Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. We consider the standard Black–Scholes model for optimal investment: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. Let X_s denote the investment portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t\in[0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E}\int_0^T \nu_s^2\, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{A}$. Let $g(x) := x^\gamma$, $\gamma \in (0,1)$ and

$$\bar{v}(t,x) := \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[g(X_T^{\nu,t,x})\right] . \tag{1}$$

a) Find a candidate for the optimal control and hence show that the solution to the corresponding Bellman PDE is
$$v(t,x) = \exp\big((T-t)\beta\big)x^\gamma ,$$
where β is a constant given in terms of σ, μ, r and γ. Give an explicit expression for β.
[7 marks]

b) Use verification theorem to check that $\bar{v} = v$ and the candidate optimal control is the true optimal control. **[8 marks]**

Comment: *This question is meant as a straightforward application of Bellman PDE and verification theorems or Pontryiagin's optimality and is available in lecture notes. Full marks will be awarded only if verification theorem was employed correctly.*

Solution:

a) We calculate (Itô formula) that $dB_t = rB_t\, dt$ and $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. We then have (with ψ_t being the number of units of risky asset we hold)

$$dX_t = \psi_t\, dS_t + \frac{X_t - \psi_t S_t}{B_t}\, dB_t = \nu_t X_t \frac{1}{S_t}\, dS_t + \frac{X_t - \nu_t X_t}{B_t}\, dB_t .$$

So
$$dX_t = X_t\left[(\mu - r)\nu_t + r\right] dt + \nu_t X_t \sigma\, dW_t .$$

We can check that the solution to this SDE is of the form $X_t = X_0 \exp(\dots) > 0$ for $X_0 > 0$. The Bellman PDE is

$$\partial_t v + \sup_u \left[\frac{1}{2}\sigma^2 u^2 x^2 \partial_{xx} v + x[(\mu - r)u + r]\partial_x v\right] = 0 \text{ on } [0,T) \times (0,\infty)$$
$$v(T,x) = x^\gamma \ \ \forall x > 0 .$$

Since $X_t > 0$ for all $t \in [0,T]$ the spatial domain is $(0,\infty)$.

The domain has to be specified and justified to get full marks.

We "guess" the form of the solution

$$v(t,x) = \lambda(t)x^\gamma$$

with $\lambda \in C^1([0,T])$ and $\lambda > 0$. Hence we have $\partial_t v = \lambda'(t)x^\gamma$, $\partial_x v = \lambda(t)\gamma x^{\gamma-1}$, $\partial_{xx} v = \gamma(\gamma-1)x^{\gamma-2}$. So we get

$$\lambda'(t)x^\gamma + \sup_{u\in\mathbb{R}} \left[\tfrac{1}{2}\sigma^2 u^2 \gamma(\gamma-1)\lambda(t)x^\gamma + \lambda(t)\left((\mu - r)u + r\right)\gamma x^\gamma\right] = 0 .$$

We can divide by $x^\gamma > 0$. The function $u \mapsto \frac{1}{2}\sigma^2 u^2 \gamma(\gamma-1)\lambda(t) + \lambda(t)\left((\mu - r)u + r\right)\gamma$ is maximized (calculus and concavity) when

$$0 = \sigma^2 u \gamma(\gamma - 1) + (\mu - r)\gamma$$

i.e.

$$u^* = \frac{\mu - r}{\sigma^2(1-\gamma)}\,.$$

The maximum itself is

$$\beta := \frac{1}{2}\sigma^2 (u^*)^2 \gamma(\gamma-1) + (\mu - r)\gamma u^* + r\gamma\,.$$

Thus

$$\lambda'(t) = -\beta\lambda(t)\,,\;\; \lambda(T) = 1 \quad \Longrightarrow \lambda(t) = \exp\left((T-t)\beta\right)\,.$$

We have established that $v(t,x) = \exp((T-t)\beta)x^\gamma$ is a solution to the Bellman PDE.

b) Let us check whether it's the value function of the control problem using verification. Moreover the Markovian optimal control $\hat{u}(t,x) = \frac{\mu - r}{\sigma^2(1-\gamma)}$ is constant and hence certainly measurable. The wealth equation with the optimal control is

$$d\hat{X}_t = \hat{X}_t\left[(\mu - r)\hat{u} + r\right]\,dt + \hat{u}\hat{X}_t\sigma\,dW_t\,.$$

This is a linear SDE with Lipschitz coefficients so it has unique solution which moreover has all the moments when started from deterministic initial value. In particular $\mathbb{E}\sup_{t\le T}|\hat{X}_t|^{2\gamma} < \infty$. We consider

$$t' \mapsto \int_t^{t'} \gamma(\hat{X}_s)^{\gamma-1}\hat{u}\hat{X}_s\sigma\,dW_s = \hat{u}\gamma\sigma\int_t^{t'}(\hat{X}_s)^\gamma\,dW_s\,.$$

Now

$$\mathbb{E}\int_0^T |\hat{X}_t|^{2\gamma}\,dt < \infty$$

because of the moment bound above. So the stochastic integral is a martingale. So the verification is complete, the constant strategy $\hat{u}$ is optimal and the optimal value for this control is $v = \bar{v}$.

2. A producer with production rate $X = X_t$ at time t may allocate a portion $\alpha = \alpha_t$ of their production rate to reinvestment (thus increasing production rate) and $1-\alpha_t$ to actual production of a storable good. Thus

$$dX_t = \gamma \alpha_t X_t \, dt\,, \quad t \in [0,T]\,, \quad X_0 = x > 0\,,$$

where $\gamma > 0$ is a constant. The admissible controls are measurable maps $t \mapsto \alpha_t \in [0,1]$. The objective is to maximize the amount of goods produced over time $[0,T]$ i.e. maximize

$$J(x,\alpha) := \int_0^T (1-\alpha_t) X_t \, dt\,.$$

i) Use Pontryagin's maximum principle to show that an optimal control is

$$\alpha_t = \begin{cases} 0 \ \text{ if } \ Y_t < \frac{1}{\gamma}\,, \\ 1 \ \text{ if } \ Y_t > \frac{1}{\gamma}\,, \end{cases}$$

where Y_t is the solution of the adjoint (backward) equation in Pontryagin optimality.
[5 marks]

ii) Assume that $T > \frac{1}{\gamma}$. Show that since $Y_T = 0$ we have

$$Y_t = \begin{cases} (T-t) \ \text{ if } \ t \in (T-\frac{1}{\gamma}, T]\,, \\ \frac{1}{\gamma}\exp\left(\gamma\left(T - \frac{1}{\gamma}\right) - \gamma t\right) \ \text{ if } \ t \in [0, T-\frac{1}{\gamma}]\,. \end{cases}$$

[5 marks]

iii) Hence show that the optimally controlled state is given by

$$X_t = \begin{cases} x e^{\gamma t} \ \text{ if } \ t \in [0, T-\frac{1}{\gamma}]\,, \\ x e^{\gamma\left(T-\frac{1}{\gamma}\right)} \ \text{ if } \ t \in (T-\frac{1}{\gamma}, T]\,. \end{cases}$$

[5 marks]

Comment: *An application of Pontryagin's optimality that's not been seen.*

Solution:

i) We can solve the controlled ODE to see that $X_t = x \exp\left(\gamma \int_0^t \alpha_r \, dr\right) > 0$.

The Hamiltonian is $H(x,y,a) = \gamma a x y + (1-a)x = ax\cdot(\gamma y - 1) + x$. This is a linear function of a. Since we only need to consider $x > 0$ this will be increasing when $\gamma y - 1 > 0$ and decreasing or flat otherwise. So, if $Y_t > \frac{1}{\gamma}$ then this is maximized by $\alpha_t = 1$ and when $Y_t < \frac{1}{\gamma}$ then this is maximized by $\alpha_t = 0$. **[5 marks]**

ii) The question is asking us to solve the backward equation

$$dY_t = -\big(\gamma \alpha_t Y_t + (1-\alpha_t)\big)\, dt\,, \quad t \in [0,T]\,, \quad Y_T = 0$$

for the optimal control. Since $Y_T = 0$ we know that at (and for t close to T, due to continuity) $Y_t < \frac{1}{\gamma}$ and so the optimal control is 0. So $dY_t = -dt$ i.e. $Y_t = T - t$. Letting time run backwards it is increasing linearly from 0 and will reach $\frac{1}{\gamma}$ when $t = T - \frac{1}{\gamma}$. Thus we have

$$Y_t = T - t \ \text{ for } \ t \in (T - \tfrac{1}{\gamma}, T]\,.$$

[2 marks]

For earlier times we have $dY_t = -\gamma Y_t \, dt$ and so $Y_t = C\exp(-\gamma t)$. Moreover $\frac{1}{\gamma} = Y_{T-\frac{1}{\gamma}}$ which implies that $\frac{1}{\gamma} = C\exp(-\gamma(T-\frac{1}{\gamma}))$ i.e. $C = \frac{1}{\gamma}\exp(\gamma(T-\frac{1}{\gamma}))$. **[3 marks]**

iii) This follows from parts i), ii) and iii) since until $T - \frac{1}{\gamma}$ the optimal control is 1 while afterwards the optimal control is 0. **[5 marks]**

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3. We consider a problem of optimal trade execution. Fix $T > 0$, $\lambda > 0$, $\sigma > 0$, $\kappa > 0$. The mid-price of an asset is

$$dS_t = \lambda \alpha_t \, dt + \sigma dW_t \,, \;\; t \in [0,T] \,, \;\; S_0 > 0 \,.$$

Our holding in the asset is given by

$$d\xi_t = \alpha_t \, dt \;, \;\; t \in [0,T] \,, \;\; \xi_0 \in \mathbb{R} \,.$$

Our cash account is

$$dB_t = -\alpha_t \left(S_t + \frac{\kappa}{2}\alpha_t \right) dt \,, \;\; t \in [0,T] \,, \;\; B_0 > 0 \,.$$

Here the control is $\alpha = \alpha_t$ representing they "buying rate". The constant $\lambda > 0$ is the "permanent price impact" while $\kappa > 0$ is the "temporary price impact".

Our task is to deliver one unit of the risky asset at time $T > 0$ and there is a quadratic penalty for missing the target. We want to do this while maximising our cash balance. Let $\mathcal{A}$ comprise processes α_t adapted to the filtration generated by W and such that $\mathbb{E}\int_0^T \alpha_t^2 \, dt < \infty$. The overall objective to maximize is, over $\alpha \in \mathcal{A}$,

$$M(S_0, \xi_0, B_0, \alpha) = \mathbb{E}\left[-\frac{1}{2}|\xi_T - 1|^2 + B_T + (\xi_T - 1)S_T \right] .$$

a) Show that

$$\max_{\alpha\in\mathcal{A}} M(S_0,\xi_0,B_0,\alpha) = B_0 - S_0 + \xi_0 S_0 + \max_{\alpha\in\mathcal{A}} J(\xi_0,\alpha) \,,$$

where

$$J(\xi_0,\alpha) = \mathbb{E}\bigg[\int_0^T \Big(-\frac{\kappa}{2}\alpha_r^2 + \lambda\alpha_t(\xi_t - 1) \Big)\, dt - \frac{1}{2}|\xi_T - 1|^2 \bigg] .$$

[8 marks]

b) Find an explicit expression for the optimal control. *Hint.* You can use either the Bellman PDE or Pontryagin optimality to solve this. **[12 marks]**

Comment: *A new question in the spirit of optimal execution. It's basically a linear-quadratic control problem (the students will need to recognise this).*

Solution:

a) Clearly we have $S_t = S_0 + \lambda\int_0^t \alpha_s \, ds + \sigma W_t$ and $B_t = B_0 + \int_0^t (-\alpha_r S_r - \frac{\kappa}{2}\alpha_r^2)\, dr$. Moreover

$$d(\xi_t S_t) = \lambda\alpha_t\xi_t \, dt + \sigma\xi_t dW_t + S_t\alpha_t \, dt \,.$$

[2 marks]

We note that with Hölder's inequality we have

$$\mathbb{E}\int_0^T \xi_t^2 \, dt = \mathbb{E}\int_0^T \left(\int_0^t \alpha_r \, dr \right)^2 dt \le \mathbb{E}\int_0^T t\int_0^t \alpha_r^2 \, dr \, dt \le T^2 \mathbb{E}\int_0^T \alpha_r^2 \, dr < \infty$$

for admissible controls. Hence $\mathbb{E}\int_0^T \xi_t \, dW_t = 0$. **[3 marks]**

We thus have that

$$\mathbb{E}\xi_T S_T = \xi_0 S_0 + \mathbb{E}\int_0^T (\lambda\alpha_t\xi_t + S_t\alpha_t)\, dt \,.$$

This leads to

$$M(S_0,\xi_0,B_0,\alpha) = B_0 - S_0 + \xi_0 S_0 + \mathbb{E}\bigg[\int_0^T \Big(-\alpha_t S_t - \frac{\kappa}{2}\alpha_r^2 - \lambda\alpha_t + \lambda\alpha_t\xi_t + S_t\alpha_t \Big)\, dt - \frac{1}{2}|\xi_T - 1|^2 \bigg] .$$

Hence

$$M(S_0, \xi_0, B_0, \alpha) = B_0 - S_0 + \xi_0 S_0 + J(\xi_0, \alpha)\,,$$

where

$$J(\xi_0, \alpha) = \mathbb{E}\bigg[\int_0^T \Big(-\frac{\kappa}{2}\alpha_r^2 + \lambda\alpha_t(\xi_t - 1)\Big)\,dt - \frac{1}{2}|\xi_T - 1|^2\bigg]\,.$$

This conveniently reduced the dimension of the underlying state space to 1. **[3 marks]**

b) Let us further set $Q_t = \xi_t - 1$ so that $dQ_t = \alpha_t\,dt$ with $Q_0 = \xi_0 - 1$. So let us maximize

$$J(Q_0, \alpha) = \mathbb{E}\bigg[\int_0^T \Big(-\frac{\kappa}{2}\alpha_r^2 + \lambda\alpha_t Q_T\Big)\,dt - \frac{1}{2}Q_T^2\bigg]\,.$$

The Hamiltonian is

$$H(Q, Y, Z, a) = aY + \lambda a Q - \frac{\kappa}{2}a^2\,.$$

We can check that this is concave as a function of (Q, a), the terminal condition $q \mapsto -\frac{1}{2}q^2$ is also concave, and so we are allowed to apply Pontryagin optimality. The adjoint equation is

$$dY_t = -\lambda\alpha_t\,dt + Z_t\,dW_t\,,\;\; Y_T = -Q_T\,.$$

We know the optimal control must locally maximize the Hamiltonian and so

$$0 = \nabla_a H = Y_t + \lambda Q_t - \kappa a_t$$

means that

$$\alpha_t = \frac{Y_t + \lambda Q_t}{\kappa}\,.$$

[7 marks]

We try the solution to the adjoint of the form $Y_t = \varphi_t Q_t$, $\varphi \in C^1$, $\varphi_T = -1$ so that

$$\alpha_t = \frac{(\lambda + \varphi_t)Q_t}{\kappa}\,.$$

We also see that (chain rule, substitute optimal control):

$$dY_t = \varphi_t\frac{(\lambda + \varphi_t)}{\kappa}Q_t\,dt + Q_t\varphi_t'\,dt$$

while at the same time (substituting optimal control):

$$dY_t = -\lambda\frac{(\lambda + \varphi_t)}{\kappa}Q_t\,dt + Z_t\,dW_t\,.$$

This can only be true if $Z_t = 0$ and if

$$\varphi_t\frac{(\lambda + \varphi_t)}{\kappa} + \varphi_t' = -\lambda\frac{(\lambda + \varphi_t)}{\kappa}$$

which leads to an ODE for φ of the form:

$$\varphi_t' = -\lambda\frac{(\lambda + \varphi_t)}{\kappa} - \varphi_t\frac{(\lambda + \varphi_t)}{\kappa} = -\frac{1}{\kappa}(\lambda + \varphi_t)^2\,.$$

So we must solve

$$\varphi_t' = -\frac{1}{\kappa}(\lambda + \varphi_t)^2\,,\;\; t \in [0, T]\,,\;\; \varphi_T = -1\,.$$

This is

$$\varphi_t = \left(\frac{t - T}{\kappa} + \frac{1}{\lambda - 1}\right)^{-1} - \lambda\,.$$

The optimal control is thus

$$\alpha_t = \kappa^{-1}\left(\frac{t - T}{\kappa} + \frac{1}{\lambda - 1}\right)^{-1}(\xi_t - 1)\,.$$

[5 marks]