Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results covered in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Let $\mu, r, \sigma, \delta, \gamma \in \mathbb{R}$ be constants s.t. $\sigma \neq 0$, $\delta > 0$, $\gamma \in (0,1)$. Consider the process

$$dX_t = X_t\big(\nu_t(\mu - r) + r - \kappa_t\big)\,dt + \nu_t \sigma X_t\,dW_t\,, t \geq 0\,, X_0 = x > 0$$

and let us write $X_t = X_t^{x,\nu,\kappa}$ to emphasize the dependence on the starting point x and the controls $\nu = (\nu_t)_{t\geq 0}$ and $\kappa = (\kappa_t)_{t\geq 0}$. We say that ν, κ are admissible if they are adapted and bounded. Let

$$v(x) = \sup_{\nu,\kappa} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\big(\kappa_t X_t^{x,\nu,\kappa}\big)^\gamma\,dt\right],$$

where the supremum is taken over all admissible ν, κ.

Using the "guess" $v(x) = \lambda x^\gamma$ for some $\lambda > 0$ solve the HJB equation giving explicit form for the constant λ depending only on μ, r, σ, δ and γ. You don't need to use the verification theorem here. **[30 marks]**

Hint. An infinite time stochastic control problem can be written as

$$v(x) = \sup_\alpha \mathbb{E}\int_0^\infty e^{-\delta t} f^{\alpha_t}(X_t^{x,\alpha})\,dt\,,$$

where the supremum is taken over admissible controls and subject to

$$dX_t^{x,\alpha} = b^{\alpha_t}(X_t^{x,\alpha})\,dt + \sigma^{\alpha_t}(X_t^{x,\alpha})\,dW_t\,,\ \ t \in [0,\infty)\,,\ \ X_0^{x,\alpha} = x.$$

The HJB equation for this infinite-time-horizon problem is

$$\sup_{a\in A}\left[\tfrac{1}{2}(\sigma^a)^2 v'' + b^a v' - \delta v + f^a\right] = 0 \ \text{ on } \ [0,\infty)\times\mathbb{R}\,.$$

Comment: *Students have seen this in the finite time setting with no consumption. With the "ansatz" for the solution is given , this shouldn't be hard.*

Solution: Let

$$L^{u,k}v = \frac{1}{2}u^2\sigma^2x^2v''(x) + [u(\mu - r) + r - k]xv'(x) - \delta v(x)\,.$$

The HJB is

$$\sup_{u,k}\Big(L^{u,k}v(x) + (kx)^\gamma\Big) = 0\,,\ \ x \in (0,\infty)\,.$$

Let us use the ansatz $v(x) = \lambda x^\gamma$ for some $\lambda > 0$. Then

$$v'(x) = \gamma v(x)x^{-1}\,,\ v''(x) = -\gamma(1-\gamma)v(x)x^{-2}\,,\ x^\gamma k^\gamma = k^\gamma v(x)\lambda^{-1}\,.$$

So the HJB becomes

$$\sup_{u,k}\bigg(-\tfrac{1}{2}u^2\sigma^2\gamma(1-\gamma)v(x) + \big[u(\mu - r) + r - k\big]\gamma v(x) - \delta v(x) + k^\gamma v(x)\lambda^{-1}\bigg) = 0\,,\ \ x \in (0,\infty)\,.$$

Since we have $v(x) > 0$ we can divide by this further simplifying the situation. By inspection this is concave in (u,k) and so we can maximize it by finding the point where the gradient vector $\nabla_{u,k}$ is zero. That is

$$-u\sigma^2\gamma(1-\gamma) + (\mu - r)\gamma = 0 \ \text{ i.e. } \ \hat{u} = \frac{\mu - r}{\sigma^2(1-\gamma)}$$

and

$$-\gamma + \frac{\gamma}{\lambda}k^{\gamma-1} = 0 \ \text{ i.e. } \ \hat{k} = \lambda^{\frac{1}{\gamma-1}}\,.$$

Note that $\hat{k}^\gamma \lambda^{-1} = \hat{k}$. The HJB equation reduces to

$$-\tfrac{1}{2}\hat{u}^2\sigma^2\gamma(1-\gamma) + \big[\hat{u}(\mu - r) + r\big]\gamma - \delta + (1-\gamma)\lambda^{\frac{1}{\gamma-1}} = 0$$

i.e.

$$\lambda = \left(\frac{1-\gamma}{\delta - C}\right)^{1-\gamma}$$

with

$$C := -\tfrac{1}{2}\hat{u}^2\sigma^2\gamma(1-\gamma) + \big[\hat{u}(\mu - r) + r\big]\gamma = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{\gamma}{1-\gamma} + r\gamma\,.$$

Hence

$$v(x) = \left(\frac{1-\gamma}{\delta - C}\right)^{1-\gamma} x^\gamma\,.$$

2.

(a) Assume that $g \in C^1(\mathbb{R})$. Show that $v(t,x) = g(x + (T-t))$ is a solution to

$$\partial_t v + \partial_x v = 0 \ \text{ on } \ [0,T]\times\mathbb{R}\,,$$
$$v(T,x) = g(x) \ \ \forall x \in \mathbb{R}\,.$$

[5 marks]

(b) Let $\mathcal{A} = \big\{\alpha : [0,T] \to \{-1,0,1\} : \alpha \ \text{ is measurable}\big\}$ and let

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} |X_T^{t,x,\alpha}|^2 \ \text{ where } \ X_T^{t,x,\alpha} = x + \int_t^T \alpha(s)\,ds\,.$$

Noting that the control has to take values in $\{-1,0,1\}$ guess an optimal Markovian control and hence solve the Bellman equation for v. Use this to verify that your guess is indeed an optimal control. **[25 marks]**

Comment: *Part (a) is there purely to help with part (b) but it should be easy for anyone who's done calculus. Part (b) is an unseen question using Bellman / HJB PDE.*

Solution:

(a) We just calculate $\partial_t v = -(\partial_x g)(x + (T-t))$, $\partial_x v = (\partial_x g)(x + (T-t))$. Hence

$$\partial_t v + \partial_x v = -(\partial_x g)(x+(T-t)) + (\partial_x g)(x + (T-t)) = 0\,.$$

Moreover $v(T,x) = g(x)$ as required. **[5 marks]**

(b) By inspection we can see that

$$a(t,x) = \begin{cases} -1 & \text{if } \ x > T - t\,,\\ \ \ 0 & \text{if } \ x \in [-(T-t), T-t]\,,\\ \ \ 1 & \text{if } \ x < -(T-t)\end{cases}$$

is an optimal control (not unique). **[10 marks]**

We now write the Bellman equation

$$\partial_t v + \inf_{a\in\{-1,0,1\}} a\partial_x v = 0 \ \text{ on } \ [0,T]\times\mathbb{R}\,, \ \ v(T,x) = x^2\,.$$

With the above control we have three cases to consider.

(i) If $x > T - t$ then the Bellman PDE becomes

$$\partial_t v - \partial_x v = 0 \text{ on } [0,T] \times \mathbb{R}, \; v(T,x) = x^2 .$$

Looking at how part (a) of the question worked it's easy to check that $v(t,x) = (x - (T-t))^2$ solves this. Indeed $\partial_t v = 2(x-(T-t))$, $\partial_x v = 2(x-(T-t))$ so the equation and terminal condition is satisfied.

(ii) If $x \in [-(T-t), T-t]$ then $v(t,x) = 0$ solves the equation.

(iii) Finally if $x < -(T-t)$ then the Bellman PDE becomes

$$\partial_t v + \partial_x v = 0 \text{ on } [0,T] \times \mathbb{R}, \; v(T,x) = x^2 .$$

Again using part (a) we see that $v(t,x) = (x + (T-t))^2$ is the solution. Indeed $\partial_t v = -2(x+(T-t)$, $\partial_x v = 2(x+(T-t))$ so the equation and terminal condition is satisfied.

We can summarize this as:

$$v(t,x) = \begin{cases} (x-(T-t))^2 & \text{if } x > T-t , \\ 0 & \text{if } x \in [-(T-t), T-t] , \\ (x+(T-t))^2 & \text{if } x < -(T-t) . \end{cases}$$

[10 marks]

Now we have our candidate and we need to check that it solves the HJB equation. There are again three cases:

(i) If $x \in [-(T-t), T-t]$ then $\partial_x v = 2(x-(T-t)) > 0$ and so $\arg\inf_{a\in\{-1,0,1\}} a\partial_x v = -1$.

(ii) If $x \in [-(T-t), T-t]$ then $\partial_x v = 0$ so $\arg\inf_{a\in\{-1,0,1\}} a\partial_x v \ni 0$.

(iii) Finally if $x < -(T-t)$ then $\partial_x v = 2(x+(T-t)) < 0$ and so $\arg\inf_{a\in\{-1,0,1\}} a\partial_x v = 1$.

This means that our v solves the HJB equation and it also confirms that a above is an optimal control. **[5 marks]**

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3. With the action space $A = \mathbb{R}$ consider the problem

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[-\frac{1}{2}\int_t^T |\alpha_s|^2\,ds + g(X_T^{t,x})\Big]\,,$$
$$dX_s^{t,x} = \alpha_s\,ds + dW_s\,,\ \ s\in[t,T]\,,\ \ X_t^{t,x} = x\in\mathbb{R}\,.$$

Here $g:\mathbb{R}\to\mathbb{R}$ is smooth and bounded. We say $\alpha\in\mathcal{A}$ if it is adapted and $\mathbb{E}\int_0^T|\alpha_t|^2dt<\infty$.

(a) Write down the HJB equation for v and hence show that

$$\partial_t v + \tfrac{1}{2}\partial_{xx}v + \tfrac{1}{2}|\partial_x v|^2 = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,,$$

with $v(T,x) = g(x)$ for $x\in\mathbb{R}$. **[10 marks]**

(b) Let $u(t,x) = e^{v(t,x)}$. Show that

$$\partial_t u + \tfrac{1}{2}\partial_{xx}u = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,,$$

with terminal condition $u(T,x) = e^{g(x)}$ for all $x\in\mathbb{R}$. **[15 marks]**

(c) Show that

$$v(t,x) = \log\int_{\mathbb{R}^d} e^{g(y)}p(T-t,y-x)\,dy\,,$$
$$p(s,z) = (2\pi s)^{-1/2}e^{-\frac{|z|^2}{2s}}\,.$$

[15 marks]

Comment: *The question goes one step beyond linear-quadratic control and as such is unseen. Many intermediate steps are given, making it hopefully accessible.*

Solution:

(a) The HJB equation is

$$\partial_t v + \sup_{a\in A}\Big[\tfrac{1}{2}\partial_{xx}v + a\cdot\partial_x v - \tfrac{1}{2}|a|^2\Big] = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,,$$

with $v(T,x) = g(x)$ for $x\in\mathbb{R}$. We note that $a\mapsto a\cdot y - \frac{1}{2}|a|^2$ is maximized (concave function), for any $y\in\mathbb{R}$ with

$$0 = y - a^* \quad\Longrightarrow\quad a^* = y\,.$$

Hence the HJB equation is

$$\partial_t v + \tfrac{1}{2}\partial_{xx}v + \tfrac{1}{2}|\partial_x v|^2 = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,, \tag{1}$$

with $v(T,x) = g(x)$ for $x\in\mathbb{R}$.

(b) We have $u(t,x) = e^{v(t,x)}$ and we calculate partial derivatives:

$$\partial_t u = e^v\partial_t v\,,\ \ \partial_x u = e^v\partial_x v\,,$$
$$\partial_{xx}u = e^v\partial_{xx}v + \partial_x v\cdot e^v\cdot\partial_x v = e^v\partial_{xx}v + e^v|\partial_x v|^2\,.$$

We multiply the HJB equation (1) by e^v to get

$$e^v\partial_t v + e^v\tfrac{1}{2}\partial_{xx}v + e^v\tfrac{1}{2}|\partial_x v|^2 = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,,$$

which, applying the derivative identities leads to

$$\partial_t u + \tfrac{1}{2}\partial_{xx}u = 0 \ \text{ on } \ [0,T)\times\mathbb{R}\,,$$

with terminal condition $u(T,x) = e^{g(x)}$ for $x\in\mathbb{R}$.

(c) There are many ways to solve the heat equation: here we write down a probabilistic one. The solution is

$$u(t,x) = \mathbb{E}\Big[e^{g(X_T^{t,x})}\Big]\,, \ \ X_s^{t,x} = x + W_{s-t}\,,$$

so

$$u(t,x) = \mathbb{E}\Big[e^{g(x+\sqrt{T-t}Z)}\Big]\,, \ \ Z \sim N(0,1)\,.$$

Hence

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{g(x+\sqrt{T-t}z)}e^{-\frac{|z|^2}{2}}\,dz$$

Introducing $y = x + \sqrt{T-t}z$ and changing variables in the integral we get

$$u(t,x) = \frac{1}{\sqrt{2\pi(T-t)}}\int_{\mathbb{R}} e^{g(y)}e^{-\frac{|y-x|^2}{2(T-t)}}\,dy = \int_{\mathbb{R}} e^{g(y)}p(T-t,y-x)\,dy.$$

Finally undoing the transform (taking log) we get

$$v(t,x) = \log\int_{\mathbb{R}} e^{g(y)}p(T-t,y-x)\,dy.$$
