There are *three* questions, each worth 25 marks. Correct answers to all questions yields the maximum of 75 marks.

Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

(1) Given $(s, y) \in [0, 1) \times \mathbb{R}$, consider the following stochastic control problem

$$\begin{aligned} V(0,0) &= \min_{\nu} J(0,0;\nu) \\ &= \min_{\nu} \mathbb{E}\left[\left(X^{\nu}_{0,0}(1)\right)^2 - \int_0^1 2\, s\, \nu(s) \mathrm{d}s\right] \end{aligned}$$

$$\text{such that} \begin{cases} \mathrm{d}X^{\nu}_{0,0}(r) &= \nu(r)\mathrm{d}W(r), \qquad r \in [0,1] \\ X^{\nu}_{0,0}(0) &= 0 \\ \nu(t) &\in \mathbb{R} \quad \forall t \in [0,1],\ (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted} \\ & \quad \text{and } \mathbb{E}[\int_0^1 \left(\nu(s)\right)^2 \mathrm{d}s] < \infty \end{cases}$$

(a) Write the dynamic version of the optimal control problem above, i.e. write down $V(t, y)$ for $t \in [0, 1]$ and $y \in \mathbb{R}$.

[2 marks]

(b) Show that one can write $V(t, x) = x^2 + g(t)$ for some function $g(t)$ you need to identify.

Compute further $\partial_x V(t, x)$ and $\partial_{xx} V(t, x)$.

[8 marks]

(c) Write down the HJB equation for this stochastic control problem.

[5 marks]

(d) Find a solution to the HJB equation and compute $V(0, 0)$.

[10 marks]

(2) (a) Denote $V(t,x)$ as the value function of the optimal control problem below starting from time t at position x, i.e.,

$$V(t,x) := \inf_{\nu} \mathbb{E}\left[\int_t^T f\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}s + h\big(X^{\nu}_{t,x}(T)\big)\right]$$

$$\text{where} \quad \begin{cases} \mathrm{d}X^{\nu}_{t,x}(s) = b\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}s + \sigma\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}W_s, & s \in [t,T] \\ X^{\nu}_{t,x}(t) = x \\ \nu \in \mathcal{U}_{ad}[t,T] \end{cases}.$$

Make the general assumption that the SDE coefficients satisfy assumption (S1) from class and $\mathcal{U}_{ad}$ is the usual admissibility set used in class implying that the SDE is well defined and has good properties, in particular, it holds that

$$\mathbb{E}\big[\sup_{s\in[t,T]} |X^{\nu}_{t,y}(s) - X^{\nu}_{t,x}(s)|^2\big] \leq C|y-x|^2.$$

Assume additionally that the function $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Assume the function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, i.e. $f(t,x,u)$ is Lipschitz in x uniformly in t,u and uniformly bounded in u and t.

Then, show that there exists constants $K > 0$ such that

$$|V(t,\, x) - V(t,\, y)| \leq K|x-y|.$$

[12 marks]

(b) A Black-Scholes market is given where there are only one stock (with drift $a \in \mathbb{R}$ and volatility $\sigma > 0$) and one bank account with constant interest rate $r > 0$.

In this market an investor, with initial wealth $x_0 > 0$ selects among *proportion strategies* ν that are *constants* and with such a strategy the *proportion of wealth invested in the stock* is a constant throughout.

The investor seeks to maximise his expected utility at time T which is a power-type utility

$$U(x) = \log x, \qquad x > 0.$$

(i) Identify the dynamics for the underlying assets.
Show that the SDE expressing the wealth process $\big(X^{\nu}(t)\big)_{t\in[0,T]}$ is of Geometric Brownian motion type and write its explicit solution.

[5 marks]

(ii) Write clearly the optimization problem and then compute the *constant optimal proportion strategy* $\nu^{\star}$ explicitly *using* the duality theory approach, in other words, determine: the deflator process $(Y_t)_{t\in[0,T]}$ and all necessary related quantities; the optimal wealth random variable $\widehat{X}_T$ using duality theory; then the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$; finally determine ν^{*}.

[8 marks]

(3) Let $\mathbf{B} := (B,\ B^{\perp})$ be a two-dimensional Brownian motion (BM) on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, with $\mathbb{F}$ denoting the filtration generated by $\mathbf{B}$.

Take a financial market with a bank account with interest rate of *zero*, a stock price S and its volatility Y follow

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + Y_t \mathrm{d}B_t), \quad \mathrm{d}Y_t = a(Y_t)\mathrm{d}t + b(Y_t)\mathrm{d}W_t,\ \mu \in \mathbb{R},$$

where W is a BM given by

$$W_t := \rho B_t + \sqrt{1-\rho^2}B_t^{\perp}, \quad \rho \in [-1, 1], \quad 0 \le t \le T.$$

The maps a, b are deterministic such that Y is positive, and the stock's market price of risk $\lambda := \mu/Y$ is square-integrable. An agent with initial wealth $x > 0$ and logarithmic utility function $U(\cdot) = \log(\cdot)$ maximises expected utility of wealth at time T, with wealth process X generated from trading S. Admissible strategies $\pi \in \mathcal{A}$ yield non-negative X. Denote by V the convex conjugate of U. Define Z as the density process with respect to $\mathbb{P}$ of any martingale measure $\mathbb{Q} \in \mathbf{M}$, where $\mathbf{M}$ denotes the set of equivalent martingale measures.

(a) Write down a stochastic exponential formula for Z involving λ and a second adapted process $\lambda^{\perp}$.

[1 marks]

(b) Derive the dynamics of ZX, deduce that $\mathbb{E}[Z_T X_T] \le x$, for any $\mathbb{Q} \in \mathbf{M}$, and show that $u(x) \le v(y) + xy$, where $u(x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T)]$ is the primal value function and $v(y) := \inf_{\mathbb{Q}\in\mathbf{M}} \mathbb{E}[V(yZ_T)]$ is the dual value function, for $y > 0$.

[5 marks]

(c) Explain the relation between the optimal terminal wealth $\hat{X}_T$ and $y\hat{Z}_T$, where $y > 0$, and $\hat{Z}_T$ is the Radon-Nikodym derivative of the dual minimiser $\hat{\mathbb{Q}} \in \mathbf{M}$, and explain how y is fixed. Hence derive a formula for $\hat{X}_T$, given that $U(\cdot) = \log(\cdot)$.

[5 marks]

(d) Derive a formula for the optimal wealth process $\hat{X} = (\hat{X}_t)_{0\le t\le T}$, the optimal portfolio proportion process $\hat{\theta}$, and characterise the dual minimiser $\hat{\mathbb{Q}} \in \mathbf{M}$.

[5 marks]

(e) Show that the primal value function is given by $u(x) = \log x + H$, where H is a constant which you should express in terms of λ.

[1 marks]

(f) Suppose $Y_t = \sigma \exp(\alpha W_t), t \in [0,\ T]$, for positive constants σ, α. Derive the form of $u(x)$ in this case, show that in the limit $\alpha \to 0$ we have

$$u(x) = \log x + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T,$$

and interpret the result.

[8 marks]

[End of Paper]