# Stochastic Differential Equations Driven by Fractional Brownian Motion - a White Noise Distribution Theory Approach

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**Abstract**

We look in detail at the construction of white noise spaces using the Bochner-Minlos theorem. We then study fractional Brownian motion with an arbitrary Hurst parameter in the white noise space setting: (i) we demonstrate that time derivative of fractional Brownian motion exists as Hida distribution; (ii) we define an integral with respect to fractional Brownian motion as a white noise integral and (iii) using the S-transform, we prove, under certain conditions, existence and uniqueness of the solution, in the weak sense, for

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB_t^H$$

with any $H \in (0, 1)$.

**Keywords:** fractional Brownian motion, white noise distributions, stochastic differential equations

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# Contents

# Chapter 1

# Introduction

Fractional Brownian motion is a family of processes $B_t^H$, first studied mathematically by Mandelbrot and Van Ness in [MN68]. They proposed the term fractional Brownian motion process for a family of continuous centered Gaussian processes with covariance function given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \in \mathbb{R}, \quad H \in (0,1). \tag{1.1}$$

One can immediately see that for $H = 1/2$, this process is just classical Brownian motion.

Fractional Brownian motion (fBm), has a many interesting features. The main property is that for $H \neq 1/2$, the increments of fBm are not independent random variables, but rather their "span of interdependence" is, in a certain sense, infinite. From the statistical point of view, fBm seems to be a better model (than ordinary Brownian motion) for many natural phenomena. Indeed the term "Hurst parameter" is named after H. E. Hurst who first described observations of statistical data, which could be modelled well with times series based on fBm. In his case it the study of successive water flows among reservoirs along the Nile river. Recently fBm has become very popular in financial modelling. For example it has been estimated that the S&P 500 stock index has a Hurst parameter of about 0.6.

We will consider fBm in the white noise space setting and prove some of it's main properties in chapter 4. For now we mention that fBm processes are not (for $H \neq 1/2$) semi-martingales (see 4.8) and hence the classical Itô stochastic integration theory cannot be applied.

A different integration theory has to be developed for fractional Brownian motion. During the last several years, several approaches have been developed. First come many path-wise integration approaches, but they only apply to fBm with $H > 1/2$. Another approach is based on ordinary path-wise product in defining the integral for simple integrands. This construction leads to an integral with the properties of Stratonovich integral rather than the Itô integral. A different approach is based on the Wick product and this has been developed by [HØ03]. Using this approach, however a different probability spaces have to be considered for different $H$ and furthermore $H$ is assumed to be greater than $1/2$. There are also approaches based on Malliavin calculus. We follow an approach, first suggested, as far as the author is aware, by [Ben03].This covers all $H$ parameters of fBm and furthermore all processes are defined on the classical white noise space.

In chapter 2, we will prove the Bochner-Minlos theorem and we will use it to define the white noise probability space. In chapter 3, the concept of generalized random variables, the Hida distributions, is introduced. This allows us to talk rigorously about the time derivatives of fBm. In this chapter, we also introduce the S-transform and Wick product, which will be the main tools we will use to establish our result about SDEs driven by fBm. In chapter 4 we define Brownian motion on the white noise space and use this to construct fBm processes on the white noise space and prove some of it's main properties. In chapter 5 we define the white noise integrals and use this to construct an integral with respect to fractional Brownian motion.

The last chapter, chapter 6, contains our main result, which is the proof of existence and uniqueness of a "weak" solution to

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB_t^H, \tag{1.2}$$

provided that $b$ and $\sigma$ satisfy certain conditions. This is, as far as the author is aware, a new result. The main strength of our result comes from the fact that it provides a unified treatment for all values of $H \in (0, 1)$. The main disadvantage of our approach arises from the fact, that we use the S-transform as the main tool in the proof. Because of that, one has to calculate the S-transform of $b$ and $\sigma$, before our result can be applied. We will give some examples of SDEs, where our theorem can be applied, and examples where we cannot apply it. We also attempt to explain where do the limitations in our result arise from.

# Chapter 2

# White noise space

In this chapter, we will construct the white noise probability space and highlight some of its main properties. Bochner-Minlos theorem will be used as the main stepping stone of the construction. We will outline the key steps needed to prove this theorem. A complete proof can be found in [Hid80].

White noise space is crucial for defining stochastic distribution processes, for example the time derivative of Brownian motion.

## 2.1 Preliminaries

The Bochner Minlos theorem says that for any characteristic functional and countably-Hilbert nuclear space, there is a unique probability measure defined on this space.

So we see that Bochner-Minlos theorem is just a non-trivial extension of Bochner's theorem, extending Bochner's result to (some) infinite dimensional vector spaces. The proof of Bochner-Minlos theorem is based on Bochner's theorem, which we now state without proof. (For proof see [GV64]).

Recall that a function $f$ is called positive-definite if it satisfies: for any $n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, and $\xi_1, \ldots, \xi_n \in E$, we have

$$\sum_{j,k} \alpha_j \overline{\alpha}_k f(\xi_j - \xi_k) \geq 0; \tag{2.1}$$

**Theorem 2.1.1** (Bochner)**.** If $\varphi$ is positive definite, uniformly continuous function such that $\varphi(0) = 1$, then there exists a unique probability measure $\mu$ on $(\mathbb{R}, \mathcal{B})$ such that

$$\varphi(z) = \int_{\mathbb{R}} e^{izx} \mu(dx) \tag{2.2}$$

The key assumption of Bochner-Minlos' theorem is that the Hilbert space $E$ over which's dual $E^*$ we wish to find our unique probability measure is a countably nuclear Hilbert space[1]. When talking about countably nuclear Hilbert spaces, one can think, for example, about the triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

[1]For definition and some properties see appendix A.2

where we will show that $\mathcal{S}(\mathbb{R})$ is a countably nuclear Hilbert space and $\mathcal{S}'(\mathbb{R})$ is the space on which we wish to define the probability measure. In our project we will only ever consider $E = \mathcal{S}(\mathbb{R})$. To prove Bochner-Minlos' theorem, will need the corollary to this result:

**Lemma 2.1.2.** If $E$ is a countably-Hilbert nuclear space and $A \subset E$ is closed and bounded then $A$ is compact.

*Proof:* Assume $A \subset E$ is closed and bounded. Since $E$ is a countably Hilbert space we have an increasing sequence of norms on $E$ ($\|.\|_0$ denotes the "usual" norm on $E$):

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \cdots \leq \|\cdot\|_n \leq \cdots$$

Therefore $A$ is bounded in $\|.\|_0$ implies that A is bounded in $\|.\|_n$ for any $n$. Also, since $E$ is nuclear, the identity operator $\mathrm{I} : E_n \to E_m$ is of Hilbert-Schmidt type (see A.2.6) and hence $\mathrm{I} : E_n \to E_m$ is completely continuous and so $A$ is also closed w.r.t. the norms $\|.\|_n$ and it's closure (which thus coincides with $A$) is compact.

Now consider a sequence $(\xi_i)_{i\in\mathbb{N}} \subset A$. It has to have a convergent sub-sequence $(\xi_{i_k})^n_{k\in\mathbb{N}}$ w.r.t. all the norms $\|.\|_n$, because A is compact with respect to all the norms $\|.\|_n$. Therefore we have:

$$\begin{array}{cccccc} \xi^1_{i_1}, & \xi^1_{i_2}, & \cdots & ,\xi^1_{i_k}, & \cdots \\ \xi^2_{i_1}, & \xi^2_{i_2}, & \cdots & ,\xi^2_{i_k}, & \cdots \\ \vdots & \vdots & & & \ddots \end{array}$$

Taking the diagonal elements $\xi^1_{i_1}, \xi^2_{i_2}, \ldots, \xi^n_{i_n}, \ldots$ we get a sub-sequence of $(\xi_i)_{i\in\mathbb{N}}$ which is convergent w.r.t all the norms $\|.\|_n$ and hence w.r.t. the topology of the nuclear space $E$. Thus $A$ is compact in $E$. □

**Corollary 2.1.3.** If the space $E$ is nuclear, then a closed bounded set $A$ in its dual space $E'$ is compact relative to weak and strong convergence.

The last thing to do, before proving the Bochner-Minlos theorem, is to define what we mean by characteristic functionals over some Hilbert space.

**Definition 2.1.4** (Characteristic Functionals)**.** The functional $C(\xi)$, $\xi \in E$, is called the characteristic functional of a generalised stochastic process X, if it satisfies the following properties:

1. $C_X$ is continuous in $\xi \in E$
2. $C_X$ is positive definite, i.e. $\forall n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $\xi_1, \ldots, \xi_n \in E$, we have
$$\sum_{j,k} \alpha_j \overline{\alpha}_k C_X(\xi_j - \xi_k) \geq 0; \qquad (2.3)$$
3. $C_X(0) = 1$.

## 2.2 Bochner-Minlos Theorem

The first step will be to construct a sigma field over the space $E^*$. Since this will be done by considering finite dimensional subspaces of $E$, the next step will be to show, that our construction extends consistently, when considering subspaces of more dimensions. The third step will be to define a finitely additive set function and the final step will be to show that this function is actually countably additive, thus defining a probability measure with the desired properties. We will first state Bochner-Minlos theorem, but then organize the proof into a sequence of lemmas.

**Theorem 2.2.1** (Bochner-Minlos Theorem)**.** If $C(\xi)$ is a characteristic functional on $E$, then there exists a unique probability measure $\mu$ on $(E^*, \mathcal{B})$ such that

$$C(\xi) = \int_{E^*} e^{i\langle x,\xi\rangle} d\mu(x). \tag{2.4}$$

**Lemma 2.2.2.** We can define an algebra and finitely additive set function on $E^*$, forming a "finitely additive probability space" $(E^*, \mathcal{U}, m)$.

*Proof.* The first step. We wish to construct an algebra. To that end, we first define *cylinder sets* in $E^*$

$$A_{\xi_1,\ldots,\xi_n} = \{x \in E^* : (\langle x,\xi_1\rangle, \ldots, \langle x,\xi_n\rangle) \in B\} \tag{2.5}$$

where $\xi_1, \ldots, \xi_n$ are "points" in $E$ and $B$ is a Borel set in $\mathbb{R}^n$. Now take a subspace $F$ of $E$ and let $\mathcal{U}_F$ be the collection of all cylinder sets such that $(\xi_i)_{i=1\ldots n} \subset F$.

Next, consider the *annihilator* $F^a$ of $F$, which is the subspace of $E^*$ defined as

$$F^a = \{x \in E^* : \langle x,\xi\rangle = 0, \quad \forall\xi \in F\}.$$

We would like to show that the quotient space ${}^{E^*}\!/_{F^a}$ is isomorphic to $F^*$. To this end, we note that

$$\begin{aligned} \overline{x} \in {}^{E^*}\!/_{F^a} &\iff \overline{x} = \{y \in E^* : x - y \in F^a\} \\ &\iff \overline{x} = \{y \in E^* : \langle x - y, \xi\rangle = 0, \quad \forall\xi \in F\} \\ &\iff \overline{x} = \{y \in E^* : \langle x,\xi\rangle = \langle y,\xi\rangle, \quad \forall\xi \in F\} \end{aligned}$$

But if $\forall\xi \in F$ we have $\langle x,\xi\rangle = \langle y,\xi\rangle$ then $\overline{x} \cong y$ for $y \in F^*$. Therefore ${}^{E^*}\!/_{F^a}$ is indeed isomorphic to $F^*$.

Now define $\langle\overline{x},\xi\rangle_F = \langle x,\xi\rangle$. Also, say $\dim F = n$, then also $\dim F^* = n$ and $F \cong \mathbb{R}^n \cong F^*$, hence $\langle .,.\rangle_F$ defines an inner product in $\mathbb{R}^n$.

We can also restrict $C(\xi)$ to $C_F(\xi)$ and by (2.3) we can view it as a characteristic function on $\mathbb{R}^n$. Thus, we can define a $\sigma$-algebra on $F^*$ by

$$\mathcal{B}_F = \mathcal{B}(\mathbb{R}^n)$$

and furthermore by Bochner's theorem 2.1.1 there exists a unique probability measure $\mu_F$ on $F^*$ (and hence on ${}^{E^*}\!/_{F^a}$) satisfying

$$C_F(\xi) = \int_{{}^{E^*}\!/_{F^a}} e^{i\langle x,\xi\rangle} d\mu_F$$

Finally let $\rho_F : E^* \to {}^{E^*}\!/_{F^a}$ be $\rho(x) = \overline{x}$. Then

$$\rho_F^{-1}(\mathcal{B}_F) = \mathcal{U}_F$$

and hence $\mathcal{U}_F$ is a $\sigma$-algebra. Now let

$$\mathcal{U} = \bigcup_{F \subset E} \mathcal{U}_F$$

where the union taken over all the finite dimensional subspaces $F$. Such $\mathcal{U}$ is only an algebra and hence we let $\mathcal{B}$ be the smallest $\sigma$-algebra containing $\mathcal{U}$. Thus, we have defined a $\sigma$-algebra over $E^*$.

The second step. Assume that $F$ and $G$ are two finite dimensional subspaces of $E$ such that $F \subset G$. Then

$${}^{E^*}\!/_{G^a} \cong G^* \supset F^* \cong {}^{E^*}\!/_{F^a}$$

and so we can define a projection map

$$T : {}^{E^*}\!/_{G^a} \to {}^{E^*}\!/_{F^a} \quad \text{as} \quad T(\overline{x}^G) = \overline{x}^F.$$

So for $B \in \mathcal{B}_F$, we have $\mu_F(B) = \mu_G(T^{-1}B)$, because $C_F(\xi) = C_G(\xi)|_F$ and by Bochner's theorem, $\mu_F$ is unique on $F^*$. Thus $(G^*, \mathcal{U}_G, \mu_G)$ extends uniquely to $(F^*, \mathcal{U}_F, \mu_F)$.

The third step. Let's now define finitely additive $m$ on

$$\mathcal{U} = \bigcup_{\substack{F \subset E \\ F \text{ f.d.}}} \mathcal{U}_F$$

as follows: for any cylinder set $A \in \mathcal{U}$, there exists a finite dimensional subspace $F$ such that $A \in \mathcal{U}_F$, so we define $m(A) = \mu_F(A)$ and by part two, this is independent of the choice of $F$.

Consider cylinder sets $A_1, \ldots, A_n \in \mathcal{U}$ which are pairwise disjoint, based on finite dimensional $F_1, \ldots, F_n$ respectively. Note that $F = \mathrm{Span}(F_1, \ldots, F_n)$ is also finite dimensional and $A_i \in F$ and hence $\mu_{F_i}(A_i) = \mu_F(A_i)$. Since $\mu_F$ is $\sigma$-additive on $\mathcal{U}_F$ we have

$$m\left(\bigcup_{i=1\ldots n} A_i\right) = \mu_F\left(\bigcup_{i=1\ldots n} A_i\right) = \sum_{i=1}^{n} \mu_F(A_i) = \sum_{i=1}^{n} m(A_i)$$

Thus, we have proved that $(E^*, \mathcal{U}, m)$ is a finitely additive probability space. $\square$

Our aim is now to extend $m$ to measure on the space $(E^*, \mathcal{B})$. We will omit the proof of the following technical lemma. Proof can be found in [Hid80].

**Lemma 2.2.3.** Let $\mu$ be a probability measure on $\mathbb{R}$ and denote by $\eta$ the ellipsoid

$$\{z = (z_1, \ldots, z_2) : \sum_1^n a_i^2 z_i^2 \leq \gamma^2\}.$$

If the characteristic function $\varphi(z)$, $z \in \mathbb{R}^n$, of $\mu$ satisfies

$$|\varphi(z) - 1| < \epsilon, \quad z \in \eta,$$

then for the ball $S(t)$ in $\mathbb{R}^n$ of radius $t$ we have the inequality

$$\mu(S(t)^c) < \beta^2 \left( \epsilon + \frac{2}{\gamma^2 t^2} \sum_{i=1}^{n} a_i^2 \right),$$

where $\beta$ is a positive constant independent of $n$ and $t$.

A further lemma gives a condition on the space $E^*$ for a measure $\mu$ to extendable.

**Lemma 2.2.4.** A finitely additive measure $m$ is extendable to a countably additive measure on $(E^*, \mathcal{B})$ if and only if the following holds:

$\forall \epsilon > 0$ there exists a natural number $n$ and a ball
$$S_n = \{x \in E^* : \|x\|_{-n} \leq \gamma_n\},$$
such that for any cylinder set $A \in \mathcal{U}$ disjoint from $S_n$ we have
$$\mu(A) < \epsilon. \tag{2.6}$$

*Proof.* Assume first, that $m$ has a $\sigma$-additive extension $\mu$. Choose a sequence $S_N$ of balls with increasing radii $\gamma_n$ such that $\gamma_n \to \infty$. Then

$$\bigcup_n S_n = E^*$$

and so, as $\mu$ is $\sigma$-additive,
$$\mu(S_n^c) < \epsilon$$
must hold for sufficiently large $n$. Hence for a cylinder set $A$ in $S_n^c$ we must have

$$\mu(A) < \epsilon.$$

Now assume that (2.6) holds. We will show that $m$ is extendable to $\sigma$-additive $\mu$ by contradiction.

Assume that $A_n$ is a sequence of pairwise disjoint elements of $\mathcal{U}$ such that $\bigcup_n A_n = E^*$. Since $m$ is finitely additive,

$$m\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} m(A_k) \leq 1$$

and so
$$\sum_{k=1}^{\infty} m(A_k) \leq 1.$$

Now assume that the above inequality is strict. Then there is $\epsilon > 0$ such that

$$\sum_{k=1}^{\infty} m(A_k) = 1 - 3\epsilon < 1.$$

From Lebesgue measure theory we know that for any open set $B \in \mathcal{B}$, we can find an open set $B'$, such that $B \subset B'$ and $\lambda(B' \setminus B) < \epsilon$. Hence for each $A_n$ we can find an open cylinder set $A'_n$ (i.e. the set $B'$ in (2.5) is open) such that $A'_n \supset A_n$ and
$$m(A'_n \setminus A_n) < \frac{\epsilon}{2^n}.$$

Since we're assuming that $\bigcup_n A_n = E^*$, it must hold that $\bigcup_j A'_j \supset S_n$. Since $S_n$ are (weakly) closed and bounded, we have by corollary 2.1.3, that they are weakly compact. Hence we can choose a finite number $A'_1, \ldots, A'_k$ of the $A'_j$s which cover $S_n$. Let $A' = \bigcup_{j=1}^n A'_j$ we have $A' \in \mathcal{U}$ and

$$1 = m(A' + A'^c) = m(A') + m(A'^c)$$

$$m(A') \leq \sum_{j=1}^{k} m(A') + \epsilon$$

also since, $\mu(S_n^c) < \epsilon$ (our hypothesis) and $A'^c \subset S_n^c$ we have

$$m(A'^c) < \epsilon$$

So combining the above three inequalities, we get

$$1 \leq \sum_{j=1}^{k} m(A_j) + \epsilon + \epsilon \leq (1 - 3\epsilon) + 2\epsilon = 1 - \epsilon,$$

clearly a contradiction. This completes the proof. □

**Lemma 2.2.5.** If $C(\xi)$, $\xi \in E$ is a characteristic functional and $E$ is a nuclear space, then there exists a unique (countably additive) extension $\mu$ of $m$ to $(E^*, B)$.

*Proof.* Since $C(\xi)$ is a characteristic functional, it is continuous in the norm $\|.\|_p$ for some $p$. Thus for any $\epsilon > 0$, there exists a ball $U$ in the space[2] $E_p$, such that

$$|C(\xi) - C(0)| = |C(\xi) - 1| < \frac{\epsilon}{2\beta^2}, \qquad \xi \in U,$$

where $\beta$ is the constant in lemma 2.2.3.

We note that since $E$ is nuclear, the projection map $\mathrm{I}_p^n : E_n \to E_p$ is of Hilbert-Schmidt type, for some $n > p$. Since $\|.\|_n \geq \|.\|_p$, there is a neighbourhood $V$ of 0 in $E_n$ such that

$$\mathrm{I}_p^n V \subset U.$$

We can then show that the ball $S_n$ in $E_n^*$ with radius

$$t = \frac{2\beta \| \mathrm{I}_p^n \|}{\sqrt{\gamma^2 \epsilon}}$$

corresponds to the ball in lemma (2.2.4). Indeed if $A$ is a cylinder set based on the finite dimensional subspace $F$ and disjoint from $S_n$, then there exists an $n$-dimensional Borel set B such that $A = \rho_F^{-1}(B)$ and

$$B \cap \rho_f(S_n) = \emptyset.$$

Now we note that since $V \cap F$ is a finite dimensional ellipsoid in the norm $\|.\|_n$, it can be expressed in Cartesian co-ordinates in the form $\sum_i a_i^2 z_i^2 \leq \gamma^2$.

[2] The same linear space as for $E$, but considered with the norm $\|.\|_p$

We also note that $\sum_i a_i^2 \leq \| \mathrm{I}_p^n \|^2$. Using lemma 2.2.3, we have

$$\begin{aligned} m_F(\rho_F(S_n^c)) &< \beta^2 \left( \frac{\epsilon}{2\beta^2} + \frac{2}{\gamma^2} t^2 \sum_{i=1}^{k} a_i^2 \right) \\ &< \frac{1}{2}\epsilon + \frac{2\beta^2}{\gamma^2 t^2} \|I_p^n\|^2 = \epsilon. \end{aligned}$$

This guarantees the existence of an extension to $m$. Uniqueness comes from general measure theory, as we're dealing with probability measures. □

*Proof of Bochner-Minlos' theorem:* By the lemma 2.2.5, we have that $\mu$ is countably additive. Let $\omega_n$ be some basis of $E^*$. Let $F_n^* = \mathrm{Span}(\omega_n)$. Then $E^* = \bigcup_{n\in\mathbb{N}} F_n$. We have shown that for finite-dimensional subspaces of $E^*$ we have, by Bochner's Theorem:

$$\int_{F_n^*} e^{\langle \bar{x},\xi\rangle} dm_F(\bar{x}) = C_{F^n}(\xi) = C(\xi)\Big|_{F^n} \tag{2.7}$$

And hence by uniqueness and by $\sigma$-additivity of $\mu$ we have:

$$\int_{E^*} e^{\langle x,\xi\rangle} d\mu(x) = \sum_{n\in\mathbb{N}} \int_{F_n^*} e^{\langle \bar{x},\xi\rangle} dm_F(\bar{x}) = \sum_{n\in\mathbb{N}} C(\xi)\Big|_{F^n} = C(\xi) \tag{2.8}$$

□

We can now apply the Bochner-Minlos theorem, to define the white noise probability space. We do this by proving that the space of Schwartz test functions is a countably Hilbert nuclear space.

## 2.3 Example of a nuclear Hilbert space

**Lemma 2.3.1.** The space of Schwartz test functions

$$\mathcal{S}(\mathbb{R}) = \left\{ \xi : \sup_{x\in\mathbb{R}} \left| x^n \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n \xi(x) \right| < \infty \quad \forall n \in \mathbb{N} \right\} \tag{2.9}$$

is a countably Hilbert nuclear space.

*Proof:* Let

$$\mathrm{A} = -\left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^2 + x^2 + 1.$$

Let

$$h_n(x) = (-1)^n e^{x^2} \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n e^{-x^2},$$

i.e. $h_n$ is the $n$th Hermite polynomial and define the Hermite functions as:

$$e_k(x) = (\pi^{1/2}(k-1)!)^{-1/2} \exp\left(-\frac{x^2}{2}\right) h_k(x) \tag{2.10}$$

The Hermite functions form an orthonormal basis of $L^2(\mathbb{R})$ (see A.3) and also it is possible to check that the Hermite functions are the eigenvectors of A:

$$\mathrm{A}\, e_n = (2n+2)e_n \tag{2.11}$$

For $p \geq 0$ define $|\xi|_p = |\,\mathrm{A}^p\,\xi|_0$ where $|.|_0$ is the $L^2(\mathbb{R})$ norm. Define the spaces

$$\mathcal{S}_p = \{\xi \in L^2(\mathbb{R}) : |\xi|_p < \infty\} \tag{2.12}$$

One can immediately see that the topology of the space $\bigcap_{p\geq 0} \mathcal{S}_p$ is equivalent to the usual topology of $\mathcal{S}(\mathbb{R})$ and hence the space is topologised by an increasing sequence of norms and so it is a countably Hilbert space.

To show that it's a nuclear space, we have to show that the identity operator $\mathrm{I} : \mathcal{S}_{p+1} \to \mathcal{S}_p$ is of Hilbert-Schmidt type. To this end we note that $(e_k)_{k\in\mathbb{N}} \subset \mathcal{S}_p$ and that

$$f_k^{(p)} = (2k+2)^{-p} e_k \tag{2.13}$$

form an orthonormal basis in each $\mathcal{S}_p$. Also

$$\sum_k |\,\mathrm{I}\, f^{(p+1)}|_p^2 = \sum_k (2k+2)^{-2} < \infty \tag{2.14}$$

So by theorem A.2.7 we have that I is of Hilbert-Schmidt type and so the space $\mathcal{S}(\mathbb{R})$ is nuclear. □

We also define the norms, for $p \geq 0$,

$$|\xi|_{-p}^2 = |\,\mathrm{A}^{-p}\,\xi|_0^2 \tag{2.15}$$

and the spaces

$$\mathcal{S}_{-p} = \{\xi \in L^2(\mathbb{R}) : |\xi|_{-p} < \infty\} \tag{2.16}$$

**Corollary 2.3.2.**

$$|f|_{-p}^2 = |A^{-p} f|_0^2 = \sum_{k=0}^{\infty} (2k+2)^{-2p} (f, e_k)_0^2$$

To summarize, we have a sequence of norms on $\mathcal{S}(\mathbb{R})$:

$$\ldots \leq |f|_{-p}^2 \leq \ldots \leq |f|_{-1}^2 \leq |f|_0^2 \leq |f|_1^2 \leq \ldots \leq |f|_p^2 \leq \ldots. \tag{2.17}$$

Note that the operator A as defined above is linear, but not self-adjoint. To see that it's not self-adjoint, consider

$$(\mathrm{A}\, e_n, e_m)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (2n+2) e_n(x) e_m(x) dx \neq (e_n, \mathrm{A}\, e_m)_{L^2(\mathbb{R})}$$

## 2.4 The construction of white noise space

Let $\Omega = S'(\mathbb{R})$, the space of all continuous liner functionals on the space of Schwartz test functions. Then by 2.2.1 we have a $\sigma$-algebra $\mathcal{B}$ on this space, and a unique probability measure $\mu$ which satisfies

$$\forall \phi \in S(\mathbb{R}) \qquad \int_{S'(\mathbb{R})} \exp(i\langle \omega, \phi \rangle) d\mu(\omega) = \exp\left(-\frac{1}{2}\|\phi\|^2\right) \tag{2.18}$$

Therefore we can define the white noise probability space over $S'(\mathbb{R})$ to be $(S'(\mathbb{R}), \mathcal{B}, \mu)$. We also note that since $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we can extend the white noise space to contain all $g \in L^2(\mathbb{R})$. Indeed, for any $g \in L^2(\mathbb{R})$ there exist $(\phi_n)_{n\in\mathbb{N}} \subset S(\mathbb{R})$ such that $\phi_n \to g$ in $L^2(\mathbb{R})$ as $n \to \infty$ and so we define

$$\langle \omega, g\rangle := \lim_{n\to\infty} \langle \omega, \phi_n\rangle$$

Immediately we get these useful corollaries:

**Corollary 2.4.1.** The random variable $\langle ., f\rangle$ is normally distributed with mean 0 and variance $\|f\|^2_{L^2(\mathbb{R})} = |f|_0^2$.

*Proof.* Let F(t) = $\mu(\langle ., f\rangle \le t)$. Then

$$\widehat{F}(\xi) = \int_{\mathbb{R}} e^{2\pi i t\xi} F(t)dt = \int_{\Omega} e^{i\langle \omega, 2\pi\xi f\rangle} d\mu(\omega) = e^{-\frac{1}{2}(2\pi)^2\xi^2|f|_0^2}$$

and hence

$$F(t) = \int_{\mathbb{R}} e^{-2\pi i t\xi} e^{-\frac{1}{2}(2\pi)^2\xi^2|f|_0^2} d\xi = \frac{1}{\sqrt{2\pi}|f|_0} e^{-\frac{t^2}{2|f|_0^2}}$$

□

**Corollary 2.4.2** (Itô's isometry)**.**

$$\forall f \in L^2(\mathbb{R}) \qquad \mathbb{E}_\mu\left(\langle ., f\rangle^2\right) = \|f\|^2_{L_2(\mathbb{R})} \tag{2.19}$$

*Proof.* This is a direct consequence of the above result. □

**Corollary 2.4.3.**

$$\forall f, g \in L^2(\mathbb{R}) \qquad \mathbb{E}_\mu\left(\langle ., f\rangle\langle ., g\rangle\right) = (f, g)_{L^2(\mathbb{R})} \tag{2.20}$$

*Proof:* This is a simple consequence of:

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{4}\left(\|f+g\|^2_{L^2(\mathbb{R})} - \|f-g\|^2_{L^2(\mathbb{R})}\right)$$

□

# Chapter 3

# Hida distributions and test functions

## 3.1 Construction

Now that we have the established some properties of the white noise space we would like to consider the random variables on this probability space. Define:

$$(L_2) = L_2(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \|\varphi\|_{L_2(\Omega)} := \int_{S'(\mathbb{R})} \varphi^2(\omega) d\mu(\omega) < \infty\} \tag{3.1}$$

Many properties of fBm (that would hold for arbitrary Hurst parameter) are difficult or even impossible to prove directly, however it is possible to prove them for generalized functionals of fBm. For instance fBm with $H \in (0,1)$ is nowhere differentiable on almost every path (see 4.6), but it is differentiable as a mapping from $I \subset \mathbb{R}$ into the space of stochastic generalized functions, the Hida distributions (see 4.7.6).

To construct the space of Hida test functions and Hida distributions we follow an approach similar to the one taken in lemma 2.3.1. Of course here, we're not trying to reconstruct an already "known" space but rather to define a new one.

**Definition 3.1.1** (Second Quantization operator)**.** The operator $\Gamma(\mathrm{A})$ is an operator on $L^2(\Omega)$ given by: For[1] $\varphi = \sum_{n=0}^{\infty} \mathrm{I_n}(f_n)$,

$$\Gamma(\mathrm{A})\,\varphi = \sum_{n=0}^{\infty} \mathrm{I_n}(\mathrm{A}^{\otimes \mathrm{n}}\, f_n) \tag{3.2}$$

where A is defined as in (2.3.1), that is $\mathrm{A} = -\frac{d^2}{dx^2} + x^2 + 1$.

A little comment on the notation (see e.g. [Coo53]): the multiple Wiener integral operator $\mathrm{I_n}$ is defined as $\mathrm{I_n} : \widehat{L^2}(\mathbb{R}^n) \to \mathbb{R}$. The space[2] $\widehat{L^2}(\mathbb{R}^n)$ is viewed as the tensor product of the Hilbert spaces $\widehat{L^2}(\mathbb{R})$ and since A is densely defined

[1]This expansion is justified by the Wiener-Itô chaos expansion theorem (see A.1.2)

[2]The space $\widehat{L^2}(\mathbb{R})$ is the subspace of $L^2(\mathbb{R})$ containing only symmetric functions. See appendix (A.1).

linear operator on $\widehat{L^2}(\mathbb{R})$, $\mathrm{A}^{\otimes \mathrm{n}}$ is also a linear a densely defined operator, but on the space $\widehat{L^2}(\mathbb{R}^n)$.

**Lemma 3.1.2.** The Second Quantization Operator $\Gamma(\mathrm{A})$ is densely defined on $L^2(\Omega)$ and the functions $\varphi_\alpha = \frac{1}{\sqrt{(\alpha_1! \ldots \alpha_n!)}} \mathrm{I_n}(e_1^{\otimes \alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{\otimes \alpha_n})$ are the eigenvectors of the operator, with eigenvalues $(2^{\alpha_1} \ldots (2n+2)^{\alpha_n})$ for all multi-indexes $\alpha$.

*Proof:* From the second version of the Wiener-Itô Chaos expansion theorem (A.1.3), we know that the functions of the form,

$$\varphi_\alpha = \frac{1}{\sqrt{(\alpha_1! \ldots \alpha_n!)}} \mathrm{I_n}(e_1^{\otimes \alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{\otimes \alpha_n}) \tag{3.3}$$

where $\alpha$ is a multi-index, form an orthonormal basis of $L^2(\Omega)$. We note that these are Wiener-Itô chaos expansions consisting of a single term and thus we may apply $\Gamma(\mathrm{A})$ directly and hence the series defining $\Gamma(\mathrm{A})$ is clearly convergent. So $\Gamma(\mathrm{A})$ is densely defined on $L^2(\Omega)$.

To see that $\varphi_\alpha$ are eigenvectors of the operator $\Gamma(\mathrm{A})$ we consider:

$$\begin{aligned}
\Gamma(\mathrm{A})\, \varphi_\alpha &= \frac{1}{\sqrt{(\alpha_1! \ldots \alpha_n!)}} \mathrm{I_n}(\mathrm{A}^{\otimes \mathrm{n}}[e_1^{\otimes \alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{\otimes \alpha_n}]) && (3.4)\\
&= \frac{2^{\alpha_1} \ldots (2n+2)^{\alpha_n}}{\sqrt{(\alpha_1! \ldots \alpha_n!)}} \mathrm{I_n}(e_1^{\otimes \alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{\otimes \alpha_n}) && (3.5)\\
&= (2^{\alpha_1} \ldots (2n+2)^{\alpha_n})\varphi_\alpha && (3.6)
\end{aligned}$$

Where the second equality follows from the fact that the Hermite functions are the eigenvectors of the operator A i.e. $\mathrm{A}\, e_j = (2j+2)e_j$. □

Thus we can define, for $p \geq 0$ and $\phi \in L^2(\Omega)$, the norms

$$\|\phi\|_p = \|\, \Gamma(\mathrm{A})^p\, \phi\|_{L^2(\Omega)}$$

and the spaces

$$(S_p) = S_p(\Omega) = \{\phi \in L^2(\Omega) : \|\phi\|_p < \infty\}$$

Finally we define the space of stochastic test functions.

**Definition 3.1.3** (Stochastic test functions)**.** The space of stochastic test functions $(S) = S(\Omega)$ is defined as

$$S(\Omega) = \bigcap_{p \geq 0} S_p(\Omega)$$

The topology on this space is defined to be the projective limit topology, that is the smallest topology such that for all $p \geq 0$, the identity map $I : S(\Omega) \to S_p(\Omega)$ is continuous.

**Corollary 3.1.4.** The spaces $S_p(\Omega)$ have an orthonormal basis formed by the functions of the form:

$$\varphi_\alpha = \frac{(2^{\alpha_1} \ldots (2n+2)^{\alpha_n})^{-p}}{\sqrt{(\alpha_1! \ldots \alpha_n!)}} \mathrm{I_n}(e_1^{\otimes \alpha_1} \widehat{\otimes} \ldots \widehat{\otimes} e_n^{\otimes \alpha_n}) \tag{3.7}$$

*Proof:* From (3.5) we see that $\|\varphi_\alpha\|_p < \infty$ and since $S_p(\Omega) \subset L^2(\Omega)$, the result follows from the second version of the Wiener-Itô Chaos expansion theorem (A.1.3). □

A more general construction, which provides a construction of test functions and distributions on different infinite dimensional spaces can be found in ([Kuo96]).

## 3.2 Some properties

The following two lemmas do not actually not use any special properties of the spaces $S_p(\Omega)$. Indeed, in [GV64] the topic is treated in a general setting. Nevertheless, stating them directly gives more intuitive understanding of the properties of $S(\Omega)$.

**Lemma 3.2.1.** The topology on $S(\Omega)$ as defined above coincides with the topology given by the metric:

$$\rho(\varphi, \psi) = \sum_{p \geq 0} 2^{-p} \frac{\|\varphi - \psi\|_p}{1 + \|\varphi - \psi\|_p} \tag{3.8}$$

*Proof:* We will show that for all $p \geq 0$ the identity map $\mathrm{I} : S(\Omega) \to S_p(\Omega)$ is continuous w.r.t. the metric (3.8). Consider a sequence $(\psi_n)_{n\in\mathbb{N}} \subset S(\Omega)$ converging to some $\psi$ in $S(\Omega)$. Assume that there is $p_0 \geq 0$ such that $\psi_n$ does not converge to $\psi$ as $n \to \infty$ w.r.t. $\|.\|_p$.

That is assume, $\exists \epsilon > 0$ such that for all $n_0 \in \mathbb{N}$ there is some $n > n_0$ for which $\|\psi - \psi_n\|_{p_0} > \epsilon$.

But this would mean that

$$\sum_{p \geq 0} 2^{-p} \frac{\|\psi_n - \psi\|_p}{1 + \|\psi_n - \psi\|_p} \geq 2^{-p_0} \frac{\epsilon}{1 + \|\psi_n - \psi\|_{p_0}} \geq 2^{-p_0} \frac{\epsilon}{2} \tag{3.9}$$

since we may, without loss of generality, assume that $\|\psi_n - \psi\|_{p_0} < 1$.

But this would clearly contradict our assumption that $(\psi_n)_{n\in\mathbb{N}} \subset S(\Omega)$ converges to some $\psi$ in $S(\Omega)$ and hence we conclude that for all $p \geq 0$ the identity map $I : S(\Omega) \to S_p(\Omega)$ is continuous w.r.t. the metric (3.8).

So the topology defined by the metric (3.8) is contained in the projective limit topology of $S(\Omega)$ and since the projective limit topology is defined as the smallest topology such that $\mathrm{I} : S(\Omega) \to S_p(\Omega)$ is continuous for all $p \geq 0$ we see that the two topologies must be equal. □

**Definition 3.2.2.** We define the *Hida distributions* as the space of all continuous linear functionals on the space of Hida test functions. We denote Hida distributions by $S'(\Omega)$ or $(S)^*$.

**Lemma 3.2.3.** The space of all continuous linear functionals on $S(\Omega)$, denoted $S'(\Omega)$ $(= (S)^*)$ is equal to the union of the spaces $S'_p(\Omega)$ (sometimes denoted $(S_p)^*$).

*Proof:* If there is $p \geq 0$ such that $F : S_p(\Omega) \to \mathbb{R}$, then for $U \subset \mathbb{R}$, $F^{-1}(U)$ is open in $S(\Omega)$, since $I : S(\Omega) \to S_p(\Omega)$ is continuous by definition. Therefore $\bigcup_{p\geq 0} S_p'(\Omega) \subset S'(\Omega)$.

Now assume that $F : S(\Omega) \to \mathbb{R}$ is continuous, i.e. assume that for any sequence $(\varphi_n)_{n\in\mathbb{N}}$ convergent to $\varphi$ in the metric $\rho$. We have to show that there is $p \geq 0$ such that $\|\varphi_n - \varphi\|_p \to 0$ as $n \to \infty$ implies that $|F(\varphi_n) - F(\varphi)| \to 0$ in $\mathbb{R}$. We will do this by contradiction.

Assume that $\forall p \geq 0$, $\|\varphi_n - \varphi\|_p \to 0$ but $\lim_{n\to\infty} |F(\varphi_n) - F(\varphi)| \neq 0$, that is $\exists \epsilon > 0$ such that $\forall n_0 \in \mathbb{R} \quad \exists n > n_0$ so that $|F(\varphi_n) - F(\varphi)| > \epsilon$. This would contradict the assumption that $\varphi_n \to \varphi$ in the metric $\rho$ implies $|F(\varphi_n) - F(\varphi)| \to 0$.

Hence there exists $p \geq 0$ such that $\|\varphi_n - \varphi\|_p \to 0$ and hence $F$ is also continuous as a map from $F : S_p(\Omega) \to \mathbb{R}$ and hence $\bigcup_{p\geq 0} S_p'(\Omega) = S'(\Omega)$. $\square$

As we mentioned above, the proofs do not use any particular properties of the spaces $S_p(\Omega)$. Indeed, exactly the same proof as above would give us this lemma:

**Lemma 3.2.4.** The Schwartz distributions $\mathcal{S}'(\mathbb{R})$, that is continuous linear functionals on $\mathcal{S}(\mathbb{R})$, is equal to the union of the spaces $\mathcal{S}_p'(\mathbb{R})$.

It can also be shown that the norms on the dual spaces $(S_p)^*$ of $(S_p)$ are given, for $p > 0$, by

$$\|\phi\|_{-p} = \|\,\Gamma(\mathrm{A})^{-p}\,\phi\|_0$$

To summarize, we have a sequence of norms on $S(\Omega) = (S)$:

$$\ldots \leq \|\phi\|_{-p}^2 \leq \ldots \leq \|\phi\|_{-1}^2 \leq \|\phi\|_0^2 \leq \|\phi\|_1^2 \leq \ldots \leq \|\phi\|_p^2 \leq \ldots. \tag{3.10}$$

The space $(S)^*$ is the dual of $(S)$ and it follows (see [GV64]) from the fact that $(S)$ is a countably Hilbert space. For $\Phi \in (S)^*$ and $\psi \in (S)$, we write

$$\Phi(\psi) = \langle\!\langle \Phi, \psi \rangle\!\rangle$$

and if $\Phi \in (L^2)$, then

$$\langle\!\langle \Phi, \psi \rangle\!\rangle = \mathbb{E}\,(\Phi\psi) = \int_{\mathcal{S}'(\mathbb{R})} \Phi(\omega)\Psi(\omega)d\mu(\omega),$$

due to the linearity and continuity of integration.

The concept of stochastic (Hida) test functions and stochastic distributions may seem rather abstract. The motivation for introducing these spaces, however, is very similar to that behind using Schwartz test functions and distributions.

## 3.3 The S transform and characterization theorems

The analogy with Schwartz distributions can be taken even further, because we can define the following “integral” transform:

**Definition 3.3.1.** For $\Phi \in (S)^*$, we define it's *S-transform* as

$$S\Phi(\psi) = \langle\langle \Phi, :e^{\langle \cdot, \psi \rangle}: \rangle\rangle \text{ for } \psi \in \mathcal{S}(\mathbb{R}), \tag{3.11}$$

where $:e^{\langle \cdot, \psi \rangle}:$ is the *Wick exponential* and is defined as

$$:e^{\langle \cdot, \psi \rangle} := \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{I_n}(\psi^{\otimes n}). \tag{3.12}$$

We now state some properties of S-transforms. They are absolutely essential for our main result on stochastic differential equations driven by fBm. The proofs can be all found in [Kuo96] and most of the white noise distribution theory rests on these results. Luckily, the results are mostly self-contained and can be used and understood without detailed understanding of the proofs. The following two theorems are simply the proposition 5.1. and theorem 8.2. in [Kuo96] respectively.

**Theorem 3.3.2** (S-transform is injective)**.** If $\Phi, \Psi \in (S)^*$ and $S\Phi = S\Psi$, then $\Phi = \Psi$.

**Theorem 3.3.3.** Assume that $\Phi \in (S)^*$. Then it's S-transform $F = S\Phi$ satisfies the following conditions:

1. For any $\xi, \nu \in \mathcal{S}(\mathbb{R})$, the function $F(z\xi + \nu)$ is an entire (analytic) function of $z \in \mathbb{C}$.

2. There exist non-negative constants $K, p$ such that

$$|F(\xi)| \leq K \exp\left(\frac{1}{2}|\xi|_p^2\right), \ \forall \xi \in \mathcal{S}(\mathbb{R}).$$

Conversely, if a function $F$ defined on $\mathcal{S}(\mathbb{R})$ satisfies the above two conditions, then there exist a unique $\Phi \in (S)^*$ such that $F = S\Phi$ and for any $q$ satisfying the condition that

$$e^2 \| \mathrm{A}^{-(q-p)} \|_{HS}^2 < 1, \tag{3.13}$$

then the following inequality holds:

$$\|\Phi\|_{-q} \leq K \left(1 - e^2 \| \mathrm{A}^{-(q-p)} \|_{HS}^2\right)^{-1/2}.$$

Assume now that $F = S\Phi$ and $G = S\Psi$. Then the product $FG$ clearly satisfies both the conditions 1. and 2. in the above theorem, for some $p$. For large $q$ the condition (3.13) also holds. Hence there is a unique element of $(S)^*$ such that it's S-transform is equal to $FG$. This justifies the following definition.

**Definition 3.3.4** (Wick product)**.** The wick product of two Hida distributions $\Phi, \Psi \in (S)^*$, denoted $\Phi \diamond \Psi$ is the unique element of $(S)^*$ such that

$$S(\Phi \diamond \Psi) = (S\Phi)(S\Psi)$$

The most important result we are going to need is the theorem about convergence of Hida distributions (theorem 8.6 in [Kuo96]).

**Theorem 3.3.5.** Assume that $\Phi_n \in (S)^*$ and let $F_n = S\Phi_n$. Then $\Phi_n$ converges strongly in $(S)^*$ if and only if the following holds:

1. $\lim_{n\to\infty} F_n(\xi)$ exists for all $\xi \in \mathcal{S}(\mathbb{R})$.
2. We can find non-negative constants $K, p$ independent of $n$, such that

$$|F_n(\xi)| \leq K \exp\left(\frac{1}{2}|\xi|_p^2\right), \ \forall n \in \mathbb{N}\ \forall \xi \in \mathcal{S}(\mathbb{R}). \tag{3.14}$$

# Chapter 4

# Stochastic processes on the white noise space

We have now a well defined white noise probability space and we have defined the spaces of Hida test functions and Hida distributions. We will now turn to the study of some basic properties of fractional Brownian motion.

First we state, without proof, the Kolmogorov-Čenstov theorem. For proof see: [KS91]. This theorem essentially states that for a certain class of stochastic processes we can find a modification which is continuous and almost surely equal to the given process.

**Theorem 4.0.6** (Kolmogorov-Čenstov)**.** If the stochastic process $X_t$, $t \in [0, t]$ satisfies

$$\mathbb{E}|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta},\ 0 \leq s, t \leq T, \tag{4.1}$$

for some positive constants $\alpha, \beta$ and $C$, then there exists a continuous modification $\widetilde{X}_t$ of $X$, which is locally Hölder continuous with Hölder exponent $\gamma$ for every $\gamma \in (0, \frac{\beta}{\alpha})$, i.e.

$$\mathbb{P}\left[\omega : \sup_{0<t-s<h(\omega)} \frac{|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)|}{|t-s|^\gamma} \leq \delta\right] = 1 \tag{4.2}$$

## 4.1 Classical Brownian motion

We have defined the white noise probability space so that for all $f \in L^2(\mathbb{R})$ the map $\langle ., f\rangle : \Omega \to \mathbb{R}$ is a random variable. We define the indicator function:

$$\mathbf{1}(a,b)(t) = \begin{cases} 1, & \text{if} \quad a \leq t < b \\ -1, & \text{if} \quad b \leq t < a \\ 0, & \quad \text{otherwise} \end{cases}$$

and we note that for any $t \in \mathbb{R}$, the function $\mathbf{1}(0,t)$ is in $L^2(\mathbb{R})$ and hence $\langle ., \mathbf{1}(0,t)\rangle$ is a random variable. Using 2.4.1 we see that it has mean 0 and variance $t$.

Simply define $\widetilde{B}_t(\omega) = \langle \omega, \mathbf{1}(0,t)\rangle$. Note that $\widetilde{B}_t - \widetilde{B}_s$ is (using 2.4.1) a Gaussian random variable and hence we have,

$$\begin{aligned}\mathbb{E}(|\widetilde{B}_t - \widetilde{B}_s|^{2n}) &= \int_{\mathbb{R}} x^{2n} \frac{1}{\sqrt{2\pi}|t-s|} \exp\left(-\frac{x^2}{2|t-s|^2}\right) dx \\ &= \frac{1}{\sqrt{\pi}} 2^n \Gamma\left(n + \frac{1}{2}\right) |t-s|^n\end{aligned}$$

thus, using Kolmogorov-Čenstov theorem, we know that there exists a $\gamma$-Hölder continuous modification of $\widetilde{B}_t$, say $B_t$, which is almost surely equal to $\widetilde{B}_t$, for any $\gamma \in (0, \frac{1}{2})$. Thus the stochastic process $B_t$ is a Brownian motion process.

We can approximate $f$ using step functions to obtain the following expression for the Wiener integral:

$$\langle ., f\rangle = \int_{\mathbb{R}} f(t) dB_t. \tag{4.3}$$

## 4.2 Extending the Wiener integral

Consider $f \in L^2(\mathbb{R})$. Then we can use (4.3) and get

$$\|\langle ., f\rangle\|_{-p} = \left\| \int_{\mathbb{R}} f(t) dB_t \right\|_{-p} = \| \mathrm{I}_1(f)\|_{-p}$$

now by definition of the $\|.\|_{-p}$ norm and due to the fact that $\mathrm{I}_1(f)$ is a chaos expansion consisting of that single term,

$$\begin{aligned}\|\langle ., f\rangle\|_{-p} &= \| \mathrm{I}_1(\mathrm{A}^{-p} f)\|_0 = \mathbb{E}\left[\left(\mathrm{I}_1(\mathrm{A}^{-p})\right)^2\right] \\ &= \mathbb{E}\left[\langle ., \mathrm{A}^{-p} f\rangle^2\right] = |\,\mathrm{A}^{-p} f|_0 = |f|_{-p}\end{aligned}$$

where we've also used (4.3) again and Itô's isometry (corollary 2.19). Hence we have, for all positive $p \in \mathbb{N}$,

$$\|\langle ., f\rangle\|_{-p} = \left\| \int_{\mathbb{R}} f(t) dB_t \right\|_{-p} = |f|_{-p}. \tag{4.4}$$

Using this isometry, we can extend the Wiener integral to $f \in \mathcal{S}'(\mathbb{R})$. However, one has to be cautious, because when $\langle ., f\rangle$ exists only as an element of $(S)_{-p}$ (and not $(L^2)$), then also $\int_{\mathbb{R}} f(t) dB_t$ is an element of $(S)^*$ but not $(L^2)$ and so it is a Hida distribution but not (necessarily) a random variable.

## 4.3 Preliminaries for a definition of fBm

We follow the same approach as in [MN68] in defining the fractional Brownian motion process, that is a stochastic process which would be almost surely continuous, Gaussian with mean 0 and have a covariance function given by (1.1). That is, for $H \in (0,1)$, we define a fractional Brownian motion as the process given by the following Wiener integral:

$$B_t^H = \frac{K_H}{\Gamma(H+1/2)} \int_{\mathbb{R}} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} dB_s, \tag{4.5}$$

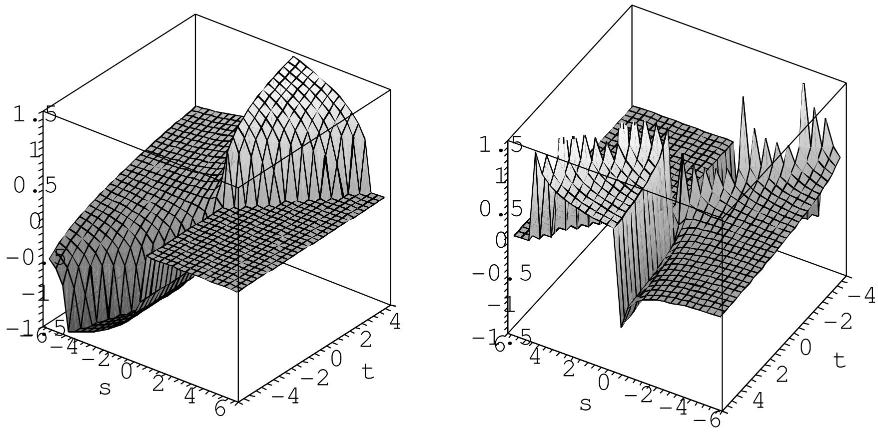

Figure 4.1: We can roughly see what the integral kernel does, for $H = 0.75$ on the left and for $H = 0.25$ on the right. As in (4.5), $t$ on the "y-axis" corresponds to $t$ in $B_t^H$.

with the normalising constant $K_H$ given by:

$$K_H = \Gamma(H + 1/2) \left( \int_{\mathbb{R}} \left( (1+s)^{H-1/2} - s^{H-1/2} \right)^2 ds + \frac{1}{2H} \right)^{-1/2}. \tag{4.6}$$

We can use the corollary 2.4.1 and (4.3) to see that $B_t^H$ are Gaussian random variables. Verifying that using this definition, we get the desired covariance function is a straightforward, if a bit tedious, matter. We will use this lemma:

**Lemma 4.3.1.**

$$\mathbb{E}\left[(B_{t+T}^H - B_t^H)^2\right] = T^{2H}$$

*Proof.* Let $\alpha = H - 1/2$. Using (4.5) and (4.3), we get

$$\begin{aligned}\mathbb{E}\left[(B_{t+T}^H - B_t^H)^2\right] = \mathbb{E}\Bigg[\Bigg( & \frac{K_H}{\Gamma(\alpha+1)} \langle ., (t+T-s)_+^{\alpha} - (-s)+^{\alpha} \rangle - \\ & - \frac{K_H}{\Gamma(\alpha+1)} \langle ., (t-s)_+^{\alpha} - (-s)+^{\alpha} \rangle \Bigg)^2 \Bigg].\end{aligned}$$

using Itô's isometry (2.19), we get

$$\mathbb{E}\left[(B_{t+T}^H - B_t^H)^2\right] = \left( \frac{K_H}{\Gamma(\alpha+1)} \right)^2 \int_{\mathbb{R}} \left( (t+T-s)_+^{\alpha} - (t-s)_+^{\alpha} \right)^2 ds.$$

Now we change the integration variable: $t - s \rightsquigarrow Tv$, to get

$$\begin{aligned}\mathbb{E}\left[(B_{t+T}^H - B_t^H)^2\right] = \left( \frac{K_H}{\Gamma(\alpha+1)} \right)^2 T^{2\alpha+1} \Bigg( & \int_{-\infty}^{0} ((1-v)^{\alpha} - (-v)^{\alpha})^2 dv + \\ & + \int_0^1 (1-v)^{2\alpha} dv \Bigg)\end{aligned}$$

Noting that $\int_0^1 (1-v)^{2\alpha} dv = \frac{1}{2H}$, we finally obtain

$$\begin{aligned}\mathbb{E}\left[(B^H_{t+T} - B^H_t)^2\right] &= T^{2H}\left(\frac{K_H}{\Gamma(\alpha+1)}\right)^2 \left(\int_0^\infty ((1+s)^\alpha - s^\alpha)^2 ds + \frac{1}{2H}\right) \\ &= T^{2H},\end{aligned}$$

where the last equality follows from the definition of $K_H$. $\square$

Now we can use the lemma to show that $B^H_t$ has the desired covariance. First we note that $2ac = a^2 + c^2 - (a-c)^2$ and hence

$$2\mathbb{E}(B^H_t B^H_s) = \mathbb{E}((B^H_t)^2) + \mathbb{E}((B^H_s)^2) - \mathbb{E}((B^H_t - B^H_s)^2)$$

Then assuming $B^H_0 = 0$ $\mu$ almost surely and using the above lemma, we see that

$$2\mathbb{E}(B^H_t B^H_s) = |t|^{2H} + |s|^{2H} - |t-s|^{2H}$$

The above lemma has another useful consequence: we can use it to show that $B^H_t$ is *self-similar* in the following sense:

**Definition 4.3.2.** A stochastic process $X_t$ is said to have self-similar increments with parameter $H \geq 0$ if and only if the random variables

$$\{X_{t+T} - X_t\} \text{ and } \frac{1}{h^H}\{X_{t+hT} - X_t\} \tag{4.7}$$

have the same distribution.

**Corollary 4.3.3.** Fractional Brownian motion processes $B^H_t$ are self-similar with the Hurst parameter $H$.

*Proof.* Thanks to lemma 2.4.1, we know that that both

$$\{B^H_{t+T} - B^H_t\} \text{ and } \frac{1}{h^H}\{B^H_{t+hT} - B^H_t\}$$

are Gaussian random variables with mean 0. Furthermore the first random variable has variance $T^{2H}$. Using lemma 4.3.1, we see that

$$\mathbb{E}\left[\left(\frac{1}{h^H} B^H_{t+hT} - B^H_t\right)^2\right] = h^{-2H}(hT)^{2H}$$

and hence the second random variable also has variance $T^{2H}$ and so fBm has self-similar increments. $\square$

All that remains to be done, in order to show that $B^H_t$ is a fractional Brownian motion process is to use the Kolmogorov-Čenstov theorem, to show that the process has a continuous modification. It seems that the most straightforward way of doing this, would be to express $B^H_t$ as $\langle ., f(t)\rangle$, for some $f \in L^2(\mathbb{R})$. That way we could use a similar approach, to show that the process satisfies the assumptions of Kolmogorov-Čenstov theorem, as we did with ordinary Brownian motion. We will use fractional integrals and fractional derivatives to do this.

## 4.4 Fraction integrals and derivatives

We would like to obtain a representation for $B_t^H$, ideally in terms of an indicator function, in some sense. We will first provide definitions of fractional integrals and derivatives.

**Definition 4.4.1** (Fractional integrals of Weyl's type)**.** Let $\alpha \in (0,1)$ and for $f$ such that the following integrals exist for any $x \in \mathbb{R}$, define:

$$(\mathrm{I}_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} f(t)(t-x)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x+t) t^{\alpha-1} dt, \tag{4.8}$$

$$(\mathrm{I}_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x-t)^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x-t) t^{\alpha-1} dt, \tag{4.9}$$

**Definition 4.4.2** (Fractional derivative of Marchaud's type)**.** Let $\alpha \in (0,1)$, let $\epsilon > 0$ and for $f$ such that the following integral and limit exist for any $x \in \mathbb{R}$, define:

$$(\mathrm{D}_{\pm,\epsilon}{}^{\alpha} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{\epsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t^{\alpha+1}} dt. \tag{4.10}$$

Then the fractional derivative of Marchaud's type is given by

$$(\mathrm{D}_{\pm}^{\alpha} f) = \lim_{\epsilon \to 0+} (\mathrm{D}_{\pm,\epsilon}{}^{\alpha} f) \tag{4.11}$$

**Lemma 4.4.3.** For $H \in (\frac{1}{2}, 1)$, one has

$$(\mathrm{I}_{-}^{H-1/2} \mathbf{1}(0,t))(s) = \frac{1}{\Gamma(H+1/2)} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \tag{4.12}$$

*Proof.* First we note that

$$\begin{aligned} \mathbf{1}(0,t)(s+x) &= \begin{cases} 1, & \text{if } 0 \le s+x < t \\ -1, & \text{if } t \le s+x < 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } -s \le x < t-s \\ -1, & \text{if } t-s \le x < -s \\ 0, & \text{otherwise} \end{cases} \\ &= \mathbf{1}(-s, t-s)(x) \end{aligned}$$

Hence we can write:

$$(\mathrm{I}_{-}^{H-1/2} \mathbf{1}(0,t))(s) = \frac{1}{\Gamma(H-1/2)} \int_0^{\infty} \mathbf{1}(-s, t-s)(x) x^{H-3/2} dx.$$

We also remark that, in general, $\Gamma(1+z) = z\Gamma(z)$ and hence

$$\Gamma(H+1/2) = \Gamma(H-1/2)(H-1/2).$$

The remaining part of the proof is tedious rather than enlightening and can be happily skipped.

Now we will consider several cases. First assume that $t > 0$ and that $s > 0$. Then

$$\begin{aligned} (\mathrm{I}_{-}^{H-1/2} \mathbf{1}(0,t))(s) &= \frac{1}{\Gamma(H-1/2)} \int_0^{t-s} x^{H-3/2} dx \\ &= \frac{1}{\Gamma(H+1/2)} (t-s)^{H-1/2} \\ &= \frac{1}{\Gamma(H+1/2)} (t-s)_+^{H-1/2} - (-s)_+^{H-1/2}, \end{aligned}$$

where the last equality follows since $s > 0$ implies $(-s)_+ = 0$ and we can assume that $t > 0$, because if it's not then $t - s < 0$ and the integral is 0 and hence $t - s = (t - s)_+$.

The second case is $t > 0$ but $s < 0$. Thus $t - s \geq 0$ and we get:

$$\begin{aligned}
(\mathrm{I}_-^{H-1/2}\,\mathbf{1}(0,t))(s) &= \frac{1}{\Gamma(H-1/2)}\int_{-s}^{t-s} x^{H-3/2}dx \\
&= \frac{1}{\Gamma(H+1/2)}(t-s)^{H-1/2} - (-s)^{H-1/2} \\
&= \frac{1}{\Gamma(H+1/2)}(t-s)_+^{H-1/2} - (-s)_+^{H-1/2},
\end{aligned}$$

where the last equality is a consequence of the fact that $s < 0$ implies $-s = (-s)_+$ and of the fact that $t - s \geq 0$.

The third case is $t \leq 0$ and $s > 0$. Here $t - s < t$ and so $t - s < 0$, giving us:

$$\begin{aligned}
(\mathrm{I}_-^{H-1/2}\,\mathbf{1}(0,t))(s) &= \frac{1}{\Gamma(H-1/2)}\int_{(0,\infty)\cap(t-s,-s)} -x^{H-3/2}dx = 0 \\
&= \frac{1}{\Gamma(H+1/2)}(t-s)_+^{H-1/2} - (-s)_+^{H-1/2},
\end{aligned}$$

since if $t - s < 0$ then $(t - s)_+ = 0$ and also if $-s < 0$ then $(-s)_+ = 0$.

The fourth and final case is assuming $t \leq 0$ and $s < 0$. We have:

$$\begin{aligned}
(\mathrm{I}_-^{H-1/2}\,\mathbf{1}(0,t))(s) &= \frac{1}{\Gamma(H-1/2)}\int_{0\wedge t-s}^{-s} -x^{H-3/2}dx \\
&= \frac{1}{\Gamma(H+1/2)}[(t-s)\wedge 0]^{H-1/2} - (-s)_+^{H-1/2} \\
&= \frac{1}{\Gamma(H+1/2)}(t-s)_+^{H-1/2} - (-s)_+^{H-1/2},
\end{aligned}$$

since $-s > 0$. We see that the four cases exhaust all the possible combinations of $s$ and $t$ and thus the proof is complete. □

**Lemma 4.4.4.** For $H \in (0, \frac{1}{2})$, one has

$$(\mathrm{D}_-^{-(H-1/2)}\,\mathbf{1}(0,t))(s) = \frac{1}{\Gamma(H+1/2)}\left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right) \tag{4.13}$$

*Proof.* We will use $\alpha = H - 1/2$. Then,

$$\begin{aligned}
\Gamma(1+\alpha)(\mathrm{D}_-^{-\alpha}\,\mathbf{1}(0,t))(s) &= \Gamma(1+\alpha)\lim_{\epsilon\to 0+}(\mathrm{D}_{-,\epsilon}\mathbf{1}(0,t))(s) \\
&= -\alpha\lim_{\epsilon\to 0+}\int_\epsilon^\infty \frac{\mathbf{1}(0,t)(s) - \mathbf{1}(0,t)(s+v)}{v^{-\alpha+1}}dv
\end{aligned}$$

As far as integration and taking the limit goes, $t$ and $s$ are fixed. And so we can consider separate cases, as in the previous proof. First assume that $t > 0$ and $0 \leq s < t$. Then we have

$$\begin{aligned}
\Gamma(1+\alpha)(\mathrm{D}_-^{-\alpha}\,\mathbf{1}(0,t))(s) &= -\alpha\lim_{\epsilon\to 0+}\int_\epsilon^\infty \frac{1 - \mathbf{1}(-s,t-s)(v)}{v^{-\alpha+1}}dv \\
&= -\alpha\lim_{\epsilon\to 0+}\int_\epsilon^\infty \frac{-\mathbf{1}(-s,t-s)(v)}{v^{-\alpha+1}}dv
\end{aligned}$$

We're assuming, that $s, t$ are fixed such that $s < t$, hence $t - s > 0$, we can assume that $\epsilon \in (0, t-s)$. Also $s \geq 0$ and hence we see that

$$\begin{aligned}\Gamma(1+\alpha)(\mathrm{D}_-^{-\alpha}\,\mathbf{1}(0,t))(s) &= -\alpha \lim_{\epsilon\to 0+}\int_\epsilon^{t-s} -\frac{1}{v^{-\alpha+1}}dv \\ &= (t-s)^\alpha = (t-s)_+^\alpha - (-s)_+^\alpha\end{aligned}$$

Now assume that $t$ is still strictly greater than 0 but $s \notin (0,t)$. We get:

$$\Gamma(1+\alpha)(\mathrm{D}_-^{-\alpha}\,\mathbf{1}(0,t))(s) = -\alpha \lim_{\epsilon\to 0+}\int_\epsilon^{\infty} -\frac{\mathbf{1}(-s,t-s)(v)}{v^{-\alpha+1}}dv$$

If $s \geq 0$, then we also have $s \geq t$ and hence $t - s < 0$ which implies that $(t-s)_+ = 0$. $s \geq 0$ would also mean that $(-s)_+ = 0$. And $s \geq 0$ would also mean that

$$(\epsilon, \infty) \cap (-s, t-s) = \emptyset$$

and so this case is fine. If on the other hand $s < 0$, then $t - s > 0$ and our integral is:

$$\Gamma(1+\alpha)(\mathrm{D}_-^{-\alpha}\,\mathbf{1}(0,t))(s) = -\alpha \lim_{\epsilon\to 0+}\int_{-s}^{t-s} -\frac{1}{v^{-\alpha+1}}dv = (t-s)_+^\alpha - (-s)_+^\alpha$$

We could indeed verify that the lemma holds also for $t < 0$, but this part we omit. It's as straightforward but as tedious as the case $t \geq 0$. $\square$

## 4.5 Concluding the construction of fBm

Using the lemmas 4.4.3 and 4.4.4 and using (4.3), we see that for $H \in (\frac{1}{2}, 1)$,

$$B_t^H = \langle ., K_H\,\mathrm{I}_-^{H-1/2}\,\mathbf{1}(0,t)\rangle.$$

For $H \in (0, \frac{1}{2})$,

$$B_t^H = \langle ., K_H\,\mathrm{D}_-^{1/2-H}\,\mathbf{1}(0,t)\rangle$$

and of course for $H = \frac{1}{2}$, $B_t^{1/2} = \langle ., \mathbf{1}(0,t)\rangle$. Thus we have a definition of fractional Brownian motion in terms of an operator and an indicator function. We can summarise this.

**Definition 4.5.1.** For $H \in (0,1)$ define the class of operators $\mathrm{M}_\pm^{\mathrm{H}}$ by

$$\mathrm{M}_\pm^{\mathrm{H}} f = \begin{cases} K_H\,\mathrm{D}_\pm^{1/2-H} f, & H \in (0, \frac{1}{2}) \\ f, & H = \frac{1}{2} \\ K_H\,\mathrm{I}_\pm^{H-1/2} f, & H \in (\frac{1}{2}, 1). \end{cases}$$

for any function $f$ such that the respective fractional derivatives or integrals make sense.

We can now show some useful properties of the operators $\mathrm{M}_\pm^{\mathrm{H}}$, before we show that fBm has a continuous modification.

**Theorem 4.5.2.** Assume $H \in (0,1)$ and $f \in \mathcal{S}(\mathbb{R})$. Then $\mathrm{M}^{\mathrm{H}}_{\pm}$ exists and there is a constant $C_H$ independent of $f$ such that

$$\sup_{x\in\mathbb{R}} |(\mathrm{M}^{\mathrm{H}}_{\pm} f)(x)| \leq C_H \left( \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |f'(x)| + \|f\|_{L^1(\mathbb{R})} \right) \tag{4.14}$$

*Proof.* For $H = \frac{1}{2}$, $\mathrm{M}^{\mathrm{H}}_{\pm}$ is the identity operator and the result is trivial.
Assume $H \in (0, \frac{1}{2})$ and let $\alpha = 1/2 - H$, then $\alpha \in (0, \frac{1}{2})$. We get:

$$|(\mathrm{M}^{\mathrm{H}}_{\pm} f)(x)| = \left| K_H \frac{\alpha}{\Gamma(\frac{1}{2}+H)} \lim_{\epsilon\to 0+} \int_{\epsilon}^{\infty} \frac{f(x) - f(x \mp y)}{y^{\alpha+1}} dy \right|$$

Now we note that

$$\begin{aligned}
\frac{\Gamma(1-\alpha)}{\alpha} |\,\mathrm{D}_{\pm} f(x)| &\leq \int_0^1 \frac{|f(x) - f(x \mp y)|}{y} y^{-\alpha} dy + \\
&\qquad + \int_1^{\infty} \frac{|f(x) - f(x \mp y)|}{y^{\alpha+1}} dy \\
&\leq \sup_{x\in\mathbb{R}} |f'(x)| \int_0^1 y^{-\alpha} dy + 2 \sup_{x\in\mathbb{R}} |f(x)| \int_1^{\infty} y^{-1-\alpha} dy \\
&= \frac{1}{1-\alpha} \sup_{x\in\mathbb{R}} |f'(x)| + \frac{2}{\alpha} \sup_{x\in\mathbb{R}} |f(x)|
\end{aligned} \tag{4.15}$$

and hence for some constant $C^1_H$, independent of $f$,

$$\sup_{x\in\mathbb{R}} |(\mathrm{M}^{\mathrm{H}}_{\pm} f)(x)| \leq C^1_H \left( \sup_{x\in\mathbb{R}} |f'(x)| + \sup_{x\in\mathbb{R}} |f(x)| \right) \tag{4.16}$$

Now assume that $H \in (\frac{1}{2}, 1)$ and let $\alpha = H - \frac{1}{2}$. Then

$$\left|(\mathrm{M}^{\mathrm{H}}_{\pm} f)(x)\right| = |K_H (\mathrm{I}_{\pm} f)(x)| \leq \left| K_H \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(x \mp y) y^{\alpha-1} dy \right|$$

and

$$\left| \int_0^{\infty} f(x \mp y) y^{\alpha-1} dy \right| \leq \int_0^1 |f(x \mp y) y^{\alpha-1}| dy + \int_1^{\infty} |f(x \mp y) y^{\alpha-1}| dy.$$

Let's note that $\alpha \in (-1, -1/2)$ and $1 - \alpha \in (1/2, 1)$. So,

$$\begin{aligned}
\left| \int_0^{\infty} f(x \mp y) y^{\alpha-1} dy \right| &\leq \int_{|y-x|<1} |f(t)||y-x|^{\alpha-1} dy + \\
&\qquad + \int_{|y-x|\geq 1} |f(y)||y-x|^{\alpha-1} dy \\
&\leq \sup_{x\in\mathbb{R}} |f(x)| \int_{|y-x|<1} |y-x|^{\alpha-1} dy + \int_{|x-y|\geq 1} |f(y)| dy \\
&\leq \frac{2}{\alpha} \sup_{x\in\mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})}
\end{aligned} \tag{4.17}$$

Thus we get, for some $C_H^2$ independent of $f$,

$$\sup_{x\in\mathbb{R}} |(\mathrm{M}^{\mathrm{H}}_{\pm} f)(x)| \le C_H^2 \left( \sup_{x\in\mathbb{R}} |f(x)| + \|f\|_{L^1(\mathbb{R})} \right) \tag{4.18}$$

and combining (4.16) and (4.18) completes the proof. □

**Theorem 4.5.3.** 1. Assume that $H \in (\frac{1}{2}, 1)$. Assume also that $f \in L^p(\mathbb{R})$, $g \in L^r(\mathbb{R})$ and $p > 1$ and $r > 1$ and

$$\frac{1}{p} + \frac{1}{r} = \frac{1}{2} + H$$

then $(f, \mathrm{M}^{\mathrm{H}}_{-} g)_0 = (\mathrm{M}^{\mathrm{H}}_{+} f, g)_0$.

2. Assume that $H \in (0, \frac{1}{2})$ and also that ${\mathrm{M}^{\mathrm{H}}_{+}}^{H} f \in L^p(\mathbb{R})$, ${\mathrm{M}^{\mathrm{H}}_{-}}^{H} g \in L^r(\mathbb{R})$, $f \in L^s(\mathbb{R})$ and $g \in L^t(\mathbb{R})$, where $p > 1$, $r > 1$ and

$$\frac{1}{p} + \frac{1}{r} = \frac{3}{2} - H \text{ and } \frac{1}{s} = \frac{1}{p} + H - \frac{1}{2} \text{ and } \frac{1}{t} = \frac{1}{r} + H - \frac{1}{2},$$

*Proof.* This is a consequence of the fractional integration by parts and differentiation by parts formula, which can be found in [SKM93]. For part one, let $\alpha = H - 1/2$, then we can use (5.16) in [SKM93] to see that

$$\int_{\mathbb{R}} f(x)(\mathrm{I}^{\alpha}_{-} g)(x)dx = \int_{\mathbb{R}} g(x)(\mathrm{I}^{\alpha}_{+} f)(x)dx$$

the conditions set out in [SKM93] for (5.16) to hold are exactly as spelled out in our theorem and hence $(f, \mathrm{M}^{\mathrm{H}}_{-} g)_0 = (\mathrm{M}^{\mathrm{H}}_{+} f, g)_0$.

For part two, let $\alpha = 1/2 - H$ and we will use (5.17) in [SKM93] to see that (again the conditions for (5.17) to hold are the same as our assumptions).

$$\int_{\mathbb{R}} f(x)(\mathrm{D}^{\alpha}_{-} g)(x)dx = \int_{\mathbb{R}} g(x)(\mathrm{D}^{\alpha}_{+} f)(x)dx$$

and hence we can again get $(f, \mathrm{M}^{\mathrm{H}}_{-} g)_0 = (\mathrm{M}^{\mathrm{H}}_{+} f, g)_0$. □

A useful consequence of the above theorem (proved in [Ben03]) is:

**Corollary 4.5.4.** If $f \in \mathcal{S}(\mathbb{R})$ then, $(f, \mathrm{M}^{\mathrm{H}}_{-} \mathbf{1}(0,t))_0 = (\mathbf{1}(0,t), \mathrm{M}^{\mathrm{H}}_{+} f)_0$.

Of course we have seen that in particular for $\mathbf{1}(0,t)$ the operator is well defined and we have that $B_t^H = \langle ., \mathrm{M}^{\mathrm{H}}_{-} \mathbf{1}(0,t) \rangle$. What still remains to be done is to show that $B_t^H$ is path-wise continuous. In fact we'll use the Kolmogorov-Čenstov theorem to show that there exists a continuous modification which as almost surely equal to $B_t^H$. Using the same approach as in the proof of lemma 4.3.1, we obtain that for $n \in \mathbb{N}$:

$$\mathbb{E}\left[(B_{t+T}^H - B_t^H)^{2n}\right] = C_{H,n} T^{2Hn} \tag{4.19}$$

Therefore, using Kolmogorov-Čenstov theorem, we see that for larger $n$ there is a $\gamma$-Hölder continuous modification, with $\gamma \in \left(0, \frac{2Hn-1}{2n}\right)$ for $\forall n \in \mathbb{N}$ and so letting $n \to \infty$ we get that $\gamma \in (0, H)$. This shows rather nicely that for small $H$ fBm has less "continuous" sample paths than for high $H$.

## 4.6 Non-differentiability of fBm

We will show that the sample paths of fBm are almost surely non-differentiable, in fact we will use the self-similarity property of fBm (corollary4.3.3) to prove that:

**Lemma 4.6.1.** If $B_t^H$ is a fBm process then

$$\mu\left\{\limsup_{t\to t_0}\left|\frac{B_t^H - B_{t_0}^H}{t - t_0}\right| = \infty\right\} = 1 \tag{4.20}$$

The prove presented here is essentially the same as in [MN68].

*Proof.* Consider $T = s - t$, so $B_{t+T}^H = B_s^H$. Let $h = (s-t)^{-1}$. Using the self-similarly property[1] we see that

$$B_s^H - B_t^H \sim (s-t)^H (B_{t+1}^H - B_t^H)$$

and thus

$$B_t^H - B_{t_0}^H \sim (t-t_0)^{H-1}(B_{t_0+1}^H - B_{t_0}^H)$$

and we also note that $B_0^H = 0$ $\mu$ almost surely, and we get

$$B_t^H - B_{t_0}^H \sim (t-t_0)^{H-1}(B_1^H).$$

Now we take some sequence $t_n \downarrow 0$ and consider the events, for some fixed $d \in \mathbb{R}$:

$$A_t = \left\{\omega : \sup_{0\le s\le t}\left|\frac{B_s^H(\omega)}{s}\right| > d\right\}$$

and we note that $A_{t_{n+1}} \subset A_{t_n}$. Thus we have

$$\mu\left(\lim_{n\to\infty} A_{t_n}\right) = \lim_{n\to\infty}\mu(A_{t_n}).$$

Finally,

$$\mu(A_{t_n}) \ge \mu\left\{\frac{|B_{t_n}^H|}{t_n} > d\right\} = \mu\left\{\frac{|B_1^H|}{t_n^{H-1}} > d\right\} = \mu\left\{|B_{t_n}^H| > t_n^{1-H} d\right\} \to 1,$$

as $n \to \infty$, because $B_1^H$ is Gaussian and $H \in (0,1)$. □

## 4.7 fBm is differentiable as a stochastic distribution process

The aim now is to show that fBm is differentiable as a stochastic distribution process. We will first define what we mean by this, but then we will have to find $\frac{\mathrm{d}}{\mathrm{dt}}\,\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t)$. This in turn depends on some properties of Hermite functions. Only then will we be able to prove our result.

[1] with $\sim$ denoting processes with the same probability distribution

**Definition 4.7.1.** Let $I \subset \mathbb{R}$ be an interval. A mapping $X : I \to (S)^*$ (i.e. a stochastic distribution process) is said to be *differentiable* if there exists $\dot{X} \in (S)^*$ such that

$$\frac{X_{t+h} - X_t}{h} \to \dot{X} \text{ as } h \to 0, \text{ convergence in } (S)^* \tag{4.21}$$

Recall that convergence in $(S)^*$ means, that there exists $p \in \mathbb{N}$ such that we have convergence with respect to the norm $\|.\|_{-p}$.

**Lemma 4.7.2.** Assume $H \in (0,1)$ and $e_n$ is the $n$th Hermite function. Then there is a positive constant $C_H$ such that

$$\sup_{x\in\mathbb{R}} |(\mathrm{M}^{\mathrm{H}}_{+}\, e_n)(x)| \leq C_H(n+1)^{5/12}.$$

*Proof.* By the theorem 4.5.2,

$$\sup_{x\in\mathbb{R}} |\, \mathrm{M}^{\mathrm{H}}_{+}\, e_n(x)| \leq C_H \left( \sup_{x\in\mathbb{R}} |e_n(x)| + \sup_{x\in\mathbb{R}} |e_n'(x)| + \|e_n\|_{L^1(\mathbb{R})} \right).$$

The result now follows from the identity (A.14) and the estimates (A.13) in appendix A.3. $\square$

**Theorem 4.7.3.** If $H \in (0,1)$ and $f \in \mathcal{S}(\mathbb{R})$, then $\mathrm{M}^{\mathrm{H}}_{+} f$ is a continuous real function.

*Proof.* This is a consequence of Lebesgue dominated convergence theorem. We will show that we can apply the dominated convergence theorem. First assume that $H \in \left(0, \frac{1}{2}\right)$ and let $\alpha = \frac{1}{2} - H$. Then

$$\frac{|f(x_n) - f(x_n - y)|}{|y|^{\alpha+1}} \leq g(x) =: \begin{cases} \sup\limits_{x\in\mathbb{R}} |f'(x)||y|^{-1-\alpha} & \text{if } |y| \in (0,1) \\ 2\sup\limits_{x\in\mathbb{R}} |f(x)||y|^{1+\alpha} & \text{otherwise} \end{cases}$$

and we can see from (4.15) that $g$ is integrable.

Now let $H \in \left(\frac{1}{2}, 1\right)$ and $\alpha = H - 1/2$. Then, by continuity of $f$ and because $x_n \to x$ as $n \to \infty$, we can say that for some $M > 0$

$$|f(x_n - y)||y|^{\alpha-1} \leq \left(|f(x-y)| + M\right)|y|^{\alpha-1}$$

and we can see from an estimate similar to (4.17) that this is integrable. $\square$

**Corollary 4.7.4.** If $H \in (0,1)$ and $f \in \mathcal{S}(\mathbb{R})$, then $(f, \mathrm{M}^{\mathrm{H}}_{-}\,\mathbf{1}(0,t))_0$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{dt}}(f, \mathrm{M}^{\mathrm{H}}_{-}\,\mathbf{1}(0,t))_0 = (\mathrm{M}^{\mathrm{H}}_{+}\, f)(t) \tag{4.22}$$

**Lemma 4.7.5.** If $H \in (0,1)$, then $\mathrm{M}^{\mathrm{H}}_{-}\,\mathbf{1}(0,.) : \mathbb{R} \to \mathcal{S}'(\mathbb{R})$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{dt}}\, \mathrm{M}^{\mathrm{H}}_{-}\,\mathbf{1}(0,t) = \sum_{k=0}^{\infty} (\mathrm{M}^{\mathrm{H}}_{+}\, e_k)(t) e_k, \tag{4.23}$$

where the limit is in $\mathcal{S}'(\mathbb{R})$.

We mention that the Hermite functions $e_k$ form an orthonormal basis in $L^2(\mathbb{R})$ and hence we can write

$$\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t) = \sum_{k=0}^{\infty}(\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t), e_k)_0 e_k = \sum_{k=0}^{\infty}\left(\int_0^t (\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s)ds\right)e_k, \tag{4.24}$$

where the last equality is a consequence of corollary 4.5.4.

*Proof of lemma 4.7.5.* In the view of (4.24), we see that

$$\begin{aligned}
&\left|\frac{\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t+h) - \mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t)}{h} - \sum_{k=0}^{\infty}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(t)e_k\right|_{-1}^2 = \\
&= \left|\frac{1}{h}\sum_{0}^{\infty}\left(\int_t^{t+h}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s)ds\right)e_k - \sum_{k=0}^{\infty}\frac{1}{h}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(t)\left(\int_t^{t+h} ds\right)e_k\right|_{-1}^2 = \\
&= \left|\frac{1}{h}\sum_{k=0}^{\infty}\left(\int_t^{t+h}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s) - (\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(t)ds\right)e_k\right|_{-1}^2 = \\
&= \sum_{k=0}^{\infty}(2k+2)^{-2}\left(\frac{1}{h}\int_t^{t+h}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s)ds - (\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(t)\right)^2
\end{aligned} \tag{4.25}$$

where the last equality followed from corollary 2.3.2, which says that

$$|f|_{-p}^2 = |A^{-p}f|_0^2 = \sum_{k=0}^{\infty}(2k+2)^{-2p}(f,e_k)_0^2$$

and from the fact that $e_k$ are orthonormal and so $(e_j, e_k)_0 = 1$ if and only if $j = k$, which effectively removes the second summation.

From lemma 4.7.2, we see that the right hand side of (4.25) converges uniformly in $h$. Thus we can pass the limit as $h \to 0$ under the summation. Finally

$$\lim_{h\to 0}\left(\frac{1}{h}\int_t^{t+h}(\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s)ds - (\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(t)\right)^2 = 0,$$

because theorem 4.7.3 tells us that $\mathrm{M}_{+}^{\mathrm{H}}\,e_k$ is continuous. So with convergence in $|.|_{-1}$,

$$\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t) = \sum_{k=0}^{\infty}(\mathrm{M}_{-}^{\mathrm{H}}\,\mathbf{1}(0,t), e_k)_0 e_k = \sum_{k=0}^{\infty}\left(\int_0^t (\mathrm{M}_{+}^{\mathrm{H}}\,e_k)(s)ds\right)e_k$$

and since, by lemma 3.2.4 the union over $p$ of $\mathcal{S}_p'(\mathbb{R})$ is equal to $\mathcal{S}'(\mathbb{R})$, we get the above equality with convergence in $\mathcal{S}'(\mathbb{R})$. This completes the proof. □

Now we very nearly have our result. Recall that by (4.4), we can apply the Itô isometry not only to $L^2(\mathbb{R})$ functions but also to tempered distributions.

Henceforth, if we assume that $F : I \to \mathcal{S}'(\mathbb{R})$ is differentiable, then we see that:

$$\begin{aligned}
&\lim_{h\to 0}\left\|\left\langle ., h^{-1}\left(F_{t+h}-F_t\right)-\frac{\mathrm{d}}{\mathrm{dt}}\,F_t\right\rangle\right\|_{-p} = \\
&= \lim_{h\to 0}\left|h^{-1}\left(F(t+h)-F(t)\right)-\frac{\mathrm{d}}{\mathrm{dt}}\,F(t)\right|_{-p} \\
&= 0,
\end{aligned} \tag{4.26}$$

by our assumption. This implies that $\langle ., F(t)\rangle$ is differentiable as a stochastic distribution process. Hence by lemma 4.7.5, we see that $B^H$ is differentiable as a stochastic distribution process and furthermore

$$\frac{\mathrm{d}}{\mathrm{dt}}\,B_t^H = \left\langle ., \sum_{k=0}^{\infty}(\mathrm{M}_+^{\mathrm{H}}\,e_k)(t)e_k\right\rangle. \tag{4.27}$$

We can find even a simpler expression for $\frac{\mathrm{d}}{\mathrm{dt}}\,B_t^H$, though. Consider

$$\begin{aligned}
&\left|\sum_{k=0}^{\infty}(\mathrm{M}_+^{\mathrm{H}}\,e_k)(t)e_k - \delta_t\circ\mathrm{M}_+^{\mathrm{H}}\right|_{-1}^2 = \\
&= \sum_{n=0}^{\infty}(2n+2)^{-2}\left\langle\sum_{k=0}^{\infty}(\mathrm{M}_+^{\mathrm{H}}\,e_k)(t)e_k - \delta_t\circ\mathrm{M}_+^{\mathrm{H}}, e_n\right\rangle^2 = \\
&= \sum_{n=0}^{\infty}(2n+2)^{-2}\left(\sum_{k=0}^{\infty}(\mathrm{M}_+^{\mathrm{H}}\,e_k)(t)(e_k,e_n)_0 - (\mathrm{M}_+^{\mathrm{H}}\,e_n)(t)\right)^2 = 0.
\end{aligned}$$

Thus we have proved the following corollary.

**Corollary 4.7.6.** $B_t^H$ is differentiable as a stochastic distribution process and

$$\frac{\mathrm{d}}{\mathrm{dt}}\,B_t^H = \langle ., \delta_t\circ\mathrm{M}_+^{\mathrm{H}}\rangle \tag{4.28}$$

and we can define the *fractional white noise* to be

$$W_t^H = \langle ., \delta_t\circ\mathrm{M}_+^{\mathrm{H}}\rangle. \tag{4.29}$$

**Theorem 4.7.7.** Let $H \in (0,1)$. Then $\forall\xi\in\mathcal{S}(\mathbb{R})$,

$$SW_t^H(\xi) = (\mathrm{M}_+^{\mathrm{H}}\,\xi)(t).$$

*Proof.* Follows from the fact that if a stochastic distribution process $X_t$ is differentiable, then

$$S\left(\frac{\mathrm{d}}{\mathrm{dt}}\,X_t\right)(\nu) = \frac{\mathrm{d}}{\mathrm{dt}}\left(SX_t(\nu)\right), \quad \forall\nu\in\mathcal{S}(\mathbb{R}), \tag{4.30}$$

which in turn follows straight from the definitions and theorem 3.3.5. □

## 4.8 fBm is not a martingale

We conclude this chapter by quickly showing that fBm process is not a martingale. This has an unfortunate consequence. We can't apply the well established martingale approach to stochastic integration (see e.g. [KS91].

**Lemma 4.8.1.** For $H \neq 1/2$, fBm is not a martingale.

*Proof:* $B_t^H$ would be a martingale if and only if

$$\forall s < t \quad \mathbb{E}(B_t^H | \mathcal{F}_s) = B_s^H \tag{4.31}$$

which is equivalent to

$$\forall s < t \quad \mathbb{E}(B_t^H - B_s^H | \mathcal{F}_s) = 0 \tag{4.32}$$

Now one notes that

$$\int_{\mathbb{R}} (t-u)_+^\alpha - (s-u)_+^\alpha dB_u = \int_{\{s \geq u\}} (t-u)_+^\alpha - (s-u)_+^\alpha dB_u \tag{4.33}$$

and hence that

$$\begin{aligned} \mathbb{E}(B_t^H - B_s^H | \mathcal{F}_s) &= \frac{K_H}{\Gamma(\alpha)} \mathbb{E}\left( \int_{\{s \geq u\}} (t-u)_+^\alpha - (s-u)_+^\alpha dB_u | \mathcal{F}_s \right) &(4.34) \\ &= \frac{K_H}{\Gamma(\alpha)} \int_{\{s \geq u\}} (t-u)_+^\alpha - (s-u)_+^\alpha dB_u &(4.35) \end{aligned}$$

since $\int_{\{s \geq u\}} (t-u)_+^\alpha - (s-u)_+^\alpha dB_u$ is $\mathcal{F}_s$ measurable. Thus for $\alpha \neq 0$ i.e. $H \neq 1/2$ $B_t^H$ is not a martingale. $\square$

# Chapter 5

# Stochastic integral with respect to fBm

We have shown in 4.8 that a fBm process is not a martingale and hence the classical integration theory does not apply.

An alternative approach to stochastic integration is to use the white noise distribution theory. This has some disadvantages as we'll demonstrate later.

## 5.1 White noise integrals

If $X : I \to (S)^*$ is a stochastic distribution process, then the integrability of X can be defined in terms of the S-transform:

**Definition 5.1.1.** A stochastic distribution process $X : I \to (S)^*$ is *integrable* in the white noise (Pettis) sense, if:

1. $SX(\eta)$ is measurable for all $\eta \in \mathcal{S}(\mathbb{R})$ and $SX(\eta) \in L^1(I)$ for all $\eta \in \mathcal{S}(\mathbb{R})$
2. there exists $\Phi \in (S)^*$ such that $\int_I SX(t)(\eta)dt = S\Phi$

The we *define* $\int_I X(t)dt = \Phi$.

However this is just a definition; it does not tell us anything about the conditions that must be satisfied for the integral to exists or what properties can we expect. The following theorems are therefore useful (a more general proof can be found in [Kuo96], chapter 13).

**Theorem 5.1.2.** Assume that $\Phi : I \to (S)^*$ satisfies:

1. $S\Phi(.)(\xi)$ is measurable for all $\xi \in \mathcal{S}(\mathbb{R})$
2. There exist positive constants $K, a$ and $p$ such that

$$\int_I |S\Phi(t)(\xi)|dt \leq K \exp[a|\xi|_p^2] \tag{5.1}$$

Then $\Phi$ is Pettis integrable and for any $E \in \mathcal{B}(I)$,

$$S\left(\int_E \Phi(t)dt\right)(\xi) = \int_E S\Phi(t)(\xi)dt \tag{5.2}$$

*Proof.* The first assumption implies that $S\Phi(.)(\xi) \in L^1(I)$ for all $\xi \in \mathcal{S}(\mathbb{R})$. Now for any $E \in \mathcal{B}(I)$, define $F(\xi) = \int_E S\Phi(t)(\xi)dt$ and note that we could use Morera's theorem to check that for all $\xi, \nu \in \mathcal{S}(\mathbb{R}$ the function $F(z\xi + \nu)$ is entire as a function of $z \in \mathbb{C}$.

Moreover we have, by the second assumption:

$$|F(\xi)| \leq \int_E |S\Phi(t)(\xi)|dt \leq \int_I |S\Phi(t)(\xi)|dt \leq K \exp[a|\xi|_p^2] \tag{5.3}$$

Thus, by theorem 3.3.3 we know that there exists a unique $\Psi \in (S)^*$ such that:

$$S\Psi(\xi) = F(\xi) = \int_E S\Phi(t)(\xi)dt \tag{5.4}$$

and so by definition $\Psi$ is the Pettis integral of $\int_E \Phi(t)dt$ and we have

$$S\left(\int_E \Phi(t)dt\right)(\xi) = S\Psi(\xi) = \int_E S\phi(t)(\xi)dt \tag{5.5}$$

□

**Example 5.1.3.** We let $H = 1/2$ and so $W_t = \frac{\mathrm{d}}{\mathrm{dt}} B_t$ in the distributional sense. We show that

$$\int_0^T e^{-(-T-t)} W_t(\omega)dt = \langle \omega, \mathbf{1}_{(0,T)} e^{-(-T-\omega)} \rangle \tag{5.6}$$

First note that

$$S(e^{-(-T-t)} W_t) = e^{-(-T-t)} \xi(t) \tag{5.7}$$

and that

$$S\langle ., \mathbf{1}_{(0,T)} e^{-(-T-.)} \rangle(\xi) = \int_0^T e^{-(-T-t)} \xi(t)dt \tag{5.8}$$

So if combine (5.7) with (5.8) and the definition (5.1.1), we get (5.6). We could have also applied the above theorem to obtain the fact, that the integral exists, indeed we have the following estimate

$$\int_0^T |e^{-(-T-t)} \xi(t)|dt \leq C_T \exp(|\xi|_0^2) \tag{5.9}$$

and hence the above theorem applies. Nonetheless, we note that

$$\mathbf{1}_{(0,T)} e^{-(-T-t)} \in L^2(\mathbb{R}).$$

So (5.6) implies that the integral is a Gaussian random variable in $(L^2)$ with mean 0 and variance $\frac{1}{2}(1 - e^{-2T})$. The shortcoming of the above theorem is that it only tells us that the integral exists as an element of $(S^*)$.

This illustrates a more general problem, one encounters, when trying to work with white noise integrals. In general we could prove a theorem similar to (5.1.2) with the second condition being that: There exist positive constants $K, a$ and $p$ such that

$$\int_I |S\Phi(t)(\xi)|dt \leq K \exp[a|\xi|_{-p}^2] \tag{5.10}$$

Then the white noise integral would again exist and furthermore for $q \in [0, p)$ we would have (under some further assumption), that the integral is an element of $(S_q)$. In particular, one might be tempted to think that it could be used that the integral is in $(S_0) = (L^2)$. This is indeed true, however even for $\int_0^T B_t dt$, one would have to be able to show that $|SB_t(\xi)| \leq K \exp[a|\xi|_{-1}^2]$, which is impossible.

In [Ben03] it is shown, in theorem 4.4, that for certain integrands the integrals are elements of $(L^2)$. It is however done indirectly, not using the characterization theorems of S-transforms.

**Example 5.1.4.** $\int_0^T W_t^H dt$ exists in the white noise sense. Recall that $W_t^H$ was defined as the distributional derivative of $B_t^H$. We would like to apply the above theorem and therefore we need to estimate $|SW_t^H(\xi)|$.

In general, we have, by definition of the S-transform and the $\|.\|_{-p}$ norm, for $p \geq 1$:

$$|S\Phi(\xi)|^2 = |\langle\langle \Phi, :e^{\langle \cdot, \xi \rangle}: \rangle\rangle|^2 \leq \|\Phi\|_{-p}^2 \| :e^{\langle \cdot, \xi \rangle}: \|_p^2 \tag{5.11}$$

We also have, by corollary 4.7.6 and the second equality by the extended Itô isometry (4.4), that

$$\begin{aligned}
\|W_t^H\|_{-p}^2 &= \|\langle \cdot, \sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k \rangle\|_{-p}^2 &(5.12)\\
&= |\mathrm{A}^{-p} \sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k|_0^2 &(5.13)\\
&= |\sum_{m=0}^{\infty} (2m+2)^{-2p} (\sum_{k=0}^{\infty} (M_+^H e_k)(t) e_k, e_m)_0 e_m|_0^2 &(5.14)\\
&= |\sum_{m=0}^{\infty} (2m+2)^{-2p} (M_+^H e_m)(t) e_m|_0^2 &(5.15)
\end{aligned}$$

Now we use the fact that $\sup_t |(M_+^H e_m)(t)| \leq C_H (m+1)^{\frac{5}{12}}$ and that $e_m$ are orthonormal in $L^2(\mathbb{R})$ and thus we get:

$$\|W_t^H\|_{-p}^2 \leq \sum_{m=0}^{\infty} (2m+2)^{-2p} C_H (m+1)^{\frac{5}{12}} \tag{5.16}$$

For $p \geq 1$ the series is convergent and thus have

$$|SW_t^H(\xi)|^2 \leq \underbrace{\left( \sum_{m=0}^{\infty} (2m+2)^{-2p} C_H (m+1)^{\frac{5}{12}} \right)}_{=:(\widetilde{C}_H)^2} \| :e^{\langle \cdot, \xi \rangle}: \|_p^2 \tag{5.17}$$

Finally we note that $\| :e^{\langle \cdot, \xi \rangle}: \|_p^2 = \exp\left(\frac{1}{2}|\xi|_p^2\right)$:

$$\int_0^T |SW_t^H(\xi)|^2 \leq T\widetilde{C}_H \exp\left(\frac{1}{2}|\xi|_p^2\right) \tag{5.18}$$

This implies, by the above theorem, that $\int_0^T W_t^H dt$ exists.

**Example 5.1.5.** Similarly, since we can estimate

$$\int_0^T |S(B_t^H \Diamond W_t^H)(\xi)| dt \leq T\widetilde{C}_H K_H \exp(|\xi|_p^2) \tag{5.19}$$

we get that $\int_0^T B_t^H \Diamond W_t^H dt$ exists.

## 5.2 Fractional "Itô" integrals

The previous two examples provide motivation for the following definition.

**Definition 5.2.1** (fractional Itô integral)**.** A stochastic (distribution) process $\Phi : [0,T] \to (S)^*$ is *fractional Itô integrable* provided that $\Phi \Diamond W^H$ is white noise (Pettis) integrable and we write:

$$\int_0^T \Phi(t) dB_t^H := \int_0^T \Phi(t) \Diamond W_t^H dt \tag{5.20}$$

**Theorem 5.2.2.** A fractional Itô integral of $\Phi : [0,T] \to (S)^*$ exists, provided that:

1. $S\Phi(.)(\xi)$ is measurable for all $\xi \in \mathcal{S}(\mathbb{R})$.
2. There exist positive constants $K, a$ and $\hat{p}$ such that

$$\int_0^T |S\Phi(t)(\xi)| dt \leq K \exp[a|\xi|_{\hat{p}}^2] \tag{5.21}$$

*Proof.* $SW_t^H(\xi) = \mathrm{M}_+^{\mathrm{H}} \xi(t)$, it is measurable and so $S(\Phi(t) \Diamond W_t^H)(\xi)$ is also measurable by the first assumption and because

$$S(\Phi(t) \Diamond W_t^H)(\xi) = S\Phi(t)(\xi) SW_t^H(\xi)$$

Now we use the same estimate for $|SW_t^H(\xi)|$ as in example 5.1.4, that is (5.17) and our second assumption to get

$$\begin{aligned} \int_0^T |S\Phi(t)(\xi) SW_t^H(\xi)| dt &\leq \widetilde{C}_H \exp\left(\frac{1}{2}|\xi|_p^2\right) \int_0^T |S\Phi(t)(\xi)| dt \\ &\leq \widetilde{C}_H \exp\left(\frac{1}{2}|\xi|_p^2\right) K \exp(a|\xi|_{\hat{p}}^2) \end{aligned} \tag{5.22}$$

In general, it holds that for $q \geq p$ that $|.|_q \geq |.|_p$, henceforth we get that

$$\int_0^T |S\Phi(t)(\xi) SW_t^H(\xi)| dt \leq \widetilde{C}_H K \exp\left(\left(\frac{1}{2} + a\right) |\xi|_{p \vee \hat{p}}^2\right) \tag{5.23}$$

So by theorem 5.1.2 the integral $\int_0^T \Phi(t) dB_t^H$ exists. □

So why do we call this construction a fractional Itô integral? Firstly, [Ben03] has shown that the integral is well defined for any functional of fractional Brownian motion, that is $\int_0^T F(t, B_t^H) dB_t^H$ exists for any $F(t,.) \in \mathcal{S}(\mathbb{R})$. Also, for $H = 1/2$

$$\int_0^T \Phi(t) dB_t = \int_0^T \Phi(t) \Diamond W_t dt = \int_0^T \Phi(t) \delta B_t \tag{5.24}$$

Where the last integral is the Skorokhod integral. See [HØUZ96] for proof. It is well known that for an $\mathcal{F}_t$ adapted process, the Skorokhod integral coincides with the Itô integral. Furthermore, if define fBm on the White noise probability space, the way it is done here, then $F(t, B_t^H)$ is $\mathcal{F}_t$ adapted (see [Rog97] for proof). Hence for $H = 1/2$ integral constructed using white noise distribution theory coincides with the classical Itô integral.

# Chapter 6

# Stochastic differential equations driven by fBm

## 6.1 Preliminary deliberations

Finally, we would like to apply the white noise distribution theory to the study of stochastic differential equations driven by fractional Brownian motion. In the classical Itô theory of SDEs, one proves the existence and uniqueness of the solution to the following SDE:

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s$$

Here, $b$ and $\sigma$ are functions from $[0,T] \times \mathbb{R}$ to $\mathbb{R}$.

It would seem natural, therefore, to strive to find the solution, in some yet to be defined sense, of the following SDE, driven by an fBm process, where the integrals are defined as white noise integrals:

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB_s^H \tag{6.1}$$

Since we have defined the integral with respect to fractional Brownian motion as a white noise integral, the solution $X(t)$ to the SDE (if it exists) will only make sense as Hida distributions.

Now imagine, that $b(t,x) = x^2$ and that $W_t^H = \frac{\mathrm{d}}{\mathrm{dt}} B_t^H$ solves the SDE. However $W_t^H$ only exists as an element of $(S)^*$ and $(W_t^H)^2$ is not really defined. Thus we have to have $b$ and $\sigma$ defined as functions from $[0,T] \times (S)^*$ to $(S)^*$.

The obvious question is, what does the fact that we have a solution in $(S)^*$ actually mean? We quote [HØUZ96] on that matter:

> We emphasize that although the space $(S)^*$ of stochastic (Hida) distributions may seem abstract, it does allow a relatively concrete interpretation. Indeed, $(S)^*$ is analogous to the classical space $S'$ of tempered distributions. the difference being that the *test function space* for $(S)^*$ is a space of "smooth" *random variables* (denoted by $(S)$). Thus, if we interpret the random element $\omega$ as a specific

"experiment" or "realization" of our system, then generic elements $F \in (S)^*$ do not have point values $F(\omega)$ for each $\omega$, but only average values $F(\eta)$ with respect to smooth random variables $\eta = \eta(\omega)$. In other words, knowing the solution $X_t$ of SDE does not tell us what the outcome of a specific realization $\omega$ would be, but rather what the average over a set of realizations would be. This seems to be appropriate for most applications, because (in most cases) each specific singleton $\omega$ has probability zero anyway.

Having said this we specify what we mean by a weak solution. The first two conditions are natural. Since the S-transform is injective, the last condition implies that if $X$ is a weak solution to (6.1), then both sides of the equation (6.1) are equal as elements of $(S)^*$.

**Definition 6.1.1** (Weak solution)**.** The mapping $X : [0,T] \to (S)^*$ is called the *weak solution* of (6.1) on $[0,T]$ if it satisfies:

1. $X$ is weakly measurable.
2. $b(s, X(s))$ is integrable with respect to $ds$ in the white noise sense and $\sigma(s, X(s))$ is integrable with respect to $dB_s^H$, in the white noise sense.
3. for all $\xi \in \mathcal{S}(\mathbb{R})$ and for almost all $t \in [0,T]$ the following holds:

$$SX(t)(\xi) = SX(0)(\xi) + \int_0^t Sb(s, X(S))(\xi)ds + \int_0^t S\sigma(s, X(s))\Diamond W_t^H(\xi)ds$$

## 6.2 Existence and uniqueness of a weak solution

We now state and prove our main result, that is existence and uniqueness for certain SDEs driven by fBm. The "inspiration" for the following theorem is taken from [Kuo96], theorem 13.43, where an existence and uniqueness of a weak solution to a general white noise integral equation is proved.

Our theorem, however, is rather different, because of the particular nature of fractional Brownian motion. In particular we need stronger assumption about $\sigma$. The method of the proof of existence will be similar to the one in [Kuo96], but it is still substantially different, because we have two functions $b$ and $\sigma$. The reason for this is that we need rather strict conditions on $\sigma$, as it is being integrated with respect a fractional Brownian motion process. That way, we don't have to restrict $b$.

**Theorem 6.2.1** (Existence and uniquencess of solution)**.** Assume that $b$ and $\sigma$ defined from $[0,T] \times (S)^* \to (S)^*$ and satisfy the following conditions:

1. *Measurability*: $b(t, X(t))$ and $\sigma(t, X(t))$ are weakly measurable for any weakly measurable stochastic distribution process $X : [0,T] \to (S)^*$.
2. *Lipshitz conditions:* For almost all $t \in [0,T]$ and for $\Phi, \Psi \in (S)^*$, $b$ must satisfy:

$$|Sb(t, \Psi)(\xi) - Sb(t, \Phi)(\xi)| \leq L_b(t, \xi)|S\Psi(\xi) - S\Phi(\xi)|, \quad \forall \xi \in \mathcal{S}(\mathbb{R})$$

where $b \geq 0$ and
$$\int_0^T L_b(t,\xi) \leq K(1+|\xi|_0^2).$$
And similarly $\sigma$ must satisfy:
$$|S\sigma(t,\Psi)(\xi) - S\sigma(t,\Phi)(\xi)| \leq L_\sigma(t,\xi)|S\Psi(\xi) - S\Phi(\xi)|, \quad \forall\xi \in \mathcal{S}(\mathbb{R})$$
where $\sigma \geq 0$ and
$$\sup_{0<t<T} L_\sigma(t,\xi) \leq K(1+|\xi|_0).$$

3. *Growth condition*: For almost all $t \in [0,T]$ and for $\Phi \in (S)^*$,
$$|Sb(t,\Phi)(\xi)| + |S\sigma(t,\Phi)(\xi) \leq \rho(t,\xi)(1+|S\Phi(\xi)|), \quad \forall\xi \in \mathcal{S}(\mathbb{R})$$
where $\rho \geq 0$ and
$$\int_0^T \rho(t,\xi)dt \leq K\exp[c|\xi|_0^2]$$
for some positive constants $K$, $p$, $c$.

Then for any $X(0) \in (S)^*$, the equation (6.1) has a unique weak solution $X$ such that
$$\operatorname*{ess-sup}_{0\leq t\leq T} |SX(t)(\xi)| < \infty, \quad \forall\xi \in \mathcal{S}(\mathbb{R})$$

We will first present a proof of existence. Uniqueness of the result will be proved separately. Before we embark on the proof of existence, we first state and prove a few auxiliary lemma's that will be used in the proof.

**Lemma 6.2.2.** If $\xi \in \mathcal{S}(\mathbb{R})$ then
$$\int_0^T |SW_t^H(\xi)|dt \leq T^H|\xi|_0$$

This result might look suspicious at first, because on the right hand side of the inequality we have $|\xi|_0$, which would generally, on the intuitive level, be "adequate" for an element of $(L^2)$. However $W_t^H$ does not belong to $(L^2)$. Of course, we're looking at the integral of the S-transform of $W_t^H$, on the left hand side. Hence there is no contradiction to our intuition.

*Proof of lemma 6.2.2.* By theorem 4.7.7 and corollary 4.5.4 we have
$$\int_0^T SW_t^H(\xi)dt = \int_0^T \mathrm{M}_+^\mathrm{H}\,\xi(t)dt = (\xi, \mathrm{M}_-^\mathrm{H}\,1(0,T))_0$$
In general $\int f d\mu = \int g d\mu$ implies that $\int |f|d\mu = \int |g|d\mu$ and hence
$$\int_0^T |SW_t^H(\xi)|dt = \int_0^T |\xi(x)||\,\mathrm{M}_-^\mathrm{H}\,1(0,T)(x)|dx$$
and by Cauchy-Schwartz inequality
$$\int_0^T |W_t^H(\xi)|dt \leq |\xi|_0|\,\mathrm{M}_-^\mathrm{H}\,1(0,T)|_0$$
to complete the proof, note that $|\,\mathrm{M}_-^\mathrm{H}\,1(0,T)|_0 = T^H$. $\square$

**Lemma 6.2.3.** Assume that $b : [0,t] \times (S)^* \to (S)^*$ and $\sigma : [0,t] \times (S)^* \to (S)^*$ are functions satisfying the measurability and Lipschitz conditions above. Assume further, that X is some weakly measurable stochastic distribution process and that there exist non-negative constants $K_1$, $c_1$ and $p_1$ such that

$$\underset{0\le t\le T}{\text{ess-sup}} |SX(t)(\xi)| \le K_1 \exp[c_1|\xi|_{p_1}^2], \quad \forall \xi \in \mathcal{S}(\mathbb{R}) \tag{6.2}$$

Then the function $b(t,X(t))+\sigma(t,X(t))\Diamond W_t^H$, $t \in [0,T]$ is white noise integrable and with $K_2 = K(C_H + K_1 + K_1 C_H)$, $c_2 = c_1 + \frac{3}{2}$ and $p_2 = p \vee p_1$, one has

$$\int_0^T |Sb(t,X(t)) + S\sigma(t,X(t))SW_t^H| dt \le K_2 \exp[c_2|\xi|_{p_2}^2], \quad \forall \xi \in \mathcal{S}(\mathbb{R}) \tag{6.3}$$

*Proof of lemma 6.2.3.* We show that the conditions in theorem 5.1.2 are satisfied for $b(t,X(t)) + \sigma(t,X(t))\Diamond W_t^H$. The first condition is immediately satisfied by our measurability assumption.

By the assumption on $X(t)$ and the second assumption, we have

$$\begin{aligned}
&\int_0^T |Sb(t,X(t))(\xi) + \sigma(t,X(t))(\xi)SW_t^H(\xi)| dt \le \\
&\le \int_0^T |Sb(t,X(t))(\xi)| dt + C_H \exp[\frac{1}{2}|\xi|_p^2] \int_0^T |\sigma(t,X(t))(\xi)| dt \le \\
&\le \left(1 + C_H \exp[\frac{1}{2}|\xi|_p^2]\right) \int_0^T \rho(t,\xi)(1 + K_1 \exp[c_1|\xi|_{p_1}^2]) dt \le \\
&\le \left(1 + C_H \exp[\frac{1}{2}|\xi|_p^2]\right) K \exp[c|\xi|_p^2](1 + K_1 \exp[c_1|\xi|_{p_1}^2]) \le \\
&\le K(C_H + K_1 + K_1 C_H) \exp\left[\left(c_1 + \frac{3}{2}\right)|\xi|_{p_1 \vee p}^2\right]
\end{aligned}$$

Thus the condition (5.1) of theorem 5.1.2 is satisfied and hence

$$b(t,X(t)) + \sigma(t,X(t))\Diamond W_t^H$$

is white noise integrable. □

*Proof of existence of weak solution.* We will use a standard iteration argument, to prove the existence. The main constraint arises from the fact the we need to have the S-transforms of the iterations in a "nice" enough shape as to be able to apply theorem 3.3.5. This in turn dictates the strictness of the Lipschitz condition.

Let $G = SX(0)$, define $X_0(t) = X(0)$. Clearly $X_0 : [0,T] \to (S)^*$ is weakly measurable and also

$$\underset{0\le t\le T}{\text{ess-sup}} |SX_0(t)(\xi)| = |G(\xi)| \le K_0 \exp[c_0|\xi|_{p_0}^2] \tag{6.4}$$

where $K_0$, $c_0$ and $p_0$ are positive constants associated with $SX(0)$ by theorem 3.3.3.

Therefore by lemma 6.2.3 $b(t,X_0(t)) + \sigma(t,X_0(t))\Diamond W_t^H$ is white noise integrable and there are non-negative constants $K_0', c_0', p_0'$ such that

$$\int_0^T |Sb(t,X_0(t))(\xi) + S\sigma(t,X_0(t))(\xi)SW_t^H(\xi)| dt \le K_0' \exp[c_0'|\xi|_{p_0'}^2] \tag{6.5}$$

So we can safely define $X_1(t)$ as

$$X_1(t) = X(0) + \int_0^t b(s, X_0(s))ds + \int_0^t \sigma(s, X_0(s))dB_s^H$$

Then $X_1$ is weakly measurable and by (6.4) and (6.5) there are positive constants $K_1, c_1, p_1$ such that

$$\text{ess-sup}\, |SX_1(t)(\xi)| \leq K_1 \exp[c_1|\xi|^2_{p_1}] \tag{6.6}$$

Therefore by lemma 6.2.3 $b(t, X_1(t)) + \sigma(t, X_1(t))\Diamond W_t^H$ is white noise integrable and there are non-negative constants $K_1', c_1', p_1'$ such that

$$\int_0^T |Sb(t, X_1(t))(\xi) + S\sigma(t, X_1(t))(\xi)SW_t^H(\xi)|dt \leq K_1' \exp[c_1'|\xi|^2_{p_1'}]$$

So, similarly as before, we can safely define $X_2$ as

$$X_2(t) = X(0) + \int_0^t b(s, X_1(s))ds + \int_0^t \sigma(s, X_1(s))dB_s^H$$

Inductively repeating this argument, we can define a sequence of weakly measurable functions $X_n : [0, T] \to (S)^*$ as

$$X_n(t) = X(0) + \int_0^t b(s, X_{n-1}(s))ds + \int_0^t \sigma(s, X_{n-1}(s))dB_s^H \tag{6.7}$$

Now define $F_n(t) = SX_n(t)$ for $n > 0$ and $F_0(t) = G$. Using the growth condition and the estimate (5.17) obtained in example (5.1.4), we get

$$\begin{aligned}
&|F_1(t)(\xi) - G(\xi)| \leq \\
&\leq \int_0^t |Sb(s, X(0))(\xi) + S\sigma(s, X(0))(\xi)SW_t^H(\xi)|dt \\
&\leq \int_0^t |Sb(s, X(0))(\xi)|dt + C_H \exp\left[\frac{1}{2}|\xi|_p^2\right] \int_0^t |S\sigma(s, X(0))(\xi)SW_t^H(\xi)|dt \\
&\leq \left(1 + C_H \exp\left[\frac{1}{2}|\xi|_p^2\right]\right) \int_0^t \rho(t, \xi)(1 + |G(\xi)|)dt \\
&\leq (K + C_H K)(1 + |G(\xi)|) \exp\left[\frac{3}{2}|\xi|_p^2\right] =: H(\xi)
\end{aligned}$$

Our aim is to find a good enough estimate for $|F_n(t)(\xi) - F_{n-1}(t)(\xi)|$. We apply the Lipschitz condition:

$$\begin{aligned}
|F_2(t)(\xi) - F_1(t)(\xi)| &\leq \int_0^t |Sb(s, X_1(s))(\xi) - Sb(s, X(0))(\xi)| + \\
&\qquad |SW_s^H(\xi)|\, |S\sigma(s, X_1(s))(\xi) - S\sigma(s, X(0))(\xi)|ds \\
&\leq \int_0^t H(\xi)L_b(s, \xi) + |SW_s^H(\xi)|H(\xi)L_\sigma(s, \xi)ds \\
&\leq H(\xi) \int_0^t L_b(s, \xi) + |SW_s^H(\xi)|L_\sigma(s, \xi)ds
\end{aligned}$$

and again, substituting for $|F_2(t)(\xi) - F_1(t)(\xi)|$:

$$\begin{aligned}|F_3(t)(\xi) - F_2(t)(\xi)| \leq \int_0^t & L_b(s,\xi)|F_2(s)(\xi) - F_1(s)(\xi)| + \\ & + |SW_s^H(\xi)|L_\sigma(s,\xi)|F_2(s)(\xi) - F_1(s)(\xi) ds \\ \leq H(\xi) \int_0^t & (L_b(s,\xi) + |SW_s^H(\xi)|L_\sigma(s,\xi)) \times \\ & \times \int_0^s L_b(s_1,\xi) + |SW_{s_1}^H(\xi)|L_\sigma(s_1,\xi) ds_1 ds\end{aligned}$$

We can repeat this argument, substituting for $|F_i(t)(\xi) - F_{i-1}(t)(\xi)|$, repeatedly, to get

$$\begin{aligned}&|F_n(t)(\xi) - F_{n-1}(t)(\xi)| \leq \\ &\leq H(\xi) \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} \left(L_b(s_1,\xi) + |SW_{s_1}^H(\xi)|L_\sigma(s_1,\xi)\right) \ldots \\ &\qquad \ldots \Big(L_b(s_{n-1},\xi) + |SW_{s_{n-1}}^H(\xi)|L_\sigma(s_{n-1},\xi)\Big) ds_1 \ldots ds_{n-1} \\ &\leq H(\xi) \frac{1}{(n-1)!} \left(\int_0^T L_b(s,\xi) + |SW_s^H(\xi)|L_\sigma(s,\xi) ds\right)^{n-1}\end{aligned}$$

Using our assumptions about $L_b$ and $L_\sigma$ and also lemma (6.2.2), we get the following estimate

$$\begin{aligned}&|F_n(t)(\xi) - F_{n-1}(t)(\xi)| \leq \\ &\leq H(\xi) \frac{1}{(n-1)!} \left(K(1 + |\xi|_0^2) + K(1 + |\xi|_0) \int_0^T |SW_s^H(\xi)| ds\right)^{n-1} \\ &\leq H(\xi) \frac{1}{(n-1)!} \left(K(1 + |\xi|_0^2) + T^{H+1}|\xi|_0 K(1 + |\xi|_0)\right)^{n-1}\end{aligned}$$

Noting that, if $|\xi|_0 < 1$ then $K(|\xi|_0 + |\xi|_0^2) \leq K(1 + |\xi|_0^2)$ and that if $|\xi|_0 \geq 1$ then $K(|\xi|_0 + |\xi|_0^2) \leq 2K|\xi|_0^2$, we see that in general

$$|\xi|_0 K(1 + |\xi|_0) \leq 2K(1 + |\xi|_0^2)$$

Finally we have the following estimate

$$|F_n(t)(\xi) - F_{n-1}(t)(\xi)| \leq H(\xi) \frac{1}{(n-1)!} \left(K(1 + 2T^{H+1})(1 + |\xi|_0^2)\right)^{n-1} \tag{6.8}$$

So from (6.8) we can see that the series

$$G(\xi) + \sum_{n=1}^{\infty} |F_n(t)(\xi) - F_{n-1}(t)(\xi)|$$

converges absolutely, uniformly for $s \in [0,T]$. Henceforth $F_n(t)(\xi)$, being a partial sum of the series, converges uniformly for $s \in [0,T]$. From (6.8) we also see that

$$\begin{aligned}|F_n(t)(\xi)| &\leq |G(\xi)| + \sum_{n=1}^{\infty} |F_n(t)(\xi) - F_{n-1}(t)(\xi)| \\ &\leq |G(\xi)| + H(\xi) \exp\left[K(1 + 2T^{H+1})(1 + |\xi|_0^2)\right]\end{aligned}$$

Recall that we had

$$G(\xi) = SX(0)(\xi) \leq K_0 \exp[c_0|\xi|^2_{p_0}]$$
$$H(\xi) = (K + C_H K)(1 + |G(\xi)|) \exp\left[\frac{3}{2}|\xi|^2_p\right]$$

and so there exist non-negative constants $K', c', p'$ such that

$$|F_n(t)(\xi)| \leq K' \exp[c'|\xi|^2_{p'}] \tag{6.9}$$

Now one can define

$$F(t)(\xi) = \lim_{n\to\infty} F_n(t)(\xi)$$

And from (6.9) we see that for all $s \in [0,T]$ and for all $\xi \in \mathcal{S}(\mathbb{R})$ we have

$$|F(t)(\xi)| \leq K' \exp[c'|\xi|^2_{p'}] \tag{6.10}$$

Therefore by theorem 3.3.5 there exist $X(t) \in (S)^*$ such that $SX(t) = F(t)$. The final step is to show that this $S$ is indeed a weak solution to (6.1) according to our definition of a weak solution (6.1.1).

$SX(.)(\xi)$ is measurable since it is a limit of measurable functions $F_n(.)(\xi)$. So $\langle\langle X(.), \varphi\rangle\rangle$ is measurable for all $\varphi \in (S)^*$ and hence $X$ is weakly measurable and the first condition (for a weak solution) is satisfied.

Now if we look at (6.9) and use lemma 6.2.3 we see that

$$b(t, X(t)) + \sigma(t, X(t))\Diamond W_t^H$$

is white noise integrable, thus satisfying the second condition (for a weak solution).

Finally we wish to verify the third condition (for a weak solution). To that end, take the S-transform of (6.7), to get

$$F_n(t)(\xi) = G(\xi) + \int_0^t Sb(s, X_{n-1}(s))(\xi)ds + \int_0^t S\sigma(s, X_{n-1}(s))(\xi)SW_s^H(\xi)ds \tag{6.11}$$

Also, for all $\xi \in \mathcal{S}(\mathbb{R})$ and by the Lipschitz condition,

$$\begin{aligned}
&\int_0^T |Sb(s, X_n(s))(\xi) - Sb(s, X(x))(\xi) + \\
&\qquad + \left(S\sigma(s, X_n(s))(\xi) - S\sigma(s, X(s))(\xi)\right) W_s^H(\xi)|ds \leq \\
&\leq \int_0^T L_b(s,\xi)|F_n(s)(\xi) - F(s)(\xi)|ds + \\
&\qquad + \int_0^T L_\sigma(s,\xi)T^H|\xi|_0|F_n(s)(\xi) - F(s)(\xi)|ds \leq \\
&\leq \sup_{0\leq s\leq T} |F_n(s)(\xi) - F(s)(\xi)| \int_0^T L_b(s,\xi) + L_\sigma(s,\xi)T^H|\xi|_0 ds \to 0 \\
&\text{as} \quad n \to \infty
\end{aligned}$$

since $F_n(s)(\xi)$ converges uniformly in $s \in [0,T]$. Thus we can let $n \to \infty$ in (6.11) and we obtain:

$$F(t)(\xi) = G(\xi) + \int_0^t Sb(s, X(s))(\xi)ds + \int_0^t S\sigma(s, X(s))(\xi)SW_s^H(\xi)ds$$

Thus the S-transform of the left hand side is equal to the S-transform of the right hand side, satisfying the third condition (for a weak solution). □

To prove the uniqueness we will use a variation on the Gromwall lemma, taken from [FR75], page 198. We will first state and prove the lemma, then proceed to the proof of uniqueness.

**Lemma 6.2.4.** Assume that $f \in L^\infty([0,T])$ and $f \geq 0$ almost everywhere. Further assume, that f satisfies the condition

$$f(t) \leq \rho(t) + \int_0^t \theta(s) f(s) ds, \quad \text{a.e. on } [0,T]$$

where $\rho \in L^\infty([0,T])$ and $\theta \in L^1([0,T)$, $\theta \geq 0$ almost everywhere. Then

$$f(t) \leq \rho(t) + \int_0^t \rho(s)\theta(s) \exp\left(\int_s^t \theta(u) du\right) ds, \quad \text{a.e. on } [0,T]$$

We just remark that $\rho$ here is different to the $\rho$ specifying the growth condition in our theorem.

*Proof.* Let $g(t) = \int_0^t \theta(s) f(s) ds$. Then we have $f(t) \leq \rho(t) + g(t)$. The function $g$ is absolutely continuous and $g'(t) = \theta(t) f(t)$ almost everywhere. Hence for almost all $t \in [0,T]$,

$$g'(t) - \theta(t) g(t) = \theta(t)(f(t) - g(t)) \leq \rho(t)\theta(t).$$

Multiplying both sides by the integrating factor $\exp\left(-\int_0^t \theta(u) du\right)$, we get

$$\frac{d}{dt}\left(g(t) \exp\left(-\int_0^t \theta(u) du\right)\right) \leq \rho(t)\theta(t) \exp\left(-\int_0^t \theta(u) du\right).$$

Which implies that

$$g(t) \exp\left(-\int_0^t \theta(u) du\right) \leq \int \rho(s)\theta(s) \exp\left(-\int_0^s \theta(u) du\right) ds.$$

Hence we obtain

$$g(t) \leq \int_0^t \rho(s)\theta(s) \exp\left(\int_s^t \theta(u) du\right) ds.$$

Finally,

$$f(t) \leq \rho(t) + g(t) \leq \rho(t) + \int_0^t \rho(s)\theta(s) \exp\left(\int_s^t \theta(u) du\right) ds.$$

□

*Proof of uniqueness of solution.* Assume $X(t)$ and $Y(t)$ are both weak solutions to (6.1), as defined above. Define $F(t) = SX(t)$ and $G(t) = SY(t)$. Then for a fixed $\xi \in \mathcal{S}(\mathbb{R})$, we have

$$
\begin{aligned}
&|F(t)(\xi) - G(t)(\xi)| \leq \\
&\leq \int_0^t |Sb(s, X(s))(\xi) - Sb(s, Y(s))(\xi)| + \\
&\qquad + |SW_s^H(\xi)|\, |S\sigma(s, X(s)) - S\sigma(s, Y(s))(\xi)| ds \\
&\leq \int_0^t \left(L_b(s, \xi) + |SW_s^H(\xi)| L_\sigma(s, \xi)\right) |F(s)(\xi) - G(s)(\xi)| ds
\end{aligned}
$$

where we have used the Lipschitz condition to get the last inequality. We also have that

$$\int_0^T L_b(s, \xi) ds \leq K(1 + |\xi|_0^2)$$

$$\int_0^T L_\sigma(s, \xi) |SW_s^H(\xi)| ds \leq T^H K(1 + |\xi|_0)|\xi|_0$$

and hence $\theta(t) := L_b(s, \xi) + L_\sigma(s, \xi)|SW_s^H(\xi)| \in L^1([0, T])$. Now define $f(t) = |F(t)(\xi) - G(t)(\xi)|$ and note that $f \in L^\infty([0, T])$ and hence the assumptions of above lemma are satisfied with $\rho \equiv 0$. Therefore $F(t)(\xi) = G(t)(\xi)$ for almost all $t \in [0, T]$.

Now we have to overcome a slight technical difficulty, namely that the null set $A_\xi = \{F(t)(\xi) \neq G(t)(\xi)\}$ could be different for each $\xi$ in $\mathcal{S}(\mathbb{R})$ and together they might have non-zero Lebesgue measure.

However, since $\mathcal{S}(\mathbb{R})$ is separable, there is a countable dense subset of $\mathcal{S}(\mathbb{R})$, say $\{\xi_n, n \geq 1\}$. Let

$$A_0 = \bigcup_{n \geq 1} A_{\xi_n}.$$

Then $A_0$ is a null subset of $[0, T]$. Hence if $t \in A_0^c$ then, $F(t)(\xi) = G(t)(\xi)$, for all $\xi \in \mathcal{S}(\mathbb{R})$. Since the S-transform is injective, $X = Y$. □

## 6.3 Examples and comments

Our theorem is very similar to theorem 13.43 [Kuo96]. However here the stochastic differential equation is of the form

$$X(t) = X(0) + \int_0^t f(s, X(s)) ds.$$

The Lipschitz condition is then similar as with our result, except that in [Kuo96] it is required that

$$\int_0^T L(t, \xi) dt \leq K(1 + |\xi|_p^2)$$

We can't achieve that with fractional white noise (time derivative of fBm). We only have (or rather, here we only proved) the estimate given by lemma 6.2.2:

$$\int_0^T |SW_t^H(\xi)| dt \leq T^H |\xi|_0,$$

which won't satisfy the above condition for $\xi$ such that $|\xi|_0 < 1$. The method of proof used above is similar to the one given in [Kuo96], but the technical details (especially the estimates) are different.

It is appropriate now to comment on the usefulness (or lack of it) of our result. Quite clearly, the most difficult conditions to check for any SDE will be the growth and Lipshitz conditions. We now give several examples of SDEs that our theorem applies to (and some where it does not apply). We begin with a trivial one.

**Example 6.3.1.** Consider the following SDE, popular in particular in mathematical finance:

$$S(t) = S(0) + \int_0^t rS(s)ds + \int_0^t \sigma S(s)dB_s^H$$

Here, the coefficients are constant and hence the growth and Lipschitz conditions are trivially satisfied. We can use the Itô formula in [Ben03], to check that the solution is given by

$$S(t) = \exp\left((r - \sigma^2 H t^{2H-1})t + \sigma B_t^H\right) \tag{6.12}$$

Our theorem tells us, that this solution is unique in $(S)^*$. Furthermore, we can see, that the right hand side of (6.12) is in $(L^2)$ and hence the solution is unique as an element of $(L^2)$.

**Example 6.3.2.**

$$X(t) = X(0) + \int_0^t B_s^H \Diamond X(s) dB_s^H$$

Here, we have $b \equiv 0$ and $\sigma(s, \Phi) = B_s^H \Diamond \Phi$. We see that

$$|S\sigma(s,\Phi)(\xi) - S\sigma(s,\Psi)(\xi)| \leq \underbrace{|SB_s^H(\xi)|}_{=:L(s,\xi)} \, |S\Phi(\xi) - S\Psi(\xi)|$$

We need to check whether $L(s, \xi)$ is "nice enough". We know that

$$|SB_s^H(\xi)| = |\left(\xi, \mathrm{M}_-^{\mathrm{H}}\, 1(0,s)\right)_0| \leq |\xi|_0 |\, \mathrm{M}_-^{\mathrm{H}}\, 1(0,t)| \leq s^H |\xi|_0 \leq T^H |\xi|_0$$

The last inequality follows since $s \in [0, T]$. Hence $L(s, \xi) \leq T^H(1 + |\xi|_0)$ and so the Lipschitz condition is satisfied. Trivially, the growth condition is also satisfied. Therefore the SDE has a unique solution in $(S)^*$.

Next we look at an example of an SDE, where our theorem tells us nothing about the solution.

**Example 6.3.3.**

$$X(t) = X(0) + \int_0^t B_s^H \Diamond B_s^H \Diamond X(s) dB_s^H$$

We proceed similarly as in the previous example, only to obtain

$$L(s, \xi) = \left(SB_s^H(\xi)\right)^2.$$

If we then used the same method as above, then the best we get is:

$$L(s,\xi) = \left(SB_s^H(\xi)\right)^2 \leq T^{2H}|\xi|_0^2$$

Hence we can't claim that the Lipschitz condition is satisfied and thus our theorem does not apply.

Perhaps we could have found a better estimate for $L(s,\xi)$ above. The general point is, though, as the example shows, that it is quite difficult to apply the theorem, because we have to calculate some S-transform, which is usually more difficult that in the case of $(B_t^H)^{\diamond 2}$. Then one has to try to find a very strict estimate of $L(s,\xi)$, which can prove to be rather difficult.

As mentioned above the strictness of the Lipschitz condition arises from the fact that we need to apply the convergence theorem 3.3.5. It does not seem likely, that there is a straightforward way, in which we could find a convergence theorem more suitable for our purpose.

# Appendix A

## A.1 Iterated Itô Integral and Wiener-Itô Chaos Expansion

A very clear exposition of the general topic of Wiener-Itô chaos expansion theorems (certainly a lot clearer than this appendix) can be found in [Nua95]. We assume that $W(t) = W(t, \omega)$ is a Wiener process on some probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ is the $\sigma$-algebra generated the Wiener process.

First we define *symmetric* functions. A function $g : [0, T]^n \to \mathbb{R}$ is called *symmetric* if and only if

$$g\left(x_{\sigma_1}, \ldots, x_{\sigma_n}\right) = g(x_1, \ldots, x_n),$$

for all permutations $\sigma$ of $(1, \ldots, n)$. If $g$ also belongs to $L^2(\mathbb{R})$, then we write $g \in \widehat{L^2}(\mathbb{R})$.

Now note that if we let

$$S_n = \{(x_1, \ldots, x_n) \in [0, T]^n : 0 \leq x_1 \leq \ldots \leq x_n \leq T\},$$

then

$$\|g\|^2_{L^2([0,t]^n)} = n! \int_{S_n} g^2(x_1, \ldots, x_n) dx_1 \ldots dx_n = n! \|g\|^2_{L^2(S_n)}.$$

If $f$ is deterministic and $f \in L^2(S_n)$ then we can form the *iterated Itô integral*:

$$\mathrm{J_n}(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \ldots, t_n) dW(t_1) dW(t_2) \ldots dW(t_{n-1}) dW(t_n),$$

because for each Itô integral with respect to $dW(t_i)$ the integrand is $\mathcal{F}_t$ adapted and square integrable with respect to $d\mathbb{P} \times dt_i$. We remark that

1. $\mathbb{E}\left(\mathrm{J_n}^2(h)\right) = \|h\|_{L^2(S_n)}$, which follows from the definition and Itô's isometry.

2. $$\mathbb{E}\left(\mathrm{J_m}(g)\, \mathrm{J_n}(g)\right) = \begin{cases} 0 & \text{if} \quad n \neq m \\ (g, h)_{L^2(S_n)} & \text{if} \quad n = m \end{cases}$$

3. For a symmetric function $g$ define $\mathrm{I_n}(g) = n! J_n(g)$.

**Theorem A.1.1.** If $e_n$ are the Hermite functions in $L^2(\mathbb{R})$ and $h_n$ the hermite polyonomials, then $\mathrm{I_n}$ satisfies:

$$\mathrm{I_n}(e_1^{\otimes\alpha_1}\widehat{\otimes}\ldots\widehat{\otimes}e_n^{\otimes\alpha_n}) = \prod_{j=1}^{n} h_{\alpha_j}(\langle .,e_j\rangle), \text{ for any multi-index } \alpha.$$

**Theorem A.1.2** (Wiener-Itô Chaos Expansion)**.** Assume that $\varphi$ is an $\mathcal{F}_t$ measurable random variable and $\varphi$ belongs to $L^2(\Omega)$. Then there exists a unique sequence $(f_n)_{n=0}^{\infty}$ of functions $f_n \in \widehat{L}^2(\mathbb{R}^n)$ such that

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n) \tag{A.1}$$

where the series is understood to converge in $L^2(\Omega)$. Furthermore we have

$$\|\varphi\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2(\mathbb{R}^n)} \tag{A.2}$$

**Theorem A.1.3** (Wiener-Itô Chaos Expansion II)**.** Assume that $X \in L^2(\Omega)$. Then there exist unique $c_\alpha \in \mathbb{R}$ such that

$$X = \sum_{\alpha} c_\alpha \,\mathrm{I_n}(e_1^{\otimes\alpha_1}\widehat{\otimes}\ldots\widehat{\otimes}e_n^{\otimes\alpha_n}) \text{ and } \|X\|^2_{L^2(\Omega)} = \sum_{\alpha} \alpha! c_\alpha^2,$$

where $e_k$ are the Hermite functions.

## A.2 Nuclear Hilbert Spaces

We briefly review some functional analytic results, mainly without proofs. See [GV64] for proofs.

**Definition A.2.1** (Countably Hilbert Space)**.** Consider some Hilbert space $\mathcal{H}$ with an inner product $(.,.)_0$. Assume that there exists a countable collection of compatible inner products $(.,.)_n$, compatible in the sense that if a sequence in $\mathcal{H}$ converges w.r.t. $(.,.)_n$ and is Cauchy w.r.t. $(.,.)_m$, then it also converges w.r.t. $(.,.)_m$.

We then say that $\mathcal{H}$ is a countably Hilbert space, if it is complete w.r.t. the metric:

$$\rho(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|\varphi - \psi\|_n}{1 + \|\varphi - \psi\|_n} \tag{A.3}$$

It follows immediately from the definition that if $\mathcal{H}$ is a countably Hilbert space, then $\mathcal{H} = \bigcap_{n\geq 1} \mathcal{H}_n$ and conversly if the topology in a linear space is defined by a countable collection of inner products, then $\mathcal{H} = \bigcap_{n\geq 1} \mathcal{H}_n$.

**Definition A.2.2** (Degenerate operators)**.** And operator is said to be degenerate if it maps an entire space onto a finite dimensional subspace.

**Definition A.2.3** (Completely continuos operators)**.** $A \in \mathcal{L}(H_1, H_2)$ is called completely continuous if it maps any bounded set into a set whose closure is compact. In other words: If $U \subset H_1$ is bounded then $\overline{A(U)} \subset H_2$ is compact.

The following result is crucial property of completely continuous linear operators.

**Theorem A.2.4.** If $A : H_1 \to H_2$ is completely continuous, then $\exists$ a positive definite $T : H_1 \to H_1$ and an isometric operator $U : Im\ T \to H_2$ s.t. $A = UT$.

Any completely continuous operator can be approximated as follows:

$$Af = \sum_{n=1}^{\infty} \lambda_n (f, e_n) h_n \tag{A.4}$$

where $\lambda_n$ are the eigenvalues of the operator $T$ in $A = UT$ and $(e_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are orthonormal bases of $H_1$ and $H_2$ respectively. Thus, we can define degenerate operators $P_k : H_1 \to H_2$ by

$$P_k := \sum_{n=1}^{k} \lambda_n (f, e_n) h_n \tag{A.5}$$

Clearly $\Im P_k$ is finitie dimensional and also $A = \lim_{k\to\infty} P_k$. Therofor one can see that the space of completely continous linear operators coincides with the completion of the space of degenerate operators.

**Definition A.2.5** (Hilbert-Schmidt operators)**.** A completely continuous operator $A = UT$ is of the Hilbert-Schmidt type iff $\sum_{n\in\mathbb{N}} \lambda_n^2 < \infty$, where $\lambda_n$ are the eigenvalues of $T$.

Clearly for any operator $A$ which is of Hilbert-Schmidt type, we get the approximation (A.4). The following two results provide equivalent definitions of Hilbert-Schmidt operators.

**Theorem A.2.6.** $A$ is of Hilbert-Schmidt type iff $\exists$ an orhtonormal basis $(f_n)_{n\in\mathbb{N}}$ of $H_1$ such that:

$$\sum_{n\in\mathbb{N}} \|Af_n\|^2 < \infty$$

**Theorem A.2.7.** $A : H_1 \to H_2$ is of Hilbert-Schmidt type iff

$$Af = \sum_{n=1}^{\infty} \lambda_n \left(f, e_n\right) h_n \tag{A.6}$$

for some $(e_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ orthonormal bases of $H_1$ and $H_2$ respectively and a sequence $(\lambda_n \geq 0)_{n\in\mathbb{N}}$ satisfying $\sum \lambda_n^2 < \infty$.

**Definition A.2.8** (Nuclear operators)**.** A completely continous operator is called nuclear iff $\sum_{n\in\mathbb{N}} \lambda_n^2 < \infty$, where $\lambda_n$ are the eigenvalues of $T$ in $A = UT$.

From the definition it follows that any nuclear operator is also of the Hilbert-Schmidt type. Nuclear operators are sometimes reffered to as trace operators.

**Lemma A.2.9.** If T is a completely continuous positive definite operator, then T is nuclear iff T has a finite trace, i.e.

$$\sum_{n\in\mathbb{N}} (Te_n, e_n) < \infty$$

for any orthonormal basis.

**Theorem A.2.10.** A product of any two Hilbert-Schmidt type operators is a nuclear operator and conversly for any nuclear operator $U$ there are Hilber-Schmidt operators $A$,$B$, such that $U = AB$.

## A.3 Hermite polynomials

**Definition A.3.1.** For $n \in \mathbb{N}$, define the *nth Hermite polynomial* as

$$h_n(x) = (-1)^n e^{x^2} \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n e^{-x^2} \tag{A.7}$$

**Theorem A.3.2.** We have the following identities:

$$h_{n+1}(x) = 2xh_n(x) + 2nh_{n-1}(x) \tag{A.8}$$

$$h_n'(x) = 2nh_{n-1}(x) \tag{A.9}$$

$$h_n''(x) - 2xh_n'(x) + 2nh_n(x) = 0 \tag{A.10}$$

**Theorem A.3.3.** The Hermite functions

$$e_n(x) = (\pi^{1/2}(n-1)!)^{-1/2} \exp\left(-\frac{x^2}{2}\right) h_n(x) \tag{A.11}$$

constiute an orthonormal basis of $L^2(\mathbb{R})$.

**Theorem A.3.4.** Hermite functions have the following properties:

$$\left(-\left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^2 + x^2 + 1\right) e_n = (2n+2)e_n \tag{A.12}$$

There is a constant $K \geq 0$ such that for $n \geq 1$

$$\sup_{x \in \mathbb{R}} |e_n(x)| \leq Kn^{-1/12} \quad \text{and} \quad \|e_n\|_{L^1(\mathbb{R})} \leq Kn^{1/4}. \tag{A.13}$$

Also,

$$e_n'(x) = \sqrt{\frac{n}{2}} e_{n-1}(x) - \sqrt{\frac{n+1}{2}} e_{n+1}(x) \tag{A.14}$$

$$xe_n(x) = \sqrt{\frac{n}{2}} e_{n-1}(x) + \sqrt{\frac{n+1}{2}} e_{n+1}(x) \tag{A.15}$$

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