Chapter Twentytwo.
Monte Carlo part III: variance reduction by control variates

Outline Solutions to odd-numbered exercises from the book:
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22.1 We know that the confidence interval width is (asymptotically) proportional to $\sqrt{\text{variance}/M}$. We can get the same accuracy from the two versions by using

$$\sqrt{\frac{\mathbb{V}ar(X)}{M}} = \sqrt{\frac{\mathbb{V}ar(Z)}{M'}} = \sqrt{R_1 \frac{\mathbb{V}ar(X)}{M'}},$$

that is, with $M' = R_1 M$ samples for Z. The number of samples changes by a factor R_1, so the cost changes by a factor $R_1 R_2$. Hence cost decreases $\Leftrightarrow R_1 R_2 < 1$.

22.3 The arguments in section 15.4 suggest that Method 1 will give a confidence interval of size

$$\frac{\text{constant}}{h\sqrt{M}}.$$

For Method 2, we are regarding $V(p+h) - V(p)$ as a single random variable, and hence the confidence interval will be determined by $\mathbb{V}ar(V(p+h) - V(p))$. Now $\mathbb{V}ar(V(p+h) - V(p)) \to 0$ as $h \to 0$, so we would expect Method 2 to give a confidence interval of the form

$$\frac{\text{something that tends to zero as } h \to 0}{h\sqrt{M}}.$$

So, Method 2 should give a smaller confidence interval than Method 1 when h is small. In fact, for smooth V we should get $\mathbb{V}ar(V(p+h) - V(p)) \approx Ch + O(h^2)$ as $h \to 0$, for some constant C, whence Method 2 gives a confidence interval of the form

$$\frac{C}{\sqrt{M}}.$$

Thus, as M increases, Method 2 should give a confidence interval for delta of the same order of magnitude as the basic Monte Carlo computation for the option value. This is what we observed in Chapter 15.