{U}$.

a) Derive the SDE satisfied by the portfolio value process $X_s = X_s^{\nu,t,x}$. **[4 marks]**

b) Show that $X_s = X_s^{\nu,t,x} > 0$ for all $s \in [t,T]$ if $X_t^{\nu,t,x} = x > 0$. **[4 marks]**

c) Consider the control problem

$$v(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\ln(X_T^{\nu,t,x})\right] . \qquad (2)$$

Write down the Bellman PDE that the function v must satisfy. **[4 marks]**

d) Show that the equation has a solution $v(t,x) = \lambda(t)\ln(\beta x) + \gamma(t)$ with $\lambda, \gamma \in C^1([0,T])$ and $\lambda > 0$. Write down the λ, γ and the optimal control explicitly. **[6 marks]**

e) Use the verification theorem to prove that the v above and the optimal control you identify are indeed the solution to the optimal control problem (2). **[7 marks]**

3. Let W be a real valued Wiener process generating a filtration $(\mathcal{F}_t)$. Consider $X_t = X_t^{\alpha,x}$ taking values in $\mathbb{R}$ given by

$$dX_t = [H(t)X_t + M(t)\alpha_t]\, dt + \sigma(t)\, dW_t \text{ for } t \in [0,T]\,, \;\; X_0 = x\,,$$

where H, M and σ are bounded deterministic functions of t. The aim will be to maximize

$$J^\alpha(x) := \mathbb{E}\left[\int_0^T \left[C(t)(X_t^{\alpha,x})^2 + D(t)\alpha_t^2\right] dt + R(X_T^{\alpha,x})^2\right]$$

over all adapted processes α such that $\mathbb{E}\int_0^T \alpha_t^2\, dt < \infty$ (we will call these admissible). We will assume that C and D are integrable deterministic functions of t and R a real constant with $C = C(t) \le 0, R \le 0$ and $D = D(t) \le -\delta < 0$ with some constant $\delta > 0$.

a) Write down the Hamiltonian $(t, x, a, y, z) \mapsto H_t(x, a, y, z)$ for this problem and explain why it is concave and differentiable as a function of (x, a) for all t, y, z. **[3 marks]**

b) Write down the adjoint BSDE. **[2 marks]**

c) Use Pontryagin's maximum principle to show that the optimal control $\hat{\alpha}$ and the adjoint BSDE $(\hat{Y}, \hat{Z})$ associated with this control process must satisfy

$$\hat{\alpha}_t = -\frac{M(t)}{2D(t)}\hat{Y}_t \text{ for } t \in [0,T]\,.$$

[3 marks]

d) Inspecting the terminal condition for the adjoint BSDE leads us to "guess" that we should have $\hat{Y}_t = 2S(t)\hat{X}_t$ for some $S \in C^1([0,T])$ with $S(T) = R$. Derive the ordinary differential equation for S. **[7 marks]**

e) Hence show that

$$J^{\hat{\alpha}}(x) = S(0)x^2 + \int_0^T S(t)\sigma^2(t)\, dt\,.$$

[10 marks]

[End of Paper]