Throughout we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports all the random variables and processes introduced. Results covered in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Let $T > 0$, $\kappa > 0$, $\lambda > 0$, $\sigma \in \mathbb{R}$, $\theta > 0$ s.t. $\theta \neq \lambda$ be fixed constants. Let W be a real-valued Wiener process. Consider the following optimal liquidation problem. The selling rate is denoted α. Admissible selling rates α are adapted to the filtration generated by W and such that $\mathbb{E}\int_0^T \alpha_u^2\, du < \infty$. The mid-price process is

$$dS_u = -\lambda \alpha_u\, du + \sigma\, dW_u\,, \quad u \in [t,T]\,, S_t = S > 0$$

and the inventory process is

$$dQ_u = -\alpha_u\, du\,, \quad u \in [t,T]\,, Q_t = q > 0\,.$$

The objective is to maximize

$$M(t,q,S,\alpha) := \mathbb{E}\left[\int_t^T (S_u - \tfrac{1}{2}\kappa\alpha_u)\alpha_u\, du + Q_T S_T - \tfrac{1}{2}\theta|Q_T|^2\right]$$

over admissable α.

(1) Find a formula for the value function $V(t,q,S) = \sup_\alpha M(t,q,S,\alpha)$ and for the optimal Markov control i.e. a function $A^* = A^*(t,q,S)$ such that if α^* is the optimal control and Q^*, S^* are the optimally controlled inventory and price processes then $\alpha_t^* = A^*(t, Q_t^*, S_t^*)$. **[25 marks]**

(2) You may have used either the HJB equation or the Pontryagin optimality principle to solve (1). If you used the HJB equation please apply the verification theorem. If you used the Pontryagin optimality please make sure to explain why the conditions for applying Pontryagin's optimality principle as a sufficient condition apply. **[10 marks]**

(3) Hence or otherwise find the optimal Markov control and the value for the case where we introduce the additional constraint that $Q_T = 0$. **[5 marks]**

2. Let W be a d'-dimensional Wiener process. Let $m \in \mathbb{N}$ and $T > 0$ be fixed. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d'}$ satisfy the condition: there is $K > 0$ such that

$$|b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leq K|x - x'| \;\; \forall x, x' \in \mathbb{R}^d\,.$$

Assume that $|b(0)| \leq K$ and $|\sigma(0)| \leq K$. Let $X^{t,x}$ be the unique solution of

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})\, dr + \int_t^s \sigma(X_r^{t,x})\, dW_r\,, \quad s \in [t,T]\,.$$

Assume you have shown that for any $m \in \mathbb{N}$ there is $c > 0$ (depending on K, m and T) such that for all $x \in \mathbb{R}^d$ we have

$$\sup_{s\in[t,T]} \mathbb{E}|X_s^{t,x}|^{2m} \leq c(1 + |x|^{2m})\,.$$

Show that for any $m \in \mathbb{N}$ there is $c > 0$ (depending on T, K, m) such that for all $x \in \mathbb{R}^d$ we have

$$\mathbb{E}|X_{s'}^{t,x} - X_s^{t,x}|^{2m} \leq c(1 + |x|^{2m})|s' - s|^m\,.$$

[30 marks]

3. Let $\sigma \in \mathbb{R}$, $T > 0$, $K > 0$ be fixed constants. Let W be a real-valued Wiener process.

(1) Let

$$dS_r = \sigma S_r \, dW_r \;\; r \in [t,T]\,, \;\; S_t = S > 0\,.$$

Let $v(t,S) = \mathbb{E}[\max(S_T - K, 1) | S_t = S]$. Use Feynman–Kac formula to write down the PDE that v satisfies. Express v using the Black–Scholes formula. **[5 marks]**

(2) Let $\mathcal{A}$ denote the class of all processes α that are adapted to the filtration generated by W and such that $\mathbb{E}\int_0^T \alpha_s^2 \, ds < \infty$. Let

$$dS_r = \sigma S_r \alpha_r \, dt + \sigma S_r \, dW_r \;\; r \in [t,T]\,, \;\; S_t = S \in \mathbb{R}$$

and denote the solution to the equation started from S at time $t \in [0,T]$ and controlled by α as $S^{t,S,\alpha}$. Let

$$u(t,S) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[-\int_t^T \tfrac{1}{2}\alpha_s^2 \, ds + g(S_T^{t,S,\alpha}) \right]\,,$$

where $g(S) = \ln\left(\max(S - K, 1)\right)$.

(a) Write down the HJB equation satisfied by u. **[4 marks]**

(b) Show that this is equivalent to

$$\begin{aligned} \partial_t u + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 u + \tfrac{1}{2}\sigma^2 S^2 \left(\partial_S u\right)^2 &= 0 \;\text{ in }\; [0,T] \times \mathbb{R} \\ u(T,\cdot) &= g \;\text{ on }\; \mathbb{R}\,. \end{aligned}$$

[6 marks]

(c) Solve the HJB equation. *Hint:* Do an exponential transformation of u and note that you thus obtain a *linear* PDE. You should recognise this linear PDE. **[15 marks]**

You do not need to carry out verification.