Throughout we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports all the random variables and processes introduced. Results covered in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Let $T > 0$, $\kappa > 0$, $\lambda > 0$, $\sigma \in \mathbb{R}$, $\theta > 0$ s.t. $\theta \neq \lambda$ be fixed constants. Let W be a real-valued Wiener process. Consider the following optimal liquidation problem. The selling rate is denoted α. Admissible selling rates α are adapted to the filtration generated by W and such that $\mathbb{E}\int_0^T \alpha_u^2\,du < \infty$. The mid-price process is

$$dS_u = -\lambda \alpha_u\,du + \sigma\,dW_u\,,\ \ u \in [t,T]\,, S_t = S > 0$$

and the inventory process is

$$dQ_u = -\alpha_u\,du\,,\ \ u \in [t,T]\,, Q_t = q > 0\,.$$

The objective is to maximize

$$M(t,q,S,\alpha) := \mathbb{E}\left[\int_t^T (S_u - \tfrac{1}{2}\kappa\alpha_u)\alpha_u\,du + Q_T S_T - \tfrac{1}{2}\theta|Q_T|^2\right]$$

over admissable α.

(a) Find a formula for the value function $V(t,q,S) = \sup_\alpha M(t,q,S,\alpha)$ and for the optimal Markov control i.e. a function $A^* = A^*(t,q,S)$ such that if α^* is the optimal control and Q^*, S^* are the optimally controlled inventory and price processes then $\alpha^*_t = A^*(t, Q^*_t, S^*_t)$. **[25 marks]**

(b) You may have used either the HJB equation or the Pontryagin optimality principle to solve (1). If you used the HJB equation please apply the verification theorem. If you used the Pontryagin optimality please make sure to explain why the conditions for applying Pontryagin's optimality principle as a sufficient condition apply. **[10 marks]**

(c) Hence or otherwise find the optimal Markov control and the value for the case where we introduce the additional constraint that $Q_T = 0$. **[5 marks]**

Comment: *Here (a) and (b) are a variation on a problem that's been solved in class. There are many ways to proceed. Part (c) is new.*

Solution:

(a) We will first note that

$$\mathbb{E}\int_0^T Q_r^2\,dr = \mathbb{E}\int_t^T \left(q + \int_t^r \alpha_u\,du\right)^2 dr \leq 2q^2(T-t) + 2\mathbb{E}\int_t^T \left(\int_t^r \alpha_u\,du\right)^2 dr$$
$$\leq 2q^2(T-t) + 2\mathbb{E}\int_t^T \left(\int_t^r du\right)\int_t^r \alpha_u^2\,du\,dr \leq 2q^2(T-t) + 2T^2\mathbb{E}\int_0^T \alpha_u^2\,du < \infty\,.$$

Hence $\mathbb{E}\int_0^T Q_r\,dW_t = 0$. Hence

$$\mathbb{E}[Q_T S_T] = qS + \mathbb{E}\int_t^T (-\lambda Q_u \alpha_u - S_u\alpha_u)\,du$$

and so

$$M(t,q,S,\alpha) = qS + \mathbb{E}\left[-\int_t^T (\tfrac{1}{2}\kappa\alpha_u^2 + \lambda Q_u\alpha_u)\,du - \tfrac{1}{2}\theta|Q_T|^2\right] =: qS + J(t,q,\alpha)\,.$$

So we see it's enough to maximize J over α and a seemingly 2 dimensional problem is just 1 dimensional.

Let $v(t,q) := \sup_\alpha J(t,q,\alpha)$. The HJB equation is

$$\partial_t v + \sup_a \Big(- a(\partial_q v + \tfrac{1}{2}\kappa a + \lambda q)\Big) = 0 \;\text{ in }\; [0,T]\times\mathbb{R}\,,\;\; v(T,q) = -\tfrac{1}{2}\theta|q|^2 \;\;\forall q\in\mathbb{R}\,.$$

Since the function $a \mapsto -a\partial_q v - \frac{1}{2}\kappa a^2 - \lambda q a$ is concave it's easy to see that the supermum is achieved for $a^* = -\frac{\lambda q + \partial_q v}{\kappa}$ and so

$$\sup_a \Big(- a(\partial_q v + \tfrac{1}{2}\kappa a + \lambda q)\Big) = \tfrac{1}{\kappa}(\lambda q + \partial_q v)(\tfrac{1}{2}\partial_q v + \tfrac{1}{2}\lambda v) = \tfrac{1}{2\kappa}(\lambda q + \partial_q v)^2\,.$$

Hence the HJB is

$$\partial_t v + \tfrac{1}{2\kappa}(\lambda q + \partial_q v)^2 = 0 \;\text{ in }\; [0,T]\times\mathbb{R}\,,\;\; v(T,q) = -\tfrac{1}{2}\theta|q|^2 \;\;\forall q\in\mathbb{R}\,.$$

As always we need to make a guess as to what form the solution will take. Since it seems like we'll have q^0, q^1 and q^2 appearing let us try

$$v(t,q) = A(t)q^0 + B(t)q^1 + \tfrac{1}{2}C(t)q^2\,,\;\; A(T)=0\,,\;\; B(T)=0\,,\;\; C(T) = -\theta\,,$$

with $A,B,C \in C^1([0,T])$. Substituting into the HJB and collecting terms we get

$$\begin{aligned} 0 &= A' + \tfrac{1}{2\kappa}B^2\,,\\ 0 &= B' + \tfrac{1}{2\kappa}2(\lambda + C)B\,,\\ 0 &= \frac{1}{2}C' + \tfrac{1}{2\kappa}(\lambda + C)^2\,. \end{aligned}$$

From this and the terminal conditions we see that $A(t) = B(t) = 0$ for $t\in[0,T]$. Letting $\gamma(t) = \lambda + C(t)$ we only have to solve

$$0 = \gamma' + \frac{1}{\kappa}\gamma^2\,,\;\; \gamma(T) = \lambda - \theta\,.$$

Separating variables we get that

$$\gamma^{-2}d\gamma = -\tfrac{1}{\kappa}\,dt$$

which is

$$-\gamma^{-1} = -\tfrac{1}{\kappa}t + \text{const}\,.$$

Hence

$$\gamma^{-1} = -\tfrac{1}{\kappa}(T-t) + (\lambda-\theta)^{-1}\,.$$

Rearranging we get

$$C(t) = \kappa\left(t - T + \tfrac{\kappa}{\lambda-\theta}\right)^{-1} - \lambda\,.$$

So that finally

$$V(t,q,S) = qS + v(t,q) = qS - \left(\tfrac{\kappa}{2}\left(T - t + \tfrac{\kappa}{\theta-\lambda}\right)^{-1} - \tfrac{1}{2}\lambda\right)q^2$$

and

$$A^*(t,q,S) = \left(\left(T - t + \tfrac{\kappa}{\theta-\lambda}\right)^{-1}\right)q\,.$$

(b) To carry out verification we note that

(i) $(t,q)\mapsto a^*(t,q) := \left(\left(T - t + \tfrac{\kappa}{\theta-\lambda}\right)^{-1}\right)q$ is clearly measurable.

(ii) The equation

$$dQ_u = -a^*(t, Q_u)\, du$$

has unique solutions since the coefficient is linear in q.

(iii) We note that there is no randomness once we've reduced the dimension so there is no stochastic integral to check.

This completes the verification.

(c) Let us re-examine the objective, this time writing its dependence on θ explicitly

$$M^\theta(t,q,S,\alpha) := \mathbb{E}\left[\int_t^T (S_u - \tfrac{1}{2}\kappa\alpha_u)\alpha_u\, du + Q_T S_T - \tfrac{1}{2}\theta|Q_T|^2\right].$$

We note that for any t, q, S, α the integrand is monotone decreasing in θ. Thus by monotone convergence theorem

$$\begin{aligned} M^\infty(t,q,S,\alpha) &= \mathbb{E}\left[\int_t^T (S_u - \tfrac{1}{2}\kappa\alpha_u)\alpha_u\, du + Q_T S_T - \lim_{\theta\to\infty}\tfrac{1}{2}\theta|Q_T|^2\right] \\ &= \begin{cases} \mathbb{E}\left[\int_t^T (S_u - \tfrac{1}{2}\kappa\alpha_u)\alpha_u\, du\right] & \text{if } Q_T = 0\,, \\ -\infty \ \text{ if } Q_T \neq 0\,. \end{cases} \end{aligned}$$

Thus the constrained case corresponds to the case when we take $\theta \to \infty$. In that case

$$V(t,q,S) = qS - \left(\tfrac{\kappa}{2}(T-t)^{-1} + \tfrac{1}{2}\lambda\right) q^2$$

and

$$A^*(t,q,S) = \frac{q}{T-t}\,.$$

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2. Let W be a d'-dimensional Wiener process. Let $m \in \mathbb{N}$ and $T > 0$ be fixed. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d'}$ satisfy the condition: there is $K > 0$ such that

$$|b(x) - b(x')| + |\sigma(x) - \sigma(x')| \le K|x - x'| \ \ \forall x, x' \in \mathbb{R}^d\,.$$

Assume that $|b(0)| \le K$ and $|\sigma(0)| \le K$. Let $X^{t,x}$ be the unique solution of

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})\, dr + \int_t^s \sigma(X_r^{t,x})\, dW_r\,, \ \ s \in [t,T]\,.$$

Assume you have shown that for any $m \in \mathbb{N}$ there is $c > 0$ (depending on K, m and T) such that for all $x \in \mathbb{R}^d$ we have

$$\sup_{s\in[t,T]} \mathbb{E}|X_s^{t,x}|^{2m} \le c(1 + |x|^{2m})\,.$$

Show that for any $m \in \mathbb{N}$ there is $c > 0$ (depending on T, K, m) such that for all $x \in \mathbb{R}^d$ we have

$$\mathbb{E}|X_{s'}^{t,x} - X_s^{t,x}|^{2m} \le c(1 + |x|^{2m})|s' - s|^m\,.$$

[30 marks]

Comment: *Related to proofs seen in the lectures but not actually given.*

Solution: We start by noting that b and σ have linear growth: for any $x \in \mathbb{R}^d$

$$|b(x)| = |b(x) - b(0) + b(0)| \le K|x| + |b(0)|$$

and similarly for σ. Clearly

$$|X_{s'}^{t,x} - X_s^{t,x}|^{2m} \leq 2^{m+1} \left| \int_s^{s'} b(X_u)\, du \right|^{2m} + 2^{m+1} \left| \int_s^{s'} \sigma(X_u)\, dW_u \right|^{2m} .$$

Applying Hölder's inequality twice we have

$$\left| \int_s^{s'} b(X_u)\, du \right|^{2m} \leq (s'-s)^m \left(\int_t^T |b(X_u)|^2\, du \right)^m \leq (s'-s)^m c_{T,m} \int_s^T |b(X_u)|^{2m}\, du\,.$$

Using the growth assumption and moment bound for solutions

$$\mathbb{E} \int_s^T |b(X_u)|^{2m}\, du \leq c_{m,T} \left(1 + \int_s^T \mathbb{E}|X_u|^{2m}\, du \right) \leq c_{T,m}(1+|x|^{2m})\,.$$

From the Burkholder–Davis–Gundy inequality, Hölder's inequality and then with the growth assumption and moment bound

$$\begin{aligned}
\mathbb{E}\left[\left| \int_s^{s'} \sigma(X_u)\, dW_u \right|^{2m} \right] &\leq c_m \mathbb{E}\left[\left(\int_s^{s'} |\sigma(X_u)|^2\, du \right)^m \right] \\
&\leq c_m \mathbb{E}\left[\left(\int_s^{s'} 1^{\frac{m}{m-1}}\, du \right)^{m-1} \left(\int_s^{s'} |\sigma(X_u)|^{2m}\, du \right) \right] \\
&\leq c_m (s'-s)^{m-1} \left(\int_s^{s'} \mathbb{E}|\sigma(X_u)|^{2m}\, du \right) \leq c_{m,T}(1+|x|^{2m})|s'-s|^m\,.
\end{aligned}$$

Altogether

$$\mathbb{E}|X_{s'}^{t,x} - X_s^{t,x}|^{2m} \leq c_{m,T}(1+|x|^{2m})|s'-s|^m\,.$$

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3. Let $\sigma \in \mathbb{R}$, $T > 0$, $K > 0$ be fixed constants. Let W be a real-valued Wiener process.

(a) Let

$$dS_r = \sigma S_r\, dW_r \;\; r \in [t,T]\,, \;\; S_t = S > 0\,.$$

Let $v(t,S) = \mathbb{E}[\max(S_T - K, 1)|S_t = S]$. Use Feynman–Kac formula to write down the PDE that v satisfies. Express v using the Black–Scholes formula. **[5 marks]**

(b) Let $\mathcal{A}$ denote the class of all processes α that are adapted to the filtration generated by W and such that $\mathbb{E}\int_0^T \alpha_s^2\, ds < \infty$. Let

$$dS_r = \sigma S_r \alpha_r\, dt + \sigma S_r\, dW_r \;\; r \in [t,T]\,, \;\; S_t = S \in \mathbb{R}$$

and denote the solution to the equation started from S at time $t \in [0,T]$ and controlled by α as $S^{t,S,\alpha}$. Let

$$u(t,S) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[-\int_t^T \tfrac{1}{2}\alpha_s^2\, ds + g(S_T^{t,S,\alpha}) \right],$$

where $g(S) = \ln\left(\max(S-K,1)\right)$.

(i) Write down the HJB equation satisfied by u. **[4 marks]**

(ii) Show that this is equivalent to

$$\partial_t u + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 u + \tfrac{1}{2}\sigma^2 S^2 \left(\partial_S u\right)^2 = 0 \text{ in } [0,T]\times\mathbb{R}$$
$$u(T,\cdot) = g \text{ on } \mathbb{R}\,.$$

[6 marks]

(iii) Solve the HJB equation. *Hint:* Do an exponential transformation of u and note that you thus obtain a *linear* PDE. You should recognise this linear PDE. **[15 marks]**

You do not need to carry out verification.

Comment: *Part a) only serves as an extended hint to part b) but should be trivial. Part b) is a variation on Cole-Hopf transform method to solve HJB equation; this has been seen but not in this context.*

Solution:

(a) From Feynman–Kac we know that

$$\partial_t v + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 v = 0 \text{ in } [0,T)\times\mathbb{R}$$
$$v(T,S) = \max(S-K,1) \;\; \forall S\in\mathbb{R}\,.$$

We note that $\max(S-K,1) = \max(S-(K+1),0)+1$ and so

$$v(t,S) = \mathbb{E}[\max(S_T-K,1)|S_t=S] = \mathbb{E}[\max(S_T-(K+1),0)|S_t=S]+1\,.$$

Thus $v(t,S)$ is 1 plus the value given by a Black–Scholes formula for call options with risk-free rate 0, volatility σ, strike $K+1$, initial asset price S and time to maturity $T-t$.

(b) (i) The HJB equation is

$$\partial_t u + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 u + \sup_{a\in\mathbb{R}}\left(\sigma S a \partial_S u - \tfrac{1}{2}a^2\right) = 0 \text{ in } [0,T)\times\mathbb{R}$$
$$u(T,\cdot) = g \text{ on } \mathbb{R}\,.$$

(ii) We note that since $a\mapsto \sigma S a\partial_S u - \frac{1}{2}a^2$ is concave regardless of the values of σ, S, $\partial_S u$ we can find the a for which supremum above is achieved simply by solving

$$0 = \sigma S\partial_S u - a$$

substituting the solution:

$$\sup_{a\in\mathbb{R}}\left(\sigma S a\partial_S u - \tfrac{1}{2}a^2\right) = \sigma^2S^2(\partial_S u)^2 - \tfrac{1}{2}\sigma^2S^2(\partial_S u)^2 = \tfrac{1}{2}\sigma^2S^2(\partial_S u)^2\,.$$

Hence we get

$$\partial_t u + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 u + \tfrac{1}{2}\sigma^2 S^2 \left(\partial_S u\right)^2 = 0 \text{ in } [0,T)\times\mathbb{R}$$
$$u(T,\cdot) = g \text{ on } \mathbb{R}\,.$$

(iii) Let $v = e^u$ so that

$$\partial_t v = e^u\partial_t u\,,\quad \partial_S v = e^u\partial_S u\,,\quad \partial_S^2 v = e^u(\partial_S u)^2 + e^u\partial_S^2 u\,.$$

Multiplying the above PDE by e^u and noting that $e^u\partial_S^2 u = \partial_S^2 v = e^u(\partial_S u)^2$ we get

$$0 = \partial_t v + \tfrac{1}{2}\sigma^2S^2\partial_S^2 v - \tfrac{1}{2}\sigma^2S^2e^u(\partial_S u)^2 + \tfrac{1}{2}\sigma^2S^2e^u(\partial_S u)^2.$$

Hence we get that v must solve

$$\partial_t v + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 v = 0 \text{ in } [0,T)\times\mathbb{R}$$
$$v(T,\cdot) = e^g \text{ on } \mathbb{R}\,.$$

But $v(T,S) = e^{g(S)} = e^{\ln(\max(S-K,1)} = \max(S-K,1)$.
Hence $u(t,S) = \ln v(t,S)$ where $v(t,S)$ is given by the solution from part (a).