Chapter Six. Asset price model: part I

Outline Solutions to odd-numbered exercises from the book:
An Introduction to Financial Option Valuation:
Mathematics, Stochastics and Computation,
by Desmond J. Higham, Cambridge University Press, 2004
ISBN 0521 83884 3 (hardback)
ISBN 0521 54757 1 (paperback)

6.1 Using

$$S(t_{i+1}) = S(t_i) + \mu\delta t S(t_i) + \sigma\delta t Y_i S(t_i)$$

we find that (6.4) changes to

$$\log\left(\frac{S(t)}{S_0}\right) \approx \sum_{i=0}^{L-1}(\mu\delta t + \sigma\delta t Y_i).$$

The RHS is a sum of i.d.d. normal random variables with mean $\mu\delta t$ and variance $\sigma^2\delta t^2$. Hence, using the result mentioned on page 26 (item (iii) in the list), the sum is a normal random variable with mean $L\mu\delta t = \mu t$ and variance $L\sigma^2\delta t^2 = \sigma^2 t\delta t$. Because the variance is tiny, this looks like a non-random quantity. As $\delta t \to 0$ (i.e. $L \to \infty$) the model looks like $\log(S(t)/S_0) = \mu t$, that is,

$$S(t) = S_0 e^{\mu t}.$$

In other words, we are not adding enough randomness to the model, and hence we get back the deterministic behaviour that we had for the interest rate.

On the other hand, using

$$S(t_{i+1}) = S(t_i) + \mu\delta t S(t_i) + \sigma\delta t^{\frac{1}{4}} Y_i S(t_i)$$

we find that (6.4) changes to (keeping only the term that looks biggest)

$$\log\left(\frac{S(t)}{S_0}\right) \approx \sum_{i=0}^{L-1}\sigma\delta t^{\frac{1}{4}} Y_i.$$

The RHS is a sum of i.d.d. normal random variables with variance $\sigma^2\delta t^{\frac{1}{2}}$. Using the result mentined above, the sum must look like a normal random variable with variance $L\sigma^2\delta t^{\frac{1}{2}} = \sigma^2 t/\sqrt{\delta t}$. This variance blows up as $\delta t \to 0$, so we do not expect a well-defined continuous model. Intuitively, we are adding too much randomness to the model: in the $\delta t \to 0$ limit the noise is swamping the behaviour.

6.3 One explanation is that because Y_i is normal with zero mean, the probability that $\sigma\sqrt{\delta t}Y_i$ lies in some interval $[a,b]$ is the same as the probability that $-\sqrt{\delta t}\sigma Y_i$ lies in that interval.

6.5 Using the fact that $Z \sim N(0,1)$, we get

$$\begin{aligned}\mathbb{P}\left(a \le S(t) \le b\right) &= \mathbb{P}\left(a \le S_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z} \le b\right)\\ &= \mathbb{P}\left(\frac{\log(a/S_0)-(\mu-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \le Z \le \frac{\log(b/S_0)-(\mu-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)\\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{\log(a/S_0)-(\mu-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}}^{\frac{\log(b/S_0)-(\mu-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}} e^{-\frac{1}{2}s^2}\,ds\end{aligned}$$

Now the subs. $x = S_0 e^{\sigma\sqrt{t}s+(\mu-\frac{1}{2}\sigma^2)t}$ converts the integral to

$$\frac{1}{\sqrt{2\pi}}\int_a^b \frac{e^{-\frac{1}{2}\left(\frac{\log(x/S_0)-(\mu-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)^2}}{x\sigma\sqrt{t}}\,dy$$

and we deduce that the required density function $f(x)$ for $x > 0$ must be

$$f(x) = \frac{\exp\left(\frac{-(\log(x/S_0)-(\mu-\sigma^2/2)t)^2}{2\sigma^2 t}\right)}{x\sigma\sqrt{2\pi t}}.$$

6.7 We have

$$\int_{-\alpha}^{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}\,ds = 0.95.$$

Hence,

$$\int_{-\infty}^{-\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}\,ds + \int_{\alpha}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}\,ds = 1-0.95 = 0.05.$$

The two integrals on the LHS are equal, so

$$\int_{-\infty}^{-\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}\,ds = \frac{0.05}{2},$$

i.e., $N(-\alpha) = 0.05/2$. The idea used to solve Exercise 13.3 shows that $\alpha = \sqrt{2}\,\mathrm{erfinv}(0.95)$.

6.9 If $Z \sim N(0,1)$ then

$$\mathbb{P}(-1.96 \le Z \le 1.96) = 0.95.$$

Hence,

$$\mathbb{P}(-1.96 \leq \frac{\log(S(t)/S_0) - (\mu - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \leq 1.96) = 0.95.$$

So

$$\mathbb{P}(S_0 e^{-1.96\sigma\sqrt{t}+(\mu-\frac{1}{2}\sigma^2)t} \leq S(t) \leq S_0 e^{1.96\sigma\sqrt{t}+(\mu-\frac{1}{2}\sigma^2)t}) = 0.95.$$

Hence,

$$[S_0 e^{-1.96\sigma\sqrt{t}+(\mu-\frac{1}{2}\sigma^2)t}, S_0 e^{1.96\sigma\sqrt{t}+(\mu-\frac{1}{2}\sigma^2)t}]$$

is a 95% confidence interval for $S(t)$.