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2023/24 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 2, for Week 3 - Last updated 13th January 2024

Exercise 2.1 (Non-existence of solution)**.**

1. Let $I = \left[0, \frac{1}{2}\right]$. Find a solution X for

$$\frac{dX_t}{dt} = X_t^2\,, \;\; t \in I\,, \;\; X_0 = 1\,.$$

2. Does a solution to the above equation exist on $I = [0,1]$? If yes, show that it satisfies the definition of an SDE solution from SAF lectures. In not, which property is violated?

Exercise 2.2 (Non-uniqueness of solution)**.** Fix $T > 0$. Consider

$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}\,, \;\; t \in [0,T]\,, \;\; X_0 = 0\,.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0,T]$ is a solution to the above ODE.
2. Show that $X_t := t^2$ for all $t \in [0,T]$ is also a solution.
3. Find at least two more solutions different from $\bar{X}$ and X.

Exercise 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be a r.v. and let $\mathcal{G} = \{\emptyset, \Omega\}$.

1. Show that there is a $c \in \mathbb{R}$ such that $\mathbb{E}[X|\mathcal{G}] = c$.
2. Show that in fact $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$.

Exercise 2.4. Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s, \;\; t \le s \le T, \;\; X_t^{t,x} = x\,.$$

Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then

$$\mathbb{P}\left[\sup_{t \le s \le T} |X_s^{t,x} - Y_s^{t,x}| > 0\right] = 0\,.$$

Show that then the *flow property* holds i.e. for $0 \le t \le t' \le T$ we have almost surely that

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}\,, \qquad \forall s \in [t', T].$$