2021/22 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 3, for Week 5 - Last updated 13th January 2024

Exercise 3.1. For any $(t,x) \in [0,T] \times \mathbb{R}$, define the stochastic process $(X^{t,x}_s)_{s\in[t,T]}$ as

$$X^{t,x}_s = x + W_s - W_t\,.$$

Let $\mathbb{E}^{t,x}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|X_t = x]$. Define a function $v = v(t,x)$ as

$$v(t,x) = \mathbb{E}^{t,x}\big[\,g\big(X_T\big)\,\big] \qquad \forall\,(t,x) \in [0,T) \times \mathbb{R}\,.$$

Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$ and that $\big(\partial_x v(s,W_s)\big)_{s\in[t,T]} \in L^2([0,T] \times \mathbb{R})$. Show that

$$\begin{aligned} \partial_t v + \frac{1}{2}\partial_{xx} v = 0 \;\text{ on }\; [0,T) \times \mathbb{R}\,,\\ v(T,\cdot) = g \;\text{ on }\; \mathbb{R}\,. \end{aligned}$$

Hints:

i) Apply Itô formula to the function v and process $X^{t,x}$.

ii) Take expectation. The condition $\big(\partial_x v(s,W_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$ ensures stochastic integral is a martingale and so it will dissapear under expectation.

iii) Convince yourself that the PDE must hold.

Exercise 3.2. For any $(t,x) \in [0,T] \times \mathbb{R}$, define the stochastic process $(X^{t,x}_s)_{s\in[t,T]}$ as the solution to the SDE

$$dX^{t,x}_s = b(X^{t,x}_s)\,ds + \sigma(X^{t,x}_s)\,dW_s, \;\; \forall\, s \in [t,T], \qquad X_t = x.$$

Let $\mathbb{E}^{t,x}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|X_t = x]$. Define a function $v = v(t,x)$ as

$$v(t,x) = e^{-r(T-t)}\mathbb{E}^{t,x}\big[\,g\big(X_T\big)\,\big] \qquad \forall\,(t,x) \in [0,T) \times \mathbb{R}\,.$$

Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$ and that $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T] \times \mathbb{R})$. Show that

$$\begin{aligned} \partial_t v + b\partial_x v + \frac{1}{2}\sigma^2\partial_{xx} v - rv = 0 \;\text{ on }\; [0,T) \times \mathbb{R}\,,\\ v(T,\cdot) = g \;\text{ on }\; \mathbb{R}\,. \end{aligned}$$