2018/19 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 3 - Friday 14th February 2020[1]

Exercise 3.1. For any $(t,x) \in [0,T] \times \mathbb{R}$, define the stochastic process $(X_s^{t,x})_{s\in[t,T]}$ as the solution to the SDE

$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s, \ \ \forall\, s \in [t,T], \qquad X_t = x.$$

Let $\mathbb{E}^{t,x}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|X_t = x]$. Define a function $v = v(t,x)$ as

$$v(t,x) = e^{-r(T-t)}\mathbb{E}^{t,x}\big[\, g\big(X_T\big)\,\big] \qquad \forall\,(t,x) \in [0,T) \times \mathbb{R}\,.$$

Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$ and that $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$. Show that

$$\partial_t v + b\partial_x v + \frac{1}{2}\sigma^2\partial_{xx} v - rv = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,,$$
$$v(T,\cdot) = g \ \text{ on } \ \mathbb{R}\,.$$

Exercise 3.2 (Unattainable optimizer)**.** Here is a simple example in which no optimal control exists, in a finite horizon setting, $T \in (0,\infty)$. Suppose that the state equation is

$$dX_s = \alpha_s\,ds + dW_s \ \ \ s \in [t,T]\,, \quad X_t = x \in \mathbb{R}.$$

A control α is admissible ($\alpha \in \mathcal{A}$) if: α takes values in $\mathbb{R}$, is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, and $\mathbb{E}\int_0^T \alpha_s^2\,ds < \infty\,.$

Let $J(t,x,\alpha) := \mathbb{E}[|X_T^{t,x,\alpha}|^2]$. The value function is $v(t,x) := \inf_{\alpha\in\mathcal{A}} J(t,x,\alpha)$. Clearly $v(t,x) \geq 0$.

i) Show that for any $t \in [0,T]$, $x \in \mathbb{R}$, $\alpha \in \mathcal{A}$ we have $\mathbb{E}[|X_T^{t,x,\alpha}|^2] < \infty$.

ii) Show that if $\alpha_t := -cX_t$ for some constant $c \in (0,\infty)$ then $\alpha \in \mathcal{A}$ and

$$J^\alpha(t,x) = J^{cX}(t,x) = \frac{1}{2c} - \frac{1-2cx^2}{2c}e^{-2c(T-t)}.$$

Hint: with such an α, the process X is an Ornstein-Uhlenbeck process, see an earlier exercise.

iii) Conclude that $v(t,x) = 0$ for all $t \in [0,T)$, $x \in \mathbb{R}$.

iv) Show that there is no $\alpha \in \mathcal{A}$ such that $J(t,x,\alpha) = 0$. *Hint:* Suppose that there is such a α and show that this leads to a contradiction.

[1] Last updated 14th February 2020