Stochastic Control and Dynamic Asset Allocation

MATH11150 **Solutions and comments** **May 2018**

Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1.

a) We consider the standard Black–Scholes model: a risk-free asset B given by

$$dB_t = rB_t\, dt\,,\ \ B_0 = 1$$

and a risky asset S given by

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t\,,\ \ S_0 > 0 \text{ fixed.}$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. An investor, with initial wealth $X_0 = x > 0$ selects among strategies ν that are *constants* and represent fraction of the wealth invested in the risky asset. The investor seeks to maximise his expected utility at time T for U given by

$$U(x) = \ln x, \qquad x > 0.$$

The optimization problem can be written as

$$u(x) = \sup_{\nu\in\mathbb{R}} \mathbb{E}\Big[U\big(X_T^\nu\big)\big|X_0 = x\Big]\,.$$

(i) Derive the SDE satisfied by the portfolio value process $X = X^\nu$. **[2 marks]**

(ii) Show that there is a risk neutral measure for this model and identify the expression for the "market price of risk", denoting it λ. **[4 marks]**

(iii) The deflator is

$$Y_t = \exp(-r\, t - \tfrac{1}{2}\lambda^2 t - \lambda\, W_t)\,.$$

Let ν be an admissible strategy and X the corresponding wealth process. Show that

$$X_t Y_t = x_0 + \int_0^t X_s Y_s (\nu\sigma - \lambda)\, dW_s\,.$$

[4 marks]

(iv) Use duality theory to identify the optimal wealth random variable $\widehat{X}_T$, the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$ and $\hat{\nu}$. **[8 marks]**

b) Let W be an $\mathbb{R}^d$-valued Wiener process, let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $\gamma = \gamma_t$ be an adapted, bounded process. Let ξ be $\mathcal{F}_T$-measurable s.t. $\mathbb{E}\xi^2 < \infty$. Find an explicit solution to

$$dY_t = \gamma_t\, dt + Z_t\, dW_t \quad t \in [0, T]\,,\ \ Y_T = \xi\,. \tag{1}$$

[7 marks]

Comment: *This is the easiest question as everything asked has been covered in class. a) Is optimization using duality theory with a little bit of the background thrown in. It is broken down into steps so that students can hopefully collect many marks (all covered in class). b) is standard example of how to obtain solutions to linear BSDEs using martingale representation (covered in class).*

Solution:

a) i) The equation of the wealth is given by

$$\begin{aligned} dX_t &= \frac{\nu X_t}{S_t}\, dS_t + \frac{(1-\nu)X_t}{B_t}\, dB_t \\ &= X_t\left[(\mu - r)\,\nu + r\right]\, dt + \nu\sigma X_t\, dW_t, \;\; X_0 = x, \end{aligned}$$

[2 marks]

ii) We need the process $\frac{S}{B}$ to be a martingale. We can calculate that $d(1/B_t) = -r(1/B_t)\,dt$ and so

$$d(\tfrac{S_t}{B_t}) = S_t\, d(\tfrac{1}{B_t}) + \tfrac{1}{B_t} dS_t = \tfrac{S_t}{B_t}\left[(\mu - r)\,dt + \sigma\, dW_t\right] .$$

Taking $W_t^{\mathbb{Q}} := W_t + \lambda t$ we have

$$d(\tfrac{S_t}{B_t}) = \tfrac{S_t}{B_t}\left[(\mu - r - \sigma\,\lambda)\,dt + \sigma\, dW_t^{\mathbb{Q}}\right] .$$

Take $\lambda = (\mu - r)\sigma^{-1}$ and $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \Lambda_T\,, \;\text{ where }\; \Lambda_t := \exp(-\lambda W_t - \tfrac{1}{2}\lambda^2 t)\,.$$

Then from e.g. direct calculation we have $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$ and so by Girsanov's theorem $W^{\mathbb{Q}}$ is a Wiener process under $\mathbb{Q}$ and so $\frac{S}{B}$ is a $\mathbb{Q}$-martingale. We have the market price of risk $\lambda = (\mu - r)\sigma^{-1}$. **[4 marks]**

iii) Let $\Lambda_t := \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right]$. The deflator is

$$Y_t := \frac{\Lambda_t}{B_t} = \exp(-r\,t - \tfrac{1}{2}\lambda^2 t - \lambda\, W_t)\,.$$

We can calculate

$$dY_t = Y_t\left(-r - \frac{1}{2}\lambda^2\right)\, dt - \lambda Y_t\, dW_t + \frac{1}{2}\lambda^2 Y_t\, dt$$

so

$$dY_t = -rY_t dt - \lambda Y_t\, dW_t\,.$$

Then from product rule we have

$$d(X_tY_t) = -\lambda X_tY_t\, dW_t + X_tY_t(\mu - r)\nu\, dt + \nu\sigma X_tY_t\, dW_t - (\mu - r)\sigma^{-1}\nu\sigma X_tY_t dt$$

$$= X_tY_t\left[\nu\sigma - \lambda\right]\, dW_t\,.$$

Integrating, we get the desired expression. **[4 marks]**

iv) For a wealth process X^ν and a control ν the optimization problem can be written as

$$u(x) = \sup_{\nu\in\mathbb{R}} \mathbb{E}\Big[U\big(X_T^\nu\big)\big|X_0 = x\Big], \qquad U(x) = \ln x.$$

We have:

$$U(x) = \ln x, \qquad U'(x) = \frac{1}{x}, \qquad I(y) := (U')^{-1}(y) = \frac{1}{y}\,.$$

Using the theory from class, the optimal wealth random variable is $\widehat{X}_T = I(yY_T)$ with $I(\cdot)$ as defined above for U and the number y is determined by the Lagrangian condition $\mathbb{E}[Y_T\widehat{X}_T] = \mathbb{E}[Y_TI(yY_T)] = x_0 \Leftrightarrow \mathcal{X}(y) = x_0$.

Hence,

$$\widehat{X}_T = I(yY_T) = \frac{1}{yY_T} \quad \text{and} \quad \mathcal{X}(y) = x_0 \Leftrightarrow \mathbb{E}[Y_T \frac{1}{yY_T}] = x_0 \Leftrightarrow \frac{1}{y} = x_0.$$

The optimal wealth at time $t = T$ is given by

$$\widehat{X}_T = I(yY_T) = \frac{1}{yY_T} = \frac{x_0}{Y_T}.$$

and the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$ is given by

$$\widehat{X}_t = \frac{1}{Y_t}\mathbb{E}[Y_T \widehat{X}_t | \mathcal{F}_t] = \frac{x_0}{Y_t}, \qquad t \in [0,T].$$

By construction, the wealth dynamics is characterized by the SDE

$$\widehat{X}_t Y_t = x_0 + \int_0^t \widehat{X}_s Y_s (\nu_s^* \sigma - \lambda) \mathrm{d}W_s$$

with the budget constraint $\mathbb{E}[Y_T X_T] = x_0$ (since the market is complete). The optimal strategy is given by $\nu = \lambda/\sigma$ as to cancel out the stochastic integral.

[8 marks]

b) Take $\hat{\xi} := \xi - \int_0^T \gamma_t \, dt$. Let

$$\hat{X}_t = \mathbb{E}\left[\hat{\xi} | \mathcal{F}_t\right] = \mathbb{E}\left[\xi - \int_0^T \gamma_t \Big| \mathcal{F}_t\right] .$$

[3 marks]

We see that $\hat{X}$ is a martingale and so there is unique $\hat{Z}$ such that $\mathrm{d}\hat{X}_t = \hat{Z}_t \, dW_t$, $\hat{X}_T = \hat{\xi}$ due to the martingale representation theorem. **[2 marks]**

Let $X_t = \hat{X}_t + \int_0^t \gamma_s \, ds$ and $Z = \hat{Z}$. Then $dX_t = d\hat{X}_t + \gamma_t dt = Z_t \, dW_t + \gamma_t \, dt$ and moreover $X_T = \xi$ so this solves the desired BDSE. We can write the solution as

$$X_t = \mathbb{E}\left[\xi - \int_t^T \gamma_s \, ds \Big| \mathcal{F}_t\right] .$$

[2 marks]

2. We consider the standard Black–Scholes model: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. We will consider the optimal investment problem with X_s denoting the portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t\in[0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E}\int_t^T \nu_s^2\, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{U}$.

a) Derive the SDE satisfied by the portfolio value process $X_s = X_s^{\nu,t,x}$. **[4 marks]**

b) Show that $X_s = X_s^{\nu,t,x} > 0$ for all $s \in [t,T]$ if $X_t^{\nu,t,x} = x > 0$. **[4 marks]**

c) Consider the control problem

$$v(t,x) := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\ln(X_T^{\nu,t,x})\right] . \tag{2}$$

Write down the Bellman PDE that the function v must satisfy. **[4 marks]**

d) Show that the equation has a solution $v(t,x) = \lambda(t)\ln(\beta x) + \gamma(t)$ with $\lambda, \gamma \in C^1([0,T])$ and $\lambda > 0$. Write down the λ, γ and the optimal control explicitly. **[6 marks]**

e) Use the verification theorem to prove that the v above and the optimal control you identify are indeed the solution to the optimal control problem (2). **[7 marks]**

Comment: *Parts a) and b) have been seen (a number of times). Parts c), d) and e) follow Merton's problem solution using Bellman PDEs students have seen in the course. The logarithmic utility function is one we haven't looked at explicitly, but since the "guess" solution the Bellman PDE is given, this should be straightforward.*

The process X^ν is the same as in Question 1 but the starting point is a bit different. Anyway, this is a happy coincidence which should let weaker students collect some easy marks.

Solution:

a) We calculate (Itô formula) that $dB_t = rB_t\, dt$ and $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. We then have (with ψ_t being the number of units of risky asset we hold)

$$dX_t = \psi_t\, dS_t + \frac{X_t - \psi_t S_t}{B_t}\, dB_t = \nu_t X_t \frac{1}{S_t}\, dS_t + \frac{X_t - \nu_t X_t}{B_t}\, dB_t .$$

So

$$dX_t = X_t\left[(\mu - r)\nu_t + r\right] dt + \nu_t X_t \sigma\, dW_t .$$

[4 marks]

b) We can guess the solution (since it's in the familiar form) and so

$$X_t = x\exp\left[\underbrace{\int_0^t \left((\mu-r)\nu_s + r - \tfrac{1}{2}\nu_s^2\sigma^2\right) ds + \int_0^t \nu_s\,\sigma\, dW_s}_{=:Y_t}\right] .$$

But we now must check that this is indeed the solution:

$$dY_t = \left((\mu - r)\nu_t + r - \tfrac{1}{2}\nu_t^2\sigma^2\right) dt + \nu_t\sigma\, dW_t .$$

Applying Itô formula to $y \mapsto e^y$ and the process Y we see that X is indeed a solution to the wealth process equation. Since $x > 0$ and $e^y > 0$ for all y we have $X_t > 0$ for all t. *Note that using Itô formula for logarithm (for anything other than an educated guess) is not valid because such step already assumes that $X_t > 0$.* **[4 marks]**

c)

$$\partial_t v + \sup_u \left[\frac{1}{2}\sigma^2 u^2 x^2 \partial_{xx} v + x[(\mu - r)u + r]\partial_x v\right] = 0 \ \text{ on } [0,T) \times (0,\infty)$$
$$v(T,x) = \ln x \ \ \forall x > 0\,.$$

The domain has to be specified to get full marks. **[4 marks]**

d) With the ansatz we have $\partial_t v = \lambda'(t)\ln x + \gamma'(t)$, $\partial_x v = \lambda(t)x^{-1}$, $\partial_{xx} v = -\lambda(t)x^{-2}$. So we get

$$\lambda'(t)\ln x + \gamma'(t) + \sup_{u\in\mathbb{R}}\left[-\tfrac{1}{2}\sigma^2 u^2 \lambda(t) + \lambda(t)\left((\mu - r)u + r\right)\right] = 0\,.$$

The function $u \mapsto -\frac{1}{2}\sigma^2 u^2\lambda(t) + \lambda(t)\left((\mu - r)u + r\right)$ is maximized (calculus and concavity) when

$$\hat{u} = \frac{\mu - r}{\sigma^2}\,.$$

Hence

$$\lambda'(t)\ln x + \gamma'(t) + \lambda(t)\left[-\tfrac{1}{2}\sigma^2\hat{u}^2 + (\mu - r)\hat{u} + r\right] = 0\,.$$

Collecting terms involving $\ln x$ and those without we get:

$$\lambda'(t) = 0\,, \ \lambda(T) = 1 \ \text{ so } \ \lambda(t) = 1 \ \text{ for all } \ t \in [0,T]$$

and

$$\gamma'(t) = \tfrac{1}{2}\sigma^2\hat{u}^2 - (\mu - r)\hat{u} - r =: \Gamma(\sigma,\mu,r)\,, \ \ \gamma(T) = 0\,.$$

Integrating this we get $\gamma(t) = -(T-t)\Gamma$. So $v(t,x) = \ln x - (T-t)\Gamma$. **[6 marks]**

e) We have established that $v(t,x) = \ln x - (T-t)\Gamma$ solves the Bellman PDE. Moreover the Markovian optimal control $\hat{u}(t,x) = \frac{\mu - r}{\sigma^2}$ is constant and hence certainly measurable. **[2 marks]**

The wealth equation with the optima control is

$$d\hat{X}_t = \hat{X}_t\left[(\mu - r)\hat{u} + r\right]\,dt + \hat{u}\hat{X}_t\sigma\,dW_t\,.$$

This is a linear SDE with Lipschitz coefficients so it has unique solution which moreover has all the moments when started from deterministic initial value. **[2 marks]**

We consider

$$t' \mapsto \int_t^{t'} \frac{1}{\hat{X}_s}\hat{u}\hat{X}_s\sigma\,dW_s = \hat{u}\sigma(W_{t'} - W_t)\,.$$

This is a martingale (Wiener process is) and so the verification is complete, the constant strategy $\hat{u}$ is optimal and the optimal value for this control is v. **[3 marks]**

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3. Let W be a real valued Wiener process generating a filtration $(\mathcal{F}_t)$. Consider $X_t = X_t^{\alpha,x}$ taking values in $\mathbb{R}$ given by

$$dX_t = [H(t)X_t + M(t)\alpha_t]\,dt + \sigma(t)\,dW_t \text{ for } t \in [0,T]\,,\ \ X_0 = x\,,$$

where H, M and σ are bounded deterministic functions of t. The aim will be to maximize

$$J^\alpha(x) := \mathbb{E}\left[\int_0^T \left[C(t)(X_t^{\alpha,x})^2 + D(t)\alpha_t^2\right]\,dt + R(X_T^{\alpha,x})^2\right]$$

over all adapted processes α such that $\mathbb{E}\int_0^T \alpha_t^2\,dt < \infty$ (we will call these admissible). We will assume that C and D are integrable deterministic functions of t and R a real constant with $C = C(t) \le 0, R \le 0$ and $D = D(t) \le -\delta < 0$ with some constant $\delta > 0$.

a) Write down the Hamiltonian $(t, x, a, y, z) \mapsto H_t(x, a, y, z)$ for this problem and explain why it is concave and differentiable as a function of (x, a) for all t, y, z. **[3 marks]**

b) Write down the adjoint BSDE. **[2 marks]**

c) Use Pontryagin's maximum principle to show that the optimal control $\hat{\alpha}$ and the adjoint BSDE $(\hat{Y}, \hat{Z})$ associated with this control process must satisfy

$$\hat{\alpha}_t = -\frac{M(t)}{2D(t)}\hat{Y}_t \text{ for } t \in [0,T]\,.$$

[3 marks]

d) Inspecting the terminal condition for the adjoint BSDE leads us to "guess" that we should have $\hat{Y}_t = 2S(t)\hat{X}_t$ for some $S \in C^1([0,T])$ with $S(T) = R$. Derive the ordinary differential equation for S. **[7 marks]**

e) Hence show that

$$J^{\hat{\alpha}}(x) = S(0)x^2 + \int_0^T S(t)\sigma^2(t)\,dt\,.$$

[10 marks]

Comment: *This is deliberately a more difficult question. Parts a), b) and c) are not hard but do not bring many marks. Parts d) and e) were not seen before in the context of BSDEs and Pontryagin's maximum principle. We solved the linear-quadratic control problem using HJB equation during the course but this isn't of much immediate help.*

Solution:

a) The Hamiltonian is

$$H_t(x, a, y, z) = H(t)\,x\,y + M(t)\,a\,y + \sigma(t)\,z + C(t)\,x^2 + D(t)\,a^2\,.$$

We see that as function of (a, x) it is a sum of linear and quadratic functions and hence differentiable. Moreover since $C \le 0$ and $D < 0$ we see that it is concave. **[3 marks]**

b) The adjoint BSDE for an admissible control process α is

$$dY_t^\alpha = -\left[H(t)Y_t^\alpha + 2C(t)X_t^\alpha\right]\,dt + Z_t^\alpha\,dW_t \text{ for } t \in [0,T]\,,\ \ Y_T^\alpha = 2RX_T^\alpha\,.$$

[2 marks]

c) Note that $x \mapsto Rx^2$ is concave (since $R \le 0$) and so the Pontryagin's maximum principle applies. If $\hat\alpha$ is the optimal control, $\hat X$ is the associated diffusion and $(\hat Y, \hat Z)$ is the solution to the adjoint BSDE for $\hat\alpha$ then the maximum principle says that

$$H_t(\hat X_t, \hat\alpha_t, \hat Y_t, \hat Z_t) = \max_{a\in\mathbb{R}} H_t(\hat X_t, a, \hat Y_t, \hat Z_t)\,.$$

In this case the maximum is achieved when (Hamiltonian is quadratic in a with negative leading coefficient so we just differentiate w.r.t. a and see for which value this is 0):

$$0 = M(t)\hat Y_t + 2D(t)\,a$$

i.e.

$$\hat\alpha = -\frac{M(t)}{2D(t)}\hat Y_t\,.$$

[3 marks]

d) Since our guess is that $\hat Y_t = 2S(t)\hat X_t$ we have, due to Itô's formula

$$d\hat Y_t = 2S'(t)\hat X_t\,dt + 2S(t)\,d\hat X_t = \left(2S'(t) + 2S(t)H(t) - 2\frac{M^2(t)S^2(t)}{D(t)}\right)\hat X_t\,dt + 2S(t)\sigma(t)\,dW_t\,.$$

[3 marks]

On the other hand the adjoint equation for $\hat Y$ gives

$$d\hat Y_t = -\left[H(t)\hat Y_t + 2C(t)\hat X_t\right]\,dt + \hat Z_t\,dW_t\,.$$

Since both must hold we get that $\hat Z_t = 2S(t)\sigma(t)$ and that

$$S'(t) = \frac{2M^2(t)}{D(t)}S^2(t) - 2S(t)H(t) - C(t)\ \text{ for }\ t\in[0,T)\,,\ \ S(T) = R\,.$$

[4 marks]

e) The optimal control is $\hat\alpha_t = -\frac{M(t)}{D(t)}S(t)\hat X_t$ and hence the equation for $\hat X$ is

$$d\hat X_t = \left[H(t)X_t - \frac{M^2(t)}{D(t)}S(t)\hat X_t\right]\,dt + \sigma(s)\,dW_s\ \text{ for }\ t\in[0,T]\,,\ \ X_0 = x\,.$$

This linear SDE clearly has unique solution and all the moments are bounded. We also have

$$J^{\hat\alpha}(x) = \mathbb{E}\left[\int_0^T \left(C(t)\hat X_t^2 - \frac{M^2(t)}{D(t)}S^2(t)\hat X_t^2\right)\,dt + R\hat X_T^2\right]\,. \tag{3}$$

[3 marks]

We observe that

$$R\hat X_T^2 = S(T)\hat X_T^2 = \frac12\hat Y_T\hat X_T = \frac12\hat Y_T\hat X_T - \frac12\hat Y_0\hat X_0 + \frac12\hat Y_0\hat X_0 = \frac12\int_0^T d(\hat Y_t\hat X_t) + S(0)\,x\,. \tag{4}$$

Moreover

$$\begin{aligned} d(\hat Y_t\hat X_t) =& 2S(t)\hat X_t d\hat X_t + \hat X_t d\hat Y_t + d\hat Y_t d\hat X_t \\ =& 2S(t)H(t)\hat X_t^2\,dt + \frac{M^2(t)}{D(t)}S^2(t)\hat X_t^2\,dt + 2S(t)\hat X_t\,\sigma(t)\,dW_t \\ & - 2S(t)H(t)\hat X_t^2\,dt - 2C(t)\hat X_t^2 + 2S(t)\sigma(t)\hat X_t\,dW_t + 2S(t)\sigma^2(t)\,dt\,. \end{aligned} \tag{5}$$

[3 marks]

We now substitute (5) into (4) and use this in (3) to see that most terms cancel and

$$J^{\hat\alpha}(x) = \mathbb{E}\left[\int_0^T S(t)\sigma^2(t)\,dt + \int_0^T 2S(t)\sigma(t)\hat X_t\,dW_t + S(0)\,x^2\right]\,.$$

Since the solution of the SDE for $\hat{X}$ has all moments bounded we have

$$\mathbb{E}\int_0^T 4S^2(t)\sigma^2(t)\hat{X}_t^2\,dt \leq N\int_0^T \mathbb{E}\hat{X}_t^2\,dt \leq N_T < \infty\,.$$

The stochastic integral is a martingale and so

$$J^{\hat{\alpha}}(x) = S(0)x^2 + \int_0^T S(t)\sigma^2(t)\,dt\,.$$

[4 marks]