Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Given $(s, y) \in [0, 1) \times \mathbb{R}$, consider the following stochastic control problem

$$\begin{aligned} V(s,y) &= \min_{\nu} J(s,y;\nu) \\ &= \min_{\nu} \mathbb{E}\left[\int_s^1 \left[\,\big(X^{\nu}_{s,y}(t)\big)^2 - \frac{1}{2}\nu^2(t)\,\right]\mathrm{d}t\right] \\ &\qquad \text{such that} \left\{ \begin{array}{rcll} \mathrm{d}X^{\nu}_{s,y}(r) &=& \nu(r)\mathrm{d}W(r), & r \in [s,T] \\ X^{\nu}_{s,y}(s) &=& y & \\ \nu(t) &\in& [0,1] & \forall t \in [0,1] \text{ and } (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted} \end{array}\right. \end{aligned}$$

(a) Let $t \in [s, 1]$. Express $\mathbb{E}\big[\big(X^{\nu}_{s,y}(t)\big)^2\big]$ in terms of the control $\nu(\cdot)$ and prove that

$$\mathbb{E}\big[\big(X^{\nu}_{s,y}(t)\big)^2\big] = y^2 + \mathbb{E}\big[\big(X^{\nu}_{s,0}(t)\big)^2\big].$$

[4 marks]

(b) Show that $V(s, y)$ can be expressed as $V(s, y) = y^2(1 - s) + g(s)$ for some function $g(s)$ you should identify and compute $\partial_y V(s, y)$ and $\partial_{yy} V(s, y)$.

[6 marks]

(c) Write down the HJB equation for this stochastic control problem.

[4 marks]

(d) Find a solution to the HJB equation.

[11 marks]

Solution:

(a) **Properties of X^ν.** The solution to the SDE is $X\nu_{s,y}(t) = y + \int_s^t \nu(r)\mathrm{d}W(r)$ and hence it is clear that

$$X^\nu_{s,y}(t) = y + X^\nu_{s,0}(t). \qquad \textbf{[2 Marks]}$$

It follows using squares, that $\mathbb{E}[X^\nu_{s,y}(t)] = y \ \forall t \geq s$ and Itô's isometry that

$$\begin{aligned}\mathbb{E}[\big(X^\nu_{s,y}(t)\big)^2] &= y^2 + 2y\mathbb{E}[\int_s^t \nu(r)\mathrm{d}W(r)] + \mathbb{E}[\Big(\int_s^t \nu(r)\mathrm{d}W(r)\Big)^2] = y^2 + \mathbb{E}[\int_s^t \nu^2(r)\mathrm{d}r] \\ &= y^2 + \mathbb{E}\Big[\big(X^\nu_{s,0}(t)\big)^2\Big]. \qquad \textbf{[2 Marks]}\end{aligned}$$

(b) **Properties of the Value function** If the value function can effectively be written as $V(s,y) = y^2(1-s) + g(s)$, then it follows immediately that $y \mapsto V(s,y)$ is twice continuously differentiable and we have

$$\begin{aligned}\partial_y V(s,y) &= 2y(1-s) && \textbf{[1 Marks]} \\ \partial_{yy} V(s,y) &= 2(1-s) > 0 \ \forall (s,y) \in [0,1) \times \mathbb{R} && \textbf{[1 Marks]}\end{aligned}$$

It remains to show the expression $V(s,y) = y^2(1-s) + g(s)$ and determine g.

From the property $X^\nu_{s,y}(t) = y + X^\nu_{s,0}(t) \ \forall t \in [s,1]$ we have

$$\begin{aligned}J(s,y,\nu) &= \mathbb{E}\left[\int_s^1 \big[\, \big(X^\nu_{s,y}(t)\big)^2 - \frac{1}{2}\nu^2(t)\,\big]\mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_s^1 \big[\, y^2 + 2yX^\nu_{s,0}(t) + \big(X^\nu_{s,0}(t)\big)^2 - \frac{1}{2}\nu^2(t)\,\big]\mathrm{d}t\right] \\ &= y^2(1-s) + 2y\int_s^1 \mathbb{E}\left[X^\nu_{s,0}(t)\right]\mathrm{d}t + \mathbb{E}\left[\int_s^1 \big[\big(X^\nu_{s,0}(t)\big)^2 - \frac{1}{2}\nu^2(t)\big]\mathrm{d}t\right]\end{aligned}$$

Recall that $\mathbb{E}[X^\nu_{s,0}(\cdot)] = 0$ then

$$\begin{aligned}J(s,y,\nu) &= y^2(1-s) + J(s,0,\nu) \\ &\Rightarrow V(s,y) = y^2(1-s) + V(s,0), \quad \text{hence} \quad g(s) = V(s,0). \qquad \textbf{[4 Marks]}\end{aligned}$$

(c) **The HJB equation** depends on the generator of the diffusion given by $\mathrm{d}X(t) = \nu(t)\mathrm{d}W(t)$, in this case $\mathcal{L}^\nu G(s,x) = \partial_t G + \frac{1}{2}\nu^2\partial_{xx}G$ with $\nu \in [0,1]$ (and no drift).

Thus, the HJB equation is given by

$$\min_{\nu\in[0,1]} \left\{\mathcal{L}^\nu V + \big(x^2 - \frac{1}{2}\nu^2\big)\right\} = 0,$$

or

$$\begin{aligned}\partial_t V + \min_{\nu\in[0,1]} \left\{\frac{1}{2}\nu^2\partial_{xx}V - \frac{1}{2}\nu^2\right\} + x^2 = 0 \quad &\text{for } (s,x) \in [0,1) \times \mathbb{R}, && \textbf{[3 Marks]} \\ V(1,x) = 0 \ &\text{for } x \in \mathbb{R}. && \textbf{[1 Marks]}\end{aligned}$$

(d) **Find a solution of the HJB equation**

Since we have $V(s,y) = y^2(1-s) + V(s,0)$ we have $\partial_{yy}V = 2(1-s)$ which can be replaced into the HJB equation to yield:

$$\partial_t V + \min_{\nu\in[0,1]} \left\{\frac{1}{2}\nu^2 2(1-s) - \frac{1}{2}\nu^2\right\} + y^2 = 0.$$

We now solve the minimization problem so that we can solve the HJB.

$$\min_{\nu\in[-1,1]}\left\{\frac{1}{2}\nu^2 2(1-s)-\frac{1}{2}\nu^2\right\}=\min_{\nu\in[-1,1]}\left\{\nu^2(\frac{1}{2}-s)\right\}$$

$$\Rightarrow \nu^\star(s)=\begin{cases}0, \text{if } s\in[0,\frac{1}{2})\\ \pm 1, \text{if } s\in[\frac{1}{2},1]\end{cases}$$ **[3 Marks] For the minimization**

the case $\nu=-1\in[0,1]$ and hence we ignore it.

The minimum reads

$$\min_{\nu\in[0,1]}\left\{\nu^2(\frac{1}{2}-s)\right\}=\begin{cases}0 & \text{, if } s\in[0,\frac{1}{2})\\ \frac{1}{2}-s & \text{, if } s\in[\frac{1}{2},1]\end{cases}$$ **[1 Marks] For the minimimum**

We obtain then two branches for the HJB equation depending on the time, $s\in[0,\frac{1}{2}]$ and $s\in[\frac{1}{2},1]$,

$$\partial_t V+\min_{\nu\in[0,1]}\left\{\frac{1}{2}\nu^2 2(1-s)-\frac{1}{2}\nu^2\right\}+y^2=0 \ \Leftrightarrow \begin{cases}\partial_t V+0+y^2=0 & \text{, if } s\in[0,\frac{1}{2})\\ \partial_t V+(\frac{1}{2}-s)+y^2=0 & \text{, if } s\in[\frac{1}{2},1]\\ V(1,y)=0\end{cases}$$

[2 Marks] Identify PDE to solve

Since we only have $V(1,0)=0$ we start with the 2nd branch $s\in[\frac{1}{2},1]$ and solve the equation by direct integration. This will also allow to identify $V(\frac{1}{2},y)$ to serve as boundary condition for the 2nd PDE. We have then

$$\begin{aligned}V(s,y)&=V(1,y)-\int_s^1[(\frac{1}{2}-r)+y^2]\mathrm{d}r\\ &=0+y^2(1-s)+\frac{1}{2}s^2-\frac{1}{2}s.\end{aligned}$$ **[3 Marks] 1st Branch**

To solve for the 2nd equation one needs to identify the appropriate boundary condition at time $s=\frac{1}{2}$, namely that $V(\frac{1}{2},y)=\frac{1}{2}y^2-\frac{1}{8}$. Hence, by solving the PDE by direct integration over $s\in[0,\frac{1}{2}]$, we have

$$\begin{aligned}V(s,y)&=V(\frac{1}{2},y)-\int_s^{\frac{1}{2}}y^2\mathrm{d}r\\ &=\frac{1}{2}y^2-\frac{1}{8}+y^2(\frac{1}{2}-s)=y^2(1-s)-\frac{1}{8}.\end{aligned}$$ **[2 Marks] 2nd Branch**

Comment:

(a) *Easy; evaluates basic Stochastic Analysis knowledge;*

(b) *Easy; evaluates basic Stochastic Analysis knowledge;*

(c) *Easy; Students must identfy the Dynkin generator and write down the HJB equation. Boundary condition must also be identified*

(d) *easy to Medium to hard. Solving the minimization problem is easy when the explicit formula for $\partial_{yy}V$ is injected; The arising HJB is slighlty different from what they have seen as the PDE has two branches; The branch $s\in[\frac{1}{2},1]$ is easy, the other brach is not so easy and requires more knowledge.*

2.

(2.a) A Black-Scholes market is given where there are only one stock (with drift $a \in \mathbb{R}$ and volatility $\sigma > 0$) and one bank account with interest rate $r \in \mathbb{R}$.

In this market an investor, with initial wealth $x_0 > 0$ selects among *proportion strategies* ν that are *constants* and with such a strategy the *proportion of wealth invested in the stock* is a constant throughout.

The investor seeks to maximise his expected utility at time T which is a power-type utility

$$U(x) = \frac{1}{\gamma}x^\gamma, \qquad \gamma \in (0,1).$$

(2.a.i) Identify explicitly the underlying, show that the SDE expressing the wealth process $\big(X^\nu(t)\big)_{t\in[0,T]}$ is of Geometric Brownian motion type and write its explicit solution.

[5 marks]

(2.a.ii) Write clearly the optimization problem and then compute the *constant optimal proportion strategy* $\nu^\star$ explicitly *without* applying the stochastic control approach. You may use without proving that $\forall\ c \in \mathbb{R}$ we have $\mathbb{E}[e^{cW(T)}] = e^{\frac{1}{2}c^2T}$.

[7 marks]

(2.b) Let $T < \infty$ and consider the following BSDE with solution $\big(Y(t), Z(t)\big)_{t\in[0,T]}$,

$$\mathrm{d}Y(t) = \big(rY(t) + aZ(t)\big)\mathrm{d}t + Z(t)\mathrm{d}W(t)\ , \qquad Y(T) = \xi. \tag{1}$$

where r, a are constants, ξ is a square-integrable, $\mathcal{F}_T$-measurable random variable in a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, and W is a one-dimensional Brownian Motion.

(2.b.i) Argue that the solution $(Y(t), Z(t))$ exists and deduce the expression yielding $Y(t)$ as a map of T, t, r, a and ξ (a so-called *closed form solution*).

[6 marks]

(2.b.ii) Denote by (Y^i, Z^i) the solution to BSDE (1) with ξ being replaced by $\xi_i, i = 1, 2$ both $\mathcal{F}_T$-adapted square-integrable RV. Suppose $\xi_1 \ge \xi_2$ a.s..
Prove that $Y^1(t) \ge Y^2(t)\ \forall t \in [0,T]$ a.s.

[7 marks]

Solution:

(2.a) (2.a.i) **The wealth process.** The stock and riskless process, denoted S and B respectively, have the following dynamics as postulated by the Black-Scholes market

$$\mathrm{d}S(t) = S(t)\left[a\mathrm{d}t + \sigma \mathrm{d}W(t)\right] \qquad \text{and} \qquad \mathrm{d}B(t) = rB(t)\mathrm{d}t.$$

If the strategies are constant proportions of wealth, ν for the proportion of wealth invested in the Stock and $1-\nu$ for the proportion of wealth invested in the bank account, then equation of the wealth is given by

$$\begin{aligned}\mathrm{d}X(t) &= \frac{\nu X(t)}{S(t)}\mathrm{d}S(t) + \frac{(1-\nu)X(t)}{B(t)}\mathrm{d}B(t)\\ &= X(t)\left[(a-r)\,\nu + r\right]\mathrm{d}t + \nu\sigma X(t)\mathrm{d}W(t), \;\; X(0) = x_0,\end{aligned}$$

As the controls are constant, $X(\cdot)$ can be computed explicitly as it is a Geometric Brownian motion. The solution is given by

$$X^\nu(t) = x_0 \cdot \exp\left\{\Big((a-r)\nu + r - \frac{1}{2}\nu^2\sigma^2\Big)t\right\} \cdot \exp\left\{\nu\sigma W(t)\right\}.$$

(2.a.ii) **The optimization** For a wealth process X^ν and a control ν the optimization problem can be written as

$$\sup_{\nu\in\mathbb{R}} \mathbb{E}\Big[U\big(X^\nu(T)\big)\Big], \qquad U(x) = \frac{1}{\gamma}x^\gamma, \qquad \gamma \in (0,1).$$

The utility is given by

$$\begin{aligned}U\big(X^\nu(T)\big) &= \frac{1}{\gamma}\left(X^\nu(T)\right)^\gamma\\ &= \frac{x_0^\gamma}{\gamma}\exp\left\{\gamma(a-r)\nu T + r\gamma T - \frac{1}{2}\gamma\nu^2\sigma^2 T\right\} \cdot \exp\left\{\nu\gamma\sigma W(T)\right\}.\end{aligned}$$

The expected utility for the strategy $\nu(t) = \nu \in \mathbb{R}$, $0 \le t \le T$ is given by

$$\mathbb{E}\Big[U\big(X^\nu(T)\big)\Big] = \frac{x_0^\gamma}{\gamma}\exp\left\{\gamma(a-r)\nu T + r\gamma T - \frac{1}{2}\gamma\nu^2\sigma^2 T + \frac{1}{2}\nu^2\gamma^2\sigma^2 T\right\},$$

or

$$\mathbb{E}\left[\frac{1}{\gamma}\left(X^\nu(T)\right)^\gamma\right] = \frac{x_0^\gamma}{\gamma}\exp\left\{\Big(\gamma(a-r)\nu + r\gamma - \frac{1}{2}\nu^2\sigma^2\gamma(1-\gamma)\Big)T\right\}, \tag{2}$$

where we used the fact that ($c = \nu\gamma\sigma$ from the question's statement)

$$\mathbb{E}\Big[\exp\left\{\nu\gamma\sigma W(T)\right\}\Big] = \mathbb{E}\Big[\exp\big\{\frac{1}{2}\nu^2\gamma^2\sigma^2 T\big\}\Big].$$

The maximum is achieved by simply maximizing the RHS of (2) wrt to ν using standard analysis techniques, namely, for

$$f(\nu) := \Big(\gamma(a-r)\nu + r\gamma - \frac{1}{2}\nu^2\sigma^2\gamma(1-\gamma)\Big)T$$

$$\frac{\mathrm{d}}{\mathrm{d}\nu}f(\nu) = f'(\nu) = \gamma(a-r) - \nu\sigma^2\gamma(1-\gamma), \qquad f''(\nu) = \sigma^2\gamma(1-\gamma) < 0,$$

and $f'(\nu) = 0$ has a unique solution $\nu^\star$ given by

$$\nu^\star = \frac{\gamma(a-r)}{\sigma^2\gamma(1-\gamma)} = \frac{a-r}{\sigma^2(1-\gamma)}.$$

for reference's sake: The optimal utility is given by

$$\frac{1}{\gamma}x_0^\gamma \cdot \exp\left(\frac{1}{2}\left(\frac{a-r}{\sigma}\right)^2 \frac{\gamma}{1-\gamma}T + r\gamma T\right).$$

(2.b) (2.b.i) **Existence & uniqueness of solutions:** The BSDE has a solution because the terminal condition ξ is a $\mathcal{F}_T$-measurable square-integrable RV; the driver function g is Lipschitz in its spatial variables and $g(0,\cdot,\cdot) = 0$. By the theorem in class there exists a unique solution to the equation in $\mathcal{H}^2 \times \mathcal{H}^2$.

Closed form solution to the equation: Using integrating factor $e(t) := e^{-rt}$ and apply Itô's formula to $e(t)Y(t)$, we have

$$\begin{aligned}
\mathrm{d}\big(e(t)Y(t)\big) &= e(t)\big(-rY(t) + rY(t) + aZ(t)\big)\mathrm{d}t + e(t)Z(t)\mathrm{d}W(t)\\
&= e(t)aZ(t)\mathrm{d}t + e(t)Z(t)\mathrm{d}W(t)\\
&= e(t)Z(t)\big(\mathrm{d}W(t) + a\mathrm{d}t\big).
\end{aligned}$$

Define now a new probability measure $\mathbb{Q}$ with Radon-Nikodym derivative given by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\big(-\int_0^T a\mathrm{d}W_r\big)$ and under which the process $\widehat{W}(t) = W(t) + a\mathrm{d}t$ is a Brownian motion. Since a is a real number the density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ is well-defined (Novikov's condition for example).

Changing the measure to $\mathbb{Q}$ and integrating over $[t,T]$ we obtain

$$\begin{aligned}
e(t)Y(t) &= e(T)\xi - \int_t^T e(s)Z(s)\mathrm{d}\widehat{W}(s)\\
\Rightarrow\quad Y(t) &= \mathbb{E}^{\mathbb{Q}}[\,e(T)e^{-1}(t)\xi\,|\mathcal{F}_t] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\,\xi\,|\mathcal{F}_t].
\end{aligned}$$

(2.b.ii) *Method #1: the students recognize that the probability measure $\mathbb{Q}$ in the previous question is the same for both BSDEs and straightforwardly use the closed form solution to conclude.*

Note that the only change in the BSDEs is the terminal condition and not the coefficients in the driver function. This means that the integrating factor $e(\cdot)$ and the probability measure $\mathbb{Q}$ are the same.

Using the closed form formula for the solution of the previous BSDEs we get that

$$Y^1(t) - Y^2(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\,\xi_1 - \xi_2\,|\mathcal{F}_t].$$

Since $\xi_1 \geq \xi_2$ $\mathbb{P}$-a.s. and the measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then it follows that $\mathbb{E}^{\mathbb{Q}}[\,\xi_1 - \xi_2\,|\mathcal{F}_t] \geq 0$ and hence $Y^1(t) \geq Y^2(t)$.

Method #2: the students compute the difference between the 2 BSDEs and use the integrating factor+measure change to reach a closed form solution for the difference $Y^1 - Y^2$. This would take a bit more time.

Comment:

This question is fairly standard and straightforward; it is the easiest question of the exam. Tests ability to manipulate the objects discussed in class.

(2.a) *Standard optimization without using the big optimization methods learned in class. Relies on the use of properties of the underlyings models which appear transversally to the whole course.*

Students are supposed to be able to write down the several quantities of interest involved in the Black-Scholes model.

2.b) *This question is standard and is an easy way for students to get some marks. Questions 2.b.i) and 2.b.ii) have been seen in class in some way or the other.*

3. In a d-dimensional complete market with zero interest rate, an agent with initial wealth $x > 0$ trades d stocks and generates wealth process X given by

$$X_t = x + \int_0^t \pi_s^{\mathrm{T}} \sigma_s (\lambda_s \mathrm{d}s + \mathrm{d}B_s)\ ,\ \ 0 \le t \le T.$$

Here, T denotes transposition, the trading strategy π is a d-dimensional vector of wealth in each stock, λ is a d-dimensional vector, σ a $d \times d$ invertible matrix, and B a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the standard augmented filtration $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$, with λ, σ, π adapted to $\mathbb{F}$.

The agent seeks to maximise $\mathbb{E}[U(X_T)]$, over the strategies such that the wealth process remains positive, and with a concave, increasing, differentiable utility function $U : (\overline{x}, \infty) \to \mathbb{R}$, for some $\overline{x} > 0$ denoting a constant below which terminal wealth is not permitted to fall. Denote by V the convex conjugate of U, and by I the inverse of U'. Denote the maximal expected utility by $u(x)$. Let $Z := \mathcal{E}(-\lambda^{\mathrm{T}} \cdot B)$ and assume Z is a martingale.

(3.a) Derive the dynamics of ZX and deduce that $\mathbb{E}[Z_T X_T] \le x$.

[3 marks]

(3.b) Show that $u(x) \le v(y) + xy$, where $v(y) := \mathbb{E}[V(yZ_T)]$, for $y > 0$.

[3 marks]

(3.c) Explain why the optimal terminal wealth, $\hat{X}_T$, is given by $\hat{X}_T = I(yZ_T)$, for some $y > 0$, and explain how y is fixed.

[3 marks]

(3.d) Suppose $U(x) = \log(x - \overline{x})$. Compute a formula for $\hat{X}_T$ in terms of x. What is the lowest value of initial wealth which guarantees that terminal wealth $\hat{X}_T > \overline{x}$?

[5 marks]

(3.e) By considering $Z\hat{X}$, where $\hat{X}$ is the optimal wealth process, show that the optimal portfolio process is given by

$$\hat{\pi}_t = (\hat{X}_t - \overline{x})(\sigma_t^{-1})^{\mathrm{T}} \lambda_t,\ \ 0 \le t \le T.$$

[5 marks]

(3.f) Suppose now that the agent also receives stochastic income at a rate $Y = (Y(t))_{0\le t\le T}$ per unit time, where Y is a bounded non-negative adapted process. By considering the dynamics of X under the unique equivalent martingale measure $\mathbb{Q}$, argue that in this case the wealth process of any strategy satisfies

$$\mathbb{E}[Z_T X_T] \le \overline{x} := x + K,$$

for some non-negative constant K that you should identify. What is the minimum initial wealth required for a feasible problem in this case? Interpret the result.

[6 marks]

Solution:

(3.a) One just applies Itô's formula to Z_tX_t which yields

$$\begin{aligned} \mathrm{d}(Z_tX_t) &= Z_t\mathrm{d}X_t + X_t\mathrm{d}Z_t + \mathrm{d}[Z,X]_t \\ &= Z_t\pi_t^{\mathrm{T}}\sigma_t(\lambda_t\mathrm{d}t + \mathrm{d}B_t) - Z_tX_t\lambda_t^{\mathrm{T}}\mathrm{d}B_t - Z_t\pi_t^{\mathrm{T}}\sigma_t\lambda_t\mathrm{d}t \\ &= Z_t(\pi_t^{\mathrm{T}}\sigma_t - X_t\lambda_t^{\mathrm{T}})\mathrm{d}B_t. \end{aligned}$$

Since ZX is a local martingale, bounded from below, it is also a super-martingale. Moreover, $X_0 = x$ and $Z_0 = 1$, this implies $\mathbb{E}[Z_TX_T] \le Z_0X_0 = x$. **[3 marks]**

(3.b)

$$\begin{aligned} \mathbb{E}[U(X_T)] &\le \mathbb{E}[U(X_T)] + y(x - \mathbb{E}[Z_TX_T]) \qquad (\text{for } y > 0) \\ &= \mathbb{E}[U(X_T) - yZ_TX_T] + xy \\ &\le \mathbb{E}[V(yZ_T)] + xy \qquad (\text{since } U(x) - xy \le V(y)\ \forall y > 0) \\ &=: v(y) + xy. \end{aligned}$$

Maximising the LHS over X_T gives $u(x) \le v(y) + xy$ **[3 marks]**

(3.c) One gets equality in part b) if $X_T = \widehat{X}_T$, such that $U'(\widehat{X}_T) = yZ_T$ with y fixed by the constraint $\mathbb{E}[Z_T\widehat{X}_T] = x$. Inverting the map U', we get $\widehat{X}_T = I(yZ_T)$ with y fixed via $\mathbb{E}[Z_TI(yZ_T)] = x$.

[3 marks]

(3.d) By direct computations

$$U'(x) = \frac{1}{x - \bar{x}}, \quad \text{so} \quad U'(\widehat{X}_T) = yZ_T \quad \text{gives} \quad \frac{1}{\widehat{X}_T - \bar{x}} = yZ_T \quad \Rightarrow \quad \widehat{X}_T = \bar{x} + \frac{1}{yZ_T}.$$

Substituting this into $\mathbb{E}[Z_T\widehat{X}_T] = x$ gives

$$\mathbb{E}[\bar{x}Z_T + \frac{1}{y}] = x \quad \Rightarrow \quad \bar{x} + \frac{1}{y} = x \Leftrightarrow \frac{1}{y} = x - \bar{x}.$$

Hence

$$\widehat{X}_T = \bar{x} + \frac{1}{yZ_T} = \bar{x} + \frac{x - \bar{x}}{Z_T}.$$

For $\widehat{X}_T > \bar{x}$ we thus require that $x > \bar{x}$.

[5 marks]

(3.e) $Z\widehat{X}$ is a martingale, so for $t \le T$

$$\begin{aligned} Z_t\widehat{X}_t &= \mathbb{E}[Z_T\widehat{X}_T|\mathcal{F}_t] \\ &= \mathbb{E}[Z_T\bar{x} + (x - \bar{x})|\mathcal{F}_t] \\ &= x - \bar{x} + \bar{x}Z_T. \end{aligned}$$

Therefore

$$\mathrm{d}(Z_t\widehat{X}_t) = \bar{x}\mathrm{d}Z_t = -\bar{x}Z_t\lambda_t^{\mathrm{T}}\mathrm{d}B_t$$

and comparing with the dynamics in part a) gives

$$\begin{aligned} \widehat{\pi}_t^{\mathrm{T}}\sigma_t - \widehat{X}_t\lambda_t^{\mathrm{T}} &= -\bar{x}\lambda_t^{\mathrm{T}} \\ &\Rightarrow \sigma_t^{\mathrm{T}}\widehat{\pi}_t = (\widehat{X}_t - \bar{x})\lambda_t \\ &\Rightarrow \widehat{\pi}_t = (\sigma_t^{-1})^{\mathrm{T}}(\widehat{X}_t - \bar{x})\lambda_t \end{aligned}$$

[5 marks]

(3.f) The wealth dynamics are

$$\mathrm{d}X_t = \pi_t^{\mathrm{T}}\sigma_t(\lambda_t \mathrm{d}t + \mathrm{d}B_t) \qquad \text{under } \mathbb{P}$$
$$\mathrm{d}X_t = \pi_t^{\mathrm{T}}\sigma_t \mathrm{d}B_t^{\mathbb{Q}} + Y_t \mathrm{d}t, \qquad \text{where } \mathrm{d}B_t^{\mathbb{Q}} = \mathrm{d}B_t + \lambda_t \mathrm{d}t \qquad B^{\mathbb{Q}} \text{ is a } \mathbb{Q}\text{-BM},$$

and hence

$$X_t - \int_0^t Y_s \mathrm{d}s = x + \int_0^t \pi_s^T \sigma_s \mathrm{d}B_s^{\mathbb{Q}}.$$

Since Y is bounded, the LHS is bounded from below, so is a $\mathbb{Q}$-super-martingale and it follows that

$$\mathbb{E}^{\mathbb{Q}}[X_T - \int_0^T Y_t \mathrm{d}t] \leq x \quad \Leftrightarrow \quad \mathbb{E}^{\mathbb{Q}}[X_T] \leq x + \mathbb{E}^{\mathbb{Q}}[\int_0^T Y_t \mathrm{d}t]$$

or equivalently under the measure $\mathbb{P}$

$$\mathbb{E}[Z_T X_T] \leq x + \mathbb{E}^{\mathbb{Q}}[\int_0^T Y_t \mathrm{d}t] =: \bar{x}.$$

By analogy with the problem analysed earlier, we replace x by $\widetilde{x}$ in part (d). So for $\widehat{X}_T$ to be greater than $\bar{x}$ we require

$$\widetilde{x} > \bar{x} \quad \Leftrightarrow \quad x + \mathbb{E}^{\mathbb{Q}}[\int_0^T Y_t \mathrm{d}t] > \bar{x} \quad \Leftrightarrow \quad x > \bar{x} - \mathbb{E}^{\mathbb{Q}}[\int_0^T Y_t \mathrm{d}t].$$

Since the agent is in receipt of income, the initial capital needed to generate a feasible terminal wealth is reduced by the fair price $\mathbb{E}^{\mathbb{Q}}[\int_0^T Y_t \mathrm{d}t]$ of the lifetime income.

[6 marks]

Comment:

(3.a) Completely standard

(3.b) Completely standard

(3.c) Completely standard

(3.d) Standard up to the formula for $\widehat{X}_T$. This utility was not seen in lectures

(3.e) Fairly standard. Even split marks for computing $Z\widehat{X}$, then using its dynamics to get $\widehat{\pi}$

(3.f) Harder. Students need to realise that $X - \int_0^{\cdot} Y_s ds$ is a $\mathbb{Q}$-super-martingale, then see that one can replace x by $\widetilde{x}$, and then also interpret the result. This exercise is unseen.

4. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, an agent trades in an incomplete market, generating wealth process X. A non-attainable claim pays a bounded random variable C at terminal time T.

An agent with initial capital x and exponential utility function $U(x) = -e^{-x}$, maximises expected utility of terminal wealth, with the random endowment of a short position in the claim. Denote the value function by $u(x)$. You may assume the interest rate is zero and that the wealth process X is a $\mathbb{Q}$-martingale, for all equivalent local martingale measures (ELMMs) $\mathbb{Q}$ with finite entropy. Denote the density process of such an ELMM $\mathbb{Q}$ by Z, and denote the relative entropy between $\mathbb{Q}$ and $\mathbb{P}$ by $H(\mathbb{Q}|\mathbb{P}) := \mathbb{E}[Z_T \log Z_T]$.

(4.a) Show that we have the inequality

$$\mathbb{E}[U(X_T - C)] \le \mathbb{E}[V(yZ_T) - yZ_TC] + xy, \quad y > 0 \tag{3}$$

for any trading strategy and any $\mathbb{Q}$, where V is the convex conjugate of U. Hence show that $u(x) \le v(y) + xy$, where v is the value function of the dual problem, which you should define.

[4 marks]

(4.b) Suppose that equality is achieved in (3) for the optimal terminal wealth $\widehat{X}_T$ and an optimal density $\widehat{Z}_T$. Show that

$$\widehat{X}_T - C = -\log(y\widehat{Z}_T) \ ,$$

for some $y > 0$. **[2 marks]**

(4.c) Explain how y is determined and hence derive a formula for $\widehat{X}_T - C$ in terms of x and $\widehat{Z}_T$, and some constants that you should identify.

[5 marks]

(4.d) Hence show that the maximal expected utility is given by

$$u(x) = -\exp\left\{-x - H(\widehat{\mathbb{Q}}|\mathbb{P}) + \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right\},$$

where $\widehat{\mathbb{Q}}$ is the ELMM corresponding to $\widehat{Z}$. **[2 marks]**

(4.e) Denote by u_0 and Z^0 the value function and density of the dual minimiser $\mathbb{Q}^0$ when there is no random endowment in the above utility maximisation problem. The utility indifference price p of the claim at time zero is defined implicitly by $u(x + p) = u_0(x)$.

Derive a formula for p. **[5 marks]**

(4.f) Now suppose the market model contains one stock S and one non-traded asset Y, following the geometric Brownian motions

$$\mathrm{d}S_t = \sigma_S S_t(\lambda_S \mathrm{d}t + \mathrm{d}B_t^S), \qquad \mathrm{d}Y_t = \sigma_Y Y_t(\lambda_Y \mathrm{d}t + \mathrm{d}B_t^Y) \ ,$$

where B^S, B^Y are correlated Brownian motions with constant correlation $\rho \in (-1, 1)$ and $\sigma_S, \sigma_Y, \lambda_S, \lambda_Y$ are constants. By deriving an expression for $H(\mathbb{Q}|\mathbb{P})$, show that in this case the indifference price has the representation

$$p = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[C - \frac{1}{2}\int_0^T \widehat{\psi}_t^2 \mathrm{d}t\right]$$

where $\widehat{\psi}$ is an adapted process. Explain how $\widehat{\psi}$ is related to $\widehat{\mathbb{Q}}$. **[7 marks]**

Solution:

(4.a) From $U(x) \le V(y) + xy$, so $U(X_T - C) \le V(yZ_T) + (X_T - C)yZ_T$.
Taking expectations and using that $\mathbb{E}[Z_T X_T] = x$ gives

$$\mathbb{E}\left[U(X_T - C)\right] \le \mathbb{E}\left[V(yZ_T) - yZ_T C\right] + xy.$$

Maximising the LHS gives $u(x)$. Defining the dual value function by

$$v(y) := \inf_{\mathbb{Q}} \mathbb{E}[V(yZ_T) - yZ_T C]$$

then we get $u(x) \le v(y) + xy$. **[4 marks]**

(4.b) We get equality if we choose $X_T = \widehat{X}_T$, $Z_T = \widehat{Z}_T$ such that

$$U'(\widehat{X}_t - C) = y\widehat{Z}_T \quad \Leftrightarrow \quad e^{-(\widehat{X}_T - C)} = y\widehat{Z}_T \quad \Leftrightarrow \quad \widehat{X}_T - C = -\log(y\widehat{Z}_T).$$

[2 marks]

(4.c) y is fixed via the constraint $\mathbb{E}[\widehat{Z}_T \widehat{X}_T] = x$, that is

$$\mathbb{E}\left[\widehat{Z}_T\left(C - \log(y\widehat{Z}_T)\right)\right] = x$$
$$\Leftrightarrow \mathbb{E}[\widehat{Z}_T C] - (\log y)\mathbb{E}[\widehat{Z}_T] - H(\widehat{\mathbb{Q}}|\mathbb{P}) = x$$

Notice that $\mathbb{E}[\widehat{Z}_T C] = \mathbb{E}^{\widehat{\mathbb{Q}}}[C]$ and $\mathbb{E}[\widehat{Z}_T] = \mathbb{E}^{\widehat{\mathbb{Q}}}[1] = 1$, and hence injecting this is the above identity yields

$$\Leftrightarrow \mathbb{E}^{\widehat{\mathbb{Q}}}[C] - \log y - H(\widehat{\mathbb{Q}}|\mathbb{P}) = x \quad \Leftrightarrow \quad -\log y = x + H(\widehat{\mathbb{Q}}|\mathbb{P}) - \mathbb{E}^{\widehat{\mathbb{Q}}}[C].$$

Finally,

$$\begin{aligned} \widehat{X}_T - C &= -\log y - \log \widehat{Z}_T \\ &= x + H(\widehat{\mathbb{Q}}|\mathbb{P}) - \mathbb{E}^{\widehat{\mathbb{Q}}}[C] - \log \widehat{Z}_T \end{aligned}$$

[5 marks]

(4.d) By directly computing the involved quantity

$$\begin{aligned} u(x) &= \mathbb{E}[-e^{-(\widehat{X}_T - C)}] \\ &= \mathbb{E}\left[-\exp\left\{-\left(x + H(\widehat{\mathbb{Q}}|\mathbb{P}) - \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right)\right\}\widehat{Z}_T\right] \\ &= -\exp\left\{-\left(x + H(\widehat{\mathbb{Q}}|\mathbb{P}) - \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right)\right\}\mathbb{E}^{\widehat{\mathbb{Q}}}[1] \\ &= -\exp\left\{-x - H(\widehat{\mathbb{Q}}|\mathbb{P}) + \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right\} \end{aligned}$$

[2 marks]

(4.e) We have $u_0(x) = -\exp\{-x - H(\mathbb{Q}^0|\mathbb{P})\}$ and from the identity defining the indifference price

$$-\exp\left\{-(x+p) - H(\widehat{\mathbb{Q}}|\mathbb{P}) + \mathbb{E}^{\widehat{\mathbb{Q}}}[C]\right\} = -\exp\{-x - H(\mathbb{Q}^0|\mathbb{P})\}$$

Applying logarithms to both sides and simplifying

$$\Leftrightarrow -p - H(\widehat{\mathbb{Q}}|\mathbb{P}) + \mathbb{E}^{\widehat{\mathbb{Q}}}[C] = -H(\mathbb{Q}^0|\mathbb{P}) \quad \Leftrightarrow \quad p = \mathbb{E}^{\widehat{\mathbb{Q}}}[C] - \left(H(\widehat{\mathbb{Q}}|\mathbb{P}) - H(\mathbb{Q}^0|\mathbb{P})\right)$$

[5 marks]

(4.f) Let $Z_T := \mathcal{E}\left(-\lambda_S B_T^S - \int_0^T \psi_t \mathrm{d}B_t^{S,\perp}\right)$ where $B^{S,\perp}$ is a BM independent of B^S (so that $B^Y = \rho B^S + \sqrt{1-\rho^2}\, B^{S,\perp}$) and ψ is an adapted process s.th. $\int_0^T \psi_t^2 \mathrm{d}t < \infty$ a.s. (and we assume that $t \mapsto \int_0^t \psi_s \mathrm{d}B_s^{S,\perp}$ is a Martingale); we have by direct computation

$$\log Z_T = -\lambda_S B_T^S - \int_0^T \psi_t \mathrm{d}B_t^{S,\perp} - \frac{1}{2}\lambda_S^2 T - \frac{1}{2}\int_0^T \psi_t^2 \mathrm{d}t.$$

Using Girsanov, we define two new Brownian motions for $0 \le t \le T$

$$B_t^{S,\mathbb{Q}} = B_t^S + \lambda_S t, \qquad B_t^{S,\perp,\mathbb{Q}} = B_t^{S,\perp} + \int_0^t \psi_s \mathrm{d}s$$

which are independent $\mathbb{Q}$-Brownian motions. Then

$$\log Z_T = -\lambda_S B_T^{S,\mathbb{Q}} - \int_0^T \psi_t \mathrm{d}B_t^{S,\perp,\mathbb{Q}} + \frac{1}{2}\lambda_S^2 T + \frac{1}{2}\int_0^T \psi_t^2 \mathrm{d}t.$$

Hence

$$H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}[Z_T \log Z_T] = \mathbb{E}^{\mathbb{Q}}[\log Z_T] = \frac{1}{2}(\lambda_S^2 T + \int_0^T \psi_t^2 \mathrm{d}t).$$

Denoting by $\widehat{\psi}$ the integrand corresponding to $\widehat{\mathbb{Q}}$, we have

$$H(\widehat{\mathbb{Q}}|\mathbb{P}) = \frac{1}{2}\left(\lambda_S^2 T + \int_0^T \widehat{\psi}_t^2 \mathrm{d}t\right) \qquad \text{and} \qquad H(\mathbb{Q}^0|\mathbb{P}) = \frac{1}{2}\lambda_S^2 T,$$

where we remark that for the $H(\mathbb{Q}^0|\mathbb{P})$ term, we have $\psi^0 = 0$ since the dual problem for $C = 0$ is to purely minimize the entropy, because

$$\mathbb{E}[V(yZ_T)] = V(y) + H(\mathbb{Q}|\mathbb{P}) \quad \text{for} \quad U(x) = -e^{-x}.$$

We have then $p = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[C - \frac{1}{2}\int_0^T \widehat{\psi}_t^2 \mathrm{d}t\right]$. **[4+3 marks]**

Comment:

(4.a) Standard bookwork, relies on knowing properties of V and the definition of v

(4.b) Very standard

(4.c) Standard work, nonetheless needs to be done properly

(4.d) Easy, but students may not be familiar with this representation

(4.e) Easy; students will not have seen this form before for indifference pricing

*(4.f) More involved; includes some similar stuff done in the course. Needs students to correctly compute $H(\mathbb{Q}|\mathbb{P})$ (**[4 marks]**) and then apply it to the indifference price p (**[3 marks]**), recognising that $\psi^0 = 0$ for problems without claim.*