Stochastic Control and Dynamic Asset Allocation*

Gonçalo dos Reis† David Šiška‡

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Contents

*Lecture notes for academic year 2018/19, School of Mathematics, University of Edinburgh.

† G.dosReis@ed.ac.uk

‡ D.Siska@ed.ac.uk

Reading these notes

The reader is expected to know basic stochastic analysis and ideally a little bit of financial mathematics. The notation and basic results used throughout the notes are in Appendix A.

Sections 1 and 2 are essential reading for what follows but Sections 3 and 4 are basically independent of each other.

Exercises

You will find a number of exercises throughout these notes. You must make an effort to solve them (individually or with friends).

Solutions to some of the exercises will be made available as time goes by but remember: no one ever learned swimming by watching other people swim (and similarly no-one ever learned mathematics by reading others' solutions).

Other reading

It is recommended that you read the relevant chapters of Pham [11, at least Chapters 1-3 and 6] as well as Touzi [13, at least Chapters 1-4 and 9].

Additionally one recommends Krylov [9] for those wishing to see everything done in full generality and with proofs that do not contain any vague arguments but it is not an easy book to read. Chapter 1 however, is very readable and much recommended. Those interested in applications in algorithmic trading should read Cartea, Jaimungal and Penalva [4] and those who would like to learn about mean field games there is Carmona and Delarue [3].

1 Introduction to stochastic control through examples

We start with some motivating examples.

1.1 Merton's problem

In this part we give a motivating example to introduce the problem of dynamic asset allocation and stochastic optimization. We will not be particularly rigorous in these calculations.

The market Consider an investor can invest in a two asset Black-Scholes market: a risk-free asset ("bank" or "Bond") with rate of return $r > 0$ and a risky asset ("stock") with mean rate of return $\mu > r$ and constant volatility $\sigma > 0$. Suppose that the price of the risk-free asset at time t, B_t, satisfies

$$\frac{dB_t}{B_t} = r\,dt \quad \text{or} \quad B_t = B_0 e^{rt}, \qquad t \geq 0.$$

The price of the stock evolves according to the following SDE:

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,$$

where $(W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion one the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

The agent's wealth process and investments Let X_t^0 denote the investor's wealth in the bank at time $t \geq 0$. Let π_t denote the wealth in the risky asset. Let $X_t = X_t^0 + \pi_t$ be the investor's total wealth. The investor has some initial capital $X_0 = x > 0$ to invest. Moreover, we also assume that the investor saves / consumes wealth at rate C_t at time $t \geq 0$.

There are three popular possibilities to describe the investment in the risky asset:

(i) Let ξ_t denote the number of units stocks held at time t (allow to be fractional and negative),

(ii) the value in units of currency $\pi_t = \xi_t S_t$ invested in the risky asset at time t,

(iii) the fraction $\nu_t = \frac{\pi_t}{X_t}$ of current wealth invested in the risky asset at time t.

The investment in the bond is then determined by the accounting identity $X_t^0 = X_t - \pi_t$. The parametrizations are equivalent as long as we consider *only* positive wealth processes (which we shall do). The gains/losses from the investment in the stock are then given by

$$\xi_t\,dS_t, \qquad \frac{\pi_t}{S_t}\,dS_t, \qquad \frac{X_t\nu_t}{S_t}\,dS_t\,.$$

The last two ways to describe the investment are especially convenient when the model for S is of the exponential type, as is the Black-Scholes one. Using (ii),

$$\begin{aligned} X_t &= x + \int_0^t \frac{\pi_s}{S_s}\,dS_s + \int_0^t \frac{X_s - \pi_s}{B_s}\,dB_s - \int_0^t C_s\,ds \\ &= x + \int_0^t \big[\pi_s(\mu - r) + rX_s - C_s\big]\,ds + \int_0^t \pi_s\sigma\,dW_s \end{aligned}$$

or in differential form

$$dX_t = \big[\pi_t(\mu - r) + rX_t - C_t\big]\,dt + \pi_t\sigma\,dW_t, \qquad X_0 = x\,.$$

Alternatively, using (iii), the equation simplifies even further.[1] Recall $\pi = \nu X$.

$$\begin{aligned} dX_t &= X_t\nu_t\frac{dS_t}{S_t} + X_t\big(1-\nu_t\big)\frac{dB_t}{B_t} - C_t\,dt \\ &= \big[X_t\big(\nu_t(\mu - r) + r\big) - C_t\big]\,dt + X_t\nu_t\sigma\,dW_t. \end{aligned}$$

We can make a further simplification and obtain an SDE in "geometric Brownian motion" format if we assume that the consumption C_t can be written as a fraction of the total wealth, i.e. $C_t = \kappa_t X_t$. Then

$$dX_t = X_t\big[\nu_t(\mu - r) + r - \kappa_t\big]\,dt + X_t\nu_t\sigma\,dW_t\,. \tag{1.1}$$

Exercise 1.1. Assuming that all coefficients in SDE (1.1) are integrable, solve the SDE for X and hence show $X > 0$ when $X_0 = x > 0$.

The optimization problem The investment allocation/consumption problem is to choose the best investment possible in the stock, bond and at the same time consume the wealth optimally. How to translate the words "best investment" into a mathematical criteria?

Classical modeling for describing the behavior and preferences of agents and investors are: *expected utility* criterion and *mean-variance* criterion.

In the *first criterion* relying on the theory of choice in uncertainty, the agent compares random incomes for which he knows the probability distributions. Under some conditions on the preferences, Von Neumann and Morgenstern show that they can be represented through the expectation of some function, called *utility*. Denoting it by U, the utility function of the agent, the random income X is preferred to a random income X' if $\mathbb{E}[U(X)] \geq \mathbb{E}[U(X')]$. The deterministic utility function U is nondecreasing and concave, this last feature formulating the risk aversion of the agent.

Example 1.2 (Examples of utility functions)**.** The most common utility functions are

- Exponential utility: $U(x) = -e^{\alpha x}$, the parameter $\alpha > 0$ is the risk aversion.
- Log utility: $U(x) - \log(x)$
- Power utility: $U(x) = (x^\gamma - 1)/\gamma$ for $\gamma \in (-\infty, 0) \cup (0, 1)$.
- Iso-elastic: $U(x) = x^{1-\rho}/(1-\rho)$ for $\rho \in (-\infty, 0) \cup (0, 1)$.

In this portfolio allocation context, the criterion consists of maximizing the expected utility from consumption and from terminal wealth. In the **the finite time-horizon case**: $T < \infty$, this is

$$\sup_{\nu, C}\mathbb{E}\left[\int_0^T U\big(C_t\big)\,dt + U\big(X_t^{\nu,C}\big)\right]\,, \text{ where (1.1) gives } X_t^{\nu,C} = X_t. \tag{1.2}$$

[1] Note that, if ν_t expresses the fraction of the total wealth X invested in the stock, then the fraction of wealth invested in the bank account is simply $1 - \nu_t$.

Without consumption, i.e. $\forall t$ we have $C(t) = 0$, the optimization problem could be written as

$$\sup_{\nu} \mathbb{E}\left[U\big(X_t^{\nu}\big)\right], \text{ where (1.1) gives } X_t^{\nu} = X_t. \tag{1.3}$$

Note that the maximization is done under the expectation.

In the **infinite time-horizon case**: $T = \infty$. In our context the optimization problem is written as (recall that $C_t = \kappa_t X_t^{\nu,\kappa}$)

$$\sup_{\kappa,\nu} \mathbb{E}\left[\int_0^{\infty} e^{-\gamma t} U\big(\kappa_t X_t^{\nu,\kappa}\big)\, dt\,, \text{ with (1.1) giving } X_t = X_t^{\nu,\kappa}.\right] \tag{1.4}$$

Let us go back to the **finite horizon case**: $T < \infty$. The *second criterion* for describing the behavior and preferences of agents and investors, the mean-variance criterion, relies on the assumption that the preferences of the agent depend only on the expectation and variance of his random incomes. To formulate the feature that the agent likes wealth and is risk-averse, the mean-variance criterion focuses on mean-variance-efficient portfolios, i.e. minimizing the variance given an expectation.

In our context and assuming that there is no consumption, i.e. $\forall t$ we have $C_t = 0$, then the optimization problem is written as

$$\inf_{\nu} \Big\{\mathrm{Var}\big(X_T^{\nu}\big) \ : \ \mathbb{E}[X_T^{\nu}] = m, \quad m \in (0,\infty)\Big\}.$$

We shall see that this problem may be reduced to the resolution of a problem in the form (1.2) for the quadratic utility function: $U(x) = \lambda - x^2$, $\lambda \in \mathbb{R}$. See Example 4.12.

1.2 Optimal liquidation problem

Trader's inventory, an $\mathbb{R}$-valued process:

$$dQ_u = -\alpha_u\, du \text{ with } Q_t = q > 0 \text{ initial inventory.}$$

Here α will typically be mostly positive as the trader should sell all the assets. We will denote this process $Q_u = Q_u^{t,q,\alpha}$ because clearly it depends on the starting point q at time t and on the trading strategy α. Asset price, an $\mathbb{R}$-valued process:

$$dS_u = \lambda\, \alpha_u\, du + \sigma\, dW_u\,, \ \ S_t = S\,.$$

We will denote this process $S_u = S_u^{t,S,\alpha}$ because clearly it depends on the starting point S at time t and on the trading strategy. Here the constant λ controls how much permanent impact the trader's own trades have on its price. Trader's execution price (for $\kappa > 0$):

$$\hat{S}_t = S_t - \kappa \alpha_t\,.$$

This means that there is a temporary price impact of the trader's trading: she doesn't receive the full price S_t but less, in proportion to her selling intensity.

Quite reasonably we wish to maximize (over trading strategies α), up to to some finite time $T > 0$, the expected amount gained in sales, whilst penalising the terminal inventory (with $\theta > 0$):

$$J(t,q,S,\alpha) := \mathbb{E}\Bigg[\underbrace{\int_t^T \hat{S}_u^{t,S,\alpha}\, \alpha_u\, du}_{\text{gains from sale}} + \underbrace{Q_T^{t,q,\alpha}\, S_T^{t,S,\alpha}}_{\text{val. of inventory}} - \underbrace{\theta\, |Q_T^{t,q,\alpha}|^2}_{\text{penalty for unsold}}\Bigg].$$

Figure 1.1: Value function for the Optimal Liquidation problem, Section 1.2, as function of time and inventory, in the case $\lambda = 0$, $T = 1$, $\theta = 10$, $\kappa = 1$ and $S = 100$.

Figure 1.2: Optimal control for the Optimal Liquidation problem, Section 1.2, as function of time and inventory, in the case $\lambda = 0$, $T = 1$, $\theta = 10$ and $\kappa = 1$.

The goal is to find

$$V(t, q, S) := \sup_{\alpha} J(t, q, S, \alpha)\,.$$

In Section 3.2 we will show that V satisfies a nonlinear partial differential equation, called the HJB equation which will allow us to solve this optimal control problem and we will see that, in the case $\lambda = 0$, the value function (see also Figure 1.1) is

$$V(t, q, S) = qS + \gamma(t)q^2\,,$$

whilst the optimal control (see also Figure 1.2) is

$$a^*(t, q, S) = -\frac{1}{\kappa}\gamma(t)q\,,$$

where

$$\gamma(t) = -\left(\frac{1}{\theta} + \frac{1}{\kappa}(T - t)\right)^{-1}\,.$$

It is possible to solve this with either the Bellman principle (see Exercise 3.12) or with Pontryagin maximum principle (see Example 4.11). Problems of this type arise in algorithmic trading. More can be found e.g. in Cartea, Jaimungal and Penalva [4].

1.3 Systemic risk - toy model

The model describes a network of N banks. We will use X^i_t to denote the logarithm of cash reserves of bank $i \in \{1, \dots, N\}$ at time $t \in [0, T]$. Let us assume that there are $N+1$ independent Wiener processes $W^0, W^1, \dots W^N$. Let us fix $\rho \in [-1, 1]$. Each bank's reserves are impacted by B^i_t where

$$B^i_t := \sqrt{1-\rho^2}W^i_t + \rho W^0_t\,.$$

We will have bank i's reserves influenced by "its own" i.e. "idiosyncratic" source of randomness W^i and also by a source of uncertainty common to all the banks, namely W^0 (the "common noise"). Let $\bar{X}_t := \frac{1}{N}\sum_{i=1}^N X^i_t$ i.e. the mean level of log-reserves. We model the reserves as

$$dX^i_u = \left[a(\bar{X}_u - X^i_u) + \alpha^i_u\right]du + \sigma dB^i_u\,,\;\; u \in [t, T]\,,\;\; X^i_t = x^i\,.$$

Let us look at the terms involved:

i) The term $a(\bar{X}_u - X^i_u)$ models inter-bank lending and borrowing; if bank i is below the average then it borrows money (the log reserves increase) whilst if bank i's level is above the average then it lends out money (the log reserves decrease). This happens at rate $a > 0$.

ii) The term α_t^i is the "control" of bank i and the interpretation is that it represents lending / borrowing outside the network of the N banks (e.g. taking deposits from / lending to individual borrowers).

iii) The term stochastic term (with $\sigma > 0$) models unpredictable gains / losses to the bank's reserves with the idiosyncratic and common noises as explained above.

iv) The initial reserve (at time t) of bank i is x_i.

v) Note that we should be really writing $X_u^{i,t,x,\alpha}$ for X_u^i since each bank's reserves depend on the starting point $x = (x^1, \dots, x^N)$ of all the banks and also on the controls $\alpha_u = (\alpha_u^1, \dots, \alpha_u^N)$ of all the individual banks. The equations are thus fully coupled.

We will say that in this model each bank tries to *minimize*

$$J^i(t,x,\alpha) := \mathbb{E}\bigg[\int_t^T \Big(\frac{1}{2}|\alpha_u^i|^2 - q\,\alpha_u^i(\bar{X}_u^{i,t,x,\alpha} - X_u^{i,t,x,\alpha}) + \frac{\varepsilon}{2}|\bar{X}_u^{i,t,x,\alpha} - X_u^{i,t,x,\alpha}|^2\Big)\,du + \frac{c}{2}|\bar{X}_T^{i,t,x,\alpha} - X_T^{i,t,x,\alpha}|^2\bigg].$$

Let's again look at the terms involved:

i) The term $\frac{1}{2}|\alpha_u^i|^2$ indicates that lending / borrowing outside the bank network carries a cost.

ii) With $-q\,\alpha_u^i(\bar{X}_u^{i,t,x,\alpha} - X_u^{i,t,x,\alpha})$ for some constant $q > 0$ we insist that bank i will want to borrow if it's below the mean ($\alpha_u^i > 0$) and vice versa.

iii) The final two terms provide a running penalty and terminal penalty for being too different from the average (think of this as the additional cost imposed on the bank if it's "too big to fail" versus the inefficiency of a bank that is much smaller than competitors).

Amazingly, under the assumption that $q^2 \le \varepsilon$ it is possible to solve this problem explicitly, using either techniques we will develop in Sections 3 or 4. This is an example from the field of N-player games, much more can be found in Carmona and Delarue [3].

1.4 Optimal stopping

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we have a d'-dimensional Wiener process $W = (W_u)_{u\in[0,T]}$ generating $\mathcal{F}_u := \sigma(W_s : s \le u)$. Let $\mathcal{T}_{t,T}$ be the set of all $(\mathcal{F}_t)$-stopping times taking values in $[t,T]$.

Given some $\mathbb{R}^d$-valued stochastic process $(X_u^{t,x})_{u\in[t,T]}$, such that $X_t^{t,x} = x$, adapted to the filtration $(\mathcal{F}_u)_{u\in[t,T]}$ and a reward function $g : \mathbb{R}^d \to \mathbb{R}$ the optimal stopping problem is to find

$$w(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\big[g(X_\tau^{t,x})\big]\,. \tag{1.5}$$

Example 1.3. A typical example is the American put option. In the Black–Scholes model for one risky asset the process $(X_u^{t,x})_{u\in[t,T]}$ is geometric Brownian motion, W

is $\mathbb{R}$-valued Wiener process (and $\mathbb{P}$ denotes the risk-neutral measure in our notation here) so that

$$dX_u = rX_u\, du + \sigma X_u\, dW_u\,,\quad u \in [t,T]\,,\quad X_t = x$$

where $r \in \mathbb{R}$ and $\sigma \in [0,\infty)$ are given constants. For the American put option $g(x) := [K-x]_+$. In this case w given by (1.6) gives the no-arbitrage price of the American put option for current asset price x at time t.

It has been shown (see Krylov [9] or Gyöngy and Šiška [6])that the optimal stopping problem (1.6) is a special case of optimal control problem given by

$$v(t,x) = \sup_{\rho\in\mathfrak{R}} \mathbb{E}\left[\int_t^T g(X_u^{t,x})\,\rho_u\, e^{-\int_t^u \rho_r\,dr} + g(X_T^{t,x})\,e^{-\int_t^T \rho_r\,dr}\right] \tag{1.6}$$

so that $w(t,x) = v(t,x)$. Here the control processes ρ_u must be adapted and such that for a given $\rho = (\rho_u)_{u\in[t,T]}$ there exists $n \in \mathbb{N}$ such that $\rho_u \in [0,N]$ for all $u \in [t,T]$.

1.5 Basic elements of a stochastic control problem

The above investment-consumption problem and its variants (is the so-called "Merton problem" and) is an example of a stochastic optimal control problem. Several key elements, which are common to many stochastic control problems, can be seen.

These include:

Time horizon. The time horizon in the investment-consumption problem may be finite or infinite, in the latter case we take the time index to be $t \in [0,\infty)$. We will also consider problems with finite horizon: $[0,T]$ for $T \in (0,\infty)$; and indefinite horizon: $[0,\tau]$ for some stopping time τ (for example, the first exit time from a certain set).

(Controlled) State process. The state process is a stochastic process which describes the state of the physical system of interest. The state process is often given by the solution of an SDE, and if the control process appears in the SDE's coefficients it is called a *controlled stochastic differential equation*. The evolution of the state process is influenced by a control. The state process takes values in a set called the state space, which is typically a subset of $\mathbb{R}^d$. In the investment-consumption problem, the state process is the wealth process $X^{\nu,C}$ in (1.1).

Control process. The control process is a stochastic process, chosen by the "controller" to influence the state of the system. For example, the controls in the investment-consumption problem are the processes $(\nu_t)_t$ and $(C_t)_t$ (see (1.1)).

We collect all the control parameters into one process denoted $\alpha = (\nu, C)$. The control process $(\alpha_t)_{t\in[0,T]}$ takes values in an action set A. The action set can be a complete separable metric space but most commonly $A \in \mathcal{B}(\mathbb{R}^m)$.

For the control problem to be meaningful, it is clear that the choice of control must allow for the state process to exist and be determined uniquely. More generally, the control may be forced satisfy further constraints like "no short-selling" (i.e. $\pi(t) \geq 0$) and or the control space varies with time. In the financial context, the control map at time t should be decided at time t based on the available information $\mathcal{F}_t$. This translates into requiring the control process to be adapted.

Admissible controls. Typically, only controls which satisfy certain "admissibility" conditions can be considered by the controller. These conditions can be both technical, for example, integrability or smoothness requirements, and physical, for example, constraints on the values of the state process or controls. For example, in the investment-consumption problem we will only consider processes $X^{\nu,C}$ for which a solution to (1.1) exists. We will also require the consumption process $C_t 0$ such that the investor has non-negative wealth at all times.

Objective function. There is some cost/gain associated with the system, which may depend on the system state itself and on the control used. The objective function contains this information and is typically expressed as a function $J(x, \alpha)$ (or in finite-time horizon case $J(t, x, \alpha)$), representing the expected total cost/gain starting from system state x (at time t in finite-time horizon case) if control process α is implemented.

For example, in the setup of (1.3) the *objective functional* (or gain/cost map) is

$$J(0, x, \nu) = \mathbb{E}\left[U\big(X^\nu(T)\big)\right], \tag{1.7}$$

as it denotes the reward associated with initial wealth x and portfolio process ν. Note that in the case of no-consumption, and given the remaining parameters of the problem (i.e. μ and σ), both x and ν determine by themselves the value of the reward.

Value function. The value function describes the value of the maximum possible gain of the system (or minimal possible loss). It is usually denoted by v and is obtained, for initial state x (or (t, x) in finite-time horizon case), by optimizing the cost over all admissible controls. The goal of a stochastic control problem is to find the value function v and find a control α^* whose cost/gain attains the minimum/maximum value: $V(x) = J(x, \alpha^*)$ for starting state x. For completeness sake, from (1.3) and (1.7), if ν^* is the optimal control, then we have the *value function*

$$V(x) = \sup_\nu \mathbb{E}\left[U\big(X^\nu(T)\big)\right] = \sup_\nu J(x, \nu) = J(x, \nu^*). \tag{1.8}$$

Typical questions of interest Typical questions of interest in Stochastic control problems include:

- Is there an optimal control?
- Is there an optimal Markov control?
- How can we find an optimal control?
- How does the value function behave?
- Can we compute or approximate an optimal control numerically?

There are of course many more and, before we start, we need to review some concepts of stochastic analysis that will help in the rigorous discussion of the material in this section so far.

1.6 Exercises

The aim of the exercises in this section is to build some confidence in manipulating the basic objects that we will be using later. It may help to browse through Section A before attempting the exercises.

Exercise 1.4. Read Definition A.18. Show that $\mathcal{H} \subset \mathcal{S}$.

Exercise 1.5 (On Gronwall's lemma)**.** Prove Gronwall's Lemma (see Lemma A.6) by following these steps:

i) Let
$$z(t) = \left(e^{-\int_0^t \lambda(r)dr}\right)\int_0^t \lambda(s)y(s)\,ds.$$
and show that
$$z'(t) \leq \lambda(t)e^{-\int_0^t \lambda(r)dr}\left(b(t) - a(t)\right).$$

ii) Integrate from 0 to t to obtain the first conclusion Lemma A.6.

iii) Obtain the second conclusion of Lemma A.6.

Exercise 1.6 (On liminf)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then the number
$$\lim_{n\to\infty}\left(\inf\{a_k : k \geq n\}\right)$$
is called *limit inferior* and is denoted by $\liminf_{n\to\infty} a_n$.

1. Show that the limit inferior is well defined, that is, the limit $\lim_{n\to\infty}\left(\inf\{a_k : k \geq n\}\right)$ exists and is finite for any bounded sequence (a_n).

2. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ has a subsequence that converges to $\lim_{n\to\infty}\inf a_n$.

   Hint: Argue that for any $n \in \mathbb{N}$ one can find $i \geq n$ such that
   $$\inf\{a_k : k \geq n\} \leq a_i < \inf\{a_k : k \geq n\} + \frac{1}{n}.$$
   Use this to construct the subsequence we are looking for.

Exercise 1.7 (Property of the supremum/infimum)**.** Let $a, b \in \mathbb{R}$. Prove that
$$\begin{aligned}\text{if } b > 0, \text{ then} \quad &\sup_{x\in X}\left\{a + bf(x)\right\} = a + b\sup_{x\in X} f(x),\\ \text{if } b < 0, \text{ then} \quad &\sup_{x\in X}\left\{a + bf(x)\right\} = a + b\inf_{x\in X} f(x).\end{aligned}$$

Exercise 1.8. Assume that $X = (X_t)_{t\geq 0}$ is a martingale with respect to a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Show that:

1. if for all $t \geq 0$ it holds that $\mathbb{E}|X_t|^2 < \infty$ then the process given by $|X_t|^2$ is a submartingale and

2. the process given by $|X_t|$ is a submartingale.

Exercise 1.9 (ODEs)**.** Assume that (r_t) is an adapted stochastic process such that for any $t \geq 0$ $\int_0^t |r_s|\,ds < \infty$ holds $\mathbb{P}$-almost surely (in other words $r \in \mathcal{A}$).

1. Solve
$$dB_t = B_t r_t dt, \quad B_0 = 1. \tag{1.9}$$

2. Is the function $t \mapsto B_t$ continuous? Why?

3. Calculate $d(1/B_t)$.

Exercise 1.10 (Geometric Brownian motion)**.** Assume that $\mu \in \mathcal{A}$ and $\sigma \in \mathcal{S}$. Let W be a real-valued Wiener martingale.

1. Solve
$$dS_t = S_t \left[\mu_t \, dt + \sigma_t \, dW_t\right], \quad S(0) = s. \tag{1.10}$$
Hint: Solve this first in the case that μ and σ are real constants. Apply Itô's formula to the process S and the function $x \mapsto \ln x$.

2. Is the function $t \mapsto S_t$ continuous? Why?

3. Calculate $d(1/S_t)$, assuming $s \neq 0$.

4. With B given by (1.9) calculate $d(S_t/B_t)$.

Exercise 1.11 (Multi-dimensional gBm)**.** Let W be an $\mathbb{R}^d$-valued Wiener martingale. Let $\mu \in \mathcal{A}^m$ and $\sigma \in \mathcal{S}^{m\times d}$. Consider the stochastic processes $S_i = (S_i(t))_{t\in[0,T]}$ given by
$$dS^i_t = S^i_t \mu^i_t \, dt + S^i_t \sum_{j=1}^{m} \sigma^{ij}_t \, dW^j_t, \; S^i_0 = s_i, \; i = 1, \dots, m. \tag{1.11}$$

1. Solve (1.11) for $i = 1, \dots, m$.
Hint: Proceed as when solving (1.10). Start by assuming that μ and σ are constants. Apply the multi-dimensional Itô formula to the process S_i and the function $x \mapsto \ln(x)$. Note that the process S_i is just $\mathbb{R}$-valued so the multi-dimensionality only comes from W being $\mathbb{R}^d$ valued.

2. Is the function $t \mapsto S^i_t$ continuous? Why?

Exercise 1.12 (Ornstein–Uhlenbeck process)**.** Let $a, b, \sigma \in \mathbb{R}$ be constants such that $b > 0, \sigma > 0$. Let W be a real-valued Wiener martingale.

1. Solve
$$dr_t = (b - ar_t) \, dt + \sigma_t \, dW_t, \quad r(0) = r_0. \tag{1.12}$$
Hint: Apply Itô's formula to the process r and the function $(t, x) \mapsto e^{at}x$.

2. Is the function $t \mapsto r_t$ continuous? Why?

3. Calculate $\mathbb{E}[r_t]$ and $\mathbb{E}[r_t^2]$.

4. What is the distribution of r_t?

Exercise 1.13. If X is a Gaussian random variable with $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \mathbb{E}[X^2 - (\mathbb{E}[X])^2] = \sigma^2$ then we write $X \sim N(\mu, \sigma^2)$. Show that if $X \sim N(\mu, \sigma^2)$ then $\mathbb{E}[e^X] = e^{\mu + \frac{\sigma^2}{2}}$.

1.7 Solutions to Exercises

Solution (Solution to Exercise 1.5)**.** Let

$$z(t) = \left(e^{-\int_0^t \lambda(r)dr}\right) \int_0^t \lambda(s)y(s)\, ds.$$

Then, almost everywhere in I,

$$z'(t) = \lambda(t)e^{-\int_0^t \lambda(r)dr} \underbrace{\left(y(t) - \int_0^t \lambda(s)y(s)\, ds\right)}_{\leq b(t)-a(t)},$$

by the inequality in our hypothesis. Hence for a.a. $s \in I$

$$z'(s) \leq \lambda(s)e^{-\int_0^s \lambda(r)dr} \left(b(s) - a(s)\right).$$

Integrating from 0 to t and using the fundamental theorem of calculus (which gives us $\int_0^t z'(s)\, ds = z(t) - z(0) = z(t)$) we obtain

$$\begin{aligned}\int_0^t \lambda(s)y(s)\, ds &\leq e^{\int_0^t \lambda(r)dr} \int_0^t \lambda(s)e^{-\int_0^s \lambda(r)dr} \left(b(s) - a(s)\right)\, ds \\ &= \int_0^t \lambda(t)e^{\int_s^t \lambda(r)dr} \left(b(s) - a(s)\right)\, ds.\end{aligned}$$

Using the left hand side of above inequality as the right hand side in the inequality in our hypothesis we get

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s)e^{\int_s^t \lambda(r)dr} \left(b(s) - a(s)\right)\, ds,$$

which is the first conclusion of the lemma. Assume now further that b is monotone increasing and a nonnegative. Then

$$\begin{aligned}y(t) + a(t) &\leq b(t) + b(t) \int_0^t \lambda(s)e^{\int_s^t \lambda(r)dr}\, ds \\ &= b(t) + b(t) \int_0^t -de^{\int_s^t \lambda(r)dr} = b(t) + b(t)\left(-1 + e^{\int_0^t \lambda(r)dr}\right) \\ &= b(t)e^{\int_0^t \lambda(r)\, dr}.\end{aligned}$$

Solution (Solution to Exercise 1.6)**.** Let $n \in \mathbb{N}$.

1. The sequence $b_n := \inf\{a_k : k \geq n\}$ is monotone increasing as $\{a_k : k \geq n+1\}$ is a subset of $\{a_k : k \geq n\}$, hence $b_n \leq b_{n+1}$. Additionally, the sequence is also bounded by the same bounds as the initial sequence (a_n). A monotone and bounded sequence of real numbers must converge and hence we can conclude that $\liminf_{n\to\infty} a_n$ exists.

2. It follows from the definition of infimum that there exists a sequence $i = i(n) \geq n$ such that

   $$b_n = \inf\{a_k : k \geq n\} \leq a_i < \inf\{a_k : k \geq n\} + \frac{1}{n} = b_n + \frac{1}{n}.$$

   The sequence of indices $\big(i(n)\big)_{n\in\mathbb{N}}$ might not be monotone, but since $i(n) \geq n$ it is always possible to select its subsequence, say $\big(j(n)\big)_{n\in\mathbb{N}}$, that is monotone.

   Since $|a_{i(n)} - b_n| \to 0$ and $(b_n)_{n\in\mathbb{N}}$ converges to $\liminf_{n\to\infty} a_n$, then so does $(a_{i(n)})_n$. As $(a_{j(n)})_n$ is a subsequence of $(a_{i(n)})_n$ the same is true for $(a_{j(n)})_n$. Hence the claim follows.

Solution (Solution to Exercise 1.7)**.** We will show the result when $b > 0$, assuming that the sup takes a finite value. Let $f^* := \sup_{x\in X} f(x)$, and $V^* := \sup_{x\in X} \big\{a + bf(x)\big\}$.

To show that $V^* = a + bf^*$, we start by showing that $V^* \leq a + bf^*$.

Note that for all $x \in X$ we have $a + bf^* \geq a + bf(x)$, that is, $a + bf^*$ is an upper bound for the set $\{y : y = a + bf(x)$ for some $x \in X\}$. As a consequence, its least upper bound V^* must be such that $a + bf^* \geq V^* = sup_{x\in X}\{a + bf(x)\}$.

To show the converse, note that from the definition of f^* as a supremum (see Definition A.1), we have that for any $\varepsilon > 0$ there must exist a $\overline{x}^\varepsilon \in X$ such that $f(\overline{x}^\varepsilon) > f^* - \varepsilon$.

Hence $a + bf(\overline{x}^\varepsilon) > a + bf^* - b\varepsilon$. Since $\overline{x}^\varepsilon \in X$, it is obvious that $V^* \geq a + bf(\overline{x}^\varepsilon)$. Hence $V^* \geq a + bf^* - b\varepsilon$. Since ε was arbitrarily chosen, we have our result: $V^* \geq a + bf^*$.

Solution (to Exercise 1.8)**.**

1. Since X_t is $\mathcal{F}_t$-measurable it follows that $|X_t|^2$ is also $\mathcal{F}_t$-measurable. Integrability holds by assumption. We further note that the conditional expectation of a non-negative random variable is non-negative and hence for $t \geq s \geq 0$ we have
$$\begin{aligned} 0 \leq \mathbb{E}[|X_t - X_s|^2|\mathcal{F}_s] &= \mathbb{E}[|X_t|^2|\mathcal{F}_s] - 2\mathbb{E}[X_tX_s|\mathcal{F}_s] + \mathbb{E}[|X_s|^2|\mathcal{F}_s] \\ &= \mathbb{E}[|X_t|^2|\mathcal{F}_s] - 2X_s\mathbb{E}[X_t|\mathcal{F}_s] + |X_s|^2 = \mathbb{E}[|X_t|^2|\mathcal{F}_s] - |X_s|^2\,, \end{aligned}$$
since X_s is $\mathcal{F}_s$-measurable and since X is a martingale. Hence $\mathbb{E}[|X_t|^2|\mathcal{F}_s] \geq |X_s|^2$ for all $t \geq s \geq 0$.
2. First note that the adaptedness and integrability properties hold. Next note that $\big|\mathbb{E}[X_t|\mathcal{F}_s]\big| \leq \mathbb{E}\left[|X_t|\middle|\mathcal{F}_s\right]$ by standard properties of conditional expectations. Since X is a martingale we have
$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s$$
and taking absolute value on both sides we see that
$$|X_s| = \big|\mathbb{E}[X_t|\mathcal{F}_s]\big| \leq \mathbb{E}\left[|X_t|\middle|\mathcal{F}_s\right]\,.$$

Solution (Solution to Exercise 1.9)**.** Let $t \in [0, \infty)$.

1. We are looking to solve:
$$B(t) = 1 + \int_0^t r(s)\,ds,$$
which is equivalent to
$$\frac{dB(t)}{dt} = r(t)B(t) \text{ for almost all } t,\ B(0) = 1.$$
Let us calculate (using chain rule and the above equation)
$$\frac{d}{dt}\left[\ln B(t)\right] = \frac{dB(t)}{dt} \cdot \frac{1}{B(t)} = r(t).$$
Integrating both sides and using the fundamental theorem of calculus
$$\ln B(t) - \ln B(0) = \int_0^t r(s)\,ds$$
and hence
$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$
2. First we note that for any function f integrable on $[0, \infty)$ we have that the map $t \mapsto \int_0^t f(x)\,dx$ is absolutely continuous in t and hence it is continuous. The function $x \mapsto e^x$ is continuous and composition of continuous functions is continuous. Hence $t \mapsto B(t)$ must be continuous.

3. There are many ways to do this. We can start with (1.9) and use chain rule:
$$\frac{d}{dt}\left[\frac{1}{B(t)}\right] = \frac{dB(t)}{dt}\cdot\left(-\frac{1}{B^2(t)}\right) = -r(t)\left(-\frac{1}{B(t)}\right)$$
and so
$$d\left(\frac{1}{B(t)}\right) = -r(t)\frac{1}{B(t)}dt.$$
Or we can start with the solution that we have calculated write
$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{B(t)}\right] &= \frac{d}{dt}\exp\left(-\int_0^t r(s)ds\right)\\ &= -r(t)\exp\left(-\int_0^t r(s)ds\right) = -r(t)\left(-\frac{1}{B(t)}\right)\end{aligned}$$
which leads to the same conclusion again.

Solution (Solution to Exercise 1.10). 1. We follow the hint (but skip directly to the general μ and σ). From Itô's formula:
$$d(\ln S(t) = \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}dS(t)\cdot dS(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \mu(t)dW(t).$$
Now we write this in the full integral notation:
$$\ln S(t) = \ln S(0) + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s).$$
Hence
$$S(t) = s\exp\left(\int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s)\right). \tag{1.13}$$
Now this is the correct result but using invalid application of Itô's formula. If we want a full proof we call (1.13) a guess and we will now check that it satisfies (1.10). To that end we apply Itô's formula to $x \mapsto s\exp(x)$ and the Itô process
$$X(t) = \int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s).$$
Thus
$$\begin{aligned}dS(t) = d(f(X(t)) &= se^{X(t)}dX(t) + \frac{1}{2}se^{X(t)}dX(t)dX(t)\\ &= S(t)\left[\left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \mu(t)dW(t)\right] + \frac{1}{2}S(t)\sigma^2(t)dt.\end{aligned}$$
Hence we see that the process given by (1.13) satisfies (1.10).

2. The continuity question is now more intricate than in the previous exercise due to the presence of the stochastic integral. From stochastic analysis in finance you know that Z given by
$$Z(t) := \int_0^t \sigma(s)dW(s)$$
is a continuous stochastic process. Thus there is a set $\Omega' \in \mathcal{F}$ such that $\mathbb{P}(\Omega') = 1$ and for each $\omega \in \Omega'$ the function $t \mapsto S(\omega, t)$ is continuous since it's a composition of continuous functions.

3. If $s \neq 0$ then $S(t) \neq 0$ for all t. We can thus use Itô's formula
$$\begin{aligned}d\left(\frac{1}{S(t)}\right) &= -\frac{1}{S^2(t)}dS(t) + \frac{1}{S^3(t)}dS(t)dS(t)\\ &= -\frac{1}{S(t)}\left[\mu(t)dt + \sigma(t)dW(t)\right] + \frac{1}{S(t)}\sigma^2(t)dt\\ &= \frac{1}{S(t)}\left[\left(-\mu(t) + \sigma^2(t)\right)dt - \sigma(t)dW(t)\right].\end{aligned}$$

4. We calculate this with Itô's product rule:

$$
\begin{aligned}
d\left(\frac{S(t)}{B(t)}\right) &= S(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dS(t) + dS(t)d\left(\frac{1}{B(t)}\right) \\
&= -r(t)\frac{S(t)}{B(t)}dt + \mu(t)\frac{S(t)}{B(t)}dt + \sigma(t)\frac{S(t)}{B(t)}dW(t) \\
&= \frac{S(t)}{B(t)}\left[(\mu(t) - r(t))\,dt + \sigma(t)dW(t)\right].
\end{aligned}
$$

Solution (Solution to Exercise 1.11)**.** 1. We Itô's formula to the function $x \mapsto \ln(x)$ and the process S_i. We thus obtain, for $X_i(t) := \ln(S_i(t))$, that

$$
\begin{aligned}
dX_i(t) = d\ln(S_i(t)) &= \frac{1}{S_i(t)}dS_i(t) - \frac{1}{2}\frac{1}{S_i^2(t)}dS_i(t)dS_i(t) \\
&= \mu_i(t)dt + \sum_{j=1}^{n}\sigma_{ij}(t)dW_j(t) - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(t)dt \\
&= \left[\mu_i(t) - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(t)\right]dt + \sum_{j=1}^{n}\sigma_{ij}(t)dW_j(t).
\end{aligned}
$$

Hence

$$
\begin{aligned}
X_i(t) - X_i(0) &= \ln S_i(t) - \ln S_i(t) \\
&= \int_0^t\left[\mu_i(s) - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(s)\right]ds + \sum_{j=1}^{n}\int_0^t\sigma_{ij}(s)dW_j(s).
\end{aligned}
$$

And so

$$
S_i(t) = S_i(0)\exp\left\{\int_0^t\left[\mu_i(s) - \frac{1}{2}\sum_{j=1}^{n}\sigma_{ij}^2(s)\right]ds + \sum_{j=1}^{n}\int_0^t\sigma_{ij}(s)\,dW_j(s)\right\}.
$$

2. Using the same argument as before and in particular noticing that for each j the function $t \mapsto \int_0^t\sigma_{ij}(s)dW_j(s)$ is continuous for almost all $\omega \in \Omega$ we get that $t \mapsto S_i(t)$ is almost surely continuous.

Solution (to Exercise 1.12)**.** 1. What the hint suggests is sometimes referred to as the "integrating factor technique." We see that

$$
d(e^{at}r(t)) = e^{at}dr(t) + ae^{at}r(t)dt = e^{at}\left[bdt + \sigma dW(t)\right].
$$

Integrating we get

$$
e^{at}r(t) = r(0) + \int_0^t e^{as}b\,ds + \int_0^t e^{as}\sigma dW(s)
$$

and hence

$$
r(t) = e^{-at}r(0) + \int_0^t e^{-a(t-s)}b\,ds + \int_0^t e^{-a(t-s)}\sigma dW(s).
$$

2. Yes. The arguments are the same as in previous exercises.

3. We know that stochastic integral of a deterministic integrand is a normally distributed random variable with mean zero and variance given via the Itô isometry. Hence

$$
\mathbb{E}r(t) = e^{-at}r(0) + \frac{b}{a}\left(1 - e^{-at}\right)
$$

and

$$\mathbb{E}r^2(t) = (\mathbb{E}r(t))^2 + e^{-2at}\sigma^2 \mathbb{E}\left[\left(\int_0^t e^{as} dW(s)\right)^2\right]$$
$$= (\mathbb{E}r(t))^2 + e^{-2at}\sigma^2 \int_0^t e^{2as} ds = (\mathbb{E}r(t))^2 + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

Hence

$$\mathrm{Var}\left[r(t)\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

4. Stochastic integral of a deterministic integrand is a normally distributed random variable. Hence for each t we know that $r(t)$ is normally distributed with mean and variance calculated above.

Solution (to Exercise 1.13)**.** Let $Y \sim N(0, 1)$. Then

$$\mathbb{E}e^X = \mathbb{E}e^{\mu+\sigma Y} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mu+\sigma z} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\left[(z-\sigma)^2 - \sigma^2 - 2\mu\right]} dz$$
$$= e^{\frac{1}{2}\sigma^2+\mu} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(z-\sigma)^2} dz = e^{\frac{1}{2}\sigma^2+\mu},$$

since $z \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma)^2}$ is a density of normal random variable with mean σ and variance 1 and thus its integral over the whole of real numbers must be 1.

2 Stochastic control of diffusion processes

In this section we introduce existence and uniqueness theory for controlled diffusion processes and building on that formulate properly the stochastic control problem we want to solve. Finally we explore some properties of the value function associated to the control problem.

2.1 Equations with random drift and diffusion

Let a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ be given. Let W be a d'-dimensional Wiener process and let ξ be a $\mathbb{R}^d$-valued random variable independent of W. Let $\mathcal{F}_t := \sigma(\xi, W_s : s \leq t)$. We consider a stochastic differential equation (SDE) of the form,

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t\,, \qquad t \in [0,T]\,, \qquad X_0 = \xi\,. \tag{2.1}$$

Equivalently, we can write this in the integral form as

$$X_t = \xi + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s\,, \qquad t \in [0,T]\,. \tag{2.2}$$

Here $\sigma : \Omega \times [0,T] \times \mathbb{R}^{d\times d'}$ and $b : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. Written component-wise, the SDE is

$$dX_t^i = b^i(t,X_t)\,dt + \sum_{j=1}^{d'} \sigma^{ij}(t,X_t)\,dW_t^j\,, \quad t \in [0,T]\,, \quad X_0^i = \xi^i, \quad i \in \{1,\cdots,m\}.$$

The drift and volatility coefficients

$$(t,\omega,x) \mapsto \big(b_t(\omega,x), \sigma_t(\omega,x)\big)$$

are progressively measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$; as usual, we suppress ω in the notation and will typically write $b_t(x)$ instead of $b_t(\omega,x)$ etc. Note that $t = 0$ plays no special role in this; we may as well start the SDE at some time $t \geq 0$ (even a stopping time), and we shall write $X^{t,x} = (X_s^{t,x})_{s\in[t,T]}$ for the solution of the SDE started at time t with initial value x (assuming it exists and is unique).

Definition 2.1 (Solution of an SDE)**.** We say that a process X is a (strong) solution to the SDE (2.16) if

i) The process X is continuous on $[0,T]$ and adapted to $(\mathcal{F}_t)_{t\in[0,T]}$,

ii) we have

$$\mathbb{P}\left[\int_0^T |b_s(X_s)|\,ds < \infty\right] = 1 \text{ and } \mathbb{P}\left[\int_0^T |\sigma_s(X_s)|^2\,ds < \infty\right] = 1\,,$$

iii) The process X satisfies (2.11) almost surely for all $t \in [0,T]$ i.e. there is $\bar{\Omega} \in \mathcal{F}$ such that $\mathbb{P}(\bar{\Omega}) = 1$ and for all $\omega \in \bar{\Omega}$ it holds that

$$X_t(\omega) = \xi(\omega) + \int_0^t b_s(\omega, X_s(\omega))\,ds + \int_0^t \sigma_s(\omega, X_s(\omega))\,dW_s(\omega)\,,\ \forall t \in [0,T]\,. \tag{2.3}$$

Given $T \geq 0$, and $m \in \mathbb{N}$, we write $\mathbb{H}^m_T$ for the set of progressively measurable processes ϕ such that

$$\|\phi\|_{\mathbb{H}^m_T} := \mathbb{E}\left[\int_0^T |\phi_t|^m \, dt\right]^{\frac{1}{m}} < \infty.$$

Proposition 2.2 (Existence and uniqueness of solutions)**.** *Assume that for all $x \in \mathbb{R}^d$ the processes $(b_t(x))_{t\in[0,T]}$ and $(\sigma_t(x))_{t\in[0,T]}$ are progressively measurable, that $\mathbb{E}|\xi|^2 < \infty$ and that there exists a constant K such that a.s. for all $t \in [0,T]$ and $x, y \in \mathbb{R}^d$ it holds that*

$$\begin{aligned} \|b(0)\|_{\mathbb{H}^2_T} + \|\sigma(0)\|_{\mathbb{H}^2_T} \leq K, \\ |b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leq K|x-y|\,. \end{aligned} \tag{2.4}$$

Then the SDE has a unique (strong) solution X on the interval $[0,T]$. Moreover, there exists a constant $C = C(K,T)$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] \leq C\big(1 + \mathbb{E}[\,|\xi|^2]\big).$$

We give an iterative scheme which we will show converges to the solution. To that end let $X^0_t = \xi$ for all $t \in [0,T]$. For $n \in \mathbb{N}$ let, for $t \in [0,T]$, the process X^n be given by

$$X^n_t = \xi + \int_0^t b_s(X^{n-1}_s)\, ds + \int_0^t \sigma_s(X^{n-1}_s)\, dW_s\,. \tag{2.5}$$

Note that here the superscript on X indicates the iteration index.[2] We can see that X^0 is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and hence (due to progressive measurability of b and σ) X^1 is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted and repeating this argument we see that each X^n is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted.

Before we prove Proposition 2.2 by taking the limit in the above iteration we will need the following result.

Lemma 2.3. *Under the conditions of Proposition 2.2 there is a constant C, depending on K and T (but independent of n) such that for all $n \in \mathbb{N}$ and $t \in [0,T]$ it holds that*

$$\mathbb{E}|X^n_t|^2 < C(1 + \mathbb{E}|\xi|^2)e^{Ct}\,.$$

Proof. We see that

$$\mathbb{E}|X^n_t|^2 \leq 4\mathbb{E}|\xi|^2 + 4\mathbb{E}\left(\int_0^t |b_s(X^{n-1}_s)|\, ds\right)^2 + 4\mathbb{E}\left(\int_0^t |\sigma_s(X^{n-1}_s)|\, dW_s\right)^2.$$

Using Hölder's inequality and Itô's isometry we can see that

$$\mathbb{E}|X^n_t|^2 \leq 4\mathbb{E}|\xi|^2 + 4t\mathbb{E}\int_0^t |b_s(X^{n-1}_s)|^2\, ds + 4\mathbb{E}\int_0^t |\sigma_s(X^{n-1}_s)|^2\, ds\,.$$

Using the Lipschitz continuity and growth assumption (2.4) we thus obtain that

$$\mathbb{E}\int_0^t |b_s(X^{n-1}_s)|^2\, ds \leq 2\mathbb{E}\int_0^t |b_s(0)|^2\, ds + 2K^2\mathbb{E}\int_0^t |X^{n-1}_s|^2\, ds \leq 2K^2\left(1 + \mathbb{E}\int_0^t |X^{n-1}_s|^2\, ds\right)$$

[2] Instead of a power or index in a vector.

and similarly

$$\mathbb{E}\int_0^t |\sigma_s(X_s^{n-1})|^2\,ds \leq 2K^2\left(1+\mathbb{E}\int_0^t |X_s^{n-1}|^2\,ds\right).$$

Thus for all $t\in[0,T]$ we have, with $L := 16K^2(t\vee 1)$, that

$$\mathbb{E}|X_t^n|^2 \leq L\left(1+\mathbb{E}|\xi|^2\right) + L\int_0^t \mathbb{E}|X_s^{n-1}|^2\,ds\,.$$

Let us iterate this. For $n=1$ we have

$$\mathbb{E}|X_t^1|^2 \leq L\left(1+\mathbb{E}|\xi|^2\right)+Lt\mathbb{E}|\xi|^2 \leq L\left(1+\mathbb{E}|\xi|^2\right)+LtL(1+\mathbb{E}|\xi|^2) = L\left(1+\mathbb{E}|\xi|^2\right)[1+Lt]\,.$$

For $n=2$ we have

$$\begin{aligned}\mathbb{E}|X_t^2|^2 &\leq L\left(1+\mathbb{E}|\xi|^2\right) + L\int_0^t \mathbb{E}|X_s^1|^2\,ds \leq L\left(1+\mathbb{E}|\xi|^2\right) + L\cdot L(1+\mathbb{E}|\xi|^2)t + L\cdot\frac{(Lt)^2}{2}\\ &\leq L(1+\mathbb{E}|\xi|^2)\left[1+Lt+\frac{(Lt)^2}{2}\right].\end{aligned}$$

If we carry on we see that

$$\mathbb{E}|X_t^n|^2 \leq L(1+\mathbb{E}|\xi|^2)\left[1+Lt+\frac{(Lt)^2}{2!}+\cdots+\frac{(Lt)^n}{n!}\right] \leq L(1+\mathbb{E}|\xi|^2)\left[\sum_{j=0}^\infty \frac{(Lt)^j}{j!}\right]$$

and hence for all $t\in[0,T]$ we have that

$$\mathbb{E}|X_t^n|^2 \leq L(1+\mathbb{E}|\xi|^2)e^{Lt}\,.$$

□

Proof of Proposition 2.2. We start with (2.5), take the difference between iteration $n+1$ and n, take the square of the $\mathbb{R}^d$ norm, take supremum and take the expectation. Then we see that

$$\begin{aligned}&\mathbb{E}\sup_{s\leq t}|X_s^{n+1}-X_s^n|^2\\ &\leq 2\mathbb{E}\sup_{s\leq t}\left|\int_0^s [b_r(X_r^n)-b_r(X_r^{n-1})]\,dr\right|^2 + 2\mathbb{E}\sup_{s\leq t}\left|\int_0^s [\sigma_r(X_r^n)-\sigma_r(X_r^{n-1})]\,dW_r\right|^2\\ &=: 2I_1(t)+2I_2(t)\,.\end{aligned}$$

We note that for all $t\in[0,T]$, having used Hölder's inequality in the penultimate step and assumption (2.4) in the final one, it holds that

$$\begin{aligned}I_1(t) &= \mathbb{E}\sup_{s\leq t}\left|\int_0^s [b_r(X_r^n)-b_r(X_r^{n-1})]\,dr\right|^2 \leq \mathbb{E}\sup_{s\leq t}\left(\int_0^s |b_r(X_r^n)-b_r(X_r^{n-1})|\,dr\right)^2\\ &\leq \mathbb{E}\left(\int_0^t |b_r(X_r^n)-b_r(X_r^{n-1})|\,dr\right)^2 \leq t\mathbb{E}\int_0^t |b_r(X_r^n)-b_r(X_r^{n-1})|^2\,dr\\ &\leq K^2 t\mathbb{E}\int_0^t |X_r^n-X_r^{n-1}|^2\,dr\,.\end{aligned}$$

Moreover $M_t = \int_0^t [\sigma_r(X_r^n) - \sigma_r(X_r^{n-1})]\, dW_r$ is a martingale and so $(|M_t|)_{t\in[0,T]}$ is a non-negative sub-martingale. Then Doob's maximal inequality, see Theorem A.14 with $p = 2$, followed by Itô's isometry implies that for all $t \in [0, T]$ it holds that

$$I_2(t) = \mathbb{E} \sup_{s\le t} \left| \int_0^s [\sigma_r(X_r^n) - \sigma_r(X_r^{n-1})]\, dW_r \right|^2 \le 4\mathbb{E} \left| \int_0^t [\sigma_r(X_r^n) - \sigma_r(X_r^{n-1})]\, dW_r \right|^2$$
$$= 4\mathbb{E} \int_0^t |\sigma_r(X_r^n) - \sigma_r(X_r^{n-1})|^2\, dr \le 4K^2 \mathbb{E} \int_0^t |X_r^n - X_r^{n-1}|^2\, dr\,.$$

Hence, with $L := 2K^2(T + 4)$ we have for all $t \in [0, T]$ that

$$\mathbb{E} \sup_{s\le t} |X_s^{n+1} - X_s^n|^2 \le L \int_0^t \mathbb{E}|X_r^n - X_r^{n-1}|^2\, dr\,. \tag{2.6}$$

Let

$$C^* := \sup_{t\in T} \mathbb{E}|X_t^1 - \xi|^2$$

and note that Lemma 2.3 implies that $C^* < \infty$. Using this and iterating the estimate (2.6) we see that for all $t \in [0, T]$ we have that

$$\mathbb{E} \sup_{s\le t} |X_s^{n+1} - X_s^n|^2 \le C^* \frac{L^n t^n}{n!}\,. \tag{2.7}$$

For $f \in C([0, T]; \mathbb{R}^d)$ let us define the norm $\|f\|_\infty := \sup_{s\in[0,T]} |f_s|$. Due to Chebychev–Markov's inequality we thus have

$$\mathbb{P}\left[\|X^{n+1} - X^n\|_\infty > \frac{1}{2^{n+1}}\right] = \mathbb{P}\left[\|X^{n+1} - X^n\|_\infty^2 > \frac{1}{2^{2(n+1)}}\right]$$
$$\le 4^{n+1} C^* \frac{L^n t^n}{n!} = 4C^* \frac{4^n L^n t^n}{n!}\,.$$

Let $E_n := \{\omega \in \Omega : \|X^{n+1}(\omega) - X^n(\omega)\|_\infty > \frac{1}{2^{n+1}}\}$. Note that clearly[3] it holds that

$$\sum_{n=0}^{\infty} \mathbb{P}E_n < \infty\,.$$

By the Borel–Cantelli Lemma it thus holds that there is $\bar\Omega \in \mathcal{F}$ and a random variable $N : \Omega \to \mathbb{N}$ such that $\mathbb{P}(\bar\Omega) = 1$ and for all $\omega \in \bar\Omega$ we have that

$$\|X^{n+1}(\omega) - X^n(\omega)\|_\infty \le 2^{-(n+1)} \quad \forall n \ge N(\omega)\,.$$

For any $\omega \in \bar\Omega$, any $m \in \mathbb{N}$ and $n \ge N(\omega)$ we then have, due to the triangle inequality, that

$$\|X^{n+m}(\omega) - X^n(\omega)\|_\infty \le \sum_{j=0}^{m-1} \|X^{n+j+1}(\omega) - X^{n+j}(\omega)\|_\infty$$
$$\le \sum_{j=0}^{m-1} 2^{-(n+j+1)} = 2^{-(n+1)} \frac{1 - \left(\frac{1}{2}\right)^m}{1 - \frac{1}{2}} \le 2^{-n}\,. \tag{2.8}$$

This means that the sequence $X^n(\omega)$ is a Cauchy sequence in the Banach space $C([0, T]; \mathbb{R}^d)$ and thus a limit $X(\omega)$ such that $X^n(\omega) \to X(\omega)$ in $C([0, T]; \mathbb{R}^d)$ as

[3] Indeed for any $x \in \mathbb{R}$ we have $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x < \infty$.

$n \to \infty$. Moreover for each $n \in \mathbb{N}$ and each $t \in [0,T]$ the random variable X_t^n is $\mathcal{F}_t$ measurable which means that $X_t = \lim_{n\to\infty} X_t^n$ is $\mathcal{F}_t$ measurable.

Finally we have to show that the limit X satisfies the SDE. On the left hand side the convergence is trivial. To take the limit in the bounded variation integral we can use simply that for all $\omega \in \Omega$ we have that $\|X(\omega) - X^n(\omega)\|_\infty < 2^{-n}$ for $n \geq N(\omega)$. This follows by taking $m \to \infty$ in (2.8) with $n \in \mathbb{N}$ fixed. Then

$$\left| \int_0^t b_s(\omega, X_s^n(\omega))\, ds - \int_0^t b_s(\omega, X_s(\omega))\, ds \right| \leq K \int_0^t |X_s^n(\omega) - X_s(\omega)|\, ds \to 0$$

as $n \to \infty$ due to Lebesgue's theorem on dominated convergence.

To deal with the stochastic integral we need to do a bit more work. We see that for any $t \in [0,T]$ it holds that

$$\mathbb{E}|X_t^{n+m} - X_t^n|^2 = \mathbb{E}\left| \sum_{j=0}^{m-1} (X_t^{n+j+1} - X_t^{n+j}) 2^{-(n+j)} 2^{n+j} \right|^2 .$$

Using Hölder's inequality we get that for any $t \in [0,T]$ it holds that

$$\mathbb{E}|X_t^{n+m} - X_t^n|^2 \leq \left(\sum_{j=0}^{m-1} 4^{-(n+j)} \right) \left(\sum_{j=0}^{m-1} \mathbb{E}|X_t^{n+j+1} - X_t^{n+j}|^2\, 4^{n+j} \right) .$$

We note that

$$\sum_{j=0}^{m-1} 4^{-j} = \frac{1 - \left(\frac{1}{4}\right)^m}{1 - \frac{1}{4}} \leq \frac{4}{3}$$

From (2.7) we thus get that for all $m \in \mathbb{N}$ and for all $t \in [0,T]$ it holds that

$$\mathbb{E}|X_t^{n+m} - X_t^n|^2 \leq \frac{4}{3} C^* 4^{-n} \sum_{j=0}^{m-1} \frac{(4Lt)^{n+j}}{(n+j)!} \leq \frac{4}{3} C^* e^{4Lt} 4^{-n} .$$

Hence for any $t \in [0,T]$ the sequence $(X_t^n)_{n\in\mathbb{N}}$ is Cauchy in $L^2(\Omega)$ and so $X_t^n \to X_t$ in $L^2(\Omega)$ as $n \to \infty$ for all $t \in [0,T]$. Finally $\mathbb{E}|X_t|^2 \leq \liminf_{n\to\infty} \mathbb{E}|X_t^n|^2 \leq C(1+|\xi|^2)e^{Lt}$ due to Lemma 2.3. Thus for each $n \in \mathbb{N}$ we have

$$\mathbb{E}|X_t^n - X_t|^2 \leq 2\mathbb{E}|X_t^n|^2 + 2\mathbb{E}|X_t|^2 \leq 4C(1+|\xi|^2)e^{Lt} =: g(t) .$$

Noting that $g \in L^1(0,T)$ we can conclude, using Lebesgue's theorem on dominated convergence that

$$\lim_{n\to\infty} \int_0^T \mathbb{E}|X_t^n - X_t|^2\, dt = \int_0^T \lim_{n\to\infty} \mathbb{E}|X_t^n - X_t|^2 = 0 .$$

This, together with Itô's isometry and assumption (2.4) allows us to take the limit in the stochastic integral term arising in (2.5). □

Remark 2.4. In the setup above the coefficients b and σ are random. In applications we will deal essentially with two settings for b and σ.

i) b and σ are deterministic, measurable, functions, i.e. $(t,x) \mapsto b_t(x)$ and $(t,x) \mapsto \sigma_t(x)$ are not random.

ii) b and σ are effectively random maps, but the randomness has a specific form. Namely, the random coefficients $b(t,\omega,x)$ and $\sigma(t,\omega,x)$ are of the form

$$b_t(\omega,x) := \bar{b}_t^{\alpha_t(\omega)}(x) \quad \text{and} \quad \sigma_t(\omega,x) := \bar{\sigma}_t^{\alpha_t(\omega)}(x)$$

where $\bar{b}$, $\bar{\sigma}$ are deterministic measurable functions on $[0,T]\times\mathbb{R}^d\times A$, A is a complete separable metric space and $(\alpha_t)_{t\in[0,T]}$ is a progressively measurable process valued in A.

This case arises in stochastic control problems that we will study later on, an example of which can already be seen with SDE (1.1).

Some properties of SDEs

In the remainder, we always assume that the coefficients b and σ satisfy the above conditions.

Proposition 2.5 (Further moment bounds)**.** *Let $m\in\mathbb{N}$, $m\geq 2$. Assume that for all $x\in\mathbb{R}^d$ the processes $(b_t(x))_{t\in[0,T]}$ and $(\sigma_t(x))_{t\in[0,T]}$ are progressively measurable, that $\mathbb{E}|\xi|^m<\infty$ and that there exists a constant K such that a.s. for all $t\in[0,T]$ and $x,y\in\mathbb{R}^d$ it holds that*

$$\|b(0)\|_{\mathbb{H}_T^m}+\|\sigma(0)\|_{\mathbb{H}_T^m}\leq K,$$
$$|b_t(x)-b_t(y)|+|\sigma_t(x)-\sigma_t(y)|\leq K|x-y|\,.$$

Then there exists a constant $C=C(K,T,m)$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^m\right]\leq C\big(1+\mathbb{E}[\,|\xi|^m]\big).$$

This can be proved using similar steps to those used in the proof of Lemma 2.3 but employing the Burkholder–Davis–Gundy inequality when estimating the expectation of the supremum of the stochastic integral term.

Proposition 2.6 (Stability)**.** *Let $m\in\mathbb{N}$, $m\geq 2$. Assume that for all $x\in\mathbb{R}^d$ the processes $(b_t(x))_{t\in[0,T]}$ and $(\sigma_t(x))_{t\in[0,T]}$ are progressively measurable, that $\mathbb{E}|\xi|^m<\infty$ and that there exists a constant K such that a.s. for all $t\in[0,T]$ and $x,y\in\mathbb{R}^d$ it holds that*

$$\|b(0)\|_{\mathbb{H}_T^m}+\|\sigma(0)\|_{\mathbb{H}_T^m}\leq K,$$
$$|b_t(x)-b_t(y)|+|\sigma_t(x)-\sigma_t(y)|\leq K|x-y|\,.$$

Let $x,x'\in\mathbb{R}^d$ and $0\leq t\leq t'\leq T$.

i) There exists a constant $C=C(K,T,m)$ such that

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X_s^{t,x}-X_s^{t,x'}|^m\right]\leq C|x-x'|^m.$$

ii) Suppose in addition that there is a constant K' such that

$$\mathbb{E}\left[\int_s^{s'}|b_r(0)|^2+|\sigma_r(0)|^2\,dr\right]\leq K'|s-s'|$$

for all $0 \le s \le s' \le T$*. Then there exists* $C = C(K,T)$ *such that*

$$\mathbb{E}\left[\sup_{t' \le s \le T} |X_s^{t,x} - X_s^{t',x}|^2\right] \le C(K + |x|^2)|t - t'|.$$

To prove the above two propositions one uses often the following inequalities: Cauchy-Schwartz, Hölder and Young's inequality; Gronwall's inequality (see Lemma A.6); Doob's maximal inequality (see Theorem A.14) and Burkholder–Davis–Gundy inequality.

Proposition 2.7 (Flow property)**.** *Let* $x \in \mathbb{R}^m$ *and* $0 \le t \le t' \le T$*. Then*

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}, \qquad s \in [t', T].$$

(This property holds even if t, t' *are stopping times.)*

See Exercise 2.17 for proof.

Proposition 2.8 (Markov property)**.** *Let* $x \in \mathbb{R}^d$ *and* $0 \le t \le t' \le s \le T$*. If* b *and* σ *are* deterministic functions*, then*

$$X_s^{t,x} \quad \text{is a function of } t,\ x,\ s,\ \text{and } \big(W_r - W_t\big)_{r \in [t,s]}.$$

Moreover,

$$\mathbb{E}\left[\Phi\big(X_r^{t,x},\ t' \le r \le s\big)|\mathcal{F}_{t'}\right] = \mathbb{E}\left[\Phi\big(X_r^{t,x},\ t' \le r \le s\big)|X_{t'}^{t,x}\right]$$

for all bounded and measurable functions $\Phi : C^0([t', s]; \mathbb{R}^m) \to \mathbb{R}$*.*

On the left hand side (LHS), the conditional expectation is on $\mathcal{F}_{t'}$ that contains all the information from time $t = 0$ up to time $t = t'$. On the right hand side (RHS), that information is replaced by the process $X_{t'}^{t,x}$ at time $t = t'$. In words, for Markovian processes the best prediction of the future, given all knowledge of the present and past (what you see on the LHS), is the present (what you see on the RHS; all information on the past can be ignored).

2.2 Controlled diffusions

We now introduce controlled SDEs with a finite time horizon $T > 0$; the infinite-horizon case is discussed later. Again, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with filtration $(\mathcal{F}_t)$ and a d'-dimensional Wiener process W compatible with this filtration.

We are given an action set A (in general separable complete metric space) and let $\mathcal{A}_0$ be the set of all A-valued progressively measurable processes, the controls. The controlled state is defined through an SDE as follows. Let

$$b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d \quad \text{and} \quad \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d \times d'}$$

be measurable functions.

Assumption 2.9. Assume that for each $t \in [0,T]$ that $(x,a) \mapsto b(t,x,a)$ and $(x,a) \mapsto \sigma(t,x,a)$ are continuous, assume that for each $t \in [0,T]$ we have $x \mapsto b(t,x,a)$ and $x \mapsto \sigma(t,x,a)$ continuous in x uniformly in $a \in A$ and that there is a constant K such that for any t,x,y,a we have

$$|b(t,x,a) - b(t,y,a)| + |\sigma(t,x,a) - \sigma(t,y,a)| \leq K|x-y|. \tag{2.9}$$

Moreover for all t,x,a it holds that

$$|b(t,x,a)| + |\sigma(t,x,a)| \leq K(1+|x|+|a|)\,. \tag{2.10}$$

We will refer to the set

$$\mathcal{A} := \{\alpha \in \mathbb{H}^2_T : \forall \omega \in \Omega, t \in [0,T]\ \alpha_t(\omega) \in A \ \text{ and } \ \alpha \ \text{ is progressively measurable}\,\}$$

set as *admissible controls.*

Given a fixed control $\alpha \in \mathcal{A}$, we consider the SDE for $0 \leq t \leq T \leq \infty$ for $s \in [t,T]$

$$dX_s = b\big(s, X_s, \alpha_s\big)\,dt + \sigma\big(s, X_s, \alpha_s\big)\,dW_s, \quad X_t = \xi. \tag{2.11}$$

With Assumption 2.9 the SDE (2.11) is a special case of an SDE with random coefficients, see (2.1). In particular, if we fix $\alpha \in \mathcal{A}$ then taking $\tilde{b}_t(x) := b(t,x,\alpha_t)$ and $\tilde{\sigma}_t(x) := \sigma(t,x,\alpha_t)$ we have the progressive measurability of $\tilde{b}$ and $\tilde{\sigma}$ (since b, σ are assumed to be measurable and α is progressively measurable. Moreover

$$\|\tilde{b}(0)\|^2_{\mathbb{H}^2_T} = \mathbb{E}\int_0^T |b(t,0,\alpha_t)|^2\,dt \leq \mathbb{E}\int_0^T K^2(1+|\alpha_t)|)^2\,dt \leq 2K^2T + 2K^2\|\alpha\|^2_{\mathbb{H}^2_T} < \infty$$

and similarly $\|\tilde{\sigma}(0)\|^2_{\mathbb{H}^2_T} < \infty$. Finally the Lipschitz continuity of the coefficients in space clearly holds and so due to Proposition 2.2 we have the following result.

Proposition 2.10 (Existence and uniqueness)**.** *Let $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t)$ and $\alpha \in \mathcal{A}_0$. Then SDE* (2.11) *has a unique (strong) Markov solution $X = X^{\alpha}_{t,\xi}$ on the interval* $[t,T]$ *such that*

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E} \sup_{s \in [t,T]} |X_s|^2 \leq c(1+\mathbb{E}|\xi|^2)\,.$$

Moreover, the solution has the properties listed in Propositions 2.5 and 2.6.

2.3 Stochastic control problem with finite time horizon

In this section we revisit the ideas of the opening one and give a stronger mathematical meaning to the general setup for optimal control problems. We distinguish the finite time horizon $T < \infty$ and the infinite time horizon $T = \infty$, the functional to optimize must differ.

In general, texts either discuss maximization or a minimization problems. Using analysis results, it is easy to jump between minimization and maximization problems: $\max_x f(x) = -\min_x -f(x)$ and the x^* that maximizes f is the same one that minimizes $-f$ (draw a picture to convince yourself).

Finite time horizon

Let

$$J(t,\xi,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{\alpha,t,\xi}, \alpha_s)\,ds + g(X_T^{\alpha,t,\xi})\right],$$

where $X_{t,\xi}$ solves (2.11) (with initial condition $X(t) = \xi$). The J here is called the *objective functional*. We refer to f as the *running gain* (or, if minimizing, *running cost*) and to g as the *terminal gain* (or *terminal cost*).

We will ensure the good behavior of J through the following assumption.

Assumption 2.11. There is $K > 0$, $m \in \{0, 1, \dots\}$ such that for all t, x, y, a we have

$$|g(x) - g(y)| + |f(t,x,a) - f(t,y,a)| \le K(1 + |x|^m + |y|^m)|x - y|,$$

$$|f(t,0,a)| \le K(1 + |a|^2)\,.$$

Note that this assumption is not the most general. For bigger generality consult e.g. [9].

The optimal control problem formulation We will focus on the following stochastic control problem. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$. Let

$$(P)\begin{cases} v(t,x) := \sup\limits_{\alpha \in \mathcal{A}[t,T]} J(t,x,\alpha) = \sup\limits_{\alpha \in \mathcal{A}[t,T]} \mathbb{E}\left[\int_t^T f\big(s, X_s^{\alpha,t,x}, \alpha_s\big)\,ds + g\big(X_T^{\alpha,t,x}\big)\right] \\ \text{and } X^{\alpha,t,x} \text{ solves (2.11) with } X_t^{\alpha,t,x} = x. \end{cases}$$

The solution to the problem (P), is the *value function*, denoted by v. One of the mathematical difficulties in stochastic control theory is that we don't even know at this point whether v is measurable or not.

In many cases there is no optimal control process α^* for which we would have $v(t,x) = J(t,x,\alpha^*)$. Recall that v is the value function of the problem (P). However there is always an ε-optimal control (simply by definition of supremum).

Definition 2.12 (ε-optimal controls)**.** Take $t \in [0,T]$ and $x \in \mathbb{R}^m$. Let $\varepsilon \ge 0$. A control $\alpha^\varepsilon \in \mathcal{A}[t,T]$ is said to be *ε-optimal* if

$$v(t,x) \le \varepsilon + J(t,x,\alpha^\varepsilon)\,. \tag{2.12}$$

Lemma 2.13 (Lipschitz continuity in x of the value function)**.** *If Assumptions 2.9 and 2.11 hold then there exists $C = C_{T,K,m} > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}^d$ we have*

$$|J(t,x,\alpha) - J(t,y,\alpha)| \le C(1 + |x|^m + |y|^m)|x - y|\,.$$

and

$$|v(t,x) - v(t,y)| \le C(1 + |x|^m + |y|^m)|x - y|\,.$$

Proof. The first step is to show that there is $C_{T,K,m} > 0$ such that for any $\alpha \in \mathcal{U}$ we have

$$I := |J(t,x,\alpha) - J(t,y,\alpha)| \le C_T(1 + |x|^m + |y|^m)|x - y|\,.$$

Using Assumption 2.11 (local Lipschitz continuity in x of f and g) we get

$$
\begin{aligned}
I &\le \mathbb{E}\left[\int_t^T |f(s,X_s^{t,x,\alpha},\alpha_s) - f(s,X_s^{t,y,\alpha},\alpha_s)|\,ds + \left|g(X_T^{t,x,\alpha}) - g(X_T^{t,x,\alpha})\right|\right] \\
&\le K\mathbb{E}\bigg[\int_t^T (1+|X_s^{t,x,\alpha}|^m + |X_s^{t,y,\alpha}|^m)|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|\,ds \\
&\qquad\qquad + (1+|X_T^{t,x,\alpha}|^m + |X_T^{t,y,\alpha}|^m)|X_T^{t,x,\alpha} - X_T^{t,y,\alpha}|\bigg].
\end{aligned}
$$

We note that due to Hölder's and Young's inequalities

$$
\begin{aligned}
I \le & C_{K,m}\left(\mathbb{E}\int_t^T (1+|X_s^{t,x,\alpha}|^{m+1} + |X_s^{t,y,\alpha}|^{m+1})\,ds\right)^{\frac{m}{m+1}}\left(\mathbb{E}\int_t^T |X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|^{m+1}\,ds\right)^{\frac{1}{m+1}} \\
&+ C_{K,m}\left(\mathbb{E}(1+|X_T^{t,x,\alpha}|^{m+1} + |X_T^{t,y,\alpha}|^{m+1})\right)^{\frac{m}{m+1}}\left(\mathbb{E}|X_T^{t,x,\alpha} - X_T^{t,y,\alpha}|^{m+1}\right)^{\frac{1}{m+1}}.
\end{aligned}
$$

Then, using Proposition 2.6, we get

$$
\begin{aligned}
I &\le C_{T,K,m}\left(\sup_{t\le s\le T}\mathbb{E}\Big[|X_s^{t,x,\alpha}|^{m+1} + |X_s^{t,y,\alpha}|^{m+1}\Big]\right)^{\frac{m}{m+1}}\left(\sup_{t\le s\le T}\mathbb{E}|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|^{m+1}\right)^{\frac{1}{m+1}} \\
&\le C_{T,K,m}(1+|x|^m + |y|^m)|x-y|\,.
\end{aligned}
$$

We now need to apply this property of J to the value function v. Let $\varepsilon > 0$ be arbitrary and fixed. Then there is $\alpha^\varepsilon \in \mathcal{U}$ such that $v(t,x) \le \varepsilon + J(t,x,\alpha^\varepsilon)$. Moreover $v(t,y) \ge J(t,y,\alpha^\epsilon)$. Thus

$$
v(t,x) - v(t,y) \le \varepsilon + J(t,x,\alpha^\varepsilon) - J(t,y,\alpha^\epsilon) \le \varepsilon + C(1+|x|^m + |y|^m)|x-y|\,.
$$

With $\varepsilon > 0$ still the same and fixed there would be $\beta^\varepsilon \in \mathcal{U}$ such that $v(t,y) \le \varepsilon + J(t,y,\beta^\varepsilon)$. Moreover $v(t,x) \ge J(t,x,\beta^\epsilon)$ and so

$$
v(t,y) - v(t,x) \le \varepsilon + J(t,y,\beta^\varepsilon) - J(t,x,\beta^\epsilon) \le \varepsilon + C(1+|x|^m + |y|^m)|x-y|\,.
$$

Hence

$$
-\varepsilon - C(1+|x|^m + |y|^m)|x-y| \le v(t,x) - v(t,y) \le \varepsilon + C(1+|x|^m + |y|^m)|x-y|\,.
$$

Letting $\varepsilon \to 0$ concludes the proof. □

An important consequence of this is that if we fix t then $x \mapsto v(t,x)$ is measurable (as continuous functions are measurable).

2.4 Exercises

Exercise 2.14 (Non-existence of solution)**.**

1. Let $I = \left[0, \frac{1}{2}\right]$. Find a solution X for

$$
\frac{dX_t}{dt} = X_t^2\,,\quad t \in I\,,\quad X_0 = 1\,.
$$

2. Does a solution to the above equation exist on $I = [0,1]$? If yes, show that it satisfies Definition 2.1. In not, which property is violated?

Exercise 2.15 (Non-uniqueness of solution)**.** Fix $T > 0$. Consider

$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}\,,\quad t \in [0,T]\,,\quad X_0 = 0\,.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0,T]$ is a solution to the above ODE.
2. Show that $X_t := t^2$ for all $t \in [0,T]$ is also a solution.
3. Find at least two more solutions different from $\bar{X}$ and X.

Exercise 2.16. Consider the SDE

$$X_t = \xi + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s\,, \qquad t \in [0,T]\,.$$

and assume that the conditions of Proposition 2.2 hold. Show that the solution to the SDE is unique in the sense that if X and Y are two solutions with $X_0 = \xi = Y_0$ then

$$\mathbb{P}\left[\sup_{0\le t\le T}|X_t - Y_t| > 0\right] = 0\,.$$

Exercise 2.17. Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s,\;\; t \le s \le T,\;\; X_t^{t,x} = x\,.$$

Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then

$$\mathbb{P}\left[\sup_{t\le s\le T}|X_s^{t,x} - Y_s^{t,x}| > 0\right] = 0\,.$$

Show that then the *flow property* holds i.e. for $0 \le t \le t' \le T$ we have almost surely that

$$X_s^{t,x} = X_s^{t',X_{t'}^{t,x}}\,, \qquad \forall s \in [t',T].$$

2.5 Solutions to Exercises

Solution (to Exercise 2.14)**.**

1. We can use the following method to get a guess: from the ODE we get $X^{-2}dX = dt$ which means that, after integrating, we get $-X^{-1} = t + C$. So $X_t = -(t+C)^{-1}$. Since $X_0 = 1$ we get $C = -1$. Thus

$$X_t = \frac{1}{1-t}\,, \quad t \in \left[0, \tfrac{1}{2}\right]\,.$$

We check by calculating that $\frac{dX_t}{dt} = (1-t)^{-2} = X_t^2$ so the equation holds in $[0,1/2]$.

2. We can see that $\lim_{t\nearrow 1} X_t = \infty$ and so the $t \mapsto X_t$ is not continuous on $[0,1]$.

Solution (to Exercise 2.15)**.**

1. Clearly $\bar{X}_0 = 0$ and for $t \in [0,T]$ we have $\frac{d\bar{X}_t}{dt} = 0 = 2\sqrt{|\bar{X}_t|}$.
2. Clearly $X_0 = 0$ and for $t \in [0,T]$ we have $\frac{dX_t}{dt} = 2t = 2\sqrt{t^2} = 2\sqrt{|X_t|}$.
3. Fix any $\tau \in (0,T)$ and define

$$X_t^{(\tau)} := \begin{cases} 0 & \text{for} \quad t \in [0,\tau)\,, \\ (t-\tau)^2 & \text{for} \quad t \in [\tau, T]\,. \end{cases}$$

Then, clearly, $dX_0^{(\tau)} = 0$ and moreover if $t \in [0,\tau)$ then we have

$$\frac{dX_t^{(\tau)}}{dt} = 0 = 2\sqrt{|X_t^{(\tau)}|}\,,$$

while if $t \in [\tau, T]$ then we have

$$\frac{dX_t^{(\tau)}}{dt} = 2(t-\tau) = 2\sqrt{|(t-\tau)^2|} = 2\sqrt{X_t^{(\tau)}}\,.$$

So, in fact, there are uncountably many different solutions.

Solution (to Exercise 2.16)**.** Using the same estimates as in the proof of Proposition 2.2, see (2.6), we get that for some constant $L > 0$

$$\mathbb{E}\sup_{s\le t}|X_s - Y_s|^2 \le L\int_0^t \mathbb{E}|X_r - Y_r|^2\,dr\,.$$

Hence

$$\mathbb{E}\sup_{s\le t}|X_s - Y_s|^2 \le L\int_0^t \mathbb{E}\sup_{s\le r}|X_s - Y_s|^2\,dr\,.$$

From Gronwall's lemma (applied with $y(t) := \mathbb{E}\sup_{s\le t}|X_s - Y_s|^2$, $a(t) = 0$, $b(t) = 0$ and $\lambda(t) = L$) we get that for all $t \in [0,T]$ we have

$$\mathbb{E}\sup_{s\le t}|X_s - Y_s|^2 \le 0\,.$$

But this means that

$$\mathbb{P}\left[\sup_{t\le T}|X_s - Y_s|^2 = 0\right] = 1\,.$$

Solution (to Exercise 2.17)**.** Let $Y_s := X_s^{t',X_{t'}^{t,x}}$ for $s \in [t',T]$ and note that the process Y solves the SDE for $s \in [t',T]$ with $Y_{t'} = X_{t'}^{t',X_{t'}^{t,x}} = X_{t'}^{t,x}$. Let $X_s := X_s^{t,x}$ for $s \in [t',T]$ and note that this also solves the SDE for $s \in [t',T]$ with

$$X_{t'} = X_{t'}^{t,x} = Y_{t'}\,.$$

Hence both Y and X solve the same SDE with the same starting point. By the pathwise uniqueness property of the solutions of this SDE we then have

$$\mathbb{P}\left[\sup_{t \leq s \leq T} |X_s - Y_s| = 0\right] = 1$$

but this means that almost surely it holds that for all $s \in [t', T]$ it holds that

$$X_s^{t', X_{t'}^{t,x}} = Y_s = X_s = X_s^{t,x}\,.$$

3 Dynamic programming and the Hamilton–Jacobi–Bellman equation

3.1 Dynamic programming principle

Dynamic programming (DP) is one of the most popular approaches to study the stochastic control problem (P). The main idea was originated from the so-called Bellman's principle, which states

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The following is the statement of Bellman's principle / dynamic programming.

Theorem 3.1 (Bellman's principle / Dynamic programming)**.** *For any* $0 \le t \le \hat{t} \le T$*, for any* $x \in \mathbb{R}^m$*, we have*

$$v(t,x) = \sup_{\alpha \in \mathcal{A}[t,\hat{t}]} \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X_s^{\alpha,t,x}, \alpha_s\big)\, ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,t,x}\big) \Big| X_s^{\alpha,t,x} = x \right]. \tag{3.1}$$

The idea behind the dynamic programming principle is as follows. The expectation on the RHS of (3.1) represents the gain if we implement the time t until time $\hat{t}$ optimal strategy and then implement the time $\hat{t}$ until T optimal strategy. Clearly, this gain will be no larger than the gain associated with using the overall optimal strategy from the start (since we can apply the overall optimal control in both scenarios and obtain the LHS).

What equation (3.1) says is that if we determine the optimal strategy separately on each of the time intervals $[t,\hat{t}]$ and $[\hat{t},T]$ we get the same answer as when we consider the whole time interval $[t,T]$ at once. Underlying this statement, hides a deeper one: that if one puts the optimal stategy over $[t,\hat{t}\,)$ together with the optimal stategy over $[\hat{t},T]$ this is still an optimal strategy.

Note that without Lemma 2.13 we would not even be allowed to write (3.1) since we need $v(\hat{t}, X_{\hat{t}}^{\alpha,t,x})$ to be a random variable (so that we are allowed to take the expectation).

Let us now prove the Bellman principle.

Proof of Theorem 3.1. We will start by showing that $v(t,x) \le$ RHS of (3.1). We note that with $\alpha \in \mathcal{A}[t,T]$ we have

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| X_t^\alpha = x \right].$$

We will use the tower property of conditional expectation and use the Markov property of the process. Let $\mathcal{F}_{\hat{t}}^{X^\alpha} := \sigma(X_s^\alpha : t \le s \le \hat{t})$. Then

$$\begin{aligned} &J(t,x,\alpha) \\ &= \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \mathbb{E}\left[\int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| \mathcal{F}_{\hat{t}}^{X^\alpha} \right] \Big| X_t^\alpha = x \right] \\ &= \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s) ds + \mathbb{E}\left[\int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s) ds + g(X_T^\alpha) \Big| X_{\hat{t}}^\alpha \right] \Big| X_t^\alpha = x \right]. \end{aligned}$$

Now, because of the flow property of SDEs,

$$\mathbb{E}\left[\int_{\hat t}^{T} f(s, X_s^{\alpha,t,x}, \alpha_s)ds + g(X_T^{\alpha,t,x})\bigg| X_{\hat t}^{\alpha,t,x}\right] = J\left(\hat t, X_{\hat t}^{\alpha,t,x}, (\alpha_s)_{s\in[\hat t,T]}\right) \le v\left(\hat t, X_{\hat t}^{\alpha,t,x}\right).$$

Hence

$$J(t,x,\alpha) \le \sup_{\alpha\in\mathcal{U}} \mathbb{E}\left[\int_t^{\hat t} f(s, X_s^{\alpha}, \alpha_s)ds + v\left(\hat t, X_{\hat t}^{\alpha,t,x}\right)\bigg| X_t^{\alpha} = x\right].$$

Taking supremum over control processes α on the left shows that $v(t,x) \le$ RHS of (3.1).

We now need to show that RHS of (3.1) $\le v(t,x)$. Fix $\varepsilon > 0$. Then there is $\alpha^\varepsilon \in \mathcal{A}[t,\hat t]$ such that

$$\text{RHS of (3.1)} \le \varepsilon + \mathbb{E}\left[\int_t^{\hat t} f\big(s, X_s^{\alpha^\varepsilon,t,x}, \alpha_s^\varepsilon\big)\, ds + v\big(\hat t, X_{\hat t}^{\alpha^\varepsilon,t,x}\big)\Big| X_t^{\alpha^\varepsilon,t,x} = x\right].$$

Let us write $X_s := X_s^{\alpha^\varepsilon,t,x}$ for brevity from now on. We now have to be careful so that we can construct an ε-optimal control which is progressively measurable on the whole $[t,T]$. To that end let $\delta = \delta(\omega) > 0$ be such that

$$2^m C(1 + |X_{\hat t}(\omega)|^m)\delta(\omega) < \varepsilon \ \text{ and } \ 2^{m-1}\delta(\omega)^m < 1.$$

where C is the constant from Lemma 2.13. Take $(x_i)_{i\in\mathbb{N}}$ dense in $\mathbb{R}^d$. By density of $(x_i)_i$ we know that for each $\delta(\omega)$ there exists $i(\omega)$ such that $|x_{i(\omega)} - X_{\hat t}(\omega)| \le \delta(\omega)$. Moreover

$$C(1 + |x_i|^m)\delta \le C(1 + 2^{m-1}|x_i - X_{\hat t}|^m + 2^{m-1}|X_{\hat t}|^m)\delta \le 2^m C(1 + |X_{\hat t}|^m)\delta < \varepsilon\,.$$

The open covering of $\mathbb{R}^d$ given by $\bigcup_{\omega\in\Omega} B_{\delta(\omega)}(x_{i(\omega)})$ has a countable sub-cover $\bigcup_{k\in\mathbb{N}} B_{\delta_k}(x_k)$. Let (Q_k) be constructed as follows:

$$Q_1 = B_{\delta_1}(x_1) \ \text{ and } \ Q_k = B_{\delta_k}(x_k) \setminus \bigcup_{k'=1}^{k-1} Q_{k'}\,.$$

Then for each x_i there is $\alpha^{\varepsilon,i} \in \mathcal{A}(\hat t, T]$ such that $v(\hat t, x_i) \le \varepsilon + J(\hat t, x_i, \alpha^{\varepsilon,i})$. Moreover if $X_{\hat t} \in Q_i$ then $|X_{\hat t}|^m \le 2^{m-1}|X_{\hat t} - x_i|^m + 2^{m-1}|x_i|^m$ and due to Lemma 2.13 we have,

$$\begin{aligned}|v(\hat t, X_{\hat t}) - v(\hat t, x_i)| &\le C(1 + |X_{\hat t}|^m + |x_i|^m)|X_{\hat t} - x_i| \\ &\le C(1 + 2^{m-1}|X_{\hat t} - x_i|^m + 2^{m-1}|x_i|^m)|X_{\hat t} - x_i| \\ &\le C(1 + 2^{m-1}\delta^m + 2^{m-1}|x_i|^m)\delta \\ &\le C(2 + 2^{m-1}|x_i|^m)\delta \le 2^m C(1 + |x_i|^m)\delta < \varepsilon\,.\end{aligned}$$

Similarly we have

$$|J(\hat t, x_i, \alpha^{\varepsilon,i}) - J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i})| \le \varepsilon.$$

Hence we get

$$v(\hat t, X_{\hat t}) \le v(\hat t, x_i) + \varepsilon \le \varepsilon + J(\hat t, x_i, \alpha^{\varepsilon,i}) + \varepsilon \le \varepsilon + J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i}) + 2\varepsilon\,.$$

And so

$$v(\hat t, X_{\hat t}) \le 3\varepsilon + J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i})\,.$$

Therefore RHS of (3.1)

$$\leq 3\varepsilon + \mathbb{E}\Bigg[\int_t^{\hat t} f\big(s, X_s^{\alpha^\varepsilon,t,x}, \alpha_s^\varepsilon\big)\, ds$$
$$+\,\mathbb{E}\left[\int_{\hat t}^T f(s, Y_s^{\alpha^{\varepsilon,i}}, \alpha_s^{\varepsilon,i})\, ds + g\left(Y_T^{\alpha^{\varepsilon,i}}\right)\Big| Y_{\hat t}^{\alpha^{\varepsilon,i}} = X_{\hat t}^{\alpha^\varepsilon,t,x}\right]\Big| X_s^{\alpha^\varepsilon,t,x} = x\Bigg].$$

Regarding controls we now have the following: $\alpha^\varepsilon \in \mathcal{A}[t,\hat t]$ and for each i we have $\alpha^{\varepsilon,i} \in \mathcal{U}(\hat t, T]$. From these we build one control process β^ε as follows:

$$\beta_s^\varepsilon := \begin{cases} \alpha_s^\varepsilon & s \in [t,\hat t] \\ \alpha_s^{\varepsilon,i} & s \in (\hat t, T] \text{ and } X_{\hat t}^{\alpha^\varepsilon,t,x} \in Q_i. \end{cases}$$

This process is progressively measurable with values in A and so $\beta^\varepsilon \in \mathcal{A}[t,T]$. Due to the flow property we may write that RHS of (3.1)

$$\leq 3\varepsilon + \mathbb{E}\left[\int_t^{\hat t} f\big(s, X_s^{\beta^\varepsilon,t,x}, \beta_s^\varepsilon\big)\, ds + \int_{\hat t}^T f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon)\, ds + g\left(X_T^{\beta^\varepsilon}\right)\Big| X_s^{\beta^\varepsilon,t,x} = x\right].$$

Finally taking supremum over all possible control strategies we see that RHS of (3.1) $\leq 3\varepsilon + v(t,x)$. Letting $\varepsilon \to 0$ completes the proof. □

Lemma 3.2 ($\frac{1}{2}$-Hölder continuity of value function in time)**.** *Let Assumptions 2.9 and 2.11 hold. Then there is a constant $C = C_{T,K,m} > 0$ such that for any $x \in \mathbb{R}^d$, $0 \leq t, \hat t \leq T$ we have*

$$|v(t,x) - v(\hat t, x)| \leq C(1 + |x|^{m+1})|t - \hat t|^{1/2}.$$

Proof. Still needs to be written down.

□

Corollary 3.3. *Let Assumptions 2.9 and 2.11 hold. Then there is a constant $C = C_{T,K,m} > 0$ such that for any $x, y \in \mathbb{R}^d$, $0 \leq s, t \leq T$ we have*

$$|v(s,x) - v(t,y)| \leq C(1 + |x|^m + |x|^{1/2})\left(|t - \hat t|^{1/2} + |x - y|\right).$$

This means that the value function v is jointly measurable in (t,x). With this we get the following.

Theorem 3.4 (Bellman's principle / Dynamic programming with stopping time)**.** *For any stopping times $t, \hat t$ such that $0 \leq t \leq \hat t \leq T$, for any $x \in \mathbb{R}^m$, we have* (3.1).

The proof uses the same arguments as before except that now have to cover the whole $[0,T] \times \mathbb{R}^d$ and we need to use the $\frac{1}{2}$-Hölder continuity in time as well.

Corollary 3.5 (Global optimality implies optimality from any time)**.** *Take $x \in \mathbb{R}$. A control $\beta \in \mathcal{U}[0,T]$ is optimal for (P) with the state process $X_s = X_s^{\beta,0,x}$ for $s \in [0,T]$ if and only if for any $\hat t \in [0,T]$ we have*

$$v(\hat t, X_{\hat t}) = J(\hat t, X_{\hat t}, \beta).$$

Proof. To ease the notation we will take $f = 0$. The reader is encouraged to prove the general case.

Due to the Bellman principle, Theorem 3.4, we have

$$v(0,x) = \sup_{\alpha\in\mathcal{U}[0,\hat{t}]} \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}^{\alpha,0,x}\right)\right] \geq \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}^{\beta,0,x}\right)\right] .$$

If β is an optimal control

$$\mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}\right)\right] \leq v(0,x) = J(0,x,\beta) = \mathbb{E}\left[g\left(X_T\right)\right] .$$

Using the tower property of conditional expectation

$$v(0,x) \leq \mathbb{E}\left[\mathbb{E}\left[g\left(X_T\right)\middle|\mathcal{F}_{\hat{t}}^X\right]\right] = \mathbb{E}\left[J\left(\hat{t}, X_{\hat{t}}, \beta\right)\right] \leq \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}\right)\right] \leq v(0,x) .$$

Since the very left and very right of these inequalities are equal we get that

$$\mathbb{E}\left[J\left(\hat{t}, X_{\hat{t}}, \beta\right)\right] = \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}\right)\right]$$

Moreover $v \geq J$ and so we can conclude that $v\left(\hat{t}, X_{\hat{t}}\right) = J\left(\hat{t}, X_{\hat{t}}, \beta\right)$ a.s. The completes the first part of the proof. The "only if" part of the proof is clear because we can take $\hat{t} = 0$ and get $v(0,x) = J(0,x,\beta)$ which means that β is an optimal control. □

From this observation we can prove the following description of optimality.

Theorem 3.6 (Martingale optimality)**.** *Let the assumptions required for Bellman's principle hold. Fix any (t,x) and let*

$$M_s := \int_t^s f_r^{\alpha_r}\left(X_r^{\alpha,t,x}\right) dr + v\big(s, X_s^{\alpha,t,x}\big) . \tag{3.2}$$

Then for any control $\alpha \in \mathcal{A}$ the process $(M_s)_{s\in[t,T]}$ is an $\mathcal{F}_s^X := \sigma(X_r^{\alpha,t,x}; t \leq r \leq s)$ super-martingale. Moreover α is optimal if and only if it is a martingale.

When comparing the subsequent argument to the deterministic case, note how "super-martingale" and "martingale" arise, respectively, as the stochastic analogues of "decreasing" and "constant" of the deterministic problem.

Proof. We have by, Theorem 3.1 (the Bellman principle) that for any $0 \leq t \leq s \leq \hat{s} \leq T$ that

$$v\big(s, X_s^{\alpha,t,x}\big) = \sup_{\hat{\alpha}\in\mathcal{A}} \mathbb{E}\left[\int_s^{\hat{s}} f_r^{\hat{\alpha}_r}\left(X_r\right) dr + v\big(\hat{s}, X_{\hat{s}}\big)\middle| X_s = X^{\alpha,t,x}\right] .$$

From the Markov property we get that

$$v\big(s, X_s^{\alpha,t,x}\big) = \sup_{\hat{\alpha}\in\mathcal{A}} \mathbb{E}\left[\int_s^{\hat{s}} f_r^{\hat{\alpha}_r}\left(X_r\right) dr + v\big(\hat{s}, X_{\hat{s}}\big)\middle| \mathcal{F}_s^X\right] .$$

Hence

$$v\big(s, X_s^{\alpha,t,x}\big) \geq \mathbb{E}\left[\int_s^{\hat{s}} f_r^{\alpha_r}\left(X_r\right) dr + v\big(\hat{s}, X_{\hat{s}}\big)\middle| \mathcal{F}_s^X\right]$$

and so

$$
\begin{aligned}
M_s &\geq \int_t^s f_r^{\alpha_r}\big(X_r^{\alpha,t,x}\big)\, dr + \mathbb{E}\left[\int_s^{\hat{s}} f_r^{\alpha_r}\big(X_r\big)\, dr + v\big(\hat{s}, X_{\hat{s}}\big)\Big|\mathcal{F}_s^X\right] \\
&= \mathbb{E}\left[M_{\hat{s}}|\mathcal{F}_s^X\right] .
\end{aligned}
$$

This means that M is a super-martingale. Moreover we see that if α is optimal then the inequalities above are equalities and hence M is a martingale.

Now assume that $M_s = \mathbb{E}[M_{\hat{s}}|\mathcal{F}_s^X]$. We want to ascertain that the control α driving M is an optimal one. But the martingale property implies that $J(t,x,\alpha) = \mathbb{E}[M_T] = \mathbb{E}[M_t] = v(t,x)$ and so α is indeed an optimal control. □

One question you may ask yourself is: How can we use the dynamic programming principle to compute an optimal control? Remember that the idea behind the DPP is that it is not necessary to optimize the control α over the entire time interval $[0,T]$ at once; we can partition the time interval into smaller sub-intervals and optimize over each individually. We will see below that this idea becomes particularly powerful if we let the partition size go to zero: the calculation of the optimal control then becomes a pointwise minimization linked to certain PDEs (see Theorem A.27). That is, for each fixed state x we compute the optimal value of control, say $a \in A$, to apply whenever $X(t) = x$.

3.2 Hamilton-Jacobi-Bellman (HJB) and verification

If the value function $v = v(t,x)$ is smooth enough, then we can apply Itô's formula to v and X in (3.2). Thus we get the *Hamilton-Jacobi-Bellman* (HJB) equation (also know and the *Dynamic Programming equation* or *Bellman* equation).

For notational convenience we will write $\sigma^a(t,x) := \sigma(t,x,a)$, $b^a(t,x) := b(t,x,a)$ and $f^a(t,x) := f(t,x,a)$. We then define

$$
L^a v := \frac{1}{2}\mathrm{tr}\left[\sigma^a(\sigma^a)^*\partial_{xx}v\right] + b^a\partial_x v\,.
$$

Recall that trace is the sum of all the elements on the diagonal of a square matrix i.e. for a matrix $(a^{ij})_{i,j=1}^d$ we get $\mathrm{tr}[a] = \sum_{i=1}^d a^{ii}$, that $\partial_{xx}v$ denotes the Jacobian matrix i.e. $(\partial_{xx}v)_{ij} = \partial_{x^i}\partial_{x^j}v$ whilst $\partial_x v$ denotes the gradient vector i.e. $(\partial_x v)_i = \partial_{x^i}v$. This means that

$$
\mathrm{tr}\left[\sigma^a(\sigma^a)^*\partial_{xx}v\right] = \sum_{i,j=1}^d [\sigma^a(\sigma^a)^*]^{ij}\partial_{x^ix^j}v \;\text{ and }\; b^a\partial_x v = \sum_{i=1}^d (b^a)^i\partial_{x^i}v\,.
$$

Theorem 3.7 (Hamilton-Jacobi-Bellman (HJB)). *If the value function v for (P) is $C^{1,2}([0,T)\times\mathbb{R}^d)$, then it satisfies*

$$
\begin{aligned}
\partial_t v + \sup_{a\in A}\Big(L^a v + f^a\Big) &= 0 \quad \text{on } [0,T)\times\mathbb{R}^d \\
v(T,x) &= g(x) \quad \forall x\in\mathbb{R}^d\,.
\end{aligned} \tag{3.3}
$$

Proof. Let $x \in \mathbb{R}$ and $t \in [0,T]$. Then the condition $v(T,x) = g(x)$ follows directly from the definition of v. Fix $\alpha \in \mathcal{A}[t,T]$ and let M be given by (3.2) i.e.

$$
M_s := \int_t^s f_r^{\alpha_r}\big(X_r^{t,x,\alpha}\big)\, dr + v\big(s, X_s^{t,x,\alpha}\big)\,.
$$

Then, Itô's formula applied to v and $X = (X_s^{t,x,\alpha})_{s\in[t,T]}$ yields

$$dM_s = \Big[\Big(\partial_t v + L^{\alpha_s} v + f^{\alpha_s}\Big)\big(s, X_s^{\alpha,t,x}\big)\Big]\, ds + \Big[(\partial_x v\, \sigma^{\alpha_s})\big(s, X_s^{\alpha,t,x})\big)\Big]\, dW_s.$$

For any $(t, x) \in [0, T] \times \mathbb{R}$ take the stopping time $\tau = \tau^{\alpha,t,x}$ given by

$$\tau := \inf \Big\{ t' \geq t : \int_t^{t'} \big(\partial_x v\, \sigma^{\alpha_s})\big(s, X_s^{\alpha,t,x})\big)^2\, ds \geq 1 \Big\} .$$

We know from Theorem 3.6 that M must be a supermartingale. On the other hand the term given by the stochastic integral is a martingale (when cosidered stopped at τ). So $(M_{t\wedge\tau})_t$ can only be a supermartingale if

$$f^{\alpha_s}(s, X_s) + (\partial_t v + L^{\alpha_s} v)(s, X_s) \leq 0 .$$

Since the starting point (t, x) and control α were arbitrary we get[4]

$$(\partial_t v + L^a v + f^a)(t, x) \leq 0 \quad \forall t, x, a .$$

Taking the supremum over $a \in A$ we get

$$\partial_t v(t, x) + \sup_{a\in A}[(L^a v + f^a)(t, x)] \leq 0 \quad \forall t, x .$$

We now need to show that in fact the inequality cannot be strict. We proceed by setting up a contradiction. Assume that there is (t, x) such that

$$\partial_t v(t_0, x_0) + \sup_{a\in A}[(L^a v + f^a)(t, x)] < 0 .$$

We will show that this contradicts the Bellman principle and hence we must have equality, thus completing the proof.

We must further assume that b and σ are right-continuous in t uniformly in the x variable[5]. Now by continuity (recall that $v \in C^{1,2}([0, T) \times \mathbb{R}^d)$ we get that there must be $\varepsilon > 0$ and an associated $\delta > 0$ such that

$$\partial_t v + \sup_{a\in A}[(L^a v + f^a)] \leq -\varepsilon < 0 \quad \text{on } [t, t + \delta) \times B_\delta(x).$$

Let us fix $\alpha \in \mathcal{A}[t, T]$ and let $X_s := X_s^{t,x,\alpha}$. We define the stopping time

$$\tau := \{s > t : |X_s - x| > \delta\} \wedge (t + \delta) .$$

Since the process X_s has a.s. continuous sample paths we get $\mathbb{E}[\tau - t] > 0$. Then

$$\begin{aligned}
&\int_t^\tau f^{\alpha_r}(r, X_r)\, dr + v(\tau, X_\tau) \\
&= v(t, x) + \int_t^\tau f^{\alpha_r}(r, X_r)\, dr + v(\tau, X_\tau) - v(t, x) \\
&= v(t, x) + \int_t^\tau \Big[\Big(\partial_t v + L^{\alpha_r} v + f^{\alpha_r}\Big)(r, X_r)\Big]\, dr + \int_t^\tau \Big[(\partial_x v)\, \sigma^{\alpha_r}\Big](r, X_r)\, dW_r \\
&\leq v(t, x) - \varepsilon(\tau - t) + \int_t^\tau \Big[(\partial_x v)\, \sigma^{\alpha_r}\Big](r, X_r)\, dW_r .
\end{aligned}$$

[4]This is not a sufficient argument as the time integral "ignores" what happens at a single point and the lecture notes should provide details in future update.

[5]Does this lead to loss of generality?

Now we take conditional expectation $\mathbb{E}^{t,x} := \mathbb{E}[\cdot|\mathcal{F}_t^X]$ on both sides of the last inequality, to get

$$\mathbb{E}^{t,x}\left[\int_t^\tau f^{\alpha_r}(r,X_r)\,dr + v(\tau,X_\tau)\right] \le v(t,x) - \varepsilon\mathbb{E}^{t,x}\left[\tau - t\right] .$$

We now can take the supremum over all controls $\alpha \in \mathcal{A}[t,\tau]$ to get

$$\sup_{\alpha\in\mathcal{A}}\mathbb{E}^{t,x}\left[\int_t^\tau f^{\alpha_s}(s,X_s)ds + v(\tau,X_\tau)\right] \le v(t,x) - \varepsilon\mathbb{E}^{t,x}\left[\tau - t\right] < 0\,.$$

But the Bellman principle states that:

$$v(t,x) = \sup_{\alpha\in\mathcal{U}}\mathbb{E}^{t,x}\left[\int_t^\tau f^{\alpha_s}(s,X_s)\,ds + v(\tau,X_\tau)\right] .$$

Hence we've obtained a contradiction and completed the proof. □

Theorem 3.8 (HJB verification)**.** *If, on the other hand, some u in $C^{1,2}([0,T)\times\mathbb{R}^d)$ satisfies* (3.3) *and we have that for all $(t,x)\in[0,T]\times\mathbb{R}^d$ there is some measurable function $a:[0,T]\times\mathbb{R}^d\to A$ such that*

$$a(t,x) \in \arg\max_{a\in A}\Big((L^a u)(t,x) + f^a(t,x)\Big)\,, \tag{3.4}$$

and if

$$dX_s^* = b\big(s,X_s^*,a(s,X_s^*)\,ds + \sigma\big(s,X_s^*,a(s,X_s^*)\,dW_s,\quad X_t^* = x$$

admits a unique solution, and if the process

$$t' \mapsto \int_t^{t'} \partial_x u\big(s,X_s^*\big)\sigma\big(s,X_s^*,a(s,X_s^*)\,dW_s \tag{3.5}$$

is a martingale in $t'\in[t,T]$, then

$$\alpha_s^* := a\big(s,X_s^*\big) \qquad s\in[t,T]$$

is optimal for problem (P) and $v(t,x) = u(t,x)$.

Proof. Let $\alpha_s^* = a(s,X_s^*)$. Apply Itô's formula to u and X^* to see that

$$\begin{aligned}
&\int_t^T f_s^{\alpha_s^*}\big(X_s^*\big)\,ds + g(X_T^*) - u(t,x) = \int_t^T f_s^{\alpha_s^*}\big(X_s^*\big)\,ds + u(T,X_T^*) - u(t,x)\\
&= \int_t^T \Big[\partial_t u(s,X_s^*) + L^{\alpha_s^*}(s,X_s^*)u(s,X_s^*) + f_s^{\alpha_s^*}\big(X_s^*\big)\Big]\,ds\\
&\quad + \int_t^T \partial_x u\big(s,X_s^*\big)\sigma\big(s,X_s^*,a(s,X_s^*)\,dW_s\\
&= \int_t^T \partial_x u\big(s,X_s^*\big)\sigma\big(s,X_s^*,a(s,X_s^*)\,dW_s\,,
\end{aligned}$$

since for all (t,x) it holds that

$$\sup_{a\in A}[L^a(t,x)u(t,x) + f^a(t,x)] = L^{a(t,x)}(t,x)u(t,x) + f^{a(t,x)}(t,x)\,.$$

Hence, as the stochastic integral is a martingale by assumption,

$$\mathbb{E}\Big[\int_t^T f_s^{\alpha_s^*}\big(X_s^*\big)\,ds + g(X_T^*) - u(t,x)\Big] = 0\,.$$

So

$$u(t,x) = \mathbb{E}\Big[\int_t^T f_s^{\alpha_s^*}\big(X_s^*\big)\,ds + g(X_T^*)\Big] \leq \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_t^T f_s^{\alpha_s}\big(X^{t,x,\alpha}\big)\,ds + g(X^{t,x,\alpha})\Big] = v(t,x)\,. \tag{3.6}$$

The same calculation with an arbitrary $\alpha \in \mathcal{A}$ and Itô formula applied to u and $X^{t,x,\alpha}$ leads to

$$\mathbb{E}\Big[\int_t^T f_s^{\alpha_s}\big(X_s^{t,x,\alpha}\big)\,ds + g(X_T^{t,x,\alpha}) - u(t,x)\Big] \leq 0\,.$$

Hence for any $\varepsilon > 0$ we have

$$v(t,x) \leq \varepsilon + \mathbb{E}\left[\int_t^T f_s^{\alpha_s^\varepsilon}\big(X_s^{t,x,\alpha^\varepsilon}\big)\,ds + g(X_T^{t,x,\alpha^\varepsilon})\right] \leq u(t,x)\,.$$

Hence $v(t,x) \leq u(t,x)$ and with (3.6) we can conclude that $v = u$.

Let

$$M_s := \int_t^s f_r^{\alpha_r^*}\big(X_r^*\big)\,dr + u\big(s,X_s^*\big)\,.$$

We would first like to see that this is a martingale. To that end, let us apply Itô's formula to v and X^* to see that

$$\begin{aligned}
dM_s &= f_s^{\alpha_s^*}\big(X_s^*\big)\,ds + dv\big(s,X_s^*\big)\\
&= \Big[\partial_t v(s,X_s^*) + L^{\alpha_s^*}(s,X_s^*)v(s,X_s^*) + f_s^{\alpha_s^*}\big(X_s^*\big)\Big]\,ds + \partial_x v\big(s,X_s^*\big)\sigma\big(s,X_s^*,a(s,X_s^*\big)\,dW_s\\
&= \partial_x v\big(s,X_s^*\big)\sigma\big(s,X_s^*,a(s,X_s^*\big)\,dW_s
\end{aligned}$$

since $v = u$ satisfies (3.3). By assumption this stochastic integral is a martingale and hence M is also a martingale. By Theorem 3.6 α^* must be an optimal control process. □

Theorem 3.8 is referred as the *verification theorem.* This is key for solving the control problem: if we know the value function v, then the dynamic optimization problem turns into a of static optimization problems at each point (t,x). Recall that (3.4) is calculated pointwise over (t,x).

Exercise 3.9. Find the HJB equation for the following problem. Let $d = 1$, $U = [\sigma_0, \sigma^1] \subset (0,\infty)$, and $k \in \mathbb{R}$. The dynamics of X are given by

$$\frac{dX_s^\alpha}{X_s^\alpha} = k\,ds + \alpha_s\,dW_s,$$

and the value function is

$$v(t,x) = \sup_{\alpha\in\mathcal{A}[t,T]} \mathbb{E}[e^{k(t-T)}g\big(X_T^{\alpha,t,x}\big)] = -\inf_{\alpha\in\mathcal{A}[t,T]} \mathbb{E}[-e^{k(t-T)}g\big(X_T^{\alpha,t,x}\big)].$$

This can be interpreted as the pricing equation for an uncertain volatility model with constant interest rate k. The equation is called Black–Scholes–Barenblatt equation and the usual way to present this problem is through a maximization problem.

3.3 Solving control problems using the HJB equation and verification theorem

Theorem 3.7 provides an approach to find optimal solutions:

1. Solve the HJB equation (3.3) (this is typically done by taking a lucky guess and in fact is rarely possible with pen and paper).
2. Find the optimal Markovian control rule $a(t,x)$ calculating (3.4).
3. Solve the optimal control and its state process (u^*, X^*).
4. Verify the martingale condition.

This approach may end up with *failures*. Step one is to solve a fully non-linear second order PDE, that may not have a solution, may have a unique solution or many solutions. If we can prove before hand that the value function for (P) is v is $C^{1,2}$, then the HJB equation admits at least one solution according to Theorem 3.7. The question of uniqueness remains.

In step two, given u that solves (3.3), the problem is a static optimization problem. This is generally much easier to solve.

If we can reach step three, then this step heavily depends on functions b and σ, for which we usually check case by case.

Example 3.10 (Merton problem with power utility and no consumption)**.** This is the classic finance application. The problem can be considered with multiple risky assets but we focus on the situation from Section 1.1.

Recall that we have risk-free asset B_t, risky asset S_t and that our portfolio has wealth given by

$$dX_s = X_s(\nu_s(\mu - r) + r)\, ds + X_s \nu_s \sigma\, dW_s\,,\ \ s \in [t,T]\,,\ \ X_t = x > 0\,.$$

Here ν_s is the control and it describes the fraction of our wealth invested in the risky asset. This can be negative (we short the stock) and it can be more than one (we borrow money from the bank and invest more than we have in the stock).

We take $g(x) := x^\gamma$ with $\gamma \in (0,1)$ a constant. Our aim is to maximize $J^\nu(t,x) := \mathbb{E}^{t,x}\left[g(X_T^\nu)\right]$. Thus our value function is

$$v(t,x) = \sup_{\nu \in \mathcal{U}} J^\nu(t,x) = \sup_{\nu \in \mathcal{U}} \mathbb{E}^{t,x}\left[g(X_T^\nu)\right]\,.$$

This should satisfy the HJB equation (Bellman PDE)

$$\partial_t v + \sup_u \left[\frac{1}{2}\sigma^2 u^2 x^2 \partial_{xx} v + x[(\mu - r)u + r]\partial_x v\right] = 0 \ \text{ on } [0,T) \times (0,\infty)$$
$$v(T,x) = g(x) = x^\gamma \ \ \forall x > 0\,.$$

At this point our best chance is to guess what form the solution may have. We try $v(t,x) = \lambda(t)x^\gamma$ with $\lambda = \lambda(t) > 0$ differentiable and $\lambda(T) = 1$. This way at least the terminal condition holds. If this is indeed a solution then (using it in HJB) we have

$$\lambda'(t) + \sup_u \left[\frac{1}{2}\sigma^2 u^2 \gamma(\gamma - 1) + (\mu - r)\gamma u + r\gamma\right]\lambda(t) = 0 \ \ \forall t \in [0,T)\,,\ \ \lambda(T) = 1\,.$$

since $x^\gamma > 0$ for $x > 0$ and thus we were allowed to divide by this. Moreover we can calculate the supremum by observing that it is quadratic in u with negative leading term $(\gamma - 1)\gamma < 0$. Thus it is maximized when $u^* = \frac{\mu - r}{\sigma^2(1-\gamma)\gamma}$. The maximum itself is

$$\beta(t) := \frac{1}{2}\sigma^2(u^*)^2\gamma(\gamma-1) + (\mu - r)\gamma u^* + r\gamma\,.$$

Thus

$$\lambda'(t) = -\beta(t)\lambda(t)\,,\ \ \lambda(T) = 1 \quad \implies \lambda(t) = \exp\left(\int_t^T \beta(s)\,ds\right)\,.$$

Thus we think that the value function and the optimal control are

$$v(t,x) = \exp\left(\int_t^T \beta(s)\,ds\right)x^\gamma \ \text{ and } \ u^* = \frac{\mu - r}{\sigma^2(1-\gamma)\gamma}\,.$$

This now needs to be verified using Theorem 3.8. First we note that the SDE for X^* always has a solution if u^* is a constant.

Next we note that $\partial_x v(s, X_s^*) = \gamma\lambda(s)(X_s^*)^{\gamma-1}$. From Itô's formula

$$\begin{aligned} dX_s^{\gamma-1} &= (\gamma-1)X_s^{\gamma-2}dX_s + \frac{1}{2}(\gamma-1)(\gamma-2)X_s^{\gamma-3}dX_s dX_s \\ &= X_s^{\gamma-1}\left[(\gamma-1)[u^*(\mu-r)+r]\,ds + \frac{1}{2}(\gamma-1)(\gamma-2)u^*\sigma\,dW_s\right]\,. \end{aligned}$$

We can either solve this (like for geometric brownian motion) or appeal to Proposition 2.6 to see that a solution will have all moments uniformly bounded in time on $[0,T]$. Moreover $\lambda = \lambda(t)$ is continuous on $[0,T]$ and thus bounded and so

$$\int_0^T \mathbb{E}\left[\lambda^2(t)|(X_s^*)^{\gamma-1}|^2\right]\,ds < \infty$$

which means that the required expression is a true martingale. This completes verification and Theorem 3.8 gives the conclusion that v is indeed the value function and u^* is indeed the optimal control.

Example 3.11 (Linear-quadratic control problem)**.** This example is a classic engineering application. Note that it can be considered in multiple spatial dimensions but here we focus on the one-dimensional case for simplicity. The multi-dimensional version is e.g. in [10, Ch. 11].

We consider

$$dX_s = [H(s)X_s + M(s)\alpha_s]\,ds + \sigma(s)dW_s\,, s \in [t,T]\,, X_t = x\,.$$

Our aim is to maximize

$$J^\alpha(t,x) := \mathbb{E}^{t,x}\left[\int_t^T (C(s)X_s^2 + D(s)\alpha_s^2)\,ds + RX_T^2\right]\,,$$

where $C = C(t) \le 0$, $R \le 0$ and $D = D(t) - \delta < 0$ are given and deterministic and bounded in t. The interpretation is the following: since we are losing money at rate C proportionally to X^2, our aim is to make X^2 as small as possible as fast as we can. However controlling X costs us at a rate D proportionally to the strength of control we apply.

The value function is $v(t,x) := \sup_\alpha J^\alpha(t,x)$.

Let us write down the Bellman PDE (HJB equation) we would expect the value function to satisfy:

$$\partial_t v + \sup_a \left[\frac{1}{2}\sigma^2 \partial_x^2 v + [H\,x + M\,a]\partial_x v + C\,x^2 + D\,a^2\right] = 0 \text{ on } [0,T)\times\mathbb{R}\,,$$
$$v(T,x) = Rx^2 \ \ \forall x \in \mathbb{R}\,.$$

Since the terminal condition is $g(x) = Rx^2$ let us try $v(t,x) = S(t)x^2 + b(t)$ for some differentiable S and b. We re-write the HJB equation in terms of S and b: (omitting time dependence in H, M, σ, C and D), for $(t,x) \in [0,T) \times \mathbb{R}$,

$$S'(t)x^2 + b'(t) + \sigma^2 S(t) + 2H\,S(t)\,x^2 + C\,x^2 + \sup_a \left[2M\,a\,S(t)\,x + D\,a^2\right] = 0\,,$$
$$S(T) = R \text{ and } b(T) = 0\,.$$

For fixed t and x we can calculate $\sup_a[2M(t)aS(t)x + D(t)a^2]$ and hence write down the optimal control function $a^* = a^*(t,x)$. Indeed since $D < 0$ and since the expression is quadratic in a we know that the maximum is reached with $a^*(t,x) = -(D^{-1}M\,S)(t)x$.

We substitute a^* back in to obtain ODEs for $S = S(t)$ and $b = b(t)$ from the HJB equation. Hence

$$\left[S'(t) + 2H\,S(t) + C - D^{-1}M^2S^2(t)\right]x^2 + b'(t) + \sigma^2 S(t) = 0\,,$$
$$S(T) = R \text{ and } b(T) = 0\,.$$

We collect terms in x^2 and terms independent of x and conclude that this can hold only if

$$S'(t) = D^{-1}M^2S^2(t) - 2H\,S(t) - C\,, \ \ S(T) = R$$

and

$$b'(t) = -\sigma^2 S(t)\,, \ \ b(T) = 0\,.$$

The ODE for S is the *Riccati equation* which has unique solution for $S(T) = R$. We can obtain the expression for $b = b(t)$ by simply integrating:

$$b(T) - b(t) = -\int_t^T \sigma^2(r)S(r)\,dr\,.$$

Then

$$\alpha^*(t,x) = -(D^{-1}MS)(t)x \text{ and } v(t,x) = S(t)x^2 + \int_t^T \sigma^2(r)S(r)\,dr \tag{3.7}$$

and we see that the control function is measurable. We will now check conditions of Theorem 3.8. The SDE with the optimal control is

$$dX_s^* = \rho(s)X_s^*\,ds + \sigma(s)dW_s\,, \ \ s \in [t,T]\,, \ \ X_t^* = x\,,$$

where $\rho := H + D^{-1}\,M^2\,S$. This is deterministic and bounded in time. The SDE thus satisfies the Lipschitz conditions and it has a unique strong solution for any t, x.

Since $\partial_x v(r, X_r^*) = 2S(r)X_s^*$, since $\sup_{r\in[t,T]} S^2(r)$ is bounded (continuous function on a closed interval) and since $\sup_{r\in[t,T]} \mathbb{E}[|X_r^*|^2] < \infty$ (moment estimate for SDEs with Lipschitz coefficients) we get

$$\mathbb{E}\int_t^T |S(r)|^2 |X_s^*|^2 \, dr < \infty$$

and thus conclude that $s \mapsto \int_t^s S(r)X_r^* \sigma(r)\, dW_r$ is a martingale. Thus Theorem 3.8 tells us that the value function and control given by (3.7) are indeed optimal.

3.4 Exercises

Exercise 3.12 (Optimal liquidation with no permanent market impact)**.** Solve the optimal liquidation problem of Section 1.2 in the case $\lambda = 0$ (i.e. there is no permanent price impact of our trading on the market price).

Exercise 3.13 (Unattainable optimizer)**.** Here is a simple example in which no optimal control exists, in a finite horizon setting, $T \in (0, \infty)$. Suppose that the state equation is

$$dX_s = \alpha_s \, ds + dW_s \quad s \in [t, T]\,, \quad X_t = x \in \mathbb{R}.$$

A control α is admissible ($\alpha \in \mathcal{A}$) if: α takes values in $\mathbb{R}$, is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, and $\mathbb{E}\int_0^T \alpha_s^2 \, ds < \infty$.

Let $J(t, x, \alpha) := \mathbb{E}[|X_T^{t,x,\alpha}|^2]$. The value function is $v(t, x) := \inf_{\alpha\in\mathcal{A}} J(t, x, \alpha)$. Clearly $v(t, x) \geq 0$.

i) Show that for any $t \in [0, T]$, $x \in \mathbb{R}$, $\alpha \in \mathcal{A}$ we have $\mathbb{E}[|X_T^{t,x,\alpha}|^2] < \infty$.

ii) Show that if $\alpha_t := -cX_t$ for some constant $c \in (0, \infty)$ then $\alpha \in \mathcal{A}$ and

$$J^\alpha(t, x) = J^{cX}(t, x) = \frac{1}{2c} - \frac{1 - 2cx^2}{2c} e^{-2c(T-t)}.$$

Hint: with such an α, the process X is an Ornstein-Uhlenbeck process, see Exercise 1.12.

iii) Conclude that $v(t, x) = 0$ for all $t \in [0, T)$, $x \in \mathbb{R}$.

iv) Show that there is no $\alpha \in \mathcal{A}$ such that $J(t, x, \alpha) = 0$. *Hint:* Suppose that there is such a α and show that this leads to a contradiction.

v) The associated HJB equation is

$$\partial_t v + \inf_{a\in\mathbb{R}} \left\{ \frac{1}{2}\partial_{xx} v + a\partial_x v \right\} = 0, \quad \text{on } [0, T) \times \mathbb{R}.$$
$$v(T, x) = x^2\,.$$

Show that there is no value $\alpha \in \mathbb{R}$ for which the infimum is attained.

Conclusions from Exercise 3.13: The value function $v(t, x) = \inf_{\alpha\in\mathcal{A}} J(t, x, \alpha)$ satisfies $v(t, x) = 0$ for all $(t, x) \in [0, T] \times \mathbb{R}$ but there is no admissible control α which attains the v (i.e. there is no $\alpha^* \in \mathcal{A}$ such that $v(t, x) = J(t, x, \alpha^*)$).

The goal in this problem is to bring the state process as close as possible to zero at the terminal time T. However, as defined above, there is no cost of actually controlling the system. We can set α arbitrarily large without any negative consequences. From a modelling standpoint, there is often a trade-off between costs incurred in applying control and our overall objective. Compare this with Example 3.11.

Exercise 3.14 (Merton problem with exponential utility and no consumption)**.** We return to the portfolio optimization problem, see Section 1.1. Unlike in Example 3.10 we consider the utility function $g(x) := -e^{-\gamma x}$, $\gamma > 0$ a constant. We will also take $r = 0$ for simplicity and assume there is no consumption ($C = 0$). With X_t denoting the wealth at time time t we have the value function given by

$$v(t,x) = \sup_{\pi \in \mathcal{U}} \mathbb{E}\left[g\left(X_T^{\pi,t,x,}\right)\right] .$$

i) Write down the expression for the wealth process in terms of π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$.

ii) Write down the HJB equation associated to the optimal control problem. Solve the HJB equation by inspecting the terminal condition and thus suggesting a possible form for the solution. Write down the optimal control explicitly.

iii) Use verification theorem to show that the solution and control obtained in previous step are indeed the value function and optimal control.

Exercise 3.15 ([12]*p252, Prob. 4.8)**.** Solve the problem

$$\max_{\nu} \mathbb{E}\big[- \int_0^T \nu^2(t) \frac{e^{-X(t)}}{2}\, dt + e^{X(T)} \big],$$

where ν takes values in $\mathbb{R}$, subject to $dX(t) = \nu(t) e^{-X(t)}\, dt + \sigma\, dW(t)$, $X(0) = x_0 \in \mathbb{R}$, $\sigma \in (0,\infty)$, σ, x_0 are fixed numbers.

Hint: Try a solution of the HJB equation of the form $v(t,x) = \phi(t)e^x + \psi(t)$.

For more exercises, see [12, Exercise 4.13, 4.14, 4.15].

3.5 Solutions to Exercises

Solution (to Exercise 3.12)**.** From Theorem 3.7 we can write down the HJB equation for $V = V(t, S, q)$:

$$\partial V_t + \frac{1}{2}\sigma^2 \partial_{SS} V + \sup_{a \in A} \{(S - \kappa a)a - a\partial_q V\} = 0 \text{ on } [0, T) \times \mathbb{R} \times \mathbb{R}\,, \tag{3.8}$$

with the terminal condition

$$V(T, q, S) = qS - \theta q^2 \quad \forall (q, S) \in \mathbb{R} \times \mathbb{R}\,. \tag{3.9}$$

Next we note that

$$a \mapsto (S - \partial_q V)a - \kappa a^2 \text{ attains its maximum with } a^* = \frac{S - \partial_q V}{2\kappa}\,.$$

Hence the HJB equation (3.8) becomes

$$\partial V_t + \frac{1}{2}\sigma^2 \partial_{SS} V + \frac{1}{4\kappa}(S - \partial_q V)^2 = 0 \text{ on } [0, T) \times \mathbb{R} \times \mathbb{R}\,. \tag{3.10}$$

We now have to "guess" an ansatz for V and, observing the similarities here with the linear-quadratic case of Example 3.11, we try

$$V(t, q, S) = \beta(t)qS + \gamma(t)q^2\,.$$

With $\beta(T) = 1$ and $\gamma(T) = -\theta$ we have the terminal condition (3.9) satisfied. To proceed we calculate the partial derivatives of V and substitute those into the HJB (3.10) to obtain

$$\beta'(t)qS + \gamma'(t)q^2 + \frac{1}{4\kappa}\big[S - \beta(t)S + 2\gamma(t)q\big]^2 = 0 \quad \forall (t, q, S) \in [0, T) \times \mathbb{R} \times \mathbb{R}\,. \tag{3.11}$$

This is equivalently

$$\begin{aligned}
&\beta'(t)qS + \gamma'(t)q^2 \\
&\qquad + \frac{1}{4\kappa}\big[S^2 - 2\beta(t)S^2 + 2\gamma(t)qS + \beta(t)^2S^2 - 4\beta(t)\gamma(t)qS + 2\gamma(t)qS + 4\gamma(t)^2q^2\big]^2 \\
&= 0 \quad \forall (t, q, S) \in [0, T) \times \mathbb{R} \times \mathbb{R}\,.
\end{aligned}$$

This has to hold for all S^2, q^2 and qS. Starting with S^2 terms we get that

$$1 - 2\beta(t) + \beta(t)^2 = 0 \quad \forall t \in [0, T)$$

which can only be true if $\beta(t) = 1$ (since $\beta(T)$ must be 1 and we need β differentiable). Considering now the qS term we have ($\beta'(t) = 0$ since we now have $\beta(t) = 1$):

$$2\gamma(t) - 4\gamma(t) + 2\gamma(t) = 0 \quad \forall t \in [0, T)$$

which holds regardless of choice of γ. Finally we have the q^2 terms which lead to

$$\gamma'(t) + \frac{1}{\kappa}\gamma(t)^2 = 0 \quad \forall t \in [0, T)\,.$$

We recall the terminal condition $\gamma(T) = -\theta$ and solve this ODE[6] thus obtaining

$$\gamma(t) = -\left(\frac{1}{\theta} + \frac{1}{\kappa}(T - t)\right)^{-1}\,.$$

This fully determines the value function

$$V(t, q, S) = qS + \gamma(t)q^2$$

and the optimal control

$$a^*(t, q, S) = -\frac{1}{\kappa}\gamma(t)q\,.$$

We note that the optimal control is independent of S and in fact the entire control problem does not depend on the volatility parameter σ.

Solution (to Exercise 3.13)**.**

[6] You can for instance recall that if $f(t) = -\frac{1}{t}$ then $f'(t) = \frac{1}{t^2}$ and so $f'(t) = f(t)^2$. Manipulating expressions of this type can lead you to the correct solution.

i) We use the fact that $\mathbb{E}\int_0^T \alpha_r^2\,dr < \infty$ for admissible control. We also use that $(a+b)^2 \leq 2a^2+2b^2$. Then for, any $s \in [t,T]$,

$$\mathbb{E}[X_s^2] \leq 4x^2 + 4\mathbb{E}\left(\int_t^s \alpha_r\,dr\right)^2 + 2\mathbb{E}(W_s - W_t)^2\,.$$

With Hölder's inequality we get

$$\mathbb{E}[X_s^2] \leq 4x^2 + 4(s-t)^{1/2}\mathbb{E}\int_t^s \alpha_r^2\,dr + 2(s-t) \leq c_T\left(1 + x^2 + \mathbb{E}\int_0^T \alpha_r^2\,dr\right) < \infty\,. \tag{3.12}$$

ii) Substitute $\alpha_s = -cX_s$. The Ornstein-Uhlenbeck SDE, see Exercise 1.12, has solution

$$X_T = e^{-c(T-t)}x + \int_t^T e^{-c(T-t)}\,dW_r\,.$$

We square this, take expectation (noting that the integrand in the stochastic integral is deterministic and square integrable):

$$\mathbb{E}X_T^2 = e^{-2c(T-t)}x^2 + \mathbb{E}\left(\int_t^T e^{-c(T-t)}\,dW_r\right)^2\,.$$

With Itô's isometry we get

$$\mathbb{E}X_T^2 = e^{-2c(T-t)}x^2 + \int_t^T e^{-2c(T-t)}\,dr\,.$$

Now we just need to integrate to obtain $J^\alpha(t,x) = J^{cX}(t,x) = \mathbb{E}X_T^2$.

iii) We know that $v(t,x) \geq 0$ already. Moreover

$$v(t,x) = \inf_{\alpha\in\mathcal{U}} J^\alpha(t,x) \leq \lim_{c\nearrow\infty} J^{cX}(t,x) = \lim_{c\nearrow\infty}\left[\frac{1}{2c} - \frac{1-2cx^2}{2c}e^{-2c(T-t)}\right] = 0\,.$$

iv) Assume that an optimal $\alpha^* \in \mathcal{U}$ exists so that $\mathbb{E}[X_T^{\alpha^*,t,x}] = J^{\alpha^*}(t,x) = 0$ for any $t < T$ and any x. We will show this leads to contradiction.

First of all, we can calculate using Itô formula that

$$dX_s^* = 2X_s^*\alpha_s^*\,ds + 2X_s^*\,dW_s + ds\,.$$

Hence

$$0 = \mathbb{E}[(X_T^*)^2] = x^2 + 2\mathbb{E}\int_t^T (X_s^*\alpha_s^* + 1)\,ds + \mathbb{E}\int_t^T X_s^*\,dW_s\,.$$

But since α^* is admissible we have $\int_t^T \mathbb{E}(X_s^*)^2\,ds < \infty$ due to (3.12). This means that the stochastic integral is a martingale and hence its expectation is zero. We now use Fatou's lemma and take the limit as $t \nearrow T$. Then

$$-x^2 = 2\liminf_{t\nearrow T}\mathbb{E}\int_t^T (X_s^*\alpha_s^* + 1)\,ds \geq 2\mathbb{E}\left[\liminf_{t\nearrow T}\int_t^T (X_s^*\alpha_s^* + 1)\,ds\right] = 0\,.$$

So $-x^2 \geq 0$. This cannot hold for all $x \in \mathbb{R}$ and so we have contradiction.

v) If $\partial_x v(t,x) \neq 0$, then $a = \pm\infty$. If $\partial_x V(t,x) = 0$, then a is undefined. One way or another there is no real number attaining the infimum.

Solution (to Exercise 3.14)**.** The wealth process (with the control expressed as π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$), is given by

$$dX_s = \pi_s\mu\,ds + \pi_s\sigma\,dW_s\,,\quad s \in [t,T]\,,\quad X_t = x > 0\,. \tag{3.13}$$

The associated HJB equation is

$$\partial_t v + \sup_{p\in\mathbb{R}}\left[\frac{1}{2}p^2\sigma^2\partial_{xx}v + p\,\mu\,\partial_x v\right] = 0 \text{ on } [0,T)\times\mathbb{R},$$
$$v(T,x) = g(x)\ \ \forall x \in \mathbb{R}\,.$$

We make a guess that $v(t,x) = \lambda(t)g(x) = -\lambda(t)e^{-\gamma x}$ for some differentiable function $\lambda = \lambda(t) \geq 0$. Then, since we can divide by $-e^{-\gamma x} \neq 0$ and since we can factor out the non-negative $\lambda(t)$, the HJB equation will hold provided that

$$\lambda'(t) + \sup_{p\in\mathbb{R}} \left[-\frac{1}{2}p^2\sigma^2\gamma^2 + p\,\mu\,\gamma \right] \lambda(t) = 0 \text{ on } [0,T),\ \lambda(T) = 1.$$

The supremum is attained for $p^* = \frac{\mu}{\sigma^2\gamma}$ since the expression we are maximizing is quadratic in p with negative leading order term. Thus $\lambda'(t) + \beta(t)\lambda(t) = 0$ and $\lambda(T) = 1$ with

$$\beta(t) := -\frac{1}{2}(p^*)^2\sigma^2\gamma^2 + p^*\,\mu\,\gamma = -\frac{1}{2}\mu\gamma + \frac{\mu^2}{\sigma^2}\,.$$

We can solve the ODE for λ to obtain

$$\lambda(t) = e^{\int_t^T \beta(r)\,dr}$$

and hence our candidate value function and control are

$$v(t,x) = e^{\int_t^T \beta(r)\,dr} g(x) \text{ and } p^* = \frac{\mu}{\sigma^2\gamma}\,.$$

We now need to use Theorem 3.8 to be able to confirm that these are indeed the value function and optimal control.

First of all the solution for optimal X^* always exists since we just need to integrate in the expression (3.13) taking $\pi_t := p^*$. We note that the resulting process is Gaussian.

Now $\partial_x v(s, X_s^*) = \lambda(t)\,\gamma e^{-\gamma X_s^*}$. We can now use what we know about moment generating functions of normal random variables to conclude that

$$\int_t^T \lambda(s)^2\, e^{-2\gamma X_s^*}\,ds < \infty.$$

The process

$$\bar{t} \mapsto \int_t^{\bar{t}} \lambda(s)\, e^{-\gamma X_s^*}\,dW_s$$

is thus a true martingale and the verification is complete.

Solution (to Exercise 3.15)**.**

$$\psi(t) = 0, \qquad \phi(t) = \frac{\sigma^2}{Ce^{\sigma^2 t/2} - 1}, \qquad C = (1+\sigma^2)e^{-\sigma^2 T/2}.$$

4 Pontryagin maximum principle and backward stochastic differential equations

In the previous part, we developed the dynamic programming theory for the stochastic control problem with Markovian system.

We introduce another approach called maximum principle, originally due to Pontryagin in the deterministic case. We will also study this approach to study the control problem (P).

4.1 Backward Stochastic Differential Equations (BSDEs)

For a deterministic differential equation

$$\frac{dx(t)}{dt} = b(t, x(t)) \;\; t \in [0,T]\,, \;\; x(T) = a$$

we can reverse the time by changing variables. Let $\tau := T - t$ and $y(\tau) = x(t)$. Then we have

$$\frac{dy(\tau)}{d\tau} = -b(T-\tau, y(\tau)) \;\; \tau \in [0,T]\,, \;\; y(0) = a\,.$$

So the *backward* ODE is equivalent to a *forward* ODE.

The same argument would fail for SDEs since the time-reversed SDE would not be adapted to the appropriate filtration and the stochastic integrals will not be well defined.

Recall the martingale representation theorem (see Theorem A.24), which says any $\xi \in L^2_{\mathcal{F}_T}$ can be uniquely represented by

$$\xi = \mathbb{E}[\xi] + \int_0^T \phi_t \, dW_t\,.$$

If we define $M_t = \mathbb{E}[\xi] + \int_0^t \phi_s dW_s$, then M_t satisfies

$$dM_t = \phi_t \, dW_t\,, \;\; M_T = \xi\,.$$

This leads to the idea that a solution to a *backward* SDE must consist of two processes (in the case above M and ϕ).

Consider the backward SDE (BSDE)

$$dY_t = g_t\big(Y_t, Z_t\big)\, dt + Z_t \, dW_t, \qquad Y(T) = \xi\,.$$

We shall give a few examples when this has explicit solution.

Example 4.1. Assume that $g = 0$. In this case, $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$ and Z is the process given by the martingale representation theorem.

Example 4.2. Assume that $g_t(y,z) = \gamma_t$. In this case, take $\hat{\xi} := \xi - \int_0^T \gamma_t \, dt$. We get the solution $(\hat{Y}, \hat{Z})$ to

$$d\hat{Y}_t = \hat{Z}_t \, dW_t\,, \;\; \hat{Y}_T = \hat{\xi}$$

as

$$\hat{Y}_t = \mathbb{E}\left[\hat{\xi}|\mathcal{F}_t\right] = \mathbb{E}\left[\xi - \int_0^T \gamma_t \, dt \middle| \mathcal{F}_t\right]$$

and we get Z from the martingale representation theorem. Then with $Y_t := \hat{Y}_t + \int_0^t \gamma_s\, ds$, $Z_t := \hat{Z}_t$ we have a solution (Y, Z) so in particular

$$Y_t = \mathbb{E}\left[\xi - \int_0^T \gamma_t\, dt \middle| \mathcal{F}_t\right] + \int_0^t \gamma_s\, ds = \mathbb{E}\left[\xi - \int_t^T \gamma_s\, ds \middle| \mathcal{F}_t\right].$$

Example 4.3. Assume that $g_t(y, z) = \alpha_t y + \beta_t z + \gamma_t$ and $\alpha = \alpha_t$, $\beta = \beta_t$, $\gamma = \gamma_t$ are adapted processes that satisfy certain integrability conditions (those will become clear). We will construct a solution using an exponential transform and a change of measure.

Consider a new measure $\mathbb{Q}$ given by the Radon–Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \beta_s^2\, ds - \int_0^T \beta_s\, dW_s\right)$$

and assume that $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$. Then, due to Girsanov's Theorem A.23, the process given by $W_t^{\mathbb{Q}} = W_t + \int_0^t \beta_s\, ds$ is a $\mathbb{Q}$-Wiener process. Consider the BSDE

$$d\bar{Y}_t = \bar{\gamma}_t\, dt + \bar{Z}_t\, dW_t^{\mathbb{Q}}, \quad \bar{Y}_T = \bar{\xi}, \tag{4.1}$$

where $\bar{\gamma}_t := \gamma_t \exp\left(-\int_0^t \alpha_s\, ds\right)$ and $\bar{\xi} := \xi \exp\left(-\int_0^T \alpha_s\, ds\right)$. We know from Example 4.2 that this BSDE has a solution $(\bar{Y}, \bar{Z})$ and in fact we know that

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-\int_0^T \alpha_s\, ds} - \int_t^T \gamma_s e^{-\int_0^s \alpha_r\, dr}\, ds \middle| \mathcal{F}_t\right].$$

We let $Y_t := \bar{Y}_t \exp\left(\int_0^t \alpha_s\, ds\right)$ and $Z_t := \bar{Z}_t \exp\left(\int_0^t \alpha_s\, ds\right)$. Now using the Itô product rule with (4.1) and the equation for $W^{\mathbb{Q}}$ we can check that

$$\begin{aligned} dY_t = d\left(\bar{Y}_t e^{\int_0^t \alpha_s\, ds}\right) &= \alpha_t Y_t\, dt + e^{\int_0^t \alpha_s\, ds}\, d\bar{Y}_t = \alpha_t Y_t\, dt + \gamma_t\, dt + Z_t\, dW_t^{\mathbb{Q}} \\ &= (\alpha_t Y_t + \beta_t Z_t + \gamma_t)\, dt + Z_t\, dW_t \end{aligned}$$

and moreover $Y_T = \xi$. In particular we get

$$Y_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-\int_t^T \alpha_s\, ds} - \int_t^T \gamma_s e^{-\int_t^s \alpha_r\, dr}\, ds \middle| \mathcal{F}_t\right]. \tag{4.2}$$

To get the solution as an expression in the original measure we need to use the Bayes formula for conditional expectation, see Proposition A.41. We obtain

$$Y_t = \frac{\mathbb{E}\left[\left(\xi e^{-\int_t^T \alpha_s\, ds} - \int_t^T \gamma_s e^{-\int_t^s \alpha_r\, dr}\, ds\right) e^{-\frac{1}{2}\int_0^T \beta_s^2\, ds - \int_0^T \beta_s\, dW_s} \middle| \mathcal{F}_t\right]}{\mathbb{E}\left[e^{-\frac{1}{2}\int_0^T \beta_s^2\, ds - \int_0^T \beta_s\, dW_s} \middle| \mathcal{F}_t\right]}.$$

Proposition 4.4 (Boundedness of solutions to linear BSDEs)**.** *Consider the linear backward SDE with $g_t(y, z) = \alpha_t y + \beta_t z + \gamma_t$. If α, β, γ and ξ are all bounded then the process Y in the solution pair (Y, Z) is bounded.*

Proof. This proof is left as exercise. □

Example 4.5 (BSDE and replication in the Black-Scholes market)**.** In a standard Black-Scholes market model we have a risk-free asset $dB_t = rB_t\,dt$ and risky assets

$$dS_t = \operatorname{diag}(\mu)S_t\,dt + \sigma S_t\,dW_t\,.$$

Here μ is the drift vector of the risky asset rate, σ is the volatility matrix.

Let π denote the cash amount invested in the risky asset and X the replicating portfolio value (so $X-\pi$ is invested in the risk-free asset). Then the self-financing property says that (interpreting $1/S$ to be $\operatorname{diag}(1/S_1,\ldots,1/S_m)$)

$$dX_t = \pi_t\frac{1}{S_t}\,dS_t + \frac{X_t - \sum_{i=1}^m \pi_t^{(i)}}{B_t}\,dB_t$$

i.e.

$$dX_t = \big[rX_t + \pi_t(\mu - r)\big]\,dt + \pi_t^\top \sigma\,dW_t\,.$$

We can define $Z_t = \sigma_t^\top \pi_t$ and if σ^{-1} exists then $\pi_t = (\sigma^\top)^{-1}Z_t = (\sigma^{-1})^\top Z_t$

$$dX_t = \big[rX_t + (\mu^\top - r)(\sigma^{-1})^\top Z_t\big]\,dt + Z_t\,dW_t.$$

For any payoff ξ at time T, the replication problem is to solve the BSDE given by this differential coupled with $X_T = \xi$. If $\xi \in L^2_{\mathcal{F}_T}$ the equation admits a unique square-integrable solution (X,Z). Hence the cash amount invested in the risky asset, required in the replicating portfolio is $\pi_t = (\sigma^{-1})^\top Z_t$, and the replication cost (contingent claim price) at time t is X_t.

We see that this is a BSDE with linear driver and so from Example 4.3 we have, see (4.2) that

$$X_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-r(T-t)}\middle|\mathcal{F}_t\right]\,,$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}|\sigma^{-1}(\mu-r)|^2T - (\mu^\top - r)(\sigma^{-1})^\top W_T}\,.$$

In other words we see that $\mathbb{Q}$ is the usual risk-neutral measure we get in Black–Scholes pricing.

A standard backward SDE (BSDE) is formulated as

$$dY_t = g_t\big(Y_t, Z_t\big)\,dt + Z_t\,dW_t, \qquad Y(T) = \xi, \tag{4.3}$$

where $g = g_t(\omega, y, z)$ must be such that $g_t(y,z)$ is at least $\mathcal{F}_t$-measurable for any fixed t, y, z. We will refer to g is called as the *generator* or *driver* of the Backward SDE.

Definition 4.6. Given $\xi \in L^2(\mathcal{F}_T)$ and a generator g, a pair of $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes (Y,Z) is called as a solution for (4.3) if

$$Y_t = \xi - \int_t^T g_s\big(Y_s, Z_s\big)\,ds - \int_t^T Z_s\,dW_s, \quad \forall t \in [0,T].$$

Theorem 4.7 (Existence and uniqueness for BSDEs)**.** *Suppose $g = g_t(y,z)$ satisfies*

(i) We have $g(0,0) \in \mathcal{H}$.

(ii) There exists a constant $L > 0$ such that

$$|g_t(y,z) - g_t(\overline{y},\overline{z})| \leq L(|y-\overline{y}| + |z-\overline{z}|) \;, a.s. \;\forall t \in [0,T]\,, \forall y, z, \bar{y}, \bar{z}\,.$$

Then for any $\xi \in L^2_{\mathcal{F}_T}$, there exists a unique $(Y,Z) \in \mathcal{H} \times \mathcal{H}$ solving the BSDE (4.3).

Recall that $\mathcal{H}$ is the space introduced in Definition A.18.

Proof. We consider the map $\Phi = \Phi(U,V)$ for (U,V) in $\mathcal{H}\times\mathcal{H}$. Given (U,V) we define $(Y,Z) = \Phi(U,V)$ as follows. Let $\hat{\xi} := \xi - \int_0^T g_s(U_s,V_s)\,ds$. Then

$$\begin{aligned}\mathbb{E}\int_0^T |g_s(U_s,V_s)|^2\,ds &\leq \mathbb{E}\int_0^T [2|g_s(U_s,V_s) - g_s(0,0)|^2 + 2|g_s(0,0)|^2]\,ds \\ &\leq \mathbb{E}\int_0^T [2L^2(|U_s|^2 + |V_s|^2) + 2|g_s(0,0)|^2]\,ds < \infty,\end{aligned} \tag{4.4}$$

since U and V and $g(0,0)$ are in $\mathcal{H}$. So $\hat{\xi} \in L^2(\mathcal{F}_T)$ and we know that for $\hat{Y}_t := \mathbb{E}[\hat{\xi}|\mathcal{F}_t]$ there is Z such that

$$d\hat{Y}_t = Z_t\,dW_t\,,\;\; \hat{Y}_T = \hat{\xi}\,.$$

Take $Y_t := \hat{Y}_t + \int_0^t g_s(U_s,V_s)\,ds$. Then

$$Y_t = \xi - \int_t^T g_s(U_s,V_s)\,ds - \int_t^T Z_s\,dW_s\,. \tag{4.5}$$

The next step is to show that $(U,V) \mapsto \Phi(U,V) = (Y,Z)$ described above is a contraction on an appropriate Banach space.

We will assume, for now, that $|\xi| \leq N$ and that $|g| \leq N$. We consider (U,V) and (U',V'). From these we obtain $(Y,Z) = \Phi(U,V)$ and $(Y',Z') = \Phi(U',V')$. We will write

$$(\bar{U},\bar{V}) := (U-U', V-V')\,,\;\; (\bar{Y},\bar{Z}) := (Y-Y', Z-Z')\,,\;\; \bar{g} := g(U,V) - g(U',V')\,.$$

Then

$$d\bar{Y}_s = \bar{g}_s\,ds + \bar{Z}_s dW_s$$

and with Itô formula we see that

$$d\bar{Y}_s^2 = 2\bar{Y}_s\bar{g}_s\,ds + 2\bar{Y}_s\bar{Z}_s\,dW_s + \bar{Z}_s^2 ds\,.$$

Hence, for some $\beta > 0$,

$$d(e^{\beta s}\bar{Y}_s^2) = e^{\beta s}\left[2\bar{Y}_s\bar{g}_s\,ds + 2\bar{Y}_s\bar{Z}_s\,dW_s + \bar{Z}_s^2\,ds + \beta\bar{Y}_s^2\,ds\right]\,.$$

Noting that, due to (4.5), we have $\bar{Y}_T = Y_T - Y'_T = 0$, we get

$$0 = \bar{Y}_0^2 + \int_0^T e^{\beta s}\left[2\bar{Y}_s\bar{g}_s + \bar{Z}_s^2 + \beta\bar{Y}_s^2\right]\,ds + \int_0^T 2e^{\beta s}\bar{Y}_s\bar{Z}_s\,dW_s\,.$$

Since $Z \in \mathcal{H}$ we have

$$\mathbb{E}\int_0^T 4e^{2\beta s}|\bar{Y}_s|^2|\bar{Z}_s|^2\,ds \leq e^{2\beta T}4N^2(1+T)^2\mathbb{E}\int_0^T |\bar{Z}_s|^2\,ds < \infty$$

and so, the stochastic integral being a martingale, we get

$$\mathbb{E}\int_0^T e^{\beta s}\left[\bar{Z}_s^2+\beta\bar{Y}_s^2\right]\,ds=-\mathbb{E}\bar{Y}_0^2-\mathbb{E}\int_0^T e^{\beta s}2\bar{Y}_s\bar{g}_s\,ds\leq 2\mathbb{E}\int_0^T e^{\beta s}|\bar{Y}_s||\bar{g}_s|\,ds\,.$$

Using the Lipschitz continuity of g and Young's inequality (with $\varepsilon=1/4$) we have

$$\begin{aligned}e^{\beta s}|\bar{Y}_s||\bar{g}_s|&\leq e^{\beta s}|\bar{Y}_s|L(|\bar{U}_s|+|\bar{V}_s|)\leq 2L^2e^{\beta s}|\bar{Y}_s|^2+\frac{1}{8}e^{\beta s}(|\bar{U}_s|+|\bar{V}_s|)^2\\&\leq 2L^2e^{\beta s}|\bar{Y}_s|^2+\frac{1}{4}e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,.\end{aligned}$$

We can now take $\beta=1+4L^2$ and we obtain

$$\mathbb{E}\int_0^T e^{\beta s}\left[\bar{Z}_s^2+\bar{Y}_s^2\right]\,ds\leq\frac{1}{2}\mathbb{E}\int_0^T e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,ds\,. \tag{4.6}$$

We now need to remove the assumption that $|\xi|\leq N$ and $|g|\leq N$. To that end consider $\xi^N:=-N\wedge\xi\vee N$ and $g^N:=-N\wedge g\vee N$ (so $|\xi^N|\leq N$ and $|g^N|\leq N$). We obtain Y^N, Z^N as before. Note that

$$Y_t=\mathbb{E}[\xi|\mathcal{F}_t]=\mathbb{E}\left[\lim_{N\to\infty}\hat{\xi}^N|\mathcal{F}_t\right]=\lim_{N\to\infty}Y_t^N$$

due to Lebesgue's dominated convergence for conditional expectations. Indeed, we have $|\hat{\xi}^N|\leq|\xi|+\int_0^T|g_s(U_s,V_s)|\,ds$ and this is in L^2 due to (4.4). Moreover

$$\begin{aligned}\mathbb{E}\int_0^T|Z_t^N-Z_t|^2\,dt&=\mathbb{E}\left(\int_0^T(Z_t^N-Z_t)\,dW_t\right)^2=\mathbb{E}\left(Y_T^N-Y_T+Y_0-Y_0^N\right)^2\\&\leq 2\mathbb{E}|Y_T^N-Y_T|^2+2\mathbb{E}|Y_0-Y_0^N|^2\to 0\ \text{ as } N\to\infty\end{aligned}$$

due to Lebesgue's dominated convergence theorem. Then from (4.6) be have, for each N,

$$\mathbb{E}\int_0^T e^{\beta s}\left[\bar{|}Z_s^N|^2+\bar{|}Y_s^N|^2\right]\,ds\leq\frac{1}{2}\mathbb{E}\int_0^T e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,ds\,.$$

But since the RHS is independent of N, we obtain (4.6) but now without the assumption that $|\xi|\leq N$ and $|g|\leq N$. Consider now the Banach space $(\mathcal{H}\times\mathcal{H},\|\cdot\|)$, with

$$\|(Y,Z)\|:=\mathbb{E}\int_0^T e^{\beta s}\left[Z_s^2+Y_s^2\right]\,ds\,.$$

From (4.6) we have

$$\|\Phi(U,V)-\Phi(U',V')\|\leq\frac{1}{2}\|(U,V)-(U',V')\|\,.$$

So the map $\Phi:\mathcal{H}\times\mathcal{H}\to\mathcal{H}\times\mathcal{H}$ is a contraction and due to Banach's Fixed Point Theorem there is a unique (Y^*,Z^*) which solves the equation $\Phi(Y^*,Z^*)=(Y^*,Z^*)$. Hence

$$Y_t^*=\xi-\int_t^T g_s(Y_s^*,Z_s^*)\,ds-\int_t^T Z_s^*\,dW_s$$

due to (4.5). □

Theorem 4.8. *Let (Y^1, Z^1) and (Y^2, Z^2) be solutions to BSDEs with generators and terminal conditions g^1, ξ^1 and g^2, ξ^2 respectively. Assume that $\xi^1 \leq \xi^2$ a.s. and that $g_t^2(Y_t^2, Z_t^2) \leq g_t^1(Y_t^1, Z_t^1)$ a.e. on $\Omega \times (0, T)$. Assume finally that the generators satisfy the assumption of Theorem 4.7 and $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$. Then $Y^1 \leq Y^2$.*

Proof. We note that the BSDE satisfied by $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$ is

$$d\bar{Y}_t = [g_t^2(Y_t^2, Z_t^2) - g_t^1(Y_t^1, Z_t^1)]\, dt + \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi} := \xi^2 - \xi^1\,.$$

This is

$$\begin{aligned} d\bar{Y}_t =& [g_t^2(Y_t^2, Z_t^2) - g_t^2(Y_t^1, Z_t^2) + g_t^2(Y_t^1, Z_t^2) - g_t^2(Y_t^1, Z_t^1) + g_t^2(Y_t^1, Z_t^1) - g_t^1(Y_t^1, Z_t^1)]\, dt \\ &+ \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi} \end{aligned}$$

which we can re-write as

$$d\bar{Y}_t = [\alpha_t \bar{Y}_t + \beta_t \bar{Z}_t + \gamma_t]\, dt + \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi}\,,$$

where

$$\alpha_t := \frac{g_t^2(Y_t^2, Z_t^2) - g_t^2(Y_t^1, Z_t^2)}{Y_t^2 - Y_t^1} \mathbb{1}_{Y_t^1 \neq Y_t^2}\,, \;\; \beta_t := \frac{g_t^2(Y_t^1, Z_t^2) - g_t^2(Y_t^1, Z_t^1)}{Z_t^2 - Z_t^1} \mathbb{1}_{Z_t^1 \neq Z_t^2}$$

and where

$$\gamma_t := g_t^2(Y_t^1, Z_t^1) - g_t^1(Y_t^1, Z_t^1)\,.$$

Due to the Lipschitz assumption on g^2 we get that α and β are bounded and since Y^i, Z^i are in $\mathcal{H}$ we get that $\gamma \in \mathcal{H}$. Thus we have an affine BSDE for $(\bar{Y}, \bar{Z})$ and the conclusion follows from (4.2) since we get

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\bar{\xi} e^{-\int_t^T \alpha_s\, ds}}_{\geq 0} - \underbrace{\int_t^T \gamma_s e^{-\int_t^s \alpha_r\, dr}\, ds}_{\leq 0} \middle| \mathcal{F}_t \right] \geq 0$$

from the assumptions that $\xi^1 \leq \xi^2$ a.s. and that $g_t^2(Y_t^2, Z_t^2) \leq g_t^1(Y_t^1, Z_t^1)$ a.e. □

4.2 Pontryagin's Maximum Principle

We now return to the optimal control problem (P). Recall that given running gain f and terminal gain g our aim is to optimally control

$$dX_t^\alpha = b_t(X_t, \alpha_t)\, dt + \sigma_t(X_t, \alpha_t)\, dW_t, \;\; t \in [0, T]\,, \;\; X_0^\alpha = x,$$

where $\alpha \in \mathcal{U}$ and we assume that Assumption 2.9 holds. Recall that by optimally controlling the process we mean a control which will maximize

$$J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \alpha_t)\, dt + g(X_T^\alpha) \right]$$

over $\alpha \in \mathcal{U}$. Unlike in Chapter 3 we can consider the process starting from time 0 (because we won't be exploiting the Markov property of the SDE) and unlike in Chapter 3 we will assume that A is a subset of $\mathbb{R}^m$.

We define the *Hamiltonian* $H : [0, T] \times \mathbb{R}^d \times A \times \mathbb{R}^d \times \mathbb{R}^{d \times d'} \to \mathbb{R}$ of the system as

$$H_t(x, a, y, z) := b_t(x, a)\, y + \mathrm{tr}[\sigma_t^\top(x, a)\, z] + f_t(x, a)\,.$$

Assumption 4.9. Assume that $x \mapsto H_t(x, a, y, z)$ is differentiable for all a, t, y, z with derivative bounded uniformly in a, t, y, z. Assume that g is differentiable in x with the derivative having at most linear growth (in x).

Consider the *adjoint BSDEs* (one for each $\alpha \in \mathcal{U}$)

$$dY_t^\alpha = -\partial_x H_t(t, X_t, \alpha_t, Y_t^\alpha, Z_t^\alpha)\, dt + Z_t\, dW_t\,,\;\; Y_T^\alpha = \partial_x g(X_T^\alpha)\,.$$

Note that under Assumption 4.9 and 2.9

$$\mathbb{E}[|\partial_x g(X_T^\alpha)|^2] \leq \mathbb{E}[(K(1 + |X_T^\alpha|)^2] < \infty,$$

Hence, due to Theorem 4.7, the adjoint BSDEs have unique solutions (Y^α, Z^α).

We will now see that it is possible to formulate a sufficient optimality criteria based on the properties of the Hamiltonian and based on the adjoint BSDEs. This is what is known as the *Pontryagin's Maximum Principle.* Consider two control processes, $\alpha, \beta \in \mathcal{U}$ and the two associated controlled diffusions, both starting from the same initial value, labelled X^α, X^β. Then

$$J(\beta) - J(\alpha) = \mathbb{E}\left[\int_0^T \left[f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t)\right] dt + g(X_T^\beta) - g(X_T^\alpha)\right]\,.$$

We will need to assume that g is concave (equivalently assume $-g$ is convex). Then $g(x) - g(y) \geq \partial_x g(x)(x - y)$ and so (recalling what the terminal condition in our adjoint equation is)

$$\mathbb{E}\left[g(X_T^\beta) - g(X_T^\alpha)\right] \geq \mathbb{E}\left[(X_T^\beta - X_T^\alpha)\partial_x g(X_T^\beta)\right] = \mathbb{E}\left[(X_T^\beta - X_T^\alpha)Y_T^\beta\right]\,.$$

We use Itô's product rule and the fact that $X_0^\alpha = X_0^\beta$. Let us write $\Delta b_t := b_t(X_t^\beta, \beta_t) - b_t(X_t^\alpha, \alpha_t)$ and $\Delta\sigma_t := \sigma_t(X_t^\beta, \beta_t) - \sigma_t(X_t^\alpha, \alpha_t)$. Then we see that

$$\mathbb{E}\left[(X_T^\beta - X_T^\alpha)Y_T^\beta\right] \geq \mathbb{E}\Bigg[\int_0^T -(X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)\, dt$$
$$+ \int_0^T \Delta b_t\, Y_t^\beta\, dt + \int_0^T \operatorname{tr}\left[\Delta\sigma_t^\top Z_t^\beta\right] dt\Bigg]\,.$$

Note that we are missing some details here, because the second stochastic integral term that we dropped isn't necessarily a martingale. However with a stopping time argument and Fatou's Lemma the details can be filled in (and this is why we have an inequality). We also have that for all y, z,

$$f(t, X_t^\beta, \beta_t) = H_t(X_t^\beta, \beta_t, y, z) - b_t(X_t^\beta, \beta_t)y - \operatorname{tr}[\sigma_t^\top(X_t^\beta, \beta_t)z]\,,$$
$$f(t, X_t^\alpha, \alpha_t) = H_t(X_t^\alpha, \alpha_t, y, z) - b_t(X_t^\alpha, \alpha_t)y - \operatorname{tr}[\sigma_t^\top(X_t^\alpha, \alpha_t)z]$$

and so

$$f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t) = \Delta H_t - \Delta b_t Y_t^\beta - \operatorname{tr}(\Delta\sigma_t^\top Z_t^\beta)$$

where

$$\Delta H_t := H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) - H_t(X_t^\alpha, \alpha_t, Y_t^\beta, Z_t^\beta)\,.$$

Thus

$$\mathbb{E}\left[\int_0^T \left[f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t)\right] dt\right] = \mathbb{E}\left[\int_0^T \left[\Delta H_t - \Delta b_t Y_t^\beta - \operatorname{tr}(\Delta\sigma_t^\top Z_t^\beta)\right] dt\right]\,.$$

Altogether

$$J(\beta) - J(\alpha) \geq \mathbb{E}\left[\int_0^T \left[\Delta H_t - (X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)\right] \, dt\right]$$

If we now assume that $(x, a) \mapsto H_t(x, a, Y_t^\beta, Z_t^\beta)$ is differentiable and concave for any t, y, z then

$$\Delta H_t \geq (X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) + (\beta_t - \alpha_t)\partial_a H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)$$

and so

$$J(\beta) - J(\alpha) \geq \mathbb{E}\left[\int_0^T (\beta_t - \alpha_t)\partial_a H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) \, dt\right] .$$

Finally we assume that β_t is a control process which satisfies

$$H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) = \max_{a \in A} H_t(X_t^\beta, a, Y_t^\beta, Z_t^\beta) < \infty \text{ a.s. for almost all } t.$$

Then $J(\beta) \geq J(\alpha)$ for arbitrary α. In other words, such control β is optimal. Hence we have proved the following theorem.

Theorem 4.10 (Pontryagin's Maximum Principle)**.** *Let Assumptions 2.9 and 4.9 holds, let $\subset \mathbb{R}^m$. Let g be concave. Let $\beta \in \mathcal{U}$ and let X^β be the associated controlled diffusion and (Y^β, Z^β) the solution of the adjoint BDSE. If $\beta \in \mathcal{U}$ is such that*

$$H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) = \max_{a \in A} H_t(X_t^\beta, a, Y_t^\beta, Z_t^\beta) < \infty \text{ a.s. for almost all } t. \tag{4.7}$$

holds and if

$$(x, a) \mapsto H_t(x, a, Y_t^\beta, Z_t^\beta)$$

is differentiable and concave then $J(\beta) = \sup_\alpha J(\alpha)$ i.e. β is an optimal control.

We can see that the Pontryagin maximum principle gives us a sufficient condition for optimality.

Example 4.11 (Linear–quadratic control revisited)**.** Take W to be $\mathbb{R}^{d'}$-valued Wiener process and let the space where controls take values to be $A = \mathbb{R}^m$. Consider $X_t = X_t^{\alpha,x}$ taking values in $\mathbb{R}^d$ given by

$$dX_t = [L(t)X_t + M(t)\alpha_t] \, dt + \sigma(t) \, dW_t \text{ for } t \in [0, T] , \;\; X_0 = x ,$$

where $L = L(t) \in \mathbb{R}^{d\times d}, M = M(t) \in \mathbb{R}^{d\times m}$ and $\sigma = \sigma(t) \in \mathbb{R}^{d\times d'}$ are bounded, measurable, deterministic functions of t.

Further let $C = C(t) \in \mathbb{R}^{d\times d}$, $D = D(t) \in \mathbb{R}^{m\times m}$, $F = F(t) \in \mathbb{R}^{d\times m}$ be deterministic, integrable functions of t and $R \in \mathbb{R}^{d\times d}$ be such that C, D and R are symmetric, $C = C(t) \leq 0, R \leq 0$ and $D = D(t) \leq -\delta < 0$ with some constant $\delta > 0$. The aim will be to maximize

$$J^\alpha(x) := \mathbb{E}^{x,\alpha}\left[\int_0^T \left[X_t^\top C(t)X_t + \alpha_t^\top D(t)\alpha_t + 2X_t^\top F(t)\alpha_t\right] \, dt + X_T^\top R X_T\right]$$

over all adapted processes α such that $\mathbb{E}\int_0^T \alpha_t^2 \, dt < \infty$ (we will call these admissible).

The Hamiltonian is

$$H_t(x, a, y, z) = x^\top L(t) \, y + y^\top M(t) \, a + \operatorname{tr}\left[\sigma(t)^\top z\right] + x^\top C(t) \, x + a^\top D(t) \, a + 2x^\top F(t) \, a .$$

We see that as function of (a, x) it is a sum of linear and quadratic functions and hence differentiable. Moreover since $C \leq 0$ and $D < 0$ we see that it is concave.

We see that

$$\partial_x H_t(x, a, y, z) = L(t)y + 2C(t)x + 2F(t)a$$

and so the adjoint BSDE $(\hat{Y}, \hat{Z})$ for the optimal control $\hat{\alpha}$ is

$$d\hat{Y}_t = -\left[L(t)\hat{Y}_t + 2C(t)\hat{X}_t + 2F(t)\hat{\alpha}\right] dt + \hat{Z}_t \, dW_t \text{ for } t \in [0, T], \;\; \hat{Y}_T = 2R\hat{X}_T .$$

Note that $x \mapsto x^\top R x$ is concave (since $R \leq 0$) and so the Pontryagin's maximum principle applies. If $\hat{\alpha}$ is the optimal control, $\hat{X}$ is the associated diffusion and $(\hat{Y}, \hat{Z})$ is the solution to the adjoint BSDE for $\hat{\alpha}$ then the maximum principle says that

$$H_t(\hat{X}_t, \hat{\alpha}_t, \hat{Y}_t, \hat{Z}_t) = \max_{a \in \mathbb{R}} H_t(\hat{X}_t, a, \hat{Y}_t, \hat{Z}_t) .$$

In this case the maximum is achieved when (Hamiltonian is quadratic in a with negative leading coefficient so we just differentiate w.r.t. a and see for which value this is 0):

$$0 = M(t)^\top \hat{Y}_t + 2D(t)\, a + 2F(t)^\top \hat{X}_t$$

i.e.

$$\hat{\alpha}_t = -\frac{1}{2} D(t)^{-1} \left(M(t)^\top \hat{Y}_t + 2F(t)^\top \hat{X}_t \right) .$$

Inspecting the terminal condition for the adjoint BSDE leads us to "guess" that we should have $\hat{Y}_t = 2S(t)\hat{X}_t$ for some $S \in C^1([0, T]; \mathbb{R}^{d\times d})$ with $S(T) = R$. We rewrite the optimal control with our guess for $\hat{Y}$:

$$\hat{\alpha}_t = -D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \hat{X}_t$$

and we can also write the optimally controlled SDE:

$$d\hat{X}_t = \left\{ L(t) + M(t) \left[-D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \right] \right\} \hat{X}_t \, dt + \sigma(t)\, dW_t . \tag{4.8}$$

Since our guess is that $\hat{Y}_t = 2S(t)\hat{X}_t$ we have, due to Itô's formula

$$\begin{aligned} d\hat{Y}_t &= 2S'(t)\hat{X}_t \, dt + 2S(t)\, d\hat{X}_t \\ &= 2S'(t)\hat{X}_t \, dt + 2S(t) \left\{ L(t) + M(t) \left[-D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \right] \right\} \hat{X}_t \, dt \\ &\quad + 2S(t)\sigma(t)\, dW_t . \end{aligned}$$

On the other hand the adjoint equation for $\hat{Y}$ gives

$$d\hat{Y}_t = -2 \left[L(t)S(t) + C(t) - F(t)D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \right] \hat{X}_t \, dt + \hat{Z}_t \, dW_t .$$

Since both must hold we get that $\hat{Z}_t = 2S(t)\sigma(t)$ and that

$$\begin{aligned} S'(t) + S(t)L(t) + S(t)M(t) \left[-D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \right] \\ = -L(t)S(t) - C(t) + F(t)D(t)^{-1} \left(M(t)^\top S(t) + F(t)^\top \right) \end{aligned}$$

so that for $t \in [0,T]$

$$S'(t) = [S(t)M(t) + F(t)]D(t)^{-1}\left(M(t)^\top S(t) + F(t)^\top\right) - L(t)S(t) - S(t)L(t) - C(t)$$

with $S(T) = R$.

The equation (4.8) for $\hat{X}$ is linear and clearly has unique solution and all the moments are bounded.

We observe (recalling $\hat{Y}_t = 2S(t)\hat{X}_t$) that

$$2\hat{X}_T^\top R\hat{X}_T = 2\hat{X}_T^\top S(T)\hat{X}_T = \hat{X}_T^\top \hat{Y}_T = \hat{X}_T^\top \hat{Y}_T - \hat{X}_0^\top \hat{Y}_0 + \hat{X}_0^\top \hat{Y}_0 = \int_0^T d(\hat{X}_t^\top \hat{Y}_t) + 2x^\top S(0)x. \tag{4.9}$$

Let us write $\psi(t) := -D(t)^{-1}\left(M(t)^\top S(t) + F(t)^\top\right)$, so that

$$\begin{aligned} d\hat{X}_t &= [L(t) + M(t)\psi(t)]\,\hat{X}_t\,dt + \sigma(t)\,dW_t\,, \\ d\hat{Y}_t &= -2\,[L(t)S(t) + C(t) + F(t)\psi(t)]\,\hat{X}_t\,dt + 2S(t)\sigma\,dW_t\,. \end{aligned}$$

Moreover

$$\begin{aligned} \frac{1}{2}d(\hat{X}_t^\top \hat{Y}_t) =& \frac{1}{2}\left(\hat{X}_t^\top S(t)d\hat{X}_t + \hat{X}_t^\top d\hat{Y}_t + d(\hat{X}_t^\top)d\hat{Y}_t\right) \\ =& \hat{X}_t^\top S(t)L(t)\hat{X}_t\,dt + \hat{X}_t^\top S(t)M(t)\psi(t)\hat{X}_t\,dt + \hat{X}_t^\top S(t)\sigma(t)\,dW_t \\ & - \hat{X}_t^\top L(t)S(t)\hat{X}_t\,dt - \hat{X}_t^\top C(t)\hat{X}_t - \hat{X}_t^\top F(t)\psi(t)\hat{X}_t \\ & + \hat{X}_t^\top S(t)\sigma(t)\,dW_t + \mathrm{tr}[\sigma(t)(S(t)\sigma(t))^\top]\,dt\,. \end{aligned}$$

Hence,[7]

$$\begin{aligned} \frac{1}{2}d(\hat{X}_t^\top \hat{Y}_t) =& + \hat{X}_t^\top S(t)M(t)\psi(t)\hat{X}_t\,dt \\ & - \hat{X}_t^\top C(t)\hat{X}_t - \hat{X}_t^\top F(t)\psi(t)\hat{X}_t \\ & + 2\hat{X}_t^\top S(t)\sigma(t)\,dW_t + \mathrm{tr}[\sigma(t)(S(t)\sigma(t))^\top]\,dt\,. \end{aligned} \tag{4.10}$$

We also have

$$J^{\hat{\alpha}}(x) = \mathbb{E}\left[\int_0^T \left(\hat{X}_t^\top C(t)\hat{X}_t + \hat{\alpha}_t^\top D(t)\hat{\alpha}_t + 2X_t^\top F(t)\hat{\alpha}_t\right)\,dt + R\hat{X}_T^2\right]. \tag{4.11}$$

Noting that $\alpha_t^\top D(t)\hat{\alpha}_t = \hat{X}_t^\top (M(t)^\top S(t) + F(t)^\top)^\top \psi(t)\hat{X}_t$ and substituting (4.10) into (4.9) and using this in (4.11) we see that most terms cancel and hence

$$J^{\hat{\alpha}}(x) = \mathbb{E}\left[\int_0^T \mathrm{tr}\left[\sigma(t)(S(t)\sigma(t))^\top\right]\,dt + \int_0^T 2\hat{X}_t^\top S(t)\sigma(t)\,dW_t + x^\top S(0)\,x\right].$$

Since the solution of the SDE for $\hat{X}$ has all moments bounded we have

$$\mathbb{E}\int_0^T 4|S(t)|^2|\sigma(t)|^4|\hat{X}_t|^2\,dt \le N\int_0^T \mathbb{E}|\hat{X}_t|^2\,dt \le N_T < \infty\,.$$

The stochastic integral is thus a martingale and so

$$v(x) = J^{\hat{\alpha}}(x) = x^\top S(0)x + \int_0^T \mathrm{tr}\left[\sigma(t)(S(t)\sigma(t))^\top\right]\,dt\,.$$

[7] We are using that $x^\top ABx = x^\top BAx$.

Example 4.12 (Minimum variance for given expected return)**.** We consider the simplest possible model for optimal investment: we have a risk-free asset B with evolution given by $dB_t = rB_t\,dt$ and $B_0 = 1$ and a risky asset S with evolution given by $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ with S_0 given. For simplicity we assume that σ, μ, r are given constants, $\sigma \neq 0$ and $\mu > r$. The value of a portfolio with no asset injections / consumption is given by $X_0 = x$ and

$$dX_t^\alpha = \frac{\alpha_t}{S_t}\,dS_t + \frac{X_t - \alpha_t}{B_t}\,dB_t\,,$$

where α_t represents the amount invested in the risky asset. Then

$$dX_t^\alpha = (rX_t + \alpha_t(\mu - r))\;dt + \sigma\alpha_t\,dW_t\,. \tag{4.12}$$

Given a desired return $m > 0$ we aim to find a trading strategy which would minimize the variance of the return (in other words a strategy that gets as close to the desired return as possible). We restrict ourselves to α such that $\mathbb{E}\int_0^T \alpha_t^2\,dt < \infty$. Thus we seek

$$V(m) := \inf_\alpha \left\{\mathrm{Var}(X_T^\alpha) : \mathbb{E}X_T^\alpha = m\right\}\,. \tag{4.13}$$

See Exercise 4.13 to convince yourself that the set over which we wish to take infimum is non-empty. Conveniently, if, for $\lambda \in \mathbb{R}$, we can calculate

$$v(\lambda) := \inf_\alpha \mathbb{E}\left[|X_T^\alpha - \lambda|^2\right]$$

then [11, Proposition 6.6.5] tells us that

$$V(m) = \sup_{\lambda\in\mathbb{R}} \left[v(\lambda) - (m-\lambda)^2\right].$$

Furthermore

$$v(\lambda) = -\sup_\alpha \mathbb{E}\left[-|X_T^\alpha - \lambda|^2\right]\,.$$

Thus our aim is to maximize

$$J_\lambda(\alpha) := \mathbb{E}\left[g(X_T^\alpha)\right] \;\text{ with }\; g(x) = -(x-\lambda)^2\,.$$

Since g is concave and differentiable we will try to apply Pontryagin's maximum principle. As there is no running gain (i.e. $f = 0$) and since X^α is given by (4.12) we have the Hamiltonian

$$H_t(x, a, y, z) = [rx + a(\mu - r)]y + \sigma a\,z\,.$$

This, being affine in (a, x), is certainly differentiable and concave. Moreover, if there is an optimal control β and if the solution of the adjoint BSDE is denoted (Y^β, Z^β) then

$$\max_a H_t(X_t^\beta, a, Y_t^\beta, Z_t^\beta) = \max_a \left[rX_t^\beta Y_t^\beta + a(\mu - r)Y_t^\beta + \sigma a Z_t^\beta\right]\,.$$

The quantity being maximized is linear in a and thus it will be finite if and only if the solution to the adjoint equation satisfies

$$(\mu - r)Y_t^\beta + \sigma Z_t^\beta = 0 \;\text{ a.s. for a.a } t. \tag{4.14}$$

From now on we omit the superscript β everywhere. Recalling the adjoint equation:

$$dY_t = -rY_t\,dt + Z_t\,dW_t \;\text{ and }\; Y_T = \partial_x g(X_T) = -2(X_T - \lambda). \tag{4.15}$$

To proceed we will need to make a guess at what the solution to the adjoint BSDE will look like. Since the terminal condition is linear in X_T we will try the ansatz $Y_t = \varphi(t)X_t + \psi(t)$ for some C^1 functions φ and ψ. Notice that this is rather different to the situation in Example 4.3, since there we obtain a solution but only in terms of an unknown process arising from the martingale representation theorem. With this ansatz we have, substituting the expression for Y on the r.h.s. of (4.15), that

$$dY_t = -r\varphi(t)X_t\,dt - r\psi(t)\,dt + Z_t\,dW_t \tag{4.16}$$

and on the other hand we can use the ansatz for Y and product rule on the l.h.s. of (4.15) to see

$$\begin{aligned} dY_t &= \varphi(t)\,dX_t + X_t\varphi'(t)\,dt + \psi'(t)\,dt \\ &= \varphi(t)\,[rX_t + \beta_t(\mu - r)]\;dt + \varphi(t)\sigma\beta_t\,dW_t + X_t\varphi'(t)\,dt + \psi'(t)\,dt\,. \end{aligned} \tag{4.17}$$

The second equality above came from (4.12) with β as the control. Then (4.16) and (4.17) can simultaneously hold only if $Z_t = \varphi(t)\sigma\beta_t$ and if

$$\varphi(t)\,[rX_t + \beta_t(\mu - r)] + X_t\varphi'(t) + \psi'(t) = -r\varphi(t)X_t - r\psi(t)\,.$$

This in turn will hold as long as

$$\beta_t = \frac{2r\varphi(t)X_t + r\psi(t) + \varphi'(t)X_t + \psi'(t)}{\varphi(t)(r-\mu)}\,. \tag{4.18}$$

On the other hand from the Pontryagin maximum principle we conculded (4.14) which, with $Y_t = \varphi(t)X_t + \psi(t)$ and $Z_t = \varphi(t)\sigma\beta_t$ says

$$(\mu - r)[\varphi(t)X_t + \psi(t)] + \sigma^2\varphi(t)\beta_t = 0,$$

i.e.

$$\beta_t = \frac{(r-\mu)[\varphi(t)X_t + \psi(t)]}{\sigma^2\varphi(t)}\,. \tag{4.19}$$

But (4.18) and (4.19) can both hold only if (collecting terms with X_t and without)

$$\begin{aligned} \varphi'(t) &= \left(\tfrac{(r-\mu)^2}{\sigma^2} - 2r\right)\varphi(t)\,,\;\; \varphi(T) = -2 \\ \psi'(t) &= \left(\tfrac{(r-\mu)^2}{\sigma^2} - r\right)\psi(t)\,,\;\; \psi(T) = 2\lambda\,. \end{aligned} \tag{4.20}$$

Note that the terminal conditions arose from Y_T (rather than from the equations for β). Also note that ψ clearly depends on λ but for now we omit this in our notation. Clearly

$$\varphi(t) = -2e^{-\left(\frac{(r-\mu)^2}{\sigma^2} - 2r\right)(T-t)} \quad\text{and}\quad \psi(t) = 2\lambda e^{-\left(\frac{(r-\mu)^2}{\sigma^2} - r\right)(T-t)}\,. \tag{4.21}$$

We note that from (4.19) we can write the control as Markov control

$$\beta(t,x) = -\frac{(\mu-r)[\varphi(t)x + \psi(t)]}{\sigma^2\varphi(t)}\,.$$

Thus X driven by this control is square integrable. Indeed β is a linear function in x and together with (4.12) and Proposition 2.6 we can conclude the square integrability. Thus we also have $\mathbb{E}\int_0^T \beta_t^2\,dt < \infty$ and so the control is admissable.

We still need to know

$$v(\lambda) = -J(\beta) = \mathbb{E}\left[|X_T - \lambda|^2\right] .$$

We cannot calculate this by solving for X as in Exercise 4.13 (try it). Instead we note that

$$\mathbb{E}|X_T - \lambda|^2 = \mathbb{E}\left[-\frac{1}{2}\varphi(T)X_T^2 - \psi(T)X_T + \lambda^2\right].$$

From Itô's formula for $\xi_t := -\frac{1}{2}\varphi(t)X_t^2 - \psi(t)X_t$ we get that

$$-d\xi_t = \left(\tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t\right)\,dt + \left[\varphi(t)X_t + \psi(t)\right]\,dX_t + \tfrac{1}{2}\varphi(t)\,dX(t)dX(t)\,.$$

And we have that

$$dX_t = (rX_t + \beta_t(\mu - r))\,dt + \sigma\beta_t\,dW_t\,.$$

Hence

$$\begin{aligned}-\mathbb{E}\xi_T = -\xi_0 + \mathbb{E}\int_0^T \Big(& \tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t \\ & + r\varphi(t)X_t^2 + r\psi(t)X_t \\ & + \beta_t(\mu - r)[\varphi(t)X_t + \psi(t)] \\ & + \tfrac{1}{2}\varphi(t)\sigma^2\beta_t^2\Big)\,dt\,.\end{aligned}$$

From the optimality condition $(\mu - r)\beta_t[\varphi(t)X_t + \psi_t] + \sigma^2\varphi(t)\beta_t^2 = 0$ we get

$$\tfrac{1}{2}\sigma^2\varphi(t)\beta_t^2 = -\tfrac{1}{2}(\mu - r)\beta_t[\varphi(t)X_t + \psi_t]$$

and so

$$\begin{aligned}-\mathbb{E}\xi_T = -\xi_0 + \mathbb{E}\int_0^T \Big(& \tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t \\ & + r\varphi(t)X_t^2 + r\psi(t)X_t \\ & + \tfrac{1}{2}\beta_t(\mu - r)[\varphi(t)X_t + \psi(t)]\Big)\,dt\,.\end{aligned}$$

This is

$$-\mathbb{E}\xi_T = -\xi_0 + \tfrac{(r-\mu)^2}{\sigma^2}\mathbb{E}\int_0^T \left(\tfrac{1}{2}\varphi(t)X_t^2 + \psi(t)X_t - \tfrac{1}{2}\tfrac{\varphi(t)^2X_t^2 + 2\varphi(t)\psi(t)X_t + \psi(t)^2}{\varphi(t)}\right)dt\,.$$

So

$$\mathbb{E}\xi_T = \xi_0 + \tfrac{1}{2}\tfrac{(r-\mu)^2}{\sigma^2}\int_0^T \tfrac{\psi(t)^2}{\varphi(t)}\,dt\,.$$

Due to (4.21) we have

$$\mathbb{E}\xi_T = \xi_0 - \lambda^2\tfrac{(r-\mu)^2}{\sigma^2}\int_0^T e^{-\frac{(r-\mu)^2}{\sigma^2}(T-t)}\,dt\,.$$

Hence

$$\mathbb{E}\xi_T = \xi_0 - \lambda^2\left[1 - e^{-\frac{(r-\mu)^2}{\sigma^2}T}\right].$$

But $\mathbb{E}|X_T - \lambda|^2 = \mathbb{E}\xi_T + \lambda^2$ and so

$$\mathbb{E}|X_T - \lambda|^2 = \xi_0 + \lambda^2 e^{-\frac{(r-\mu)^2}{\sigma^2}T}.$$

Moreover $\xi_0 = -\frac{1}{2}\varphi(0)x^2 - \psi(x)x$ and so

$$\xi_0 = x^2 e^{-\left(\frac{(r-\mu)^2}{\sigma^2}-2r\right)T} - 2x\lambda e^{-\left(\frac{(r-\mu)^2}{\sigma^2}-r\right)T}.$$

Finally

$$\mathbb{E}|X_T - \lambda|^2 = e^{-\frac{(r-\mu)^2}{\sigma^2}T}\left[x^2 e^{2rT} - 2x\lambda e^{rT} + \lambda^2\right] = e^{-\frac{(r-\mu)^2}{\sigma^2}T}\left(\lambda - xe^{rT}\right)^2.$$

which means that

$$v(\lambda) = -\kappa\left(\lambda - xe^{rT}\right)^2,$$

where $\kappa := e^{-\frac{(r-\mu)^2}{\sigma^2}T} > 0$. We thus get

$$V(m) = \sup_{\lambda\in\mathbb{R}}\left[-\kappa\left(\lambda^2 - 2\lambda xe^{rT} + x^2e^{2rT}\right) - \lambda^2 + 2\lambda m - m^2\right].$$

This is achieved when

$$0 = -\kappa\lambda + \kappa xe^{rT} - \lambda + m$$

i.e. when $\lambda = \frac{\kappa xe^{rT}+m}{\kappa+1}$.

4.3 Exercises

Exercise 4.13 (To complement Example 4.12)**.** Show that, under the assumptions of Example 4.12, the set $\{\mathrm{Var}(X_T^\alpha) : \mathbb{E}X_T^\alpha = m\}$ is nonempty.

Exercise 4.14 (Merton's problem with exponential utility, no consumption, using Pontryagin's Maximum Principle)**.** Consider a model with a risky asset $(S_t)_{t\in[0,T]}$ and a risk-free asset $(B_t)_{t\in[0,T]}$ given by

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t \;\; t\in[0,T], S_0 = S,\\ dB_t &= rB_t\,dt \;\; t\in[0,T], S_0 = S, B_0 = 1, \end{aligned}$$

where $\mu, r \in \mathbb{R}$ and $\sigma > 0$ are given constants. Let $(X_t)_{t\in[0,T]}$ denote the value of a self-financing investment portfolio with $X_0 = x > 0$ and let α_t denote the fraction of the portfolio value X_t invested in the risky asset. We note that X_t depends on the investment strategy α_t and so we write $X_t = X_t^\alpha$. We will only consider α that are real-valued, adapted and such that $\mathbb{E}\int_0^T \alpha_t^2\,dt < \infty$, denoting such strategies $\mathcal{A}$ and calling them admissable.

Our aim is to find the investment strategy $\hat\alpha$ which maximizes, over $\alpha\in\mathcal{A}$,

$$J(\alpha) = \mathbb{E}\left[-\exp(-\gamma X_T^\alpha)\right],$$

for some $\gamma > 0$.

i) Use the definition of a self-financing portfolio to derive the equation for the portfolio value:

$$dX_t = X_t\left[\alpha_t(\mu - r) + r\right]dt + X_t\alpha_t\sigma\,dW_t.$$

ii) Write down the Hamiltonian for the problem and the adjoint BSDE for the optimal portfolio (use $\hat{\alpha}$ to denote the optimal control, $(\hat{Y}, \hat{Z})$ to denote the BSDE).

iii) Justify the use of Pontryagin's maximum principle and show that it implies that

$$\hat{Z}_t = -\frac{\mu - r}{\sigma}\hat{Y}_t\,.$$

iv) Noting that $\hat{Y}_T = \gamma e^{-\gamma \hat{X}_T}$ use the "ansatz" $\hat{Y}_t = \phi_t e^{-\psi_t \hat{X}_t}$ with some $\phi, \psi \in C^1([0,T])$ such that $\phi_T = \gamma$ and $\psi_T = \gamma$. Hence show that

$$\hat{X}_t \hat{\alpha}_t = e^{-r(T-t)}\frac{\mu - r}{\gamma\sigma^2}\,.$$

4.4 Solutions to the exercises

Solution (to Exercise 4.13)**.** We start by solving (4.12) for some $\alpha_t = a$ constant. Note that (with $X = X^\alpha$)

$$d(e^{-rt}X_t) = e^{-rt}\left[dX_t - rX_t\,dt\right] = e^{-rt}\left[a(\mu-r)\,dt + \sigma a\,dW_t\right].$$

Thus

$$e^{-rT}X_T = x + \int_0^T e^{-rt}a(\mu-r)\,dt + \int_0^T \sigma a e^{-rt}\,dW_t\,.$$

Since the stochastic integral is a true martingale

$$\mathbb{E}X_T = e^{rT}x + e^{rT}a(\mu-r)\int_0^T e^{-rt}\,dt = e^{rT}x + a(\mu-r)\frac{1}{r}\left(e^{rT}-1\right).$$

Thus with

$$a = r\frac{m - e^{rT}x}{(\mu-r)(e^{rT}-1)}$$

we see that $\mathbb{E}X_T = m$ and so the set is non-empty.

Solution (to Exercise 4.14)**.** i) We have

$$dX_t = \frac{\alpha_t X_t}{S_t}\,dS_t + \frac{X_t - \alpha_t X_t}{B_t}\,dB_t = \alpha_t X_t \mu\,dt + \alpha_t X_t \sigma\,dW_t + X_t r\,dt - \alpha_t X_t r\,dt$$

so

$$dX_t = X_t\left[\alpha_t(\mu-r)+r\right]dt + X_t\alpha_t\sigma\,dW_t\,.$$

ii) Let us write down the Hamiltonian:

$$H_t(x,a,y,z) = x[a(\mu-r)+r]y + x\,a\,\sigma\,z$$

so

$$\partial_x H_t(x,a,y,z) = [a(\mu-r)+r]y + a\,\sigma\,z\,.$$

The adjoint BSDE for the optimal portfolio $\hat{X}$, which we denote $(\hat{Y},\hat{Z})$ then is

$$dY_t = -[\hat{\alpha}_t(\mu-r)+r]\hat{Y}_t\,dt + \hat{\alpha}_t\sigma\hat{Z}_t\,dt + \hat{Z}_t\,dW_t \quad t\in[0,T]\,,\; \hat{Y}_T = \gamma\exp(-\gamma\hat{X}_T)\,. \tag{4.22}$$

We can show that $\hat{X}_t > 0$ since $x>0$. Hence $|\hat{Y}_T|^2 = \gamma^2\exp(-2\gamma\hat{X}_T) \le \gamma^2$ and so $\hat{Y}_T \in L^2(\mathcal{F}_T)$. The above affine BSDE thus has a unique solution $(\hat{Y},\hat{Z})$ and we may proceed.

iii) We note that the terminal reward function $g(x) = -e^{-\gamma x}$ is concave. We can check that the Hamiltonian is concave in x as well as in a. Thus the optimal control $\hat{\alpha}$ must satisfy

$$H_t(\hat{X}_t,\hat{\alpha}_t,\hat{Y}_t,\hat{Z}_t) = \max_{a\in\mathbb{R}}\left[\hat{X}_t\big(a(\mu-r)-r\big)\hat{Y}_t + \hat{X}_t a\sigma\hat{Z}_t\right].$$

We need the Hamiltonian to be finite which in turns means that it must hold that

$$\hat{X}_t\hat{Y}_t(\mu-r) + \hat{X}_t\hat{Z}_t\sigma = 0\,.$$

Hence

$$\hat{Z}_t = -\frac{\mu-r}{\sigma}\hat{Y}_t\,. \tag{4.23}$$

iv) We will use the "ansatz" $\hat{Y}_t = \phi_t e^{-\psi_t\hat{X}_t}$ with some $\phi,\psi \in C^1([0,T])$ such that $\phi_T = \gamma$ and $\psi_T = \gamma$. We note that

$$d(-\psi_t\hat{X}_t) = -\psi_t\hat{X}_t[\hat{\alpha}_t(\mu-r)+r]\,dt - \psi_t\hat{X}_t\hat{\alpha}_t\sigma\,dW_t$$

so that

$$\begin{aligned}
d\hat{Y}_t &= \phi_t d(e^{-\psi_t\hat{X}_t}) + e^{-\psi_t\hat{X}_t}d\phi_t\\
&= e^{-\psi_t\hat{X}_t}\left[\phi_t d(-\psi_t\hat{X}_t) + \frac{1}{2}\phi_t d(-\psi_t\hat{X}_t)d(-\psi_t\hat{X}_t) + d\phi_t\right]\\
&= e^{-\psi_t\hat{X}_t}\left[-\psi_t\phi_t d\hat{X}_t - \hat{X}_t\phi_t d\psi_t + \frac{1}{2}\phi_t\psi_t^2 d\hat{X}_t d\hat{X}_t + d\phi_t\right]\\
&= e^{-\psi_t\hat{X}_t}\left[-\psi_t\phi_t\hat{X}_t\big[\left(\hat{\alpha}_t(\mu-r)+r\right)dt + \hat{\alpha}_t\sigma\,dW_t\big] - \hat{X}_t\phi_t\psi_t'\,dt + \frac{1}{2}\phi_t\psi_t^2\hat{\alpha}_t^2\hat{X}_t^2\sigma^2\,dt + \phi_t'\,dt\right].
\end{aligned}$$

If we now go to the adjoint BSDE (4.22) and substitute for $\hat{Z}_t$ from (4.23) we see that we must also have

$$d\hat{Y}_t = -r\hat{Y}_t dt - \frac{\mu-r}{\sigma}\hat{Y}_t\,dW_t\,.$$

Equating the "dW terms" leads to

$$\frac{\mu - r}{\sigma^2} = \psi_t \hat{X}_t \hat{\alpha}_t \implies \hat{X}_t \hat{\alpha}_t = \frac{\mu - r}{\sigma^2 \psi_t}\,.$$

Equating the "dt terms" will let us identify ψ and ϕ. Indeed we get

$$-r\phi_t e^{-\psi_t \hat{X}_t} = e^{-\psi_t \hat{X}_t}\Big[- \psi_t \phi_t \hat{X}_t \big[(\hat{\alpha}_t(\mu - r) + r) \big] - \hat{X}_t \phi_t \psi_t' + \frac{1}{2}\phi_t \psi_t^2 \hat{\alpha}_t^2 \hat{X}_t^2 \sigma^2 + \phi_t' \Big]\,.$$

Substituting the control and dividing by the exponential term leads to:

$$-r\phi_t = -\phi_t \left(\tfrac{(\mu-r)^2}{\sigma^2} + \psi_t \hat{X}_t r \right) - \hat{X}_t \phi_t \psi_t' + \tfrac{1}{2}\phi_t \tfrac{(\mu-r)^2}{\sigma^2} + \phi_t'\,.$$

This simplifies to

$$-r\phi_t = -\psi_t \phi_t \hat{X}_t r - \tfrac{1}{2}\phi_t \tfrac{(\mu-r)^2}{\sigma^2} - \hat{X}_t \phi_t \psi_t' + \phi_t'\,.$$

From this we get (equating the terms with X_t and without):

$$\begin{aligned} \phi_t' &= \left(\tfrac{1}{2} \tfrac{(\mu-r)^2}{\sigma^2} - r \right) \phi_t\,, \;\; \phi_T = \gamma \\ \psi_t' &= -r\psi_t\,, \;\; \psi_T = \gamma\,. \end{aligned}$$

Hence

$$\begin{aligned} \phi_t &= \gamma \exp\left((T-t) \left(\tfrac{1}{2} \tfrac{(\mu-r)^2}{\sigma^2} - r \right) \right)\,, \;\; t \in [0,T]\,, \\ \psi_t &= \gamma \exp(r(T-t))\,, \;\; t \in [0,T]\,. \end{aligned}$$

So finally the optimal control is:

$$\hat{X}_t \hat{\alpha}_t = e^{-r(T-t)} \frac{\mu - r}{\gamma \sigma^2}\,.$$

A Appendix

A.1 Basic notation and useful review of analysis concepts

Here we set the main notation for the rest of the course. These pages serve as an easy reference.

General For any two real numbers x, y,

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \quad x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\}.$$

Sets, metrics and matrices $\mathbb{N}$ is the set of strictly positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbb{R}^d$ denotes the d-dimensional Euclidean space of real numbers. For any $x = (x_1, \cdots, x_d)$, $y = (y_1, \cdots, y_d)$ in $\mathbb{R}^d$, we denote the inner product by xy and by $|\cdot|$ the Euclidean norm i.e.

$$xy := \sum_{i=1}^{d} x_i y_i \quad \text{and} \quad |x| := \left(\sum_{i=1}^{d} x_i^2 \right)^{\frac{1}{2}}$$

$\mathbb{R}^{d\times n}$ denotes the set of real valued $d \times n$-matrices; I_n denotes the $n \times n$-identity matrix. For any $\sigma \in \mathbb{R}^{n\times d}$, $\sigma = (\sigma_{ij})_{1\le i\le n, 1\le j\le d}$ we write the transpose of σ as $\sigma^\top = (\sigma_{ji})_{1\le j\le d, 1\le i\le n} \in \mathbb{R}^{d\times n}$. We write the trace operator of an $n \times n$-matrix σ as $\mathrm{Tr}(\sigma) = \sum_{i=1}^n \sigma_{ii}$. For a matrices we will use the norm $|\sigma| := \big(\mathrm{Tr}(\sigma\sigma^\top)\big)^{1/2}$.

Definition A.1 (Supremum/Infimum)**.** Given a set $S \subset \mathbb{R}$, we say that μ is the supremum of S if *(i)* $\mu \ge x$ for each $x \in S$ and if *(ii)* for every $\varepsilon > 0$ there exists an element $y \in S$ such that $y > \mu - \varepsilon$. We write $\mu = \sup S$.

The infimum is defined symmetrically as follows: λ is the infimum if *(i)* $\lambda \le x$ for each $x \in S$ and if *(ii)* for every $\varepsilon > 0$ there exists an element $y \in S$ such that $y < \lambda + \varepsilon$. We write $\lambda = \inf S$.

Note that supremum is the *least upper bound*, i.e. the smallest real number greater than or equal to all the elements of the set S. Infimum is the *greatest lower bound*, i.e. the largest number smaller than or equal to all the elements of the set S. It is also important to note that the infimum (or supremum) do not necessarily have to belong to the set S.

Functions, derivatives For any set A, the indicator function of A is

$$\mathbb{1}_A(x) = 1 \text{ if } x \in A, \qquad \text{otherwise } \mathbb{1}_A(x) = 0 \text{ if } x \notin A.$$

We write $C^k(A)$ is the space of all real-valued continuous functions on A with continuous derivatives up to order $k \in \mathbb{N}_0$, $A \subset \mathbb{R}^n$. In particular $C^0(A)$ is the space of real-valued functions on A that are continuous.

For a real-valued function functions $f = f(t, x)$ defined $I \times A$ we write $\partial_t f$, $\partial_{x_i} f$ and $\partial_{x_i x_j} f$ for $1 \le i, j \le n$ for its partial derivatives. By Df we denote the gradient vector of f and by $D^2 f$ the Hessian matrix of f (whose entries $1 \le i, j \le d$ are given by $\partial_{x_i x_j} f(t, x)$).

Consider an interval I (and think of I as a time interval $I = [0,T]$ or $I = [0,\infty)$). Then $C^{1,2}(I \times A)$ is the set of real valued functions $f = f(t,x)$ on $I \times A$ whose partial derivatives $\partial_t f$, $\partial_{x_i} f$ and $\partial_{x_i x_j} f$ for $1 \le i,j \le n$ exist and are continuous on $I \times A$.

Integration and probability We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a probability space with $\mathbb{P}$ being the probability measure and $\mathcal{F}$ the σ-algebra.

"$\mathbb{P}$-a.s." denotes "almost surely for the probability measure $\mathbb{P}$" (we often omit the reference to $\mathbb{P}$). "μ-a.e." denotes "almost everywhere for the measure μ"; here μ will not be a probability measure. This means is that a statement Z made about $\omega \in \Omega$ holds $\mathbb{P}$-a.s. if there is a set $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ and Z is true for *all* $\omega \in E^c = \Omega \setminus E$.

$\mathcal{B}(U)$ is the Borel σ-algebra generated by the open sets of the topological space U.

$\mathbb{E}[X]$ is the expectation of the random variable X with respect to a probability $\mathbb{P}$. $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$. The variance of the random variable X, possibly vector valued, is denoted by $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^\top]$.

Since we may define different measures on the same σ-algebra we must sometimes distinguish which measure is used for expectation, conditional expectation or variance. We thus sometimes write $\mathbb{E}^{\mathbb{Q}}[X]$, $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ or $\mathrm{Var}^{\mathbb{Q}}$ to show which measure was used.

General analysis definitions and inequalities

Definition A.2 (Convex function)**.** A function $f : \mathbb{R} \to (-\infty, \infty]$ is called convex if

$$\forall\ \lambda \in [0,1]\ \forall\ x,y \in \mathbb{R} \qquad f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y).$$

If a function f is convex then it is differentiable a.e. and (with f'_- denoting its left-derivative, f'_+ its right-derivative) and we have

$$f'_+(x) := \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \inf_{y > x} \frac{f(y) - f(x)}{y - x}\,,$$

$$f'_-(x) := \lim_{y \nearrow x} \frac{f(y) - f(x)}{y - x} = \sup_{y < x} \frac{f(y) - f(x)}{y - x}\,.$$

So, from the expression with infimum we see that,

$$\text{if } y > x \text{ then } f'_+(x) \le \frac{f(y) - f(x)}{y - x} \text{ which implies } f(y) \ge f(x) + f'_+(x)(y-x) \text{ for } y > x.$$

Moreover, from the expression with supremum we see that[8],

$$\text{if } y < x \text{ then } f'_-(x) \ge \frac{f(y) - f(x)}{y - x} \text{ which implies } f(y) \ge f(x) + f'_-(x)(y-x) \text{ for } y < x.$$

We review a few standard analysis inequalities, some not named and some others named: Cauchy-Schwarz, Holder, Young and Gronwall's inequality.

$$\begin{aligned} \forall\, x \in \mathbb{R} \qquad & x \le 1 + x^2 \\ \forall a,b \in \mathbb{R} \qquad & 2ab \le a^2 + b^2 \\ \forall\ n \in \mathbb{N}\ \forall a,b \in \mathbb{R} \qquad & |a+b|^n \le 2^{n-1}\big(|a|^n + |b|^n\big) \end{aligned}$$

[8] As $y < x$ we multiply by negative number, flipping the inequality.

Lemma A.3 (Cauchy–Schwarz inequality). *Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. If $x,y\in H$ then $(x,y)\leq |x|_H|y|_H$.*

Example A.4. i) If $x,y\in\mathbb{R}^d$ then $xy<|x||y|$.

ii) We can check that $L^2(\Omega)$ with inner product given by $\mathbb{E}[XY]$ for $X,Y\in L^2(\Omega)$ is a Hilbert space. Hence the Cauchy–Schwarz inequality is $\mathbb{E}[XY]\leq \big(\mathbb{E}[X^2]\big)^{1/2}\big(\mathbb{E}[Y^2]\big)^{1/2}$.

Lemma A.5 (Young's inequality). *Let $a,b\in\mathbb{R}$. Then for any $\varepsilon\in(0,\infty)$ for any $p,q\in(1,\infty)$ such that $1/p+1/q=1$ it holds that*

$$ab\leq \varepsilon\frac{|a|^p}{p}+\frac{1}{\varepsilon}\frac{|b|^q}{q}\,.$$

The above inequality is not the original Young's inequality, that is for the choice $\varepsilon=1$. The one here is the original Young's inequality with the choice $(ab)=(\varepsilon a)(b/\varepsilon)$.

Lemma A.6 (Gronwall's lemma / inequality). *Let $\lambda=\lambda(t)\geq 0$, $a=a(t)$, $b=b(t)$ and $y=y(t)$ be locally integrable, real valued functions defined on I (with $I=[0,T]$ or $I=[0,\infty)$) such that λy is also locally integrable and for almost all $t\in[0,T]$*

$$y(t)+a(t)\leq b(t)+\int_0^t\lambda(s)y(s)\,ds.$$

Then

$$y(t)+a(t)\leq b(t)+\int_0^t\lambda(s)e^{\int_s^t\lambda(r)dr}(b(s)-a(s))\,ds\quad \text{for almost all } t\in I.$$

Furthermore, if b is monotone increasing and a is non-negative, then

$$y(t)+a(t)\leq b(t)e^{\int_0^t\lambda(r)\,dr},\quad \text{for almost all } t\in I.$$

If the function y in Gronwall's lemma is continuous then the conclusions hold for all $t\in I$. For proof see Exercise 1.5.

Some fundamental probability results

(Following the notation established in SAF) we define lim inf and lim sup.

Definition A.7 (limsup & liminf). Let $(a_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$

$$\liminf_{n\to\infty}a_n:=\lim_{n\to\infty}\lim_{k\to\infty}\min\{a_n,a_{n+1},a_{n+2},\ldots,a_k\}=\inf_n\sup_{k\geq n}a_k,$$
$$\limsup_{n\to\infty}a_n:=\lim_{n\to\infty}\lim_{k\to\infty}\max\{a_n,a_{n+1},a_{n+2},\ldots,a_k\}=\sup_n\inf_{k\geq n}a_k.$$

Clearly $\liminf_{n\to\infty}a_n\leq\limsup_{n\to\infty}a_n$ and if $\lim_{n\to\infty}a_n=:a$ exists, then $\liminf_{n\to\infty}a_n=\limsup_{n\to\infty}a_n=a$. On the other hand, if $\liminf_{n\to\infty}a_n\geq\limsup_{n\to\infty}a_n$, then $\lim_{n\to\infty}a_n=a$ exists.

Exercise A.8 (lim sup and lim inf of RV are RV). Show that $\liminf_{n\to\infty}X_n$ and $\limsup_{n\to\infty}X_n$ are random variables for any sequence of random variables X_n.

Lemma A.9 (Fatou's lemma)**.** *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative random variables. Then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

Moreover,

i) If there exists a r.v. Y *such that* $\mathbb{E}[\,|Y|\,] < \infty$ *and* $Y \leq X_n$ $\forall n$ *(allows* $X_n < 0$*), then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

ii) If there exists a r.v. Y *such that* $\mathbb{E}[\,|Y|\,] < \infty$ *and* $Y \geq X_n$ $\forall n$*, then*

$$\mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \geq \limsup_{n\to\infty} \mathbb{E}\left[X_n\right].$$

The first part of the above lemma does not require integrability of the sequence of $(X_n)_{n\in\mathbb{N}}$ due to the use of the Monotone Convergence Theorem in its proof. The enumerated statements follow as a corollary of the first statement. Of course, a version of Fatou's lemma using conditional expectations also exists (simply replace $\mathbb{E}[\cdot]$ with $\mathbb{E}[\cdot|\mathcal{F}_t]$).

Lemma A.10 (Hölder's inequality)**.** *Let* $(X, \mathcal{X}, \mu)$ *be a measure space (i.e.* X *is a set,* $\mathcal{X}$ *a* σ*-algebra and* μ *a measure). Let* $p, q > 1$ *be real numbers s.t.* $1/p + 1/q = 1$ *or let* $p = 1, q = \infty$*. Let* $f \in L^p(X, \mu)$*,* $g \in L^q(X, \mu)$*. Then*

$$\int_X |fg|\, d\mu \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu\right)^{\frac{1}{q}}$$

In particular if p, q are such that $1/p+1/q = 1$ and $X \in L^p(\Omega)$, $Y \in L^q(\Omega)$ are random variables then

$$\mathbb{E}[\,|XY|\,] \leq \mathbb{E}[\,|X|^p\,]^{\frac{1}{p}} \mathbb{E}[\,|Y|^q\,]^{\frac{1}{q}}\,.$$

Lemma A.11 (Minkowski's inequality or triangle inequality)**.** *Let* $(X, \mathcal{X}, \mu)$ *be a measure space (i.e.* X *is a set,* $\mathcal{X}$ *a* σ*-algebra and* μ *a measure). For any* $p \in [1, \infty]$ *and* $f, g \in L^p(X, \mu)$

$$\left(\int_X |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\, d\mu\right)^{\frac{1}{p}}.$$

Lemma A.12 (Jensen's inequality)**.** *Let* f *be a convex function and* X *be any random variable with* $\mathbb{E}[|X|] < \infty$*. Then*

$$f\big(\mathbb{E}[X]\big) \leq \mathbb{E}\big[f(x)\big].$$

A.2 Some useful results from stochastic analysis

For convenience we state some results from stochastic analysis. Proofs can be found for example in Stochastic Analysis for Finance lecture notes, in [11], [2] or [7].

Probability Space

Let us always assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space. We assume that $\mathcal{F}$ is complete which means that all the subsets of sets with probability zero are included in $\mathcal{F}$. We assume there is a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ (which means $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$) such that $\mathcal{F}_0$ contains all the sets of probability zero.

Stochastic Processes, Martingales

A stochastic process $X = (X_t)_{t\geq 0}$ is a collection of random variables X_t which take values in $\mathbb{R}^d$.

We will always assume that stochastic processes are *measurable*. This means that $(\omega, t) \mapsto X(\omega)_t$ taken as a function from $\Omega \times [0, \infty)$ to $\mathbb{R}^d$ is measurable with respect to σ-algebra $\mathcal{F} \otimes \mathcal{B}([0, \infty))$.[9] This product is defined as the σ-algebra generated by sets $E \times B$ such that $E \in \mathcal{F}$ and $B \in \mathcal{B}([0, \infty))$. From Theorem A.30 we then get that

$$t \mapsto X_t(\omega) \text{ is measurable for all } \omega \in \Omega.$$

We say X is $(\mathcal{F}_t)_{t\geq 0}$ *adapted* if for all $t \geq 0$ we have that X_t is $\mathcal{F}_t$-measurable.

Definition A.13. Let X be a stochastic process that is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and such that for every $t \geq 0$ we have $\mathbb{E}[|X_t|] < \infty$. If for every $0 \leq s < t \leq T$ we have

i) $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s.then the process is called *submartingale*.

ii) $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ a.s.then the process is called *supermartingale*.

iii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s.then the process is called *martingale*.

For submartingales we have Doob's maximal inequality:

Theorem A.14 (Doob's submartingale inequality)**.** *Let $X \geq 0$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-submartingale and $p > 1$ be given. Assume $\mathbb{E}\left[X_T^p\right] < \infty$. Then*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} X_t^p\Big] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right].$$

Definition A.15 (Local Martingale)**.** A stochastic process X is called a *local martingale* if is there exists a sequence of stopping time $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \leq \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ and if the *stopped process* $(X(t \wedge \tau_n))_{t\geq 0}$ is a martingale for every n.

Lemma A.16 (Bounded from below local martingales are supermartingales)**.** *Let $(M_t)_{t\in[0,T]}$ be a local Martingale and assume it is positive or more generally bounded from below. Then M is a super-martingale.*

Proof. The proof makes use of Fatou's Lemma A.9 above. Since M is a local Martingale then there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity

[9] If the process is almost surely continuous i.e. if the map $[0, \infty) \ni t \mapsto X_t(\omega) \in \mathbb{R}^d$ is continuous for almost all $\omega \in \Omega$ then $\Omega \times [0, \infty) \ni (\omega, t) \mapsto X_{(}\omega) \in \mathbb{R}^d$ is a so-called Carathéodory map the stochastic process will be measurable due to e.g. Aliprantis and Border [1, Lemma 4.51].

a.s. such that the stopped process $M^n_t := M_{t\wedge\tau_n}$ is a Martingale. We have then, using Fatou's lemma for any $0 \leq s \leq t \leq T$

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\liminf_{n\to\infty} M^n_t|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[M^n_t|\mathcal{F}_s] = \liminf_{n\to\infty} M^n_s = M_s,$$

and hence M is a supermartingale. □

Exercise A.17 (Submartingale)**.** In view of the previous lemma, is a bounded from above local martingale a submartingale?

Integration Classes and Itô's Formula

Definition A.18. By $\mathcal{H}$ we mean all $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\|g\|^2_{\mathcal{H}_T} := \mathbb{E}\left[\int_0^T |g_s|^2 ds\right] < \infty.$$

By $\mathcal{S}$ we mean all $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\mathbb{P}\left[\int_0^T |g_s|^2 ds < \infty\right] = 1.$$

The importance of these two classes is that stochastic integral with respect to W is defined for all integrands in class $\mathcal{S}$ and this stochastic integral is a continous *local* martingale. For the class $\mathcal{H}$ the stochastic integral with respect to W is a martingale.

Definition A.19. By $\mathcal{A}$ we denote $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\mathbb{P}\left[\int_0^T |g_s| ds < \infty\right] = 1.$$

By $\mathcal{H}^{d\times n}$, $\mathcal{S}^{d\times n}$ we denote processes taking values the space of $d\times n$-matrices such that each component of the matrix is in $\mathcal{H}$ or $\mathcal{S}$ respectively. By $\mathcal{A}^d$ we denote processes taking values in $\mathbb{R}^d$ such that each component is in $\mathcal{A}$

Itô processes and Itô Formula

We will need the multi-dimensional version of the Itô's formula. Let W be an n-dimensional Wiener martingale with respect to $(\mathcal{F})_{t\geq 0}$. Let $\sigma \in \mathcal{S}^{m\times d}$ and let $b \in \mathcal{A}^m$. We say that the d-dimensional process X has the stochastic differential

$$dX_t = b_t\, dt + \sigma_t\, dW_t \tag{A.1}$$

for $t \in [0, T]$, if

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW(s).$$

Such a process is also called an *Itô process.*

The Itô formula or chain rule for stochastic processes Before we go into the main result, let us go over an example from classic analysis. Take three functions, $u = u(t,x)$, $g = g(t)$ and $h = h(t)$ given by $h(t) := u\big(t, g(t)\big)$. Let us compute $\frac{d}{dt}h(t)$. Since h is given as a composition of functions, we use here is the standard chain for functions of several variables (this takes into account that the variation of h arising from changes in t comes from the variation of g and also from the first component in u). Thus we have

$$\frac{d}{dt}h(t) = \big(\partial_t u\big)\big(t, g(t)\big) + \big(\partial_x u\big)\big(t, g(t)\big)\frac{d}{dt}g(t).$$

We want to see the contrast with Itô formula, which has to be written in integral form (since W has almost everywhere non-differentiable paths). To that end, we integrate

$$\int_0^t \frac{d}{dt}h(s)\,ds = \int_0^t \big(\partial_t u\big)\big(s, g(s)\big)\,ds + \int_0^t \big(\partial_x u\big)\big(s, g(s)\big)\frac{d}{dt}g(s)\,ds$$

and use the Fundamental theorem of calculus

$$h(t) - h(0) = \int_0^t \big(\partial_t u\big)\big(s, g(s)\big)\,ds + \int_0^t \big(\partial_x u\big)\big(s, g(s)\big)\,dg(s)$$

which can be written in the differential notation as

$$dh(t) = \partial_t f\big(t, g(t)\big)\,dt + \partial_x f\big(t, g(t)\big)\,dg(t). \tag{A.2}$$

Compare (A.2) with (A.3) below. You see a fundamental difference: the second derivative term! It appears there exactly because the Wiener process has non-differentiable paths and hence a correction to (A.2) is needed.

We have then the following important result.

Theorem A.20 (Multi-dimensional Itô formula)**.** *Let X be a m-dimensional Itô process given by (A.1). Let $u \in C^{1,2}([0,T]\times\mathbb{R}^m)$. Then the process given by $u(t, X_t)$ has the stochastic differential*

$$\begin{aligned} du\big(t, X_t\big) = \partial_t u\big(t, X_t\big)\,dt &+ \sum_{i=1}^d \partial_{x_i} u\big(t, X_t\big)\,dX_t^i \\ &+ \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i x_j} u\big(t, X_t\big)\,dX_t^i\,dX_t^j, \end{aligned} \tag{A.3}$$

where for $i, j = 1, \ldots, m$

$$dt\,dt = dt\,dW_t^i = 0, \quad dW_t^i\,dW_t^j = \delta_{ij}\,dt.$$

We now consider a very useful special case. Let X and Y be $\mathbb{R}$-valued Itô processes. We will apply to above theorem with $f(x,y) = xy$. Then $\partial_x f = y$, $\partial_y f = x$, $\partial_{xx} f = \partial_{yy} f = 0$ and $\partial_{xy} f = \partial_{yx} f = 1$. Hence from the multi-dimensional Itô formula we have

$$df\big(X_t, Y_t\big) = Y_t\,dX_t + X_t\,dY_t + \frac{1}{2}\,dY_t\,dX_t + \frac{1}{2}\,dX_t\,dY_t.$$

Hence we have the following corollary

Corollary A.21 (Itô's product rule)**.** *Let X and Y be $\mathbb{R}$-valued Itô processes. Then*

$$d\big(X_t Y_t\big) = X_t\,dY_t + Y_t\,dX_t + dX_t\,dY_t.$$

Martingale Representation Formula and Girsanov's theorem

Theorem A.22 (Lévy characterization)**.** *Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Let $X = (X_t)_{t\in[0,T]}$ be a continuous m-dimensional process adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ such that for $i = 1,\dots,d$ the processes*

$$M_t^i := X_t^i - X_0^i$$

are local martingales with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and $dM_t^i\, dM_t^j = \delta_{ij}\, dt$ for $i,j = 1,\dots,d$. Then X is a Wiener martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

So essentially any continuous local martingale with the right quadratic variation is a Wiener process.

Theorem A.23 (Girsanov)**.** *Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Let $W = (W_t)_{t\in[0,T]}$ be a d-dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Let $\varphi = (\varphi_t)_{t\in[0,T]}$ be a d-dimensional process adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ such that*

$$\mathbb{E}\Big[\int_0^T |\varphi_s|^2\, ds\Big] < \infty.$$

Let

$$L_t := \exp\left\{ -\int_0^t \varphi_s^\top\, dW(s) - \frac{1}{2}\int_0^t |\varphi_s|^2\, ds \right\} \tag{A.4}$$

and assume that $\mathbb{E}[L_T] = 1$. Let $\mathbb{Q}$ be a new measure on $\mathcal{F}_T$ given by the Radon-Nikodym derivative $d\mathbb{Q} = L(T)\, d\mathbb{P}$. Then

$$W_t^{\mathbb{Q}} := W_t + \int_0^t \varphi_s\, ds$$

is a $\mathbb{Q}$-Wiener martingale.

We don't give proof but only make some useful observations.

1. Clearly $L_0 = 1$.
2. The Novikov condition is a useful way of establishing that $\mathbb{E}[L_T] = 1$: if

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\varphi_t|^2\, dt}\right] < \infty$$

   then L is a martingale (and hence $\mathbb{E}[L_T] = \mathbb{E}[L_0] = 1$).
3. Applying Itô's formula to $f(x) = \exp(x)$ and

$$dX_t = -\varphi_t^\top\, dW_t - \frac{1}{2}|\varphi_t|^2\, dt$$

   yields

$$dL_t = -L_t \varphi_t^\top\, dW_t.$$

Theorem A.24 (Martingale representation)**.** *Let $W = (W_t)_{t\in[0,T]}$ be a d-dimensional Wiener martingale and let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $M = (M_t)_{t\in[0,T]}$ be a continuous real valued martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.*

Then there exists unique adapted d-dimensional process $h = (h_t)_{t\in[0,T]}$ such that for $t \in [0,T]$ we have

$$M_t = M_0 + \sum_{i=1}^d \int_0^t h_s^i\, dW_s^i.$$

If the martingale M is square integrable then h is in $\mathcal{H}$.

Essentially what the theorem is saying is that we can write continuous martingales as stochastic integrals with respect to some process as long as they're adapted to the filtration generated by the process.

A.2.1 PDEs and Feynman-Kac Formula

(This section can be traced back to either [11] or SAF notes (Section 16).)

In the case of deterministic maps b and σ in (2.1), the so-called *diffusion SDE*, we can give the following definition of Infinitesimal generator.

Definition A.25 (Infinitesimal generator (associated to an SDE))**.** Let b and σ be deterministic functions in (2.1). For all $t \in [0,T]$, the following second order differential operator $\mathcal{L}$ is called the *infinitesimal generator associated to the diffusion* (2.1),

$$\mathcal{L}\varphi(t,x) = b(t,x)D\varphi(t,x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top D^2\varphi)(t,x), \qquad \varphi \in C^{0,2}([0,T]\times\mathbb{R}^m).$$

Although the above definition does seems weird and unfamiliar, the operator $\mathcal{L}$ appears every time one uses the Itô formula to $\varphi(t,X_t)$ where the process $(X_t)_{t\in[0,T]}$ is the solution to (2.1).

Exercise A.26. Let $(X_t)_{t\in[0,T]}$ be the solution to (2.1).

Show that for $\varphi \in C^{1,2}([0,T]\times\mathbb{R})$, we have

$$d\varphi(t,X_t) = \big(\partial_t\varphi + \mathcal{L}\varphi\big)(t,X_t)\,dt + \big(\partial_x\varphi\sigma\big)(t,X_t)\,dW_t.$$

It is possible, for certain classes of SDE and differential equations, to write the solution to a PDE as an expectation of (a function of) the solution to the SDE associated to the differential operator appearing in the PDE; it is not surprising that the PDE differential operator must be the infinitesimal generator. That is the core message of the next result.

Theorem A.27 (Feynman-Kac formula in 1-dim)**.** *Assume that the function* $v : [0,T]\times\mathbb{R}\to\mathbb{R}$ *belongs to* $C^{1,2}([0,T)\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ *and is a solution to the following boundary value problem*

$$\partial_t v(t,x) + b(t,x)\partial_x v(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_{xx}v(t,x) - rv(t,x) = 0, \tag{A.5}$$

$$v(T,x) = h(x), \tag{A.6}$$

where b *and* σ *are deterministic functions.*

For any $(t,x) \in [0,T]\times\mathbb{R}$*, define the stochastic process* $(X_s)_{s\in[t,T]}$ *as the solution to the SDE*

$$dX_s = b(s,X_s)\,ds + \sigma(s,X_s)\,dW_s, \quad \forall\, s\in[t,T], \qquad X_t = x. \tag{A.7}$$

Assume that the stochastic process $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$*.*

Then the solution v *of* (A.5)-(A.6) *can be expressed as (with* $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t = x]$*)*

$$v(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}\big[\,h\big(X_T\big)\,\big] \qquad \forall\,(t,x)\in[0,T\times\mathbb{R}.$$

Proof. The proof is rather straightforward and is based on a direct application of Itô's formula.

Define the process $(Y_s)_{s\in[t,T]}$ as $Y_s = e^{-rs}v(s,X_s)$ where X is given by (A.7). Applying Itô's formula to Y, i.e. computing dY_s gives

$$\begin{aligned}
dY_s &= d\Big(e^{-rs}v(s,X_s)\Big)\\
&= (-r)e^{-rs}v\,ds + e^{-rs}\partial_s v\,ds + e^{-rs}\partial_x v\,dX_s + \frac{1}{2}e^{-rs}\partial_{xx}v\big(dX_s\big)^2\\
&= e^{-rs}\big[-rv + \partial_t v + b\partial_x v + \frac{1}{2}\sigma^2\partial_{xx}v\big]\,ds + e^{-rs}\big[\sigma\partial_x v\big]\,dW_s,
\end{aligned}$$

where the v function is evaluated in point (s,X_s). Using the equality given by (A.5) we see that the ds term disappears completely leaving

$$dY_s = d\Big(e^{-rs}v(s,X_s)\Big) = e^{-rs}\big[\ \sigma(s,X_s)\partial_x v(s,X_s)\ \big]\,dW_s.$$

Integrating both sides from $s=t$ to $s=T$ gives

$$\begin{aligned}
\big[e^{-rs}v(s,X_s)\big]\Big|_{s=t}^{s=T} &= \int_t^T e^{-ru}\sigma(u,X_u)\partial_x v(u,X_u)\,dW_u\\
\Leftrightarrow e^{-rt}v(t,X_t) &= e^{-rT}v(T,X_T) - \int_t^T e^{-ru}\sigma(u,X_u)\partial_x v(u,X_u)\,dW_u,\\
\Leftrightarrow v(t,X_t) &= e^{-r(T-t)}v(T,X_T) - \int_t^T e^{-r(u-t)}\sigma(u,X_u)\partial_x v(u,X_u)\,dW_u.
\end{aligned}$$

Taking expectations $\mathbb{E}_{(t,x)}[\cdot]$ on both sides (recall that the process X starts at time t in position x; this is the meaning of the subscript (t,x) in the expectation sign),

$$v(t,X_t) = e^{-r(T-t)}\mathbb{E}_{t,x}\big[v(T,X_T)\big] = e^{-r(T-t)}\mathbb{E}_{t,x}\big[h(X_T)\big],$$

where the expectation of the stochastic integral disappears due to the properties of the stochastic integral, since by assumption we have $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$. □

Exercise A.28 (Two extensions of the Feynman-Kac formula)**.** a) Redo the previous proof when the constant r is replaced by a function $r:[0,T]\times\mathbb{R}\to\mathbb{R}$; assume r to be bounded and continuous. Hint instead of e^{-rs}, use $\exp\{-\int_t^s r(u,X_u)\,du\}$.

b) Redo the previous proof when the PDE (A.5) is equal to some $f(t,x)$ instead of being equal to zero.

A.3 Useful Results from Other Courses

The aim of this section is to collect, mostly without proofs, results that are needed or useful for this course but that cannot be covered in the lectures i.e. prerequisites. You are expected to be able to use the results given here.

A.3.1 Linear Algebra

The inverse of a square real matrix A exists if and only if $\det(A) \neq 0$.

The inverse of square real matricies A and B exists if and only if the inverse of AB exists and moreover $(AB)^{-1} = B^{-1}A^{-1}$.

The inverse of a square real matrix A exists if and only if the inverse of A^T exists and $(A^T)^{-1} = (A^{-1})^T$.

If x is a vector in $\mathbb{R}^d$ then $\operatorname{diag}(x)$ denotes the matrix in $\mathbb{R}^{d\times d}$ with the entries of x on its diagonal and zeros everywhere else. The inverse of $\operatorname{diag}(x)$ exists if and only if $x_i \neq 0$ for all $i = 1, \ldots, d$ and moreover

$$\operatorname{diag}(x)^{-1} = \operatorname{diag}(1/x_1, 1/x_2, \ldots, 1/x_d).$$

A.3.2 Real Analysis and Measure Theory

Let $(X, \mathcal{X}, \mu)$ be a measure space (i.e. X is a set, $\mathcal{X}$ a σ-algebra and μ a measure).

Lemma A.29 (Fatou's Lemma)**.** *Let $f_1, f_2, \ldots$ be a sequence of non-negative and measurable functions. Then the function defined point-wise as*

$$f(x) := \liminf_{k\to\infty} f_k(x)$$

is $\mathcal{X}$-measurable and

$$\int_X f\,d\mu \leq \liminf_{k\to\infty} \int_X f_k\,d\mu.$$

Consider sets X and Y with σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. By $\mathcal{X} \times \mathcal{Y}$ we denote the collection of sets $C = A \times B$ where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$. By $\mathcal{X} \otimes \mathcal{Y} = \sigma(\mathcal{X} \times \mathcal{Y})$, which is the σ-algebra generated by $\mathcal{X} \times \mathcal{Y}$.

Theorem A.30. *Let $f : X \times Y \to \mathbb{R}$ be a measurable function, i.e. measurable with respect to the σ-algebras $\mathcal{X} \otimes \mathcal{Y}$ and $\mathcal{B}(\mathbb{R})$. Then for each $x \in X$ the function $y \mapsto f(x, y)$ is measurable with respecto to $\mathcal{Y}$ and $\mathcal{B}(\mathbb{R})$. Similarly for each $y \in Y$ the function $x \mapsto f(x, y)$ is measurable with respecto to $\mathcal{X}$ and $\mathcal{B}(\mathbb{R})$.*

The proof is short and so it's easiest to just include it here.

Proof. We first consider functions of the form $f = \mathbb{1}_C$ with $C \in \mathcal{X} \otimes \mathcal{Y}$. Let

$$\mathcal{H} = \{C \in \mathcal{X} \otimes \mathcal{Y} : y \mapsto \mathbb{1}_C(x, y) \text{ is } \mathcal{F} - \text{measurable for each fixed } x \in E\}.$$

It is easy to check that $\mathcal{H}$ is a σ-algebra. Moreover if $C = A \times B$ with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ then

$$y \mapsto \mathbb{1}_C(x, y) = \mathbb{1}_A(x)\mathbb{1}_B(y).$$

As x is fixed $\mathbb{1}_A(x)$ is just a constant and since $B \in \mathcal{Y}$ the function $y \mapsto \mathbb{1}_A(x)\mathbb{1}_B(y)$ must be measurable. Hence $\mathcal{X} \times \mathcal{Y} \subseteq \mathcal{H}$ and thus $\mathcal{X} \otimes \mathcal{Y} \subseteq \mathcal{H}$. But $\mathcal{H} \subseteq \mathcal{X} \otimes \mathcal{Y}$ and so $\mathcal{H} = \mathcal{X} \otimes \mathcal{Y}$. Hence if f is a simple function then the conclusion of the theorem holds.

Now consider $f \geq 0$ and let f_n be a sequence of simple functions increasing to f. Then for a fixed x the function $y \mapsto g_n(y) = f_n(x, y)$ is measurable. Moreover since $g(y) = \lim_{n\to\infty} g_n(y) = f(x, y)$ and since the limit of measurable functions is measurable we get the result for $f \geq 0$. For general $f = f^+ - f^-$ the result follows using the result for $f^+ \geq 0, f^- \geq 0$ and noting that the difference of measurable functions is measurable. $\square$

Consider measure spaces $(X, \mathcal{X}, \mu_x)$, $(Y, \mathcal{Y}, \mu_y)$. That is, X and Y are sets, $\mathcal{X}$ and $\mathcal{Y}$ are σ-algebras and μ_x and μ_y are measures on $\mathcal{X}$ and $\mathcal{Y}$ respectively. For all details on Fubini's Theorem we refer to Kolmogorov and Fomin [8].

Theorem A.31 (Fubini)**.** *Let μ be the Lebesgue extension of $\mu_x \otimes \mu_y$. Let $A \in \mathcal{X} \otimes \mathcal{Y}$. and let $f : A \to \mathbb{R}$ be a measurable function (considering $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$). If f is integrable i.e. if*

$$\int_A |f(x, y)| d\mu < \infty$$

then

$$\int_A f(x, y) d\mu = \int_X \left[\int_{A_x} f(x, y) d\mu_y \right] d\mu_x = \int_Y \left[\int_{A_y} f(x, y) d\mu_x \right] d\mu_y,$$

where $A_x := \{y \in Y : (x, y) \in A\}$ and $A_y := \{x \in X : (x, y) \in A\}$.

Remark A.32. The conclusion of Fubini's theorem implies that for μ_x-almost all x the integral $\int_{A_x} f(x, y) d\mu_y$ exists which in turn implies that the function $f(x, \cdot) : A_x \to \mathbb{R}$ must be measurable. This statement also holds if we exchange x for y.

A.3.3 Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given.

Theorem A.33. *Let X be an integrable random variable. If $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra then there exists a unique $\mathcal{G}$ measurable random variable Z such that*

$$\forall G \in \mathcal{G} \quad \int_G X d\mathbb{P} = \int_G Z d\mathbb{P}.$$

The proof can be found in xxxx xxxx.

Definition A.34. Let X be an integrable random variable. If $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra then $\mathcal{G}$-random variable from Theorem A.33 is called the *conditional expectation* of X given $\mathcal{G}$ and write $\mathbb{E}(X|\mathcal{G}) := Z$.

Conditional expectations are rather abstract notion so two examples might help.

Example A.35. Consider $\mathcal{G} := \{\emptyset, \Omega\}$. So $\mathcal{G}$ is just the trivial σ-algebra. For a random variable X we then have, by definition, that Z is the conditional expectation (denoted $\mathbb{E}[X|\mathcal{G}]$), if and only if

$$\int_\Omega Z d\mathbb{P} = \int_\Omega X d\mathbb{P}.$$

The right hand side of the above expression is in fact just $\mathbb{E}X$ and so the equality would be satisfied if we set $Z = \mathbb{E}X$ (just a constant). Indeed then (going right to left)

$$\mathbb{E}X = \int_\Omega X d\mathbb{P} = \int_\Omega Z d\mathbb{P} = \int_\Omega \mathbb{E}X d\mathbb{P} = \mathbb{E}X \int_\Omega d\mathbb{P} = \mathbb{E}X.$$

Example A.36. Let $X \sim N(0,1)$. Let $\mathcal{G} = \{\emptyset, \{X \leq 0\}, \{X > 0\}, \Omega\}$. One can (and should) check that this is a σ-algebra. By definition the conditional expectation is a unique random variable that satisfies

$$\begin{aligned} \int_\Omega \mathbb{1}_{\{X>0\}} Z d\mathbb{P} &= \int_\Omega \mathbb{1}_{\{X>0\}} X d\mathbb{P}, \\ \int_\Omega \mathbb{1}_{\{X\leq 0\}} Z d\mathbb{P} &= \int_\Omega \mathbb{1}_{\{X\leq 0\}} X d\mathbb{P}, \\ \int_\Omega Z d\mathbb{P} &= \int_\Omega X d\mathbb{P}. \end{aligned} \tag{A.8}$$

It is a matter of integrating with respect to normal density to find out that

$$\int_\Omega \mathbb{1}_{\{X>0\}} X d\mathbb{P} = \int_0^\infty x\phi(x)dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}, \quad \int_\Omega \mathbb{1}_{\{X\leq 0\}} X d\mathbb{P} = -\frac{1}{2}\sqrt{\frac{2}{\pi}}. \tag{A.9}$$

Since Z must be $\mathcal{G}$ measurable it can only take two values:

$$Z = \begin{cases} z_1 & \text{on} \quad \{X > 0\}, \\ z_2 & \text{on} \quad \{X \leq 0\}, \end{cases}$$

for some real constants z_1 and z_2 to be yet determined. But (A.8) and (A.9) taken together imply that

$$\frac{1}{2}\sqrt{\frac{2}{\pi}} = \int_\Omega \mathbb{1}_{\{X>0\}} Z d\mathbb{P} = \int_\Omega \mathbb{1}_{\{X>0\}} z_1 d\mathbb{P} = z_1 \mathbb{P}(X > 0) = \frac{1}{2} z_1.$$

Hence $z_1 = \sqrt{2/\pi}$. Similarly we calculate that $z_2 = -\sqrt{2/\pi}$. Finally we check that the third equation in (A.8) holds. Thus

$$\mathbb{E}[X|\mathcal{G}] = Z = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{on} \quad \{X > 0\}, \\ -\sqrt{\frac{2}{\pi}} & \text{on} \quad \{X \leq 0\}. \end{cases}$$

Here are some further important properties of conditional expectations which we present without proof.

Theorem A.37 (Properties of conditional expectations)**.** *Let X and Y be random variables. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.*

1. *If $\mathcal{G} = \{\emptyset, \Omega\}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.*
2. *If $X = x$ a. s. for some constant $x \in \mathbb{R}$ then $\mathbb{E}(X|\mathcal{G}) = x$ a.s..*
3. *For any $\alpha, \beta \in \mathbb{R}$*

   $$\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha \mathbb{E}(X|\mathcal{G}) + \beta \mathbb{E}(Y|\mathcal{G}).$$

   This is called linearity*.*

4. *If $X \leq Y$ almost surely then $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$ a.s..*

5. $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X|\,|\mathcal{G})$.

6. *If $X_n \to X$ a. s. and $|X_n| \leq Z$ for some integrable Z then $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ a. s. . This is the "dominated convergence theorem for conditional expectation".*

7. *If Y is $\mathcal{G}$ measurable then $\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G})$.*

8. *Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{G}$. Then*

$$\mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}).$$

This is called the tower property. *A special case is $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X|\mathcal{G}))$.*

9. *If $\sigma(X)$ is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.*

Definition A.38. Let X and Y be two random variables. The *conditional expectation of X given Y* is defined as $\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$, that is, it is the conditional expectation of X given the σ-algebra generated by Y.

Definition A.39. Let X a random variables and $A \in \mathcal{F}$ an event. The *conditional expectation of X given A* is defined as $\mathbb{E}(X|A) := \mathbb{E}(X|\sigma(A))$. This means it is the conditional expectation of X given the sigma algebra generated by A i.e. $\mathbb{E}(X|A) := \mathbb{E}(X|\{\emptyset, A, A^c, \Omega\})$.

We can immediately see that $\mathbb{E}(X|A) = \mathbb{E}(X|\mathbb{1}_A)$.

Recall that if X and Y are jointly continuous random variables with joint density $(x, y) \mapsto f(x, y)$ then for any measurable function $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbb{E}|\rho(X, Y)| < \infty$ we have

$$\mathbb{E}\rho(X, Y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x, y) f(x, y) dy dx.$$

Moreover the marginal density of X is

$$g(x) = \int_{\mathbb{R}} f(x, y) dy$$

while the marginal density of Y is

$$h(y) = \int_{\mathbb{R}} f(x, y) dx.$$

Theorem A.40. *Let X and Y be jointly continuous random variables with joint density $(x, y) \mapsto f(x, y)$. Then for any measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}|\varphi(Y)| < \infty$ the conditional expectation of $\varphi(Y)$ given X is*

$$\mathbb{E}(\varphi(Y)|X) = \psi(X)$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is given by

$$\psi(x) = \mathbb{1}_{\{g(x)>0\}} \frac{\int_{\mathbb{R}} \varphi(y) f(x, y) dy}{g(x)}.$$

Proof. Every A in $\sigma(X)$ must be of the form $A = \{\omega \in \Omega : X(\omega) \in B\}$ for some B in $\mathcal{B}(\mathbb{R})$. We need to show that for any such A

$$\int_A \psi(X) d\mathbb{P} = \int_A \varphi(Y) d\mathbb{P}.$$

But since $\mathbb{E}|\varphi(Y)| < \infty$ we can use Fubini's theorem to show that

$$\begin{aligned}\int_A \psi(X) d\mathbb{P} &= \mathbb{E}\mathbb{1}_A \psi(X) = \mathbb{E}\mathbb{1}_{\{X\in B\}}\psi(X) = \int_B \psi(x) g(x) dx \\ &= \int_B \int_{\mathbb{R}} \varphi(y) f(x,y) dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(x) \varphi(y) f(x,y) dx dy \\ &= \mathbb{E}\mathbb{1}_{\{X\in B\}}\varphi(Y) = \int_A \varphi(Y) d\mathbb{P}.\end{aligned}$$

□

Let on $(\Omega, \mathcal{F})$ be a measurable space. Recall that we say that a measure $\mathbb{Q}$ is absolutely continuous with respect to a measure $\mathbb{P}$ if $\mathbb{P}(E) = 0$ implies that $\mathbb{Q}(E) = 0$. We write $\mathbb{Q} << \mathbb{P}$.

Proposition A.41. *Take two probability measures $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{Q} << \mathbb{P}$ with*

$$d\mathbb{Q} = \Lambda d\mathbb{P}.$$

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then $\mathbb{Q}$ almost surely $\mathbb{E}[\Lambda|\mathcal{G}] > 0$. Moreover for any $\mathcal{F}$-random variable X we have

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \frac{\mathbb{E}[X\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]}. \tag{A.10}$$

Proof. Let $S := \{\omega : \mathbb{E}[\Lambda|\mathcal{G}](\omega) = 0\}$. Then $S \in \mathcal{G}$ and so by definition of conditional expectation

$$\mathbb{Q}(S) = \int_S d\mathbb{Q} = \int_S \Lambda d\mathbb{P} = \int_S \mathbb{E}[\Lambda|\mathcal{G}] d\mathbb{P} = \int_S 0 \, d\mathbb{P} = 0.$$

Thus $\mathbb{Q}$-a.s. we have $\mathbb{E}[\Lambda|\mathcal{G}](\omega) > 0$.

To prove the second claim assume first that $X \geq 0$. We note that by definition of conditional expectation, for all $G \in \mathcal{G}$:

$$\int_G \mathbb{E}[X\Lambda|\mathcal{G}] d\mathbb{P} = \int_G X\Lambda d\mathbb{P} = \int_G X d\mathbb{Q} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] d\mathbb{Q} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda d\mathbb{P}.$$

Now we use the definition of conditional expectation to take *another* conditional expectation with respect to $\mathcal{G}$. Since $G \in \mathcal{G}$:

$$\int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda d\mathbb{P} = \int_G \mathbb{E}\left[\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda|\mathcal{G}\right] d\mathbb{P}.$$

But $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable and so

$$\int_G \mathbb{E}\left[\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda|\mathcal{G}\right] d\mathbb{P} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\mathbb{E}\left[\Lambda|\mathcal{G}\right] d\mathbb{P}.$$

Thus, since in particular $\Omega \in \mathcal{G}$, we get

$$\int_\Omega \mathbb{E}[X\Lambda|\mathcal{G}]d\mathbb{P} = \int_\Omega \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\mathbb{E}\left[\Lambda|\mathcal{G}\right]d\mathbb{P}.$$

Since $X \geq 0$ (and $\Lambda \geq 0$) this means that $\mathbb{P}$-a.s. and hence $\mathbb{Q}$-a.s. we have (A.10).

$$\mathbb{E}[X\Lambda|\mathcal{G}] = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\mathbb{E}\left[\Lambda|\mathcal{G}\right].$$

For a general X write $X = X^+ - X^-$, where $X^+ = \mathbb{1}_{\{X\geq 0\}}X \geq 0$ and $X^- = -\mathbb{1}_{\{X<0\}}X \geq 0$. Then

$$\mathbb{E}^{\mathbb{Q}}[X^+ - X^-|\mathcal{G}] = \frac{\mathbb{E}[X^+\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]} - \frac{\mathbb{E}[X^-\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]} = \frac{\mathbb{E}[X^+ - X^-\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]}.$$

□

A.3.4 Multivariate normal distribution

There are a number of ways how to define a multivariate normal distribution. See e.g. [5, Chapter 5] for a more definite treatment. We will define a multivariate normal distribution as follows. Let $\mu \in \mathbb{R}^d$ be given and let Σ be a given symmetric, invertible, positive definite $d\times d$ matrix (it is also possible to consider positive semi-definite matrix Σ but for simplicity we ignore that situation here).

A matrix is positive definite if, for any $x \in \mathbb{R}^d$ such that $x \neq 0$, the inequality $x^T\Sigma x > 0$ holds. From linear algebra we know that this is equivalent to:

1. The eigenvalues of the matrix Σ are all positive.
2. There is a unique (up to multiplication by -1) matrix B such that $BB^T = \Sigma$.

Let B be a $d \times k$ matrix such that $BB^T = \Sigma$.

Let $(X_i)_{i=1}^d$ be independent random variables with $N(0,1)$ distribution. Let $X = (X_1,\ldots,X_d)^T$ and $Z := \mu + BX$. We then say $Z \sim N(\mu,\Sigma)$ and call Σ the covariance matrix of Z.

Exercise A.42. Show that $\mathrm{Cov}(Z_i,Z_j) = \mathbb{E}((Z_i - \mathbb{E}Z_i)(Z_j - \mathbb{E}Z_j)) = \Sigma_{ij}$. This justifies the name "covariance matrix" for Σ.

It is possible to show that the density function of $N(\mu,\Sigma)$ is

$$f(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}}\exp\left(-\frac{1}{2}((x-\mu)^T\Sigma^{-1}(x-\mu))\right). \tag{A.11}$$

Note that if Σ is symmetric and invertible then Σ^{-1} is also symmetric.

Exercise A.43. You will show that $Z = BX$ defined above has the density f given by (A.11) if $\mu = 0$.

i) Show that the characteristic function of $Y \sim N(0,1)$ is $t \mapsto \exp(-t^2/2)$. In other words, show that $\mathbb{E}(e^{itY}) = \exp(-t^2/2)$. *Hint.* complete the squares.

ii) Show that the characteristic function of a random variable Y with density f given by (A.11) is

$$\mathbb{E}\left(e^{i(\Sigma^{-1}\xi)^T Y}\right) = \exp\left(-\frac{1}{2}\xi^T\Sigma^{-1}\xi\right).$$

By taking $y = \Sigma^{-1}\xi$ conclude that

$$\mathbb{E}\left(e^{iy^T Y}\right) = \exp\left(-\frac{1}{2}y^T\Sigma^{-1}y\right).$$

Hint. use a similar trick to completing squares. You can use the fact that since Σ^{-1} is symmetric $\xi^T\Sigma^{-1}x = (\Sigma^{-1}\xi)^T x$.

iii) Recall that two distributions are identiacal if and only if their characteristic functions are identical. Compute $\mathbb{E}\left(e^{iy^T Z}\right)$ for $Z = BX$ and $X = (X_1, \dots, X_d)^T$ with $(X_i)_{i=1}^d$ independent random variables such that $X_i \sim N(0,1)$. Hence conclude that Z has density given by (A.11) with $\mu = 0$.

You can now also try to show that all this works with $\mu \neq 0$.

A.3.5 Stochastic Analysis Details

The aim of this section is to collect technical details in stochastic analysis needed to make the main part of the notes correct but perhaps too technical to be of interest to many readers.

Definition A.44. We say that a process X is called *progressively measurable* if the function $(\omega, t) \mapsto X(\omega, t)$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}([0,t])$ for all $t \in [0,T]$.

We will use Prog_T to denote the σ-algebra generated by all the progressively measurable processes on $\Omega \times [0,T]$.

If X is progressively measurable then the processes $\left(\int_0^t X(s)ds\right)_{t\in[0,T]}$ and $(X(t \wedge \tau))_{t\in[0,T]}$ are adapted (provided the paths of X are Lebesgue integrable and provided τ is a stopping time). The important thing for us is that any left (or right) continuous adapted process is progressively measurable.

A.3.6 More Exercises

Exercise A.45. Say $f : \mathbb{R} \to \mathbb{R}$ is smooth and $W = (W(t))_{t\in[0,T]}$ is a Wiener process. Calculate

$$\mathbb{E}\left[f'(W(T))W(T)\right].$$

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