PART B Please complete in BLOCK CAPITALS

Student Number ______________ **Name** ______________________________ **Signature** ______________________________

This information will not be visible to the marker.

PART A Please complete clearly

Exam Number ______________________
as shown on your university card

THE UNIVERSITY *of* EDINBURGH

Stochastic Control and Dynamic Asset Allocation
MATH11150

Sunday 25 December 2022
13:00–15:00

Number of questions: 3
Total number of marks: 100

IMPORTANT PLEASE READ CAREFULLY

Before the examination

1. Put your university card face up on the desk.
2. **Complete PART A and PART B above**. By completing PART B you are accepting the University Regulations on student conduct in an examination (see back cover).
3. Complete the attendance slip and leave it on the desk.
4. In this examination, candidates are allowed to have three sheets (six sides) of A4 paper with whatever notes they desire written or printed on one or both sides of the paper. Magnifying devices to enlarge the contents of the sheets for viewing are not permitted. No further notes, printed matter or books are allowed.
5. A scientific calculator is permitted in this examination. It must not be a graphic calculator. It must not be able to communicate with any other device.

During the examination

1. Write clearly, in ink, in the space provided after each question. If you need more space then please use the extra pages at the end of the examination script or ask an invigilator for additional paper.
2. **Attempt all questions**.
3. If you have rough work to do, simply include it within your overall answer – put brackets at the start and end of it if you want to highlight that it is rough work.

At the end of the examination

1. This examination script must not be removed from the examination venue.
2. There are extra pages for working at the end of this examination script. If used, you should clearly label your working with the question to which it relates.
3. Additional paper and graph paper, if used, should be attached to the back of this examination script. Write your examination number on the top of each additional sheet.

Throughout we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports all the random variables and processes introduced. Results covered in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Let $T > 0$, H, M, $C \leq 0$, $D < 0$, $R \leq 0$, κ, $\lambda > 0$ be constants with values in $\mathbb{R}$. Consider the controlled jump-diffusion

$$dX_s = (HX_s^\alpha + M\alpha_s)\,ds + \kappa(dN_s^u - dN_s^d)\,, \quad s \in [t,T]\,, \;\; X_t = x \in \mathbb{R}\,,$$

where $N^u = (N_t^u)_{t\in[0,T]}$ and $N^d = (N_t^d)_{t\in[0,T]}$ are Poisson processes with arrival intensity λ.

The optimization objective which is to maximize

$$J(t,x,\alpha) := \mathbb{E}_{t,x,\alpha}\left[\int_t^T (CX_s^2 + D\alpha_s^2)\,ds + RX_T^2\right]\,,$$

over all square integrable predictable processes α. Here $\mathbb{E}_{t,x,\alpha}[\cdot]$ denotes the conditional expectation where the process X starts from x at time t and the control process α is used.

(a) Write down the Bellman equation for $v(t,x) = \sup_\alpha J(t,x,\alpha)$. [7 marks]

(b) Assume that S and b are in $C^1([0,T])$. Use the ansatz $v(t,x) = S(t)x^2 + b(t)$ to derive a solution to the Bellman equation i.e. derive the ordinary differential equations that S and b satisfy. Write down an expression for $v(t,x)$ in terms of t, x, R and S only (i.e. solve the equation for b). [23 marks]

(c) Take $H = 0$, $M = 1$, $C = 0$, $D = -1$, $R = -1$. Solve the ODE for S and hence get an exact expression the optimal Markov control. [5 marks]

Hint: If you wish you can take $\kappa = 0$ throughout and still collect up to 80% of the marks.

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If you have used additional space for working then please tick here: ☐

2. Let W be a real-valued Wiener process, $x \in \mathbb{R}$, $b : \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : \mathbb{R} \times A \to \mathbb{R}$ be Lipschitz continuous in $(x, a) \in \mathbb{R} \times A$ with $A \subseteq \mathbb{R}$. Let $\mathcal{A}$ denote A-valued processes that are square integrable and adapted to the filtration generated by W. For $\alpha \in \mathcal{A}$ consider the controlled SDE

$$dX_s^{t,x,\alpha} = b\big(X_s^{t,x,\alpha}, \alpha_s\big)\, ds + \sigma\big(X_s^{t,x,\alpha}, \alpha_s\big)\, dW_s, \,,\; t \in [t,T]\,,\;\; X_t = x\,.$$

(a) State the tower property of conditional expectation. [2 marks]

(b) State the flow property that solutions to the above controlled SDE satisfy. [2 marks]

(c) State the Markov property that solutions to the above controlled SDE satisfy. [3 marks]

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and of polynomial growth. Let $f : \mathbb{R} \times A \to \mathbb{R}$ be continuous, of polynomial growth in x and at most square growth in a. Let

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(X_s^{\alpha,t,x}, \alpha_s)\, ds + g(X_T^{\alpha,t,\xi})\right] \quad \text{and} \quad v(t,x) := \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha)\,.$$

Prove that for $t \le \hat{t} \le T$ we have

$$v(t,x) \le \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_t^{\hat{t}} f\big(X_s^{\alpha,t,x}, \alpha_s\big)\; ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,t,x}\big)\Big| X_t^{\alpha,t,x} = x\right]\,.$$

Justify each step, in particular by stating when you're applying a), b) or c) above. [18 marks]

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If you have used additional space for working then please tick here: ☐

3. Consider the following control problem for a jump diffusion:

- Let $dS_r^{t,S} = \sigma dW_r$ for $r \in [t,T]$ with initial value $S_t = S \in \mathbb{R}$ given.
- Let $M = (M_t)_{t\geq 0}$ be a Poisson jump process with intensity λ.
- Let $(U_i)_{i\in\mathbb{N}}$ be iid r.v.s with uniform distribution on $[0,1]$.
- Let $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \sigma(M_s : s \leq t) \vee \sigma(U_i : i \leq M_t)$. Let $\mathcal{A}$ denote $\mathbb{R}^+$-valued stochastic processes that are square integrable and progressively measurable w.r.t $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. We shall call these admissible controls. Let $\delta \in \mathcal{A}$.
- Let N_t^δ be a stochastic process satisfying $N_t^\delta = N_{t-}^\delta + (M_t - M_{t-})\mathbf{1}_{U_{M_t} \geq e^{-\kappa\delta_t}}$. In other words it is a doubly stochastic Poisson process with stochastic intensity given by $\lambda e^{-\kappa\delta_t}$.
- Let $Q_r^{t,q,\delta} = q - N_r^\delta$ for $r \in [t,T]$ with initial value $Q_t = q \in \{0\} \cup \mathbb{N}$ given.
- Let $dX_r^{t,q,S,x,\delta} = (S_r + \delta_r)dN_r^\delta$ for $r \in [t,T]$ with initial value $X_t = x \in \mathbb{R}$ given.
- Let $\tau^{t,q,\delta} := T \wedge \inf\{r \geq t : Q_r^{t,q,\delta} \leq 0\}$.

Our aim is to maximize

$$J(t,q,S,x,\delta) = \mathbb{E}_{t,q,S,x,\delta}\Big[X_\tau + S_\tau Q_\tau - \alpha Q_\tau^2\Big]$$

over $\mathcal{A}$. Here $\mathbb{E}_{t,q,S,x,\delta}[\cdot]$ denotes the conditional expectation given $S_t = S$, $X_t = x$, $Q_t = q$ and given the process control δ is used. For simplicity we will assume $q \in \{0,1,2\}$. Let

$$v(t,q,S,x) = \sup_{\delta\in\mathcal{A}} J(t,q,S,x,\delta)\,.$$

(a) Write down the Bellman equation for v, including all boundary and terminal conditions and carefully specifying the domain. [7 marks]

(b) Assume that $\theta \in C^1([0,T])$. Use the ansatz $v(t,q,S,x) = x + Sq + \theta(t,q)$ and the Bellman PDE to derive that

$$\begin{aligned}\partial_t\theta + \frac{1}{\kappa}\lambda e^{-1}\exp\Big(\kappa(\theta(t,q-1) - \theta)\Big) = 0 \ \text{ on } \ [0,T) \times \{1,2\}\,,\\ \theta(t,0) = 0 \ \ \forall t \in [0,T),\\ \theta(T,q) = -\alpha q^2 \ \ \forall q \in \{0,1,2\}\,.\end{aligned}$$

[23 marks]

(c) Let $w(t,q) := e^{\kappa\theta(t,q)}$ and write down the ODE w satisfies. [5 marks]

(d) Solve the ODE for w with $q = 1$ and thus derive an explicit expression for $\theta(t,1)$. [5 marks]

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If you have used additional space for working then please tick here: ☐

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Exam Hall Regulations

The following is a copy of a Notice which is displayed in Edinburgh University Examination Halls for the information of students and staff.

The University of Edinburgh Exam Hall Regulations

1. An examination attendance sheet is laid on the desk for each student to complete upon arrival. These are collected by an invigilator after thirty minutes have elapsed from the start of the examination. Students are not normally allowed to enter the examination hall more than thirty minutes after the start of the examination.
2. Students arriving after the start of the examination are required to complete a "Late arrival form" which requires them to sign a statement that they understand that they are not entitled to any additional time. Students are not allowed to leave the examination hall less than thirty minutes after the commencement of the examination or within the last fifteen minutes of the examination.
3. Books, papers, briefcases and cases must be left at the back or sides of the examination room. It is an offence against University discipline for a student to have in their possession in the examination any material relevant to the work being examined unless this has been authorised by the examiners.
4. Students must take their seats within the block of desks allocated to them and must not communicate with other students either by word or sign, nor let their papers be seen by any other student.
5. Students are prohibited from deliberately doing anything that might distract other students. Students wishing to attract the attention of an invigilator shall do so without causing a disturbance. Any student who causes a disturbance in an examination room may be required to leave the room, and shall be reported to the University Secretary.
6. Personal handbags must be placed on the floor at the student's feet; they should be opened only in full view of an invigilator.
7. An announcement will be made to students that they may start the examination. Students must stop writing immediately when the end of the examination is announced.
8. Answers should be written in the script book provided. Rough work, if any, should be completed within the script book and subsequently crossed out. Script books must be left in the examination hall.
9. During an examination, students will be permitted to use only such dictionaries, other reference books, computers, calculators and other electronic technology as have been issued or specifically authorised by the examiners. Such authorisation must be confirmed by the Registry.
10. The use of mobile telephones is not permitted and mobile telephones must be switched off during an examination.
11. It is an offence against University discipline for any student knowingly
    - to make use of unfair means in any University examination
    - to assist a student to make use of such unfair means
    - to do anything prejudicial to the good conduct of the examination, or
    - to impersonate another student or allow another student to impersonate them
12. Students will be required to display their University Card on the desk throughout all written degree examinations and certain other examinations. If a card is not produced, the student will be required to make alternative arrangements to allow their identity to be verified before the examination is marked.
13. Smoking and eating are not allowed inside the examination hall.
14. If an invigilator suspects a student of cheating, they shall impound any prohibited material and shall inform the Examinations Office as soon as possible.
15. Cheating is an extremely serious offence, and any student found by the Discipline Committee to have cheated or attempted to cheat in an examination may be deemed to have failed that examination or the entire diet of examinations, or be subject to such penalty as the Discipline Committee considers appropriate.