Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

(1) Given $(s, y) \in [0,1) \times \mathbb{R}$, consider the following stochastic control problem

$$\begin{aligned} V(s,y) &= \min_{\nu} J(s,y;\nu) \\ &= \min_{\nu} \mathbb{E}\left[\int_s^1 \left[\,\big(X^{\nu}_{s,y}(t)\big)^2 - \frac{1}{2}\nu^2(t)\,\right]\mathrm{d}t\right] \end{aligned}$$

$$\text{such that } \begin{cases} \mathrm{d}X^{\nu}_{s,y}(r) &= \nu(r)\mathrm{d}W(r), \qquad r \in [s,T] \\ X^{\nu}_{s,y}(s) &= y \\ \nu(t) &\in [0,1] \quad \forall t \in [0,1] \text{ and } (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted} \end{cases}$$

(a) Let $t \in [s,1]$. Express $\mathbb{E}[\big(X^{\nu}_{s,y}(t)\big)^2]$ in terms of the control $\nu(\cdot)$ and prove that

$$\mathbb{E}[\big(X^{\nu}_{s,y}(t)\big)^2] = y^2 + \mathbb{E}[\big(X^{\nu}_{s,0}(t)\big)^2].$$

[4 marks]

(b) Show that $V(s,y)$ can be expressed as $V(s,y) = y^2(1-s) + g(s)$ for some function $g(s)$ you should identify and compute $\partial_y V(s,y)$ and $\partial_{yy} V(s,y)$.

[6 marks]

(c) Write down the HJB equation for this stochastic control problem.

[4 marks]

(d) Find a solution to the HJB equation.

[11 marks]

(2) (2.a) A Black-Scholes market is given where there are only one stock (with drift $a \in \mathbb{R}$ and volatility $\sigma > 0$) and one bank account with interest rate $r \in \mathbb{R}$.

In this market an investor, with initial wealth $x_0 > 0$ selects among *proportion strategies* ν that are *constants* and with such a strategy the *proportion of wealth invested in the stock* is a constant throughout.

The investor seeks to maximise his expected utility at time T which is a power-type utility

$$U(x) = \frac{1}{\gamma} x^\gamma, \qquad \gamma \in (0,1).$$

(2.a.i) Identify explicitly the underlying, show that the SDE expressing the wealth process $\big(X^\nu(t)\big)_{t\in[0,T]}$ is of Geometric Brownian motion type and write its explicit solution.

[5 marks]

(2.a.ii) Write clearly the optimization problem and then compute the *constant optimal proportion strategy* $\nu^\star$ explicitly *without* applying the stochastic control approach.

You may use without proving that $\forall\, c \in \mathbb{R}$ we have $\mathbb{E}[e^{cW(T)}] = e^{\frac{1}{2}c^2 T}$.

[7 marks]

(2.b) Let $T < \infty$ and consider the following BSDE with solution $\big(Y(t), Z(t)\big)_{t\in[0,T]}$,

$$\mathrm{d}Y(t) = \big(rY(t) + aZ(t)\big)\mathrm{d}t + Z(t)\mathrm{d}W(t)\ , \qquad Y(T) = \xi. \tag{1}$$

where r, a are constants, ξ is a square-integrable, $\mathcal{F}_T$-measurable random variable in a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, and W is a one-dimensional Brownian Motion.

(2.b.i) Argue that the solution $(Y(t), Z(t))$ exists and deduce the expression yielding $Y(t)$ as a map of T, t, r, a and ξ (a so-called *closed form solution*).

[6 marks]

(2.b.ii) Denote by (Y^i, Z^i) the solution to BSDE (1) with ξ being replaced by $\xi_i, i = 1, 2$ both $\mathcal{F}_T$-adapted square-integrable RV. Suppose $\xi_1 \ge \xi_2$ a.s..

Prove that $Y^1(t) \ge Y^2(t)\ \forall t \in [0,T]$ a.s.

[7 marks]

(3) In a d-dimensional complete market with zero interest rate, an agent with initial wealth $x > 0$ trades d stocks and generates wealth process X given by

$$X_t = x + \int_0^t \pi_s^{\mathrm{T}} \sigma_s (\lambda_s \mathrm{d}s + \mathrm{d}B_s) \ , \ 0 \le t \le T.$$

Here, T denotes transposition, the trading strategy π is a d-dimensional vector of wealth in each stock, λ is a d-dimensional vector, σ a $d \times d$ invertible matrix, and B a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the standard augmented filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$, with λ, σ, π adapted to $\mathbb{F}$.

The agent seeks to maximise $\mathbb{E}[U(X_T)]$, over the strategies such that the wealth process remains positive, and with a concave, increasing, differentiable utility function $U : (\overline{x}, \infty) \to \mathbb{R}$, for some $\overline{x} > 0$ denoting a constant below which terminal wealth is not permitted to fall. Denote by V the convex conjugate of U, and by I the inverse of U'. Denote the maximal expected utility by $u(x)$. Let $Z := \mathcal{E}(-\lambda^{\mathrm{T}} \cdot B)$ and assume Z is a martingale.

(3.a) Derive the dynamics of ZX and deduce that $\mathbb{E}[Z_T X_T] \le x$.

[3 marks]

(3.b) Show that $u(x) \le v(y) + xy$, where $v(y) := \mathbb{E}[V(yZ_T)]$, for $y > 0$.

[3 marks]

(3.c) Explain why the optimal terminal wealth, $\hat{X}_T$, is given by $\hat{X}_T = I(yZ_T)$, for some $y > 0$, and explain how y is fixed.

[3 marks]

(3.d) Suppose $U(x) = \log(x - \overline{x})$. Compute a formula for $\hat{X}_T$ in terms of x. What is the lowest value of initial wealth which guarantees that terminal wealth $\hat{X}_T > \overline{x}$?

[5 marks]

(3.e) By considering $Z\hat{X}$, where $\hat{X}$ is the optimal wealth process, show that the optimal portfolio process is given by

$$\hat{\pi}_t = (\hat{X}_t - \overline{x})(\sigma_t^{-1})^{\mathrm{T}} \lambda_t, \ 0 \le t \le T.$$

[5 marks]

(3.f) Suppose now that the agent also receives stochastic income at a rate $Y = (Y(t))_{0 \le t \le T}$ per unit time, where Y is a bounded non-negative adapted process. By considering the dynamics of X under the unique equivalent martingale measure $\mathbb{Q}$, argue that in this case the wealth process of any strategy satisfies

$$\mathbb{E}[Z_T X_T] \le \overline{x} := x + K,$$

for some non-negative constant K that you should identify. What is the minimum initial wealth required for a feasible problem in this case? Interpret the result.

[6 marks]

(4) On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, an agent trades in an incomplete market, generating wealth process X. A non-attainable claim pays a bounded random variable C at terminal time T.

An agent with initial capital x and exponential utility function $U(x) = -e^{-x}$, maximises expected utility of terminal wealth, with the random endowment of a short position in the claim. Denote the value function by $u(x)$. You may assume the interest rate is zero and that the wealth process X is a $\mathbb{Q}$-martingale, for all equivalent local martingale measures (ELMMs) $\mathbb{Q}$ with finite entropy. Denote the density process of such an ELMM $\mathbb{Q}$ by Z, and denote the relative entropy between $\mathbb{Q}$ and $\mathbb{P}$ by $H(\mathbb{Q}|\mathbb{P}) := \mathbb{E}[Z_T \log Z_T]$.

(4.a) Show that we have the inequality

$$\mathbb{E}[U(X_T - C)] \leq \mathbb{E}[V(yZ_T) - yZ_T C] + xy, \quad y > 0 \tag{2}$$

for any trading strategy and any $\mathbb{Q}$, where V is the convex conjugate of U. Hence show that $u(x) \leq v(y) + xy$, where v is the value function of the dual problem, which you should define.

[4 marks]

(4.b) Suppose that equality is achieved in (2) for the optimal terminal wealth $\widehat{X}_T$ and an optimal density $\widehat{Z}_T$. Show that

$$\widehat{X}_T - C = -\log(y\widehat{Z}_T) \ ,$$

for some $y > 0$. **[2 marks]**

(4.c) Explain how y is determined and hence derive a formula for $\widehat{X}_T - C$ in terms of x and $\widehat{Z}_T$, and some constants that you should identify.

[5 marks]

(4.d) Hence show that the maximal expected utility is given by

$$u(x) = -\exp\left\{-x - H(\widehat{\mathbb{Q}}|\mathbb{P}) + \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right\},$$

where $\widehat{\mathbb{Q}}$ is the ELMM corresponding to $\widehat{Z}$. **[2 marks]**

(4.e) Denote by u_0 and Z^0 the value function and density of the dual minimiser $\mathbb{Q}^0$ when there is no random endowment in the above utility maximisation problem. The utility indifference price p of the claim at time zero is defined implicitly by $u(x+p) = u_0(x)$.
Derive a formula for p. **[5 marks]**

(4.f) Now suppose the market model contains one stock S and one non-traded asset Y, following the geometric Brownian motions

$$\mathrm{d}S_t = \sigma_S S_t(\lambda_S \mathrm{d}t + \mathrm{d}B_t^S), \qquad \mathrm{d}Y_t = \sigma_Y Y_t(\lambda_Y \mathrm{d}t + \mathrm{d}B_t^Y) \ ,$$

where B^S, B^Y are correlated Brownian motions with constant correlation $\rho \in (-1, 1)$ and $\sigma_S, \sigma_Y, \lambda_S, \lambda_Y$ are constants. By deriving an expression for $H(\mathbb{Q}|\mathbb{P})$, show that in this case the indifference price has the representation

$$p = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[C - \frac{1}{2}\int_0^T \widehat{\psi}_t^2 \mathrm{d}t\right]$$

where $\widehat{\psi}$ is an adapted process. Explain how $\widehat{\psi}$ is related to $\widehat{\mathbb{Q}}$. **[7 marks]**

[End of Paper]