Chapter Fourteen. Implied volatility

Outline Solutions to odd-numbered exercises from the book:
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14.1 We know that $\frac{\partial C}{\partial \sigma}$ has a turning point when $\frac{\partial^2 C}{\partial \sigma^2} = 0$. Using (14.4), this gives $d_1 = 0$ or $d_2 = 0$. Now, using $\frac{\partial d_1}{\partial \sigma} = -\frac{d_2}{\sigma}$ and the analogous identity $\frac{\partial d_2}{\partial \sigma} = -\frac{d_1}{\sigma}$, we have, from (14.4),

$$\begin{aligned}
\frac{\partial^3 C}{\partial \sigma^3} &= \frac{\partial d_1}{\partial \sigma}\frac{d_2}{\sigma}\frac{\partial C}{\partial \sigma} + \frac{\partial d_2}{\partial \sigma}\frac{d_1}{\sigma}\frac{\partial C}{\partial \sigma} + \frac{d_1\, d_2}{\sigma}\frac{\partial^2 C}{\partial \sigma^2} - \frac{d_1\, d_2}{\sigma^2}\frac{\partial C}{\partial \sigma} \\
&= -\frac{d_2^2}{\sigma}\frac{\partial C}{\partial \sigma} - \frac{d_1^2}{\sigma}\frac{\partial C}{\partial \sigma} + \frac{d_1\, d_2}{\sigma}\frac{\partial^2 C}{\partial \sigma^2} - \frac{d_1\, d_2}{\sigma^2}\frac{\partial C}{\partial \sigma}.
\end{aligned}$$

Since $\frac{\partial C}{\partial \sigma} > 0$ it follows that $\frac{\partial^3 C}{\partial \sigma^3} < 0$ at $d_1 = 0$ and at $d_2 = 0$.

Hence $d_1 = 0$ and $d_2 = 0$ give max. values.

Solving $d_1 = 0$ for σ gives $\sigma^2 = -2\left[\frac{\log(S/E) + r(T-t)}{T-t}\right]$.

Solving $d_2 = 0$ for σ gives $\sigma^2 = 2\left[\frac{\log(S/E) + r(T-t)}{T-t}\right]$.

Since $\sigma^2 \ge 0$, we conclude that $\frac{\partial C}{\partial \sigma}$ has a unique max. over $(0, \infty)$ given by $\sigma = \widehat{\sigma}$ in (10).

14.3 We have $\widehat{\sigma} > \sigma^*$ and (for $\sigma_0 = \widehat{\sigma}$)

$$0 < \frac{\sigma_1 - \sigma^*}{\sigma_0 - \sigma^*} < 1.$$

This tells us that $\sigma_0 > \sigma_1 > \sigma^*$.

Now, we know that $F''(\sigma) > 0$ for all $\sigma < \sigma_0 = \widehat{\sigma}$, hence $0 < F'(\xi_1) < F'(\sigma_1)$. (It might help to draw a picture.)

So, in (14.8),

$$0 < \frac{\sigma_2 - \sigma^*}{\sigma_1 - \sigma^*} < 1.$$

The same argument now applies for $n = 2, 3, \ldots$ and hence (14.10) holds.