Stochastic Control and Dynamic Asset Allocation

MATH11150 **Solutions and comments** **April and May 2019**

Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. We consider the standard Black–Scholes model for optimal investment: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. Let X_s denote the investment portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t\in[0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E}\int_t^T \nu_s^2\, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{A}$.

a) Derive the SDE satisfied by the portfolio value process $X_s = X_s^{\nu,t,x}$. **[3 marks]**

b) Consider the control problem

$$v(t,x) := \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\ln(X_T^{\nu,t,x})\right] . \tag{1}$$

Write down the Bellman PDE that the function v must satisfy. **[3 marks]**

c) Show that

$$v(t,x) = \ln x - (T-t)\left[\tfrac{1}{2}\sigma^2\hat{u}^2 - (\mu - r)\hat{u} - r\right] ,$$

where

$$\hat{u} := \frac{\mu - r}{\sigma^2} .$$

[7 marks]

d) Use the verification theorem to prove that the v above and the optimal control you identified are indeed the solution to the optimal control problem (1). **[4 marks]**

Comment: *Parts a) and b) have been seen (a number of times). Parts c), d) and e) follow Merton's problem solution using Bellman PDEs students have seen in the course. This question is meant as a straightforward application of Bellman PDE and verification theorems. All student should be getting nearly full marks on this question.*

Solution:

a) We calculate (Itô formula) that $dB_t = rB_t\, dt$ and $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. We then have (with ψ_t being the number of units of risky asset we hold)

$$dX_t = \psi_t\, dS_t + \frac{X_t - \psi_t S_t}{B_t}\, dB_t = \nu_t X_t \frac{1}{S_t}\, dS_t + \frac{X_t - \nu_t X_t}{B_t}\, dB_t .$$

So

$$dX_t = X_t\left[(\mu - r)\nu_t + r\right]\, dt + \nu_t X_t \sigma\, dW_t .$$

[3 marks]

b)

$$\partial_t v + \sup_u \left[\frac{1}{2}\sigma^2 u^2 x^2 \partial_{xx} v + x[(\mu - r)u + r]\partial_x v\right] = 0 \text{ on } [0,T)\times(0,\infty)$$
$$v(T,x) = \ln x \;\; \forall x > 0 .$$

The domain has to be specified to get full marks. **[3 marks]**

c) We "guess" the ansatz

$$v(t,x) = \lambda(t)\ln(\beta x) + \gamma(t)$$

with $\lambda, \gamma \in C^1([0,T])$ and $\lambda > 0$. Hence we have $\partial_t v = \lambda'(t)\ln x + \gamma'(t)$, $\partial_x v = \lambda(t)x^{-1}$, $\partial_{xx} v = -\lambda(t)x^{-2}$. So we get

$$\lambda'(t)\ln x + \gamma'(t) + \sup_{u\in\mathbb{R}}\left[-\tfrac{1}{2}\sigma^2 u^2 \lambda(t) + \lambda(t)\left((\mu - r)u + r\right)\right] = 0\,.$$

The function $u \mapsto -\frac{1}{2}\sigma^2 u^2\lambda(t) + \lambda(t)\left((\mu-r)u + r\right)$ is maximized (calculus and concavity) when

$$\hat{u} = \frac{\mu - r}{\sigma^2}\,.$$

Hence

$$\lambda'(t)\ln x + \gamma'(t) + \lambda(t)\left[-\tfrac{1}{2}\sigma^2\hat{u}^2 + (\mu - r)\hat{u} + r\right] = 0\,.$$

Collecting terms involving $\ln x$ and those without we get:

$$\lambda'(t) = 0\,,\ \lambda(T) = 1 \text{ so } \lambda(t) = 1 \text{ for all } t \in [0,T]$$

and

$$\gamma'(t) = \tfrac{1}{2}\sigma^2\hat{u}^2 - (\mu - r)\hat{u} - r =: \Gamma(\sigma,\mu,r)\,,\ \ \gamma(T) = 0\,.$$

Integrating this we get $\gamma(t) = -(T-t)\Gamma$. So $v(t,x) = \ln x - (T-t)\Gamma$. **[7 marks]**

d) We have established that $v(t,x) = \ln x - (T-t)\Gamma$ solves the Bellman PDE. Moreover the Markovian optimal control $\hat{u}(t,x) = \frac{\mu - r}{\sigma^2}$ is constant and hence certainly measurable. **[1 mark]**

The wealth equation with the optimal control is

$$d\hat{X}_t = \hat{X}_t\left[(\mu - r)\hat{u} + r\right]\,dt + \hat{u}\hat{X}_t\sigma\,dW_t\,.$$

This is a linear SDE with Lipschitz coefficients so it has unique solution which moreover has all the moments when started from deterministic initial value. **[1 mark]**

We consider

$$t' \mapsto \int_t^{t'} \frac{1}{\hat{X}_s}\hat{u}\hat{X}_s\sigma\,dW_s = \hat{u}\sigma(W_{t'} - W_t)\,.$$

This is a martingale (Wiener process is) and so the verification is complete, the constant strategy $\hat{u}$ is optimal and the optimal value for this control is v. **[2 marks]**

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2. We consider a simple model for optimal liquidation of an asset via market orders on an exchange over a finite time interval $[0, T]$. Let W be a real-valued Wiener process generating the filtration $\mathbb{F}$, let the process $Q = (Q_t)_{t\in[0,T]}$ represent the inventory level and let the process $S = (S_t)_{t\in[0,T]}$ represent the asset price. The control is $\alpha = (\alpha_t)_{t\in[0,T]}$ which represents the selling rate at t (if $\alpha_t > 0$) or, buying rate at t (if $\alpha_t < 0$). In our model

$$\begin{aligned} dQ_t &= -\alpha_t\, dt \\ dS_t &= \lambda\alpha_t\, dt + \sigma\, dW_t\,, \quad t \in [0, T]\,, \quad Q_0 = q > 0\,, \quad S_0 = S > 0\,. \end{aligned}$$

Here $\sigma > 0$ is the volatility of the asset and $\lambda < 0$ captures the permanent price impact of our trading. There is temporary price impact captured by the "slippage" parameter $\kappa > 0$ and the price at which we actually sell is $S_t - \kappa\alpha_t$. Finally, there is a penalty for unsold inventory at time T given by $\theta \geq 0$.

Let the set of real-valued, $\mathbb{F}$-adapted processes $\alpha = (\alpha_s)_{s\in[0,T]}$ such that $\mathbb{E}\int_0^T |\alpha_s|^2\, ds < \infty$ be denoted by $\mathcal{A}$. We wish to maximize, over $\alpha \in \mathcal{A}$, the functional

$$M(q, S, \alpha) = \mathbb{E}^{q,S,\alpha}\left[\int_0^T (S_t\alpha_t - \kappa\alpha_t^2)\, dt + Q_T S_T - \theta Q_T^2\right].$$

a) Prove that if $\alpha \in \mathcal{A}$ then $\mathbb{E}\left[\int_0^T Q_t^2\, dt\right] < \infty$. *Hint:* Use Hölder's inequality. **[3 marks]**

b) Hence show that with $\gamma := \theta + \frac{1}{2}\lambda$

$$M(q, S, \alpha) = qS - \theta q^2 + J(q, \alpha)\,, \text{ where } J(q, \alpha) := \mathbb{E}^{q,\alpha}\left[\int_0^T \left(2\gamma\alpha_t Q_t - \kappa\alpha_t^2\right)\, dt\right].$$

Hint: Use product rule to calculate $d(Q_t S_t) = \ldots$ and $d(Q_t^2) = \ldots$. **[3 marks]**

c) Write down the Hamiltonian and the adjoint BSDE $(\hat{Y}, \hat{Z})$ associated to the optimal control $\hat{\alpha}$ for the control problem

$$\max_{\alpha\in\mathcal{A}} J(q, \alpha) \text{ subject to } Q_t = q - \int_0^t \alpha_s\, ds\,.$$

[3 marks]

d) Use the Pontryagin maximum principle to show that the optimal control is

$$\hat{\alpha}_t = \frac{1}{\kappa}\left(\frac{1}{\gamma} + \frac{1}{\kappa}(T - t)\right)^{-1}\hat{Q}_t\,.$$

Hint: Make the "ansatz" that $\hat{Y}_t = 2\xi(t)\hat{Q}_t$ for some $\xi \in C^1([0, T]; \mathbb{R})$. **[8 marks]**

Comment: *Parts a) and b) are very simple but strictly speaking unseen but should not be hard given the hints provided.*

Parts d) and e) are a special case of the general linear-quadratic control problem solved using Pontryagin maximum principle and this has been done in the lectures.

Solution:

a) Let $\|\alpha\|^2 := \mathbb{E}\int_0^T \alpha_s^2\,ds$. We see that $Q_t = q - \int_0^t \alpha_s\,ds$ and hence using Hölder's inequality we get

$$Q_t^2 \leq 2q^2 + 2\left(\int_0^t \alpha_s\,ds\right)^2 \leq 2q^2 + 2T\int_0^T \alpha_s^2\,ds\,.$$

Hence

$$\mathbb{E}\left[\int_0^T Q_t^2\,dt\right] = \int_0^T \mathbb{E}Q_t^2\,dt \leq 2q^2T + 2T^2\|\alpha\|^2 < \infty\,.$$

[3 marks]

b) We calculate, using Itô's formula, that

$$Q_T^2 = q^2 - 2\int_0^T \alpha_t Q_t\,dt$$

and, using Itô's product rule, that

$$Q_T S_T = qS + \int_0^T (\lambda\alpha_t Q_t - S_t\alpha_t)\,dt + \sigma\int_0^T Q_t\,dW_t\,.$$

Since $\mathbb{E}\left[\int_0^T Q_t^2\,dt\right] < \infty$ we know that $\mathbb{E}\left[\int_0^T Q_t\,dW_t\right] = 0$. Then

$$\begin{aligned}
M(q,S,\alpha) &= \mathbb{E}^{q,S,\alpha}\left[\int_0^T (S_t\alpha_t - \kappa\alpha_t^2)\,dt + Q_T S_T - \theta Q_T^2\right] \\
&= \mathbb{E}^{q,S,\alpha}\left[\int_0^T (S_t\alpha_t - \kappa\alpha_t^2)\,dt + qS + \int_0^T (\lambda\alpha_t Q_t - S_t\alpha_t)\,dt - \theta q^2 + 2\theta\int_0^T \alpha_t Q_t\,dt\right] \\
&= qS - \theta q^2 + \mathbb{E}^{q,\alpha}\left[\int_0^T (\gamma\alpha_t Q_t - \kappa\alpha_t^2)\,dt\right]
\end{aligned}$$

as required. **[3 marks]**

c) The Hamiltonian is $H(q,a,y) = -ay - \kappa a^2 + 2\gamma qa$ and the adjoint BSDE is

$$d\hat{Y}_t = -\partial_q H(\hat{Q}_t, \hat{\alpha}_t, \hat{Y}_t)\,dt + \hat{Z}_t\,dW_t\,,\ \ t\in[0,T]\,,\ \ \hat{Y}_T = 0$$

i.e.

$$d\hat{Y}_t = -2\gamma\hat{\alpha}_t + \hat{Z}_t\,dW_t\,,\ \ t\in[0,T]\,,\ \ \hat{Y}_T = 0\,.$$

[3 marks]

d) We note that the Pontryagin maximum principle does apply (terminal condition is concave as a function of q since it's constant and the Hamiltonian is differentiable and concave in (q,a)).

We maximize the Hamiltonian as a function of $a \in \mathbb{R}$, since it's concave it's enough to solve for a in

$$-y - 2\kappa a + 2\gamma q = 0\,.$$

By the Pontryagin's maximum principle

$$\hat{\alpha}_t = \frac{2\gamma\hat{Q}_t - \hat{Y}_t}{2\kappa} = \frac{\gamma - \xi(t)}{\kappa}\hat{Q}_t$$

where we used the suggested guess for $\hat{Y}$. **[3 marks]**

So the adjoint BSDE becomes

$$d\hat{Y}_t = 2\gamma \frac{1}{\kappa}(\xi(t) - \gamma)\hat{Q}_t\,dt + \hat{Z}_t\,dW_t\,.$$

But our guess also forces us to conclude, since $d\hat{Q}_t = \frac{1}{\kappa}(\xi(t) - \gamma)\hat{Q}_t\,dt$, that

$$d\hat{Y}_t = 2d(\xi(t)\hat{Q}_t) = 2\xi'(t)\hat{Q}_t\,dt + 2\xi(t)\,d\hat{Q}_t = 2\xi'(t)\hat{Q}_t\,dt + \frac{2}{\kappa}(\xi(t) - \gamma)\xi(t)\hat{Q}_t\,dt\,.$$

[2 marks]

As both equations for $\hat{Y}$ must hold simultaneously we conclude that $\hat{Z} = 0$ and moreover

$$\xi'(t) = -\frac{1}{\kappa}(\xi(t) - \gamma)^2\,,\ \ t \in [0, T]\,,\ \xi(T) = 0\,.$$

[2 marks]

To solve this take $\psi(t) := \xi(t) - \gamma$ so that $\psi'(t) = -\frac{1}{\kappa}\psi(t)^2$ with $\psi(T) = -\gamma$. Then

$$\psi(t) = -\left(\frac{1}{\gamma} + \frac{1}{\kappa}(T - t)\right)^{-1}\,.$$

[1 mark]

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3. Let W be an $\mathbb{R}$-valued Wiener process generating the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ i.e $\mathcal{F}_t := \sigma(W_s : s \leq t)$. Let the set of real-valued, $\mathbb{F}$-adapted processes $\alpha = (\alpha_s)_{s\in[0,T]}$ such that $\mathbb{E}\int_0^T |\alpha_s|^2\,ds < \infty$ be denoted by $\mathcal{A}$. For $x \in \mathbb{R}$ and $\alpha \in \mathcal{A}$ let

$$X_s^{t,x,\alpha} = x + \int_t^s \alpha_r\,dW_r\,.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be such that for some constants $K \geq 0$, $m \in \mathbb{N}$, it holds for all $x \in \mathbb{R}$ that $|g(x)| \leq K(1 + |x|^m)$. For $t \in [0,T] \times \mathbb{R}$ let

$$v(t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[g(X_T^{t,x,\alpha})\right]\,.$$

Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$.

a) Prove that

$$0 \geq \partial_t v + \frac{1}{2}a^2\partial_x^2 v \text{ on } [0,T) \times \mathbb{R}\,.$$

Hint: Use the Bellman principle (DPP) and then Itô's formula. **[10 marks]**

b) Hence prove that for any $t \in [0,T)$ the function $x \mapsto v(t,x)$ is concave. **[3 marks]**

c) Hence prove that if g is concave then $v(t,\cdot) = g$. **[3 marks]**

Comment: *This is a deliberately harder question. To get full marks in part a) students need to carefully use a stopping time argument, continuity of v and the process and mean value theorem as well as justify each step. However some marks can be collected from just DPP and Ito formula. Part b) has not been seen as such but it requires only a) and characterization of concave functions in terms of 2nd derivative. Part c) only requires the use of Jensen's inequality and students would have seen similar argument when showing that American call options are no more valuable than European call options.*

Solution:

a) Let $a \in \mathbb{R}$ and $(t,x) \in [0,T) \times \mathbb{R}$ be fixed. Let $\alpha_s := a$ for all $s \in [t,T]$. Let $X_s := X_s^{t,x,\alpha}$ for all $s \in [t,T]$. For $h > 0$ define

$$\tau_h := \inf\{s > t : s > t + h \text{ or } X_s \notin (x-1, x+1)\}\,.$$

Then $\tau_h \to t$ as $h \to 0$. Moreover for almost all ω, due to continuity of $s \mapsto X_s(\omega)$, we have some $\bar{h}(\omega)$ such that for $h < \bar{h}(\omega)$ it holds that $X_s(\omega) \in (x-1, x+1)$ for all $s \in [t, t+h)$ and so for $h < \bar{h}(\omega)$ it holds that $\tau_h(\omega) = t + h$.

From the dynamic programming principle

$$v(t,x) \geq \mathbb{E}\left[v(\tau_h, X_{\tau_h}^{t,x,a})\right]\,.$$

By Itô's formula applied to the function v and process X we have that

$$\mathbb{E}\left[v(\tau_h, X^{t,x,a}) - v(t,x)\right] = \mathbb{E}\int_t^{\tau_h}\left[\partial_t v + \frac{1}{2}a^2\partial_x^2 v\right](r, X_r)\,dr + \mathbb{E}\int_t^{\tau_h}\partial_x v(r, X_r)a\,dW_r\,.$$

[3 marks]

Since $v \in C^{1,2}([0,T) \times \mathbb{R})$ we have that $\partial_x v$ is bounded on $[t, t+h] \times [x-1, x+1]$ and hence the stochastic integral is a martingale. Thus

$$0 \geq \mathbb{E} h \int_t^{\tau_h} \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](r, X_r)\, dr\,. \tag{2}$$

[1 mark]

By Fatou's lemma

$$0 \geq \liminf_{h \to 0} \mathbb{E} h \int_t^{\tau_h} \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](r, X_r)\, dr \geq \mathbb{E}\left[\liminf_{h \to 0} h \int_t^{\tau_h} \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](r, X_r)\, dr\right].$$

[1 mark]

Now fix ω and $h < \bar{h}(\omega)$ so that $\tau_h = t + h$. By the mean value theorem there is $\xi \in [t, t+h]$ such that

$$h \int_t^{t+h} \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](r, X_r(\omega))\, dr = \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](\xi, X_\xi(\omega))\,.$$

Sending $h \to 0$ and using that $v \in C^{1,2}([0,T) \times \mathbb{R})$ and $r \mapsto X_r(\omega)$ is continuous we get that

$$\liminf_{h \to 0} h \int_t^{\tau_h} \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](r, X_r)\, dr = \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](t, x)\,.$$

Hence

$$0 \geq \left[\partial_t v + \frac{1}{2} a^2 \partial_x^2 v\right](t, x)\,.$$

Noting that $(t, x) \in [0,T) \times \mathbb{R}$ was arbitrary completes the argument. **[5 marks]**

Note on marking: Students who use the DPP, correctly apply Itô's formula and then (without justification) write down (2) *and claim that they've answered should get about 6-7 marks. The idea is there, the proof is not.*

b) Next we show that v is concave. Indeed, fix (t, x). Then

$$\partial_x^2 v(t, x) \leq -\frac{2}{a^2} \partial_t v(t, x) \to 0 \text{ as } a \to \infty\,.$$

Hence for each t, x we have $\partial_x^2 v(t, x) \leq 0$ i.e. v is concave. **[3 marks]**

c) First we note that

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[g(X_T^{t,x,\alpha})] \geq \mathbb{E}[g(X_T^{t,x,0})] = g(x)\,.$$

Using concavity of v we note that by Jensen's inequality it holds that

$$\mathbb{E}[g(X_T^{t,x,\alpha})] \leq g\left(\mathbb{E}[X_T^{t,x,\alpha}]\right) = g(x)$$

since $X^{t,x,\alpha}$ is a martingale for all $\alpha \in \mathcal{A}$. Hence

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}[g(X_T^{t,x,\alpha})] \leq g(x)\,.$$

But then $v(t, \cdot) = g$. **[3 marks]**
