Cruz-Gonzalez

~~APPENDIX~~

ON THE STATISTICS OF DOUBLE STARS

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The observed distribution of eccentricities among double stars with known orbits is far from proving (despite the opinion of Jeans) the presence of equipartition of energy among them. Direct examination of the distribution of the internal energies of stellar pairs (of the semimajor axes of the orbits) shows, on the contrary, that equipartition of energy has not approached even among wide pairs. This circumstance, together with the absence of dissociative equilibrium between double and individual stars leads to an age of the aggregate of double stars not over 10^{10} years.

It has been shown by a series of authors that the study of the distribution law of the elements of double star orbits, as well as of other statistical dependences for these objects, can give interesting results for cosmogony in general and for the solution of the problem of the age of our stellar system in particular*. However, as was shown by the author in a preliminary note[1], erroneous conclusions often arise from the observational data. The goal of the present investigation is to show the incorrectness of several earlier inferences that have received rather wide circulation in the literature[2], and to point out several new consequences of the observational material that concerns double stars.

*Everywhere in the present article we discuss the age of our stellar system, and not the age of the universe as a whole.

1. The distribution of orbital eccentricities among double stars

The question of the eccentricity of orbits is fairly often discussed from the observational regularities. It is established that among double stars with determined orbits the number of pairs with eccentricities less than e is proportional to e^2.

On the other hand, Jeans showed that in the presence of statistical equilibrium (Boltzmann distribution) this very dependence should be observed. Hence there arose the conclusion that we already have the situation with the most probable distribution, which leads directly to a long time scale. According to the more careful formulation of Jeans, cited in answer to the preliminary note of the author[3], equipartition was established at least in several respects.

First of all it is necessary to understand clearly that the quoted distribution of eccentricities can to a considerable degree differ from the real one as a consequence of the selectivity of the observational material. So far we know only the orbits of pairs with relatively short periods. On the other hand, the average eccentricity, as the observations reveal, undoubtedly increases with the period. Therefore in fact the relative number of all pairs with larger eccentricities is greater than the relative number of these pairs among double stars with known orbits.

In order to introduce clarity into the question under consideration, we examine from the theoretical side the distribution of states of the companions in phase space. For the coordinates in phase space we may use the three components of position of the companion and the three components of its momentum, referred to the primary star.

For statistical equilibrium we should have the number of companions dN in the element of volume $dx\,dy\,dz\,dp_x\,dp_y\,dp_z$ equal to

$$dN = Ce^{-\frac{E(x,y,z,p_x,p_y,p_z)}{\theta}}\,dx\,dy\,dz\,dp_x\,dp_y\,dp_z \qquad (1)$$

where

$$E = \frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right) - \frac{\gamma Mm}{\sqrt{x^2+y^2+z^2}} \qquad (2)$$

is the energy of the companion, M and m are the masses of the primary star and companion, γ is the gravitational constant, and θ the modulus of the Boltzmann distribution.

Let us consider, however, instead of this most probable distribution the considerably more general type of distribution when the density in phase space does not have the special form $Ce^{-\frac{E}{\theta}}$ but appears as an arbitrary function f(E) of the energy E. Then

$$dN = f(E)\,dx\,dy\,dz\,dp_x\,dp_y\,dp_z$$

We now introduce in phase space a canonical transformation, passing from the variables x, y, z, p_x, p_y, p_z to the variables of Delaunay's[4] lunar theory: L, G, H, l, g, and h. As for the first three of these quantities, they are expressed through the usual elements of elliptic motion: the semimajor axis a, the inclination i, and the eccentricity e, in the following manner:

$$L = m\sqrt{\gamma M}\,a^{\frac{1}{2}}$$
$$G = m\sqrt{\gamma M}\,a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}$$
$$H = m\sqrt{\gamma M}\,a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}\cos i$$

However, the angular coorddnates l, g, and h appear in terms of other elements, such as the mean anomaly, the distance from periastron to the node, and the longitude of the ascending node.

As is well known, under a canonical transformation in phase space the element of volume preserves its magnitude (the Jacobian of the transformation is equal to unity). In other words:

$$dx\, dy\, dz\, dp_x\, dp_y\, dp_z = dL\, dG\, dH\, dl\, dg\, dh$$

On the other hand,

$$E = -\frac{\gamma^2 M^2 m^3}{2L^2} = -\frac{1}{2}\frac{\gamma M m}{a}$$

That is, the energy depends only on the element L. Consequently, in our case also the density in phase space depends only on L, and we may write

$$dN = f(L)\, dL\, dG\, dH\, dl\, dg\, dh$$

From this it follows that the number of pairs for which L is contained between L and L+dL, and G is greater than some given value G_0, is equal to

$$8\pi^3 f(L)\, dL \int_{G_0}^{L} dG \int_0^G dH$$

as l, g, and h vary independently of each other from 0 to 2π, H takes values from 0 to G, and G by definition varies from 0 to L. The expression written above is thus equal to

$$4\pi^3 f(L)\,(L^2 - G_0^2)\, dL$$

but

$$L^2 - G_0^2 = m^2 \underbrace{\gamma^2 M^2}_{} a e_0^2 = L^2 e_0^2$$

This should be γM; the original is misprinted where e is the eccentricity corresponding to the orbit with the given L and $G = G_0$.

So the number of stars with $e < e_0$ (i.e., $G > G_0$) and L contained in the limits L and L+dL is equal to

$$4\pi^3 f(L) L^2 e_0^2 \, dL$$

from which it follows that the number of all orbits for which $e < e_0$ is equal to

$$N(e_0) = 4\pi^3 e_0^2 \int_0^\infty f(L) L^2 \, dL$$

The integral on the right is a constant number, and we therefore obtain the following theorem:

If the density in phase space is an arbitrary function of L; that is, of the total energy, and only of its magnitude, then the number of all stars with eccentricities less than e_0 is proportional to ${e_0}^2$.

From this it follows that if we even consider that the observed $N(e_0)$ is also proportional to ${e_0}^2$ (although, as has been pointed out above, the selectivity of the material strongly compels us to doubt this) then from this it is completely impossible to conclude that the phase density is proportional to $e^{-\frac{\varepsilon}{\delta}}$, i.e. that equipartition of energy occurs. On the contrary, for <u>any distribution</u> of energy with the condition only that the phase density does not depend on the other elements, we should have $N(e) \sim e^2$.

Thus even if we accept that in actuality $N(e) \sim e^2$, nevertheless from this it is impossible to draw conclusions about the equipartition of energy, and especially about the duration of the life of the stellar system.

The following circumstance, however, is worthy of attention. According to what we have said, in the case where the phase density depends only on L (that is, on E, or, what is the same, on the semimajor axis) for each interval dL we have that the number of orbits with eccentricities contained between e and e+de should be proportional to $e\,de$, and independent of a. Thus also the average value of the eccentricity for each interval of size of the semimajor axis

$$\bar{e} = \frac{\int_0^1 e^2\,de}{\int_0^1 e\,de} = \frac{2}{3} \qquad (4)$$

should not depend on a, but should be equal to two thirds. The observational material is in contradiction to this. The following table, obtained by Aitken[5], points this out:

$\bar{P}$	$\bar{e}$	n
16.8 years	0.43	14
37.1	0.40	24
73.0	0.53	24
138	0.57	23
200	0.62	18

In this table are given the average values of the eccentricities for stars grouped according to period. In the first column is the average period of each group, and in the last, the number of stars in the group. If to this table we add Russell's statistical result that for stars with periods around 5000 years the average eccentricity is equal to 0.76, then we should conclude that e depends on P. It is known that $P \sim L^3$. Therefore $\bar{e}$ depends on L. It is perfectly obvious that the fundamental assumption made above is untrue, and the phase density does not depend only on the semimajor axis. This means that not only is it impossible to say that the phase density is proportional to $e^{-\frac{E}{\theta}}$, but also that in general it is impossible to consider that the phase density depends only on the energy. However, there are indications that the cited dependence of ϖ on P is strongly subjedt to the effect of observational selection.[10,11,12]

It is possible, however, that for distant components (P greater than 100 years) the change in ϖ is small, and that for these the phase density depends only on E. Thus it is of interest which dependence of the phase density on E the observations point to, and to what extent the existing dependence is close to the Boltzmann.

2. Derivation of the phase density from observational data

In the present paragraph we assume that the phase density depends only on E, and try to obtain from the empirical material the form of this dependence. We have seen that at least for small L the phase density perhaps depends also on the other elements. Therefore we must interpret the result we have obtained only as a result of averaging over the other elements. Even in this form our result will have some value, especially as for distant components our assumption is probably correct. Thus, let the phase density be equal to f(L). This means that in the volume element $dx\,dy\,dz\,dp_x dp_y dp_z$ the number of stars will be equal to

$$f\left(\gamma M m \sqrt{\frac{m}{\frac{2Mm\gamma}{r} - \frac{p^2}{m}}}\right) dx\,dy\,dz\,dp_x\,dp_y\,dp_z$$

where

$$r = \sqrt{x^2+y^2+z^2} \quad ; \quad p = \sqrt{p_x^2+p_y^2+p_z^2}$$

Therefore the density of the distribution in ordinary space is equal to

$$\rho = \iiint f\left(\gamma M m \sqrt{\frac{m}{\frac{2Mm\gamma}{r} - \frac{p^2}{m}}}\right) dp_x\, dp_y\, dp_z$$

$$= 4\pi \int_0^{m\sqrt{\frac{2M\gamma}{r}}} f\left(\gamma M m \sqrt{\frac{m}{\frac{2Mm\gamma}{r} - \frac{p^2}{m}}}\right) p^2 dp$$

The "distribution in ordinary space" is here understood to be the spatial distribution of all companions, when the primary stars are all combined into one point. The upper limit of integration in the last integral is obtained from the condition that we are examining only physical companions, moving in elliptical orbits; that is, those systems for which the total energy is negative.

We now introduce into the last integral instead of p the variable of integration

$$L = \gamma M m \sqrt{\frac{m}{\frac{2Mm\gamma}{r} - \frac{p^2}{m}}}$$

We obtain

$$\rho(r) = \int_{m\sqrt{\frac{\gamma M}{2}}\, r^{1/2}}^{\infty} f(L) \sqrt{\frac{2Mm^2\gamma}{r} - \frac{\gamma^2 M^2 m^4}{L^2}} \cdot \frac{4\pi^2\gamma^2 M^2 m^4\, dL}{L^3}$$

If we designate

$$K = m\sqrt{\frac{\gamma M}{2}}\, r^{1/2}$$

then

$$\rho(r) = 4\pi^2\gamma^3 M^3 m^6 \int_K^{\infty} f(L) \sqrt{\frac{1}{K^2} - \frac{1}{L^2}}\ \frac{dL}{L^3}$$

or

$$\rho(K) = C \int_K^{\infty} f(L) \sqrt{L^2 - K^2}\ \frac{dL}{KL^4} \qquad (5)$$

Making use of these integral equations, we can find for a known density in ordinary space ρ the phase density f(L). In a well-known study, Öpik[6] showed that the existing observational material, corrected for observational selection, gives

$$\rho \sim \frac{1}{r^3} \qquad (6)$$

or

$$\rho \sim \frac{1}{K^6}$$

It is evident that for this special form of the function ρ, equation (5) is satisfied by the function

$$f(L) \sim \frac{1}{L^3} \qquad (7)$$

Let us now compare this "observed" density in phase space with that which should exist for statistical equilibrium. For statistical equilibrium we should have

$$f(L) = Ce^{-\frac{E}{\theta}} = Ce^{\frac{\chi^2 M^2 m^3}{\kappa L^2 \theta}} \qquad (8)$$

We see that to elucidate the question it is necessary to know the quantity θ. If the set of double stars has arrived at statistical equilibrium as a result of encounters with other stars, then θ should be of the order of two thirds of the average kinetic energy of the translational motion of the circling star. If we assume that the average speed of translational motion of the stars is of the order of 25 km/sec, then already for about a > 20 A.U. the exponent on the right side of formula (8) is very small compared to unity. Thus for all larger values of L (and in the same way of a) we can to a high degree of approximation rewrite (8) in the form

$$f(L) = \text{constant} \qquad (9)$$

At the same time, the result obtained by Öpik applies just to the distant components. Thus (7) applies to large values of L.

We see that the "observed" phase density varies according to a law sharply different from that which exists for statistical equilibrium.

It is possible to show that the "distribution in ordinary space" for the case of statistical equilibrium also differs sharply from observation. In fact, from (9) and (5) it follows that for statistical equilibrium

$$\rho \sim \frac{1}{r^{3/2}} \qquad (10)$$

in contradiction to the observed distribution (6). The difference between (10) and (6) is so great that there cannot be any doubt that (10) is not observed in actuality if only with a small degree of approximation. Since the regularity (3) was extablished by Öpik for distant components with distances up to 10,000 A.U., we may conclude that even for such distant components the influence of encounters has not yet led to statistical equilibrium (i.e., to the most probable distribution) in the distribution of the semimajor axes; that is, of the energies. As we shall see, this strongly decreases the upper bound for the lifetime of the stellar system.

3. Verification of the Opik inverse cube law for new observational material

In the present paragraph we examine one very simple method of verification of the law obtained by Öpik for the "distribution of components in ordinary space." We shall see that a quite new method of analysis of the question confirms the approximate accuracy of formula (6).

The fact is that if the conponents of the distribution about the central stars follow the law $\frac{1}{r^n}$, where n is any number, then the distribution of density in projection on the celestial sphere will be determined by the law $\frac{1}{r^{n-1}}$.

If we have some set of double stars with such a distribution, contained in some volume element, and if this volume element is removed, then the distribution of visible separations in projection will, it is evident, continue to satisfy the law $\frac{1}{r^{n-1}}$. The summing up of such distributions for different volume elements along the line of sight and for different directions on the celestial sphere also leads to the proportionality $\frac{1}{r^{n-1}}$. Thus for every visible distribution in projection we should obtain--for as large a section of the sky as we choose--the same law.

In particular with the alternative assumptions

$$\rho \sim \frac{1}{r^3} \quad \text{and} \quad \rho \sim \frac{1}{r^{3/2}}$$

we should obtain for the distribution of densities in projection

$$\rho \sim \frac{1}{r^2} \quad \text{and} \quad \rho \sim \frac{1}{r^{1/2}}$$

from which it follows that the number of stars with visible separations contained between r_1 and r_2 should be proportional to

$$\ln \frac{r_2}{r_1} \quad \text{and} \quad r_2^{3/2} - r_1^{3/2} \qquad (11)$$

For the solution of the problem there were taken all the stars down to visual magnitude 9.0, situated in the northern hemisphere, that appear in Aitken's catalogue[7] (4640 stars). In these limits Aitken's catalogue may be considered sufficiently uniform, because all the stars to magnitude 9.0 were checked for duality by Aitken himself at the Lick Observatory. In the following table are given the number of pairs with separations from 0".5 to 1", from 1" to 2", from 2" to 4", and from 4" to 8". In the second and third lines are given the numbers proportional to $\ln(r_2/r_1)$ and $r_2^{3/2} - r_1^{3/2}$, correspondingly; the coefficient of proportionality C is expressed in such a way that the total number in each of the three lines was the same.

Interval	0".5-1"	1"-2"	2"-4"	4"-8"	Total
Observed no. of pairs	883	1160	1283	1314	4640
$C \ln(r_2/r_1)$	1160	1160	1160	1160	4640
$C (r_2^{3/2} - r_1^{3/2})$	136	382	1080	3040	4638

From this comparison it is clear that the formula $C \ln(r_2/r_1)$ strikes a certain approximation (with an accuracy of about 10%) to the observed figures, while the formula $C (r_2^{3/2} - r_1^{3/2})$ is not justified to any degree. The existing deviations from the formula $C \ln (r_2/r_1)$ undoubtedly should decrease in the case of the exclusion of optical pairs.

It is clear from this that Öpik's law $\rho \sim \frac{1}{R^3}$ is confirmed to a first approximation. In the same way it is shown anew that the energgies of stellar pairs are not distributed according to the Boltzmann formula.

In answer to the author's preliminary note on the subjects considered above, Jeans admitted[3] that equipartition of energy does not exist, but immediately after this he added "in certain respects there is a tolerably good approximation to equipartition." In what then does Jeans see if only a distant approximation to equipartition in the field of double stars? After the cited consequences of Öpik's law it is evident that it is not possible to speak of any approximation whatever to equipartition.

4. Relaxation time of a set of double stars

Let us examine the problem of the time necessary that a set of double stars which have entered a stellar system may have arrived at statistical equilibrium with the surrounding stars. It is evident that in a state of statistical equilibrium two opposite and mutually compensating processes are possible: there should occur on the one hand the destruction of physical pairs owing to the passing of a third body, and, on the other hand, the formation of pairs during the approach of three mutually unbound stars, in which the third body carries away a surplus of energy freed as a result of the formation of a physical pair. As we shall see farther on, in our stellar system, owing to the absence of statistical equilibrium, the full mutual compensation of these processes does not take place, in the sense that the number of pairs formed is insignificantly small in comparison with the number of destroyed pairs.

Side by side with the destruction of pairs may also occur, as a consequence of approaches with formed stars, small changes of the energy of the system which may, in summation, also entail disruption. It is these processes of change of energy (of the semimajor axis) as a consequence of encounters, as well as the destruction of pairs, that lead to the establishment of statistical equilibrium in the sense of establishing a Boltzmann distribution.

It is obvious that the average time of destruction of a stellar pair, which we shall calculate at once, is perfectly sufficient for the establishment of a Boltzmann distribution.

Indeed, the establishment of a Boltzmann distribution takes place by means of changes in the energy less that those associated with disruption. Therefore the time necessary for this is not greater than the average time of destruction of a pair. In this way the average time of destruction of a pair will give us the order of the "relaxation time" of a system of double stars.

For such calculations we go no further than the "distant pairs"; that is, those for which the distance between the components exceeds 100 A.U., and on the average is of the order of thousands of astronomical units.

We can divide encounters of a third star with the stellar pair into two types: 1) encounters for which the minimum distance of the passing star from the center of mass of the system is very large in comparison with the semimajor axis of the orbit, 2) encounters for which the distance of the third body from one of the components of the pair becomes small in comparison with the semimajor axis of the system. We shall call such encounters "distant" and "close" respectively. There can also arise encounters of an intermediate type, but we shall not pause over these, because they do not have a significant value.

Bohr[8,9] has already shown (while considering the encounter of a particle interacting according to Coulomb's law, around an atom) that the role of "distant" encounters is insignificantly small in comparison with the role of "close" encounters. Therefore we shall examine only the "close" encounters. The consideration, in addition, of the "distant" encounters as well can only lessen somewhat the relaxation time without changing its order.

For pairs of the type considered the orbital speed of motion around the center of mass is of the order of one or at most several (2-3) kilometers per second. Meanwhile the relative speeds in the stellar system are generally of the order of 30 km/sec. Therefore, in practice, in the coordinate system related to the center of mass, the companion may be considered almost motionless. The time during which the passing star will exert the major part of its influence on the companion will be insignificantly small in comparison with the time of revolution of the pair both as a consequence of the indicated small ratio of speeds, and as a consequence of the "closeness" of the encounter. Therefore each time as a result of a perturbation of the companion there will appear an additional kinetic energy with respect to the constant potential energy; i.e., there will arise either some increase of the energy (of the semi-major axis) or the complete disruption of the pair. In this way the change of energy, evidently, will always take place in the direction of increase. Only in a few cases, when the kinetic energy of the passing star relative to the center of mass is small in comparison with the kinetic energy of the companion, can we have the reverse picture. But there will be a comparatively insignificant number of such passing stars.

On the other hand, the conditions written above allow us to consider the companion as "free," since the influence of the central star on the companion is noticeable only in the course of a time interval much greater than the duration of the collision. Thus the question reduces to the calculation of the change in kinetic energy of the companion during its encounter with a passing star, in the coordinate system connected with the center of mass of the pair.

A simple calculation shows that the increment of energy at the time of such a distant encounter is equal to

$$\Delta E = \frac{mv^2}{2} \frac{1}{1+\frac{p^2 v^4}{4 m^2 \gamma^2}} \tag{12}$$

if we assume that the masses of the passing star and of the companion are the same. Here p is the "impact parameter," i.e. the distance of the companion from the initial straight line along which the passing star was moving before the approach. The number of encounters for which the impact parameter is contained between p and p+dp for the timedt, and the speed of the passing star lies between v and v+dv, is equal to

$$2\pi\, p\, dp\, v \cdot dt\, dn,$$

where dn is the number of stars per unit volume, the velocities of which lie between v and v+dv. Thus the increase in energy over a time t will be equal to

$$\pi t \int m v^3 dn \int \frac{p\,dp}{1+\frac{p^2 v^4}{4 m^2 \gamma^2}} \;;$$

the integration over p must be carried out over the region in which the encounter may be considered "close." The limit of close encounters is p = a (i.e., the semimajor axis of the orbit).

Therefore

$$\Delta E = 2\pi t m^3 \gamma^2 \int \frac{\ln\left(1+\frac{a^2 v^4}{4m^2\gamma^2}\right)}{v}\, dn$$

or

$$\Delta E = 2\pi t m^3 \gamma^2 \frac{n}{\bar{v}} \ln\left(1+\frac{a^2 \bar{v}^4}{4m^2\gamma^2}\right) \tag{13}$$

where n is the total number of stars per unit volume (stellar density) and $\bar{v}$ is some average speed. If we take for the time of relaxation the time during which the absolute magnitude of ΔE becomes equal to the total energy of the system $-\frac{\chi m^2}{2a}$, then we may write

$$t = \frac{\bar{v}}{4\pi m\chi a n \ln\left(1+\frac{a^2v^4}{4m^2\chi^2}\right)} \qquad (14)$$

Here we must use, of course, some average value of a for the period considered. This average value is very close to the initial value, because the major part of the time goes into the increase of energy for small values of a.

Let us here set $\bar{v} = 3 \cdot 10^6$ cm/sec, and use for m the mass of the sun; recognizing that the observed values of a reach 1/20 of a parsec, and $n = 0.1/pc^3$, we find that $t = 5 \cdot 10^9$ years. For smaller a we find values of the order of 10^{10} and 10^{11} years.

Thus for double stars with separations of the components reaching ten thousand astronomical units, a Boltzmann distribution should be established during a time of the order of 10^{10} years. The Öpik distribution ($\rho \sim \frac{1}{x^3}$) was deduced exactly for stars with a large separation of the components (up to ten thousand A.U.). Therefore for such stars a Boltzmann distribution does not in fact occur. From this we must conclude that not more than 10^{10} years have elapsed from the moment of formation of these pairs. Thus the distribution of the semimajor axes of double stars speaks in a most definite way in favor of a short time scale.

The argument cited has been indicated in our preliminary note, but our calculations of the relaxation time were not introduced in it. This gave Jeans ground to write that "I cannot see that Prof. Ambarzumian's remarks in any way challenge this position, so that, it seems to me that the observational data he mentions are not opposed to the long time scale of 10^{13} years, but only to an infinitely long time scale."

Meanwhile we see that simple calculations point to the fact that the observational data under discussion not only contradict a scale of 10^{13} years, but even a scale of 10^{10} years, and in the same way speak wholly in favor of a short time scale.

5. Dissociative equilibrium for double stars

A further highly important fact, showing that the encounters have not yet succeeded in establishing statistical equilibrium for pairs with separations of the order of 10^4 A.U., appears in the data on the deviation of the number of such pairs observed from the formula of dissociative equilibrium.

If δn_D designates the number of pairs for which the companions are located with the element $\delta\Gamma$ of phase space mentioned above, then, according to the theory of gases, for dissociative equilibrium we should have

$$\frac{\delta n_D}{n^2} = \frac{\delta\Gamma}{(\pi m \theta)^{3/2}} e^{-\frac{E}{\theta}}, \qquad (15)$$

where E, as always, is the internal energy of the pair, when the companion is located in the element $\delta\Gamma$, θ is the modulus of the Boltzmann distribution for the translational motion of the stars, n is the number of individual stars per unit volume. If we take $\delta\Gamma$ in regions of phase space where $a > 100$ A.U., then the factor $e^{-\frac{E}{\theta}}$ can be set equal to unity. Then

$$\frac{\delta n_D}{n^2} = \frac{\delta\Gamma}{(\pi m \theta)^{3/2}} \qquad (16)$$

Summing up, we see that this formula holds good even when the