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CONVERGENCE AND STABILITY ANALYSIS FOR IMPLICIT SIMULATIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH RANDOM JUMP MAGNITUDES

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Abstract. Stochastic differential equations with Poisson driven jumps of random magnitude are popular as models in mathematical finance. Strong, or pathwise, simulation of these models is required in various settings and long time stability is desirable to control error growth. Here, we examine strong convergence and mean-square stability of a class of implicit numerical methods, proving both positive and negative results. The analysis is backed up with numerical experiments.

1. **Introduction.** Stochastic differential equations (SDEs) arise in many disciplines. In particular, they are used in mathematical finance in order to simulate asset prices, interest rates and volatilities. Advanced models frequently incorporate jumps. In the financial setting a key early reference is [21] and up to date treatments can be found in [3, 6]. Jump models also arise in many other application areas and have proved successful at describing unexpected, abrupt changes of state [24]. Typically, these models do not admit analytical solutions and hence must be simulated numerically.

There is extensive literature on the numerical simulation of SDEs without jumps, and efforts are now being made to bring jump SDEs up a similar level. In [14] strong convergence and mean-square stability properties were analysed in the case of Poisson-driven jumps of deterministic magnitude. In this work we extend that analysis to the case where jump magnitudes are random—a situation that is now common in financial models [9, 10, 17, 25, 26]. We also give what appears to be the first stability analysis for this class of jump-SDE simulations, and show that an A-stability property from the constant jump setting does not generalize.

We emphasize that this work treats convergence in the strong sense. Although many classical problems in mathematical finance require only weak convergence, there are important instances where strong convergence is relevant. An interesting recent example is the multi-level Monte Carlo method in [5], where the order

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of strong convergence is a vital ingredient in the error analysis and the resulting algorithm.

We study fixed timestep methods, rather than jump-adapted methods with precomputed jump times embedded within a standard time grid [7, 8, 18, 19, 22]. A fixed timestep offers advantages when the jump intensity is high [6] and also fits in with the natural parallelisability of Monte Carlo.

A closely related and more general convergence theory has recently been developed by Bruti-Liberati and Platen, [2], with related earlier work appearing in [4, 18, 19, 20]. Our convergence results in section 3 could be derived using their techniques, which also apply to higher order methods. However, we believe that our treatment is of independent interest as it is based on the style of analysis in [16] and hence has potential to be extended to the case of nonlinear coefficients that are not globally Lipschitz. In this context the use of implicit methods that can be guaranteed to produce moment-bounded solutions appears to be an essential ingredient in the error analysis.

2. Preliminaries.

We examine the following class of scalar jump SDEs

$$\mathrm{d}X(t) = f(X(t^-))\,\mathrm{d}t + g(X(t^-))\,\mathrm{d}W(t) + h(X(t^-), \gamma_{N(t^-)+1})\,\mathrm{d}N(t), \qquad (1)$$

$X(0) = X_0$, for $t > 0$, where $W(t)$ is a standard 1-dimensional Wiener process; $N(t)$ is a Poisson process with mean λt; $X(t^-) := \lim_{s \nearrow t} X(s)$; and γ_i, $i = 1, 2, \ldots$ are independent, identically distributed random variables representing magnitudes for each jump. Our notation is consistent with that in [6, p. 363].

The analysis in [14] deals with the case where γ is a deterministic parameter. Our aim here is to generalize to the case of random jump sizes, thereby covering a wider range of models in finance [9, 10, 11, 17, 25]. We assume that for some $q \geq 1$ there is a constant $B = B_q$ such that

$$\mathbb{E}\left[|\gamma_i|^{2q}\right] \leq B, \qquad (2)$$

that is, the $2q$th moment of the jump magnitude is bounded. In many cases, including the standard log-normal model, such a $B = B_q$ exists for any q.

We will impose global Lipschitz bounds on f, g and h, that is,

$$\max\Big(\big(f(x_1) - f(x_2)\big)^2, \big(g(x_1) - g(x_2)\big)^2\Big) \;\leq\; K(x_1 - x_2)^2, \qquad (3)$$

$$\big(h(x_1, y_1) - h(x_2, y_2)\big)^2 \;\leq\; K\Big((x_1 - x_2)^2 + (y_1 - y_2)^2\Big), \qquad (4)$$

where K is a constant independent of x_1, x_2, y_1 and y_2. This implies the linear growth bounds

$$\max\left(f(x)^2, g(x)^2\right) \;\leq\; L\left(1 + x^2\right), \qquad (5)$$

$$h(x, y)^2 \;\leq\; L\left(1 + x^2 + y^2\right), \qquad (6)$$

where L is a constant independent of x and y.

We note for later reference that (1) involves the jump process

$$\gamma(t) := \gamma_{N(t^-)+1} = \sum_j \gamma_{j+1} \mathbf{1}_{[\tau_j, \tau_{j+1})}(t), \qquad (7)$$

where $\tau_0 = 0$ and τ_i, $i = 1, 2, \ldots$ are the jump times and $\mathbf{1}_G$ denotes the indicator function for the set G.

We discretize (1) by extending the definition of the theta method [14] in a natural way, to get

$$Y_{n+1} = Y_n + (1-\theta) f(Y_n)\,\Delta t + \theta f(Y_{n+1})\,\Delta t + g(Y_n)\,\Delta W_n + h(Y_n, \gamma_{N(t_n)+1})\,\Delta N_n, \quad (8)$$

where θ is a fixed parameter. Typically, $0 \le \theta \le 1$, although values outside this range may also be useful for stochastic problems [12]. Here, $Y_n \approx X(t_n)$, for $t_n = n\Delta t$, where Δt is a fixed stepsize, and $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta N_n := N(t_{n+1}) - N(t_n)$ are the Brownian and Poisson increments, respectively.

As in [14], we extend our discrete numerical solution to continuous time. First we define the step functions

$$\begin{aligned} Z_1(t) &:= \sum_i Y_i \mathbf{1}_{[i\Delta t,(i+1)\Delta t)}(t), \\ Z_2(t) &:= \sum_i Y_{i+1} \mathbf{1}_{[i\Delta t,(i+1)\Delta t)}(t), \\ \bar{\gamma}(t) &:= \sum_i \gamma(t_i) \mathbf{1}_{[i\Delta t,(i+1)\Delta t)}(t). \end{aligned}$$

We note that $Z_2(t)$ is not adapted to the natural filtration that measures $W(t)$ and $N(t)$, but this will not pose a significant limitation in our analysis. We then define the continuous-time approximation

$$\begin{aligned} Y(t) =& Y_0 + \int_0^t (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s + \int_0^t g(Z_1(s))\,\mathrm{d}W(s) \\ &+ \int_0^t h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}N(s), \end{aligned} \quad (9)$$

which interpolates the discrete numerical approximation (8). So a convergence result for $Y(t)$ immediately provides a result for $\{Y_k\}$.

A key difficulty that we address in this work is that relative to previous analysis [14] a new source of error arises from the approximation of $\gamma(t)$ by $\bar{\gamma}(t)$. This occurs because the exponentially distributed jump times do not coincide with the grid points. Hence there will be small intervals of time where the discretization method picks up the wrong jump magnitude. Figure 1 shows a graphical representation of this issue.

We state here two results that are used in our analysis.

Result 2.1 (Young's Inequality)**.** [23]

$$ab \le \frac{q-1}{q} \varepsilon^{1/(q-1)} a^{q/(q-1)} + \frac{1}{q\varepsilon} b^{q}, \quad (10)$$

where $a, b, \varepsilon > 0$ and $1 < q < \infty$.

Result 2.2 (Martingale Isometry for Compensated Poisson Process)**.** [4]
For the compensated Poisson process, $\widetilde{N}(t) := N(t) - \lambda t$, which is a martingale, we have the following isometry

$$\mathbb{E}\left|\int_a^b h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2 = \lambda \int_a^b \mathbb{E}|h(Z_1(s), \bar{\gamma}(s))|^2\,\mathrm{d}s. \quad (11)$$

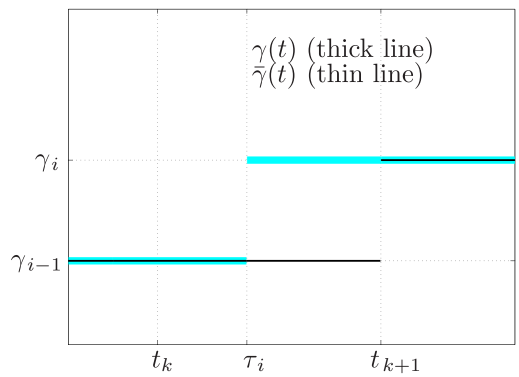

FIGURE 1. *Illustration of the exact process $\gamma(t)$ and the approximation $\bar{\gamma}(t)$ available to the numerical method, over a timestep containing a jump.*

3. **Convergence Analysis.** In this section we derive our strong convergence result, Theorem 3.4. The analysis proceeds in a similar vein to that in [14], with appropriate extensions to deal with the more general random jump magnitudes. We have aimed to make this treatment self-contained, while referring to [14] for some extra details.

We note that for $\theta \neq 0$, an implicit equation determines the numerical solution in (8). The global Lipschitz condition (3) ensures that a unique solution exists with probability one for sufficently small stepsizes; see, for example, [15, Lemma A.1].

Throughout our analysis, $C_i, D_i,\ i = 1, 2, ...$ denote generic constants, independent of Δt.

First, we show that the discrete numerical solution has bounded second moments.

Lemma 3.1. *Under the above assumptions, there exists $\Delta t^\star > 0$ such that for all $0 < \Delta t \leq \Delta t^\star$,*

$$\mathbb{E}|Y_k|^2 \leq C_1\big(1 + \mathbb{E}|X(0)|^2\big)\,, \quad \text{for } k\Delta t \leq T\ .$$

Proof. By construction, we have,

$$Y_{k+1} = Y_0 + \int_0^{(k+1)\Delta t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s + \int_0^{(k+1)\Delta t} g(Z_1(s))\,\mathrm{d}W(s) + \int_0^{(k+1)\Delta t} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}N(s).$$

So, for $(k+1)\Delta t \leq T$,

$$\begin{aligned} \mathbb{E}|Y_{k+1}|^2 \leq 4\mathbb{E}|Y_0|^2 &+ 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s\right|^2 \\ &+ 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} g(Z_1(s))\,\mathrm{d}W(s)\right|^2 \\ &+ 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}N(s)\right|^2. \end{aligned} \tag{12}$$

Now, it follows from [14] that

$$\mathbb{E}\bigg|\int_0^{(k+1)\Delta t}(1-\theta)f(Z_1(s))+\theta f(Z_2(s))\,\mathrm{d}s\bigg|^2 \le 4T^2L+4TL\Delta t\sum_{j=0}^{k}\mathbb{E}|Y_j|^2 + 2TL\Delta t\,\mathbb{E}|Y_{k+1}|^2 \tag{13}$$

and

$$\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} g(Z_1(s))\,\mathrm{d}W(s)\bigg|^2 \le LT+L\Delta t\sum_{j=0}^{k}\mathbb{E}|Y_j|^2. \tag{14}$$

For the jump integral term in (12), we can transform to the compensated Poisson process and use the martingale isometry (11) to obtain,

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}N(s)\bigg|^2\\
=\;&\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}\widetilde N(s)+\lambda\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}s\bigg|^2\\
\le\;&2\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}\widetilde N(s)\bigg|^2+2\lambda^2\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}s\bigg|^2\\
\le\;&2\lambda\int_0^{(k+1)\Delta t}\mathbb{E}\big|h(Z_1(s),\bar\gamma(s))\big|^2\mathrm{d}s+2\lambda^2T\int_0^{(k+1)\Delta t}\mathbb{E}\big|h(Z_1(s),\bar\gamma(s))\big|^2\mathrm{d}s\\
=\;&2\lambda(1+\lambda T)\Delta t\sum_{j=0}^{k}\mathbb{E}\big|h(Y_j,\gamma(t_j))\big|^2.
\end{aligned}$$

Applying the linear growth bound (6) gives,

$$\begin{aligned}
&\mathbb{E}\bigg|\int_0^{(k+1)\Delta t} h(Z_1(s),\bar\gamma(s))\,\mathrm{d}N(s)\bigg|^2\\
\le\;&2\lambda(1+\lambda T)\Delta t\,L\sum_{j=0}^{k}\mathbb{E}\Big(1+|Y_j|^2+|\gamma(t_j)|^2\Big)\\
=\;&2\lambda(1+\lambda T)\Delta t\,L\bigg(k+\sum_{j=0}^{k}\mathbb{E}|Y_j|^2+\sum_{j=0}^{k}\mathbb{E}|\gamma(t_j)|^2\bigg)\\
\le\;&2\lambda(1+\lambda T)L\Bigg(T+\Delta t\bigg(\sum_{j=0}^{k}\mathbb{E}|Y_j|^2+\sum_{j=0}^{k}\mathbb{E}|\gamma(t_j)|^2\bigg)\Bigg).
\end{aligned} \tag{15}$$

Combining (13), (14) and (15) with (12) yields,

$$\begin{aligned}
\mathbb{E}|Y_{k+1}|^2 \;\le\;& 4\Big[\mathbb{E}|Y_0|^2+4T^2L+LT+2\lambda T(1+\lambda T)L\Big]\\
&+4\Delta t\Big[4LT+L+2\lambda(1+\lambda T)L\Big]\sum_{j=0}^{k}\mathbb{E}|Y_j|^2+8LT\Delta t\,\mathbb{E}|Y_{k+1}|^2\\
&+8\lambda\Delta t\,(1+\lambda T)L\sum_{j=0}^{k}\mathbb{E}|\gamma(t_j)|^2.
\end{aligned}$$

Now choosing Δt sufficiently small such that $1 - 8TL\Delta t \geq \frac{1}{2}$, we obtain,

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 \quad &\leq \quad 8\Big[\mathbb{E}|Y_0|^2 + LT\big(4T + 1 + 2\lambda(1+\lambda T)\big)\Big] \\ &\quad + 8\Delta t\, L\big(4T + 1 + 2\lambda(1+\lambda T)\big)\sum_{j=0}^{k}\mathbb{E}|Y_j|^2 \\ &\quad + 8\Delta t\, L\lambda(1+\lambda T)\sum_{j=0}^{k}\mathbb{E}|\gamma(t_j)|^2.\end{aligned}$$

Assumption (2) implies that each $\mathbb{E}|\gamma(t_j)|^2 \leq B_1$, and hence

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 \quad &\leq \quad 8\Big[\mathbb{E}|Y_0|^2 + LT\big(4T + 1 + 2\lambda(1+\lambda T)\big)\Big] \\ &\quad + 8L\lambda(1+\lambda T)B_1 T \\ &\quad + 8\Delta t\, L\big(4T + 1 + 2\lambda(1+\lambda T)\big)\sum_{j=0}^{k}\mathbb{E}|Y_j|^2.\end{aligned}$$

We may then apply the discrete Gronwall inequality and the result follows. ☐

Next, we show the boundedness of the continuous-time approximation.

Lemma 3.2. *There exists $\Delta t^* > 0$ such that for all $0 < \Delta t \leq \Delta t^*$,*

$$\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 \leq C_2(1 + \mathbb{E}|X(0)|^2). \tag{16}$$

Proof. From (9), we have

$$|Y(t)|^2 \leq 4|Y_0|^2 + 4\left|\int_0^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s\right|^2 + 4\left|\int_0^t g(Z_1(s))\,\mathrm{d}W(s)\right|^2 + 4\left|\int_0^t h(Z_1(s),\bar{\gamma}(s))\,\mathrm{d}N(s)\right|^2.$$

Thus, using the Cauchy-Schwarz inequality and the definition of $\widetilde{N}(s)$,

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 \\ \leq\quad & 4\mathbb{E}|Y_0|^2 + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s\right|^2 \\ & + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t g(Z_1(s))\,\mathrm{d}W(s)\right|^2 \\ & + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t h(Z_1(s),\bar{\gamma}(s))\,\mathrm{d}N(s)\right|^2 \\ \leq\quad & 4\mathbb{E}|Y_0|^2 4\mathbb{E}\sup_{t\in[0,T]}\left(\int_0^t \mathrm{d}s\int_0^t 2|f(Z_1(s))|^2 + 2|f(Z_2(s))|^2\,\mathrm{d}s\right) \\ & + 4\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t g(Z_1(s))\,\mathrm{d}W(s)\right|^2 \\ & + 8\mathbb{E}\sup_{t\in[0,T]}\left(\left|\int_0^t h(Z_1(s),\bar{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2 + \lambda^2\left|\int_0^t h(Z_1(s),\bar{\gamma}(s))\,\mathrm{d}s\right|^2\right).\end{aligned}$$

We may now apply the Doob martingale inequality, then Itô and martingale isometries, and lastly growth bounds, to get

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 \;\le\;& 4\mathbb{E}|Y_0|^2 + 8T\int_0^T \mathbb{E}\big(|f(Z_1(s))|^2 + |f(Z_2(s))|^2\big)\,\mathrm{d}s\\
&+ 16\mathbb{E}\left|\int_0^T g(Z_1(s))\,\mathrm{d}W(s)\right|^2\\
&+ 32\mathbb{E}\left|\int_0^T h(Z_1(s),\bar{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2\\
&+ 8\lambda^2 T\int_0^T \mathbb{E}|h(Z_1(s),\bar{\gamma}(s))|^2\,\mathrm{d}s\\
\le\;& 4\mathbb{E}|Y_0|^2 + 8LT\int_0^T \big(2 + \mathbb{E}|Z_1(s)|^2 + \mathbb{E}|Z_2(s)|^2\big)\,\mathrm{d}s\\
&+ 16L\int_0^T \big(1 + \mathbb{E}|Z_1(s)|^2\big)\,\mathrm{d}s\\
&+ 32\lambda L\int_0^T \big(1 + \mathbb{E}|Z_1(s)|^2 + \mathbb{E}|\bar{\gamma}(s)|^2\big)\,\mathrm{d}s\\
&+ 8\lambda^2 LT\int_0^T \big(1 + \mathbb{E}|Z_1(s)|^2 + \mathbb{E}|\bar{\gamma}(s)|^2\big)\,\mathrm{d}s.
\end{aligned}
$$

Collecting like terms, we obtain

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 \;\le\;& 4\mathbb{E}|Y_0|^2 + 8LT\big(2T + 2 + \lambda(4+\lambda T)\big)\\
&+ 8L\big(T + 2 + \lambda(4+\lambda T)\big)\int_0^T \mathbb{E}|Z_1(s)|^2\,\mathrm{d}s\\
&+ 8LT\int_0^T \mathbb{E}|Z_2(s)|\,\mathrm{d}s\\
&+ 8L\lambda(4+\lambda T)\int_0^T \mathbb{E}|\bar{\gamma}(s)|^2\,\mathrm{d}s.
\end{aligned}
$$

If we now apply Lemma 3.1 over an interval $[0, T+\Delta t]$ (as some $Z_2(t)$ may extend beyond T), the result (16) follows. □

The following shows that the continuous-time approximation remains close to the step functions in a strong sense.

Lemma 3.3. *There exists $\Delta t^* > 0$ such that for all $0 < \Delta t \le \Delta t^*$*

$$
\mathbb{E}\sup_{t\in[0,T]}|Y(t) - Z_1(t)|^2 \le C_3\Delta t\big(1 + \mathbb{E}|X(0)|^2\big) \tag{17}
$$

and

$$
\mathbb{E}\sup_{t\in[0,T]}|Y(t) - Z_2(t)|^2 \le C_4\Delta t\big(1 + \mathbb{E}|X(0)|^2\big). \tag{18}
$$

Proof. Consider $t \in [k\Delta t, (k+1)\Delta t] \subseteq [0,T]$. In this interval we have

$$\begin{aligned} Y(t) - Z_1(t) &= Y(t) - Y_k \\ &= \int_{k\Delta t}^{t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s + \int_{k\Delta t}^{t} g(Z_1(s))\,\mathrm{d}W(s) \\ &\quad + \int_{k\Delta t}^{t} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}N(s). \end{aligned}$$

So,

$$\begin{aligned} |Y(t) - Z_1(t)|^2 \le 3\left|\int_{k\Delta t}^{t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s\right|^2 + 3\left|\int_{k\Delta t}^{t} g(Z_1(s))\,\mathrm{d}W(s)\right|^2 \\ + 3\left|\int_{k\Delta t}^{t} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}N(s)\right|^2. \end{aligned}$$

Thus, for each $t \in [k\Delta t, (k+1)\Delta t],\ k \in \mathbb{N}$,

$$\begin{aligned} &\sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \\ \le\ &\max_{0\le k\le T/\Delta t - 1} \sup_{\tau\in[k\Delta t,(k+1)\Delta t]} \Bigg\{ 3\left|\int_{k\Delta t}^{\tau} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\,\mathrm{d}s\right|^2 \\ &+ 3\left|\int_{k\Delta t}^{\tau} g(z_1(s))\,\mathrm{d}W(s)\right|^2 \\ &+ 6\left|\int_{k\Delta t}^{\tau} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2 \\ &+ 6\lambda^2\left|\int_{k\Delta t}^{\tau} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}s\right|^2 \Bigg\}. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\left|\int_{k\Delta t}^{\tau} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}s\right|^2 \le \Delta t \int_{k\Delta t}^{\tau} |h(Z_1(s), \bar{\gamma}(s))|^2\ \mathrm{d}s.$$

Therefore, after applying the Doob martingale inequality, we have

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \\ \le\ &\max_{0\le k\le T/\Delta t - 1} \Bigg\{ 6\Delta t \mathbb{E} \int_{k\Delta t}^{(k+1)\Delta t} |f(Z_1(s))|^2 + |f(Z_2(s))|^2\,\mathrm{d}s \\ &+ 12\mathbb{E}\left|\int_{k\Delta t}^{(k+1)\Delta t} g(Z_1(s))\,\mathrm{d}W(s)\right|^2 \\ &+ 24\mathbb{E}\left|\int_{k\Delta t}^{(k+1)\Delta t} h(Z_1(s), \bar{\gamma}(s))\,\mathrm{d}\widetilde{N}(s)\right|^2 \\ &+ 6\Delta t \lambda^2 \mathbb{E} \int_{k\Delta t}^{(k+1)\Delta t} |h(Z_1(s), \bar{\gamma}(s))|^2\,\mathrm{d}s \Bigg\}. \end{aligned}$$

Then applying Itô and martingale isometries, Fubini's theorem and the growth bounds, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}|Y(t)-Z_1(t)|^2\\
\leq\ &\max_{0\leq k\leq T/\Delta t-1}\Bigg\{6\Delta tL\int_{k\Delta t}^{(k+1)\Delta t}\Big(2+\mathbb{E}|Z_1(s)|^2+\mathbb{E}|Z_2(s)|^2\Big)\,\mathrm{d}s\\
&+12\int_{k\Delta t}^{(k+1)\Delta t}\mathbb{E}|g(Z_1(s))|^2\,\mathrm{d}s+24\lambda\int_{k\Delta t}^{(k+1)\Delta t}\mathbb{E}|h(Z_1(s),\bar{\gamma}(s))|^2\,\mathrm{d}s\\
&+6\Delta t\lambda^2\int_{k\Delta t}^{(k+1)\Delta t}\mathbb{E}|h(Z_1(s),\bar{\gamma}(s))|^2\,\mathrm{d}s\Bigg\}\\
\leq\ &\max_{0\leq k\leq T/\Delta t-1}\Bigg\{6\Delta tL\int_{k\Delta t}^{(k+1)\Delta t}2+\mathbb{E}|Z_1(s)|^2+\mathbb{E}|Z_2(s)|^2\,\mathrm{d}s\\
&+12L\int_{k\Delta t}^{(k+1)\Delta t}1+\mathbb{E}|Z_1(s)|^2\,\mathrm{d}s\\
&+24\lambda L\int_{k\Delta t}^{(k+1)\Delta t}1+\mathbb{E}|Z_1(s)|^2+\mathbb{E}|\bar{\gamma}(s)|^2\,\mathrm{d}s\\
&+6\Delta t\lambda^2L\int_{k\Delta t}^{(k+1)\Delta t}1+\mathbb{E}|Z_1(s)|^2+\mathbb{E}|\bar{\gamma}(s)|^2\,\mathrm{d}s\Bigg\}.
\end{aligned}$$

Now, on $[k\Delta t,(k+1)\Delta t]$, $Z_1\equiv Y_k,\ Z_2\equiv Y_{k+1}$ and $\bar{\gamma}\equiv\gamma_k$. So

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|Y(t)-Z_1(t)|^2&\leq 6\Delta tL(2\Delta t+2+4\lambda+\Delta t\lambda^2+4\lambda B_1+\Delta t\lambda^2B_1)\\
&\quad+6\Delta tL(2\Delta t+4\lambda+\Delta t\lambda^2)C_1(1+\mathbb{E}|X(0)|^2)\\
&\leq C_3\Delta t(1+\mathbb{E}|X(0)|^2),
\end{aligned}$$

showing (17).

A similar analysis gives (18). □

We are now in a position to prove our strong convergence result.

Theorem 3.4. *Under assumption (2) for some $q>1$ and assumptions (3) and (4), there exists $\Delta t^\star>0$ and $C_5=C_5(q)$ such that for all $0<\Delta t\leq\Delta t^\star$,*

$$\mathbb{E}\sup_{t\in[0,T]}|Y(t)-X(t)|^2\leq C_5\Delta t^{1-\frac{1}{q}}. \tag{19}$$

Proof. By construction,

$$
\begin{aligned}
Y(t) - X(t) = &\int_0^t (1-\theta)\big[f(Z_1(s)) - f(X(s^-))\big] + \theta\big[f(Z_2(s)) - f(X(s^-))\big]\,\mathrm{d}s \\
&+ \int_0^t g(Z_1(s)) - g(X(s^-))\,\mathrm{d}W(s) \\
&+ \int_0^t h(Z_1(s),\bar\gamma(s)) - h(X(s^-),\gamma(s^-))\,\mathrm{d}N(s) \\
= &\int_0^t (1-\theta)\big[f(Z_1(s)) - f(X(s^-))\big] + \theta\big[f(Z_2(s)) - f(X(s^-))\big]\,\mathrm{d}s \\
&+ \int_0^t g(Z_1(s)) - g(X(s^-))\,\mathrm{d}W(s) \\
&+ \int_0^t h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}N(s) \\
&+ \int_0^t h(Z_1(s),\gamma(s^-)) - h(X(s^-),\gamma(s^-))\,\mathrm{d}N(s).
\end{aligned} \tag{20}
$$

Now for any $0 \le \hat t \le T$ we have from [14], the bounds

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t (1-\theta)\big[f(Z_1(s)) - f(X(s^-))\big] + \theta\big[f(Z_2(s)) - f(X(s^-))\big]\,\mathrm{d}s\Big|^2 \\
\le 2TK\int_0^{\hat t} \mathbb{E}|Z_1(s)-X(s^-)|^2 + \mathbb{E}|Z_2(s)-X(s^-)|^2\,\mathrm{d}s,
\end{aligned} \tag{21}
$$

$$
\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t g(Z_1(s)) - g(X(s^-))\,\mathrm{d}W(s)\Big|^2 \le 4K\int_0^{\hat t}\mathbb{E}|Z_1(s)-X(s^-)|^2\,\mathrm{d}s \tag{22}
$$

and

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\gamma(s^-)) - h(X(s^-),\gamma(s^-))\,\mathrm{d}N(s)\Big|^2 \\
\le (8\lambda + 2T\lambda^2)K\int_0^{\hat t}\mathbb{E}|Z_1(s)-X(s^-)|^2\,\mathrm{d}s
\end{aligned} \tag{23}
$$

We also have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}N(s)\Big|^2 \\
\le\;& 2\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}\widetilde N(s)\Big|^2 \\
&+ 2\lambda^2\,\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}s\Big|^2 \\
\le\;& 8\mathbb{E}\Big|\int_0^{\hat t} h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}\widetilde N(s)\Big|^2 \\
&+ 2\lambda^2\hat t\,\mathbb{E}\Big[\int_0^{\hat t} |h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))|^2\,\mathrm{d}s\Big],
\end{aligned}
$$

where we have applied the Doob martingale inequality and the Cauchy-Schwarz inequality.

We may now apply the martingale isometry and then Lipschitz condition (4), to get

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\hat{t}]}\left|\int_0^t h(Z_1(s),\bar{\gamma}(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}N(s)\right|^2 \\
\leq\;& 8\lambda\mathbb{E}\left[\int_0^{\hat{t}}\left|h(Z_1(s),\bar{\gamma}(s)) - h(Z_1(s),\gamma(s^-))\right|^2\mathrm{d}s\right] \\
&+ 2\lambda^2 T\mathbb{E}\left[\int_0^{\hat{t}}\left|h(Z_1(s),\bar{\gamma}(s)) - h(Z_1(s),\gamma(s^-))\right|^2\mathrm{d}s\right] \\
\leq\;& 2\lambda(4+\lambda T)K\mathbb{E}\left[\int_0^{\hat{t}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right] \\
\leq\;& 2\lambda(4+\lambda T)K\left(\sum_{n=0}^{\hat{M}-1}\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right]\right),
\end{aligned} \tag{24}$$

where $\hat{M}$ is the smallest integer such that $\hat{M}\Delta t \geq \hat{t}$.

Now the number of nonzero terms in the summation in (24) is a random variable that is not independent of the summands. To obtain a useful bound we apply Young's inequality (10) to get

$$\begin{aligned}
&\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right] = \mathbb{E}\left[\mathbf{1}_{\{\Delta N_n\geq 1\}}\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right] \\
&\leq \left(\frac{q-1}{q}\right)\varepsilon^{1/(q-1)}\mathbb{E}\left[\mathbf{1}_{\{\Delta N_n\geq 1\}}\right] + \frac{1}{q\varepsilon}\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right)^q\right].
\end{aligned} \tag{25}$$

To bound the integral under the second expectation in (25), we can apply the Hölder inequality as follows

$$\begin{aligned}
\left(\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right)^q &= \left(\int_{t_n}^{t_{n+1}}1\cdot|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right)^q \\
&\leq \left(\int_{t_n}^{t_{n+1}}\mathrm{d}s\right)^{q-1}\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^{2q}\,\mathrm{d}s \\
&= \Delta t^{q-1}\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^{2q}\,\mathrm{d}s.
\end{aligned}$$

Hence, taking expectations,

$$\mathbb{E}\left[\left(\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^2\,\mathrm{d}s\right)^q\right] \leq \Delta t^{q-1}\mathbb{E}\left[\int_{t_n}^{t_{n+1}}|\bar{\gamma}(s)-\gamma(s^-)|^{2q}\,\mathrm{d}s\right]. \tag{26}$$

Using (26) in (25), applying Fubini's theorem and the bound

$$\mathbb{E}\left[|\bar{\gamma}(s)-\gamma(s^-)|^{2q}\right] \leq 2^{2q-1}\left(\mathbb{E}\left[|\bar{\gamma}(s)|^{2q}\right] + \mathbb{E}\left[|\gamma(s^-)|^{2q}\right]\right),$$

which follows from Hölder's inequality, we find that

$$\begin{aligned}
&\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} |\bar\gamma(s)-\gamma(s^-)|^2\,\mathrm{d}s\Big] \\
\le\;& \Big(\frac{q-1}{q}\Big)\varepsilon^{1/(q-1)}\mathbb{E}\Big[\mathbf{1}_{\{\Delta N_n\ge 1\}}\Big] + \frac{\Delta t^{q-1}}{q\varepsilon}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} |\bar\gamma(s)-\gamma(s^-)|^{2q}\,\mathrm{d}s\Big] \\
\le\;& \Big(\frac{q-1}{q}\Big)\varepsilon^{1/(q-1)}\mathbb{P}\big(\Delta N_n\ge 1\big) + \frac{\Delta t^{q-1}}{q\varepsilon}\int_{t_n}^{t_{n+1}} \mathbb{E}|\bar\gamma(s)-\gamma(s^-)|^{2q}\,\mathrm{d}s \\
\le\;& \Big(\frac{q-1}{q}\Big)\varepsilon^{1/(q-1)}\lambda\Delta t + \frac{\Delta t^{q-1}}{q\varepsilon}2^{2q-1}\int_{t_n}^{t_{n+1}} \mathbb{E}|\bar\gamma(s)|^{2q}+\mathbb{E}|\gamma(s^-)|^{2q}\,\mathrm{d}s \\
\le\;& \Big(\frac{q-1}{q}\Big)\varepsilon^{1/(q-1)}\lambda\Delta t + \frac{2^{2q}B}{q\varepsilon}\Delta t^{q}.
\end{aligned}$$

Then, by choosing $\varepsilon = \Delta t^{(q-1)^2/q}$ and simplifying, we get

$$\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} |\bar\gamma(s)-\gamma(s^-)|^2\,\mathrm{d}s\Big] \le \frac{1}{q}\Big((q-1)\lambda + 2^{2q}B\Big)\Delta t^{2-1/q}. \tag{27}$$

Inserting (27) into (24) gives

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\bar\gamma(s)) - h(Z_1(s),\gamma(s^-))\,\mathrm{d}N(s)\Big|^2 \\
\le\;& 2\lambda(4+\lambda T)K\sum_{n=0}^{\hat M-1}\frac{1}{q}\Big((q-1)\lambda+2^{2q}B\Big)\Delta t^{2-1/q} \\
=\;& \frac{2\lambda(4+\lambda T)K}{q}\Big((q-1)\lambda+2^{2q}B\Big)\hat M\Delta t^{2-1/q} \\
\le\;& \frac{2\lambda(4+\lambda T)TK}{q}\Big((q-1)\lambda+2^{2q}B\Big)\Delta t^{1-1/q},
\end{aligned} \tag{28}$$

as $\hat M\Delta t \le M\Delta t = T$.

Now, using the bounds (21), (22), (23) and (28) it follows from (20) that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\hat t]}|Y(t)-X(t)|^2 \\
\le\;& 4\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t (1-\theta)\big(f(Z_1(s))-f(X(s^-))\big)-\theta\big(f(Z_2(s))-f(X(s^-))\big)\,\mathrm{d}s\Big|^2 \\
&+4\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t g(Z_1(s))-g(X(s^-))\,\mathrm{d}W(s)\Big|^2 \\
&+4\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\gamma(s^-))-h(X(s^-),\gamma(s^-))\,\mathrm{d}N(s)\Big|^2 \\
&+4\mathbb{E}\sup_{t\in[0,\hat t]}\Big|\int_0^t h(Z_1(s),\bar\gamma(s))-h(Z_1(s),\gamma(s^-))\,\mathrm{d}N(s)\Big|^2
\end{aligned}$$

That is,

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\hat{t}]} |Y(t)-X(t)|^2 \\
\leq\quad & 8KT \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-X(s^-)|^2 + \mathbb{E}|Z_2(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + 16K \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + 32\lambda K \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-X(s^-)|^2 \,\mathrm{d}s + 8\lambda^2 KT \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + \frac{8\lambda KT(4+\lambda T)}{q}\Big((q-1)\lambda + 2^{2q}B\Big)\Delta t^{1-1/q}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\hat{t}]} |Y(t)-X(t)|^2 \\
\leq\quad & (8KT+16K+32\lambda K+8\lambda^2 KT) \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + 8KT \int_0^{\hat{t}} \mathbb{E}|Z_2(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + \frac{8\lambda KT(4+\lambda T)}{q}\Big((q-1)\lambda + 2^{2q}B\Big)\Delta t^{1-1/q} \\
\leq\quad & 16K(T+2+4\lambda+\lambda^2 T) \int_0^{\hat{t}} \mathbb{E}|Z_1(s)-Y(s)|^2 + \mathbb{E}|Y(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + 16KT \int_0^{\hat{t}} \mathbb{E}|Z_2(s)-Y(s)|^2 + \mathbb{E}|Y(s)-X(s^-)|^2 \,\mathrm{d}s \\
& + \frac{8\lambda KT(4+\lambda T)}{q}\Big((q-1)\lambda + 2^{2q}B\Big)\Delta t^{1-1/q}.
\end{aligned}
$$

Lemma 3.3 then gives

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,\hat{t}]} |Y(t)-X(t)|^2 \\
\leq\quad & 16K(T+2+2\lambda+\lambda^2 T)TC_3\Delta t(1+\mathbb{E}|X(0)|^2) \\
& + 16KT^2 C_4 \Delta t(1+\mathbb{E}|X(0)|^2) \\
& + \frac{8\lambda KT(4+\lambda T)}{q}\Big((q-1)\lambda + 2^{2q}B\Big)\Delta t^{1-1/q} \\
& + 16K(2T+2+4\lambda+\lambda^2 T) \int_0^{\hat{t}} \mathbb{E} \sup_{t\in[0,s]} |Y(t)-X(t^-)|^2 \,\mathrm{d}s \\
\leq\quad & 16K(2T+2+4\lambda+\lambda^2 T) \int_0^{\hat{t}} \mathbb{E} \sup_{t\in[0,s]} |Y(t)-X(t^-)|^2 \,\mathrm{d}s \\
& + D_1 \Delta t^{1-1/q}.
\end{aligned}
$$

The result (19) follows from an application of Gronwall's inequality. ☐

3.1. **Numerical Test.** In this section we give a numerical experiment that corroborates the strong convergence theory. For convenience, we look at the strong end point error

$$\mathrm{e}_{\Delta t} := \mathbb{E}|Y_M - X(T)|, \quad \text{where } M\Delta t = T.$$

From Theorem 3.4 we have

$$\mathrm{e}_{\Delta t} \le C_5 \Delta t^{\frac{1}{2}-\delta}, \tag{29}$$

where $\delta = 1/q$, for sufficiently small Δt.

For simplicity, we have chosen the case where f, g and h are multiplicatively linear; that is, $f(X(t)) \equiv \mu X(t)$, $g(X(t)) \equiv \sigma X(t)$ and $h(\gamma(t), X(t)) \equiv \gamma(t)X(t)$. We chose the distribution for γ to be log-normal, specifically $\log Y \sim N(0.1, 0.01)$. In this case all $2q$th moments are bounded, so (2) holds for any q. We note that this type of jump SDE is a standard asset price model in mathematical finance. We chose drift and volatility parameters of $\mu = 1.07$ and $\sigma = 0.4$, and initial value $X(0) = 1$. As our numerical method we use the $\theta = 0$ version, that is, the natural extension of the explicit Euler-Maruyama scheme.

We simulated 20,000 discretized jump-SDE paths over $[0, 1]$ with at discretization level $h = 2^{-20}$, retaining the same jump profile for each path. For each sample path, Euler–Maruyama was applied for four different stepsizes: $\Delta t = 2^{q-1}h$ for $1 \le q \le 4$. We then took the sample mean of the endpoint errors over the sample paths, thus approximating the strong error $\mathrm{e}_{\Delta t}$ for each stepsize.

If the inequality (29) holds with approximate equality, then taking logs gives

$$\log \mathrm{e}_{\Delta t} \approx \log C + \left(\frac{1}{2} - \delta\right) \log \Delta t \ .$$

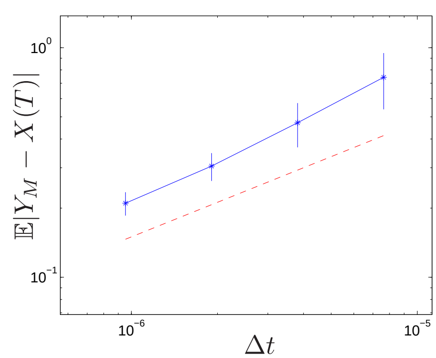

FIGURE 2. *Strong error plot for jump-SDE Euler-Maruyama approximation with random jump magnitudes*

Figure 2 shows the plot of our approximation to $\mathrm{e}_{\Delta t}$ against Δt on a log–log scale (solid line), we have also superimposed a reference slope of one-half (broken line). The vertical lines give approximate 95% confidence intervals for the sample means. We can see that there is a good match to the reference line. If we further assume a power law $\mathrm{e}_{\Delta t} = C\Delta t^{\alpha}$ for some constants C and α, so that $\log \mathrm{e}_{\Delta t} = \log C + \alpha \log \Delta t$, we can compute a least squares fit for α. Doing so yields the value $\alpha = 0.6092$ with least squares residual of 0.0427. Overall, this is consistent with our result that the strong error is of order $\frac{1}{2} - \delta$ for any $\delta > 0$.

4. **Mean-Square Stability.** In this section we consider stability issues, extending the analysis in [14] from a constant jump magnitude to the random case. Throughout this section, our use of the term *stability* is always to be interpreted in the mean-square sense, rather than, for example, pathwise [1].

We consider the linear multiplicative test equation where $f(x) = \mu x$, $g(x) = \sigma x$ and $h(x, y) = yx$ in (1), so that

$$\mathrm{d}X(t) = \mu X(t^-)\,\mathrm{d}t + \sigma X(t^-)\,\mathrm{d}W(t) + \gamma(t)X(t^-)\,\mathrm{d}N(t), \tag{30}$$

assuming $X(0) \neq 0$ with probability 1. Here μ and σ are constants and we recall that $\gamma(t)$ is defined in (7). This problem has explicit solution

$$X(t) = X(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)\prod_{i=1}^{N(t)}(1+\gamma_i).$$

It can easily be shown that

$$\mathbb{E}\left[\prod_{i=1}^{N(t)}(1+\gamma_i)^2\right] = e^{\lambda t\mathbb{E}[\gamma(2+\gamma)]}$$

and so,

$$\mathbb{E}\left[X(t)^2\right] = \mathbb{E}\left[X(0)^2\right]e^{\left(2\mu+\sigma^2+2\lambda\mathbb{E}[\gamma]+\lambda\mathbb{E}[\gamma^2]\right)t}.$$

Hence, mean-square stability (of the zero solution) in (30) may be characterised by

$$\lim_{t\to\infty}\mathbb{E}\left[X(t)^2\right] = 0 \qquad \Longleftrightarrow \qquad 2\mu + \sigma^2 + 2\lambda\mathbb{E}[\gamma] + \lambda\mathbb{E}[\gamma^2] < 0. \tag{31}$$

Next we look at the corresponding mean-square stability property $\lim_{n\to\infty}\mathbb{E}\left[X_n^2\right] = 0$ for the theta method applied to (30). Following the traditional numerical analysis viewpoint, our aim is to quantify the range of stepsizes for which the numerical method matches the SDE.

4.1. **Stability of the Theta Method.** Applying the theta method (8) to (30) gives the recurrence

$$Y_{n+1} = Y_n + ((1-\theta)\mu Y_n + \theta\mu Y_{n+1})\,\Delta t + \sigma Y_n\Delta W_n + \gamma_n Y_n\Delta N_n. \tag{32}$$

For the implicit case, $\theta > 0$, we require $\mu\Delta t\theta \neq 1$ in order for the method to be well defined. Rearranging, then squaring and taking expectations of both sides gives

$$(1-\theta\mu\Delta t)^2\mathbb{E}\left[Y_{n+1}^2\right] = \mathbb{E}\left[Y_n^2\right]\mathbb{E}\left[(1+(1-\theta)\mu\Delta t + \sigma\Delta W_n + \gamma_n\Delta N_n)^2\right].$$

The Wiener increments satisfy $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}\left[\Delta W_n^2\right] = \Delta t$, and the Poisson increments satisfy $\mathbb{E}[\Delta N_n] = \lambda\Delta t$ and $\mathbb{E}\left[\Delta N_n^2\right] = \lambda\Delta t(1+\lambda\Delta t)$. Hence, by the independence of the increments, we find that

$$\begin{aligned}(1-\theta\mu\Delta t)^2\mathbb{E}\left[Y_{n+1}^2\right] =& \mathbb{E}\left[Y_n^2\right]\Big(1+\Delta t\left(2(1-\theta)\mu+\sigma^2+\lambda\mathbb{E}\left[\gamma(2+\gamma)\right]\right) \\ &+ \Delta t^2\left((1-\theta)^2\mu^2 + 2(1-\theta)\mu\lambda\mathbb{E}[\gamma] + \lambda^2\mathbb{E}\left[\gamma^2\right]\right)\Big).\end{aligned}$$

This leads to the mean-square stability characterisation

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[Y_n^2\right] = 0 \qquad \Longleftrightarrow \\ &\Delta t\Big((1-2\theta)\mu^2 + 2(1-\theta)\mu\lambda\mathbb{E}[\gamma] + \lambda^2\mathbb{E}\left[\gamma^2\right]\Big) < -\Big(2\mu+\sigma^2+2\lambda\mathbb{E}[\gamma]+\lambda\mathbb{E}\left[\gamma^2\right]\Big).\end{aligned} \tag{33}$$

We observe that the right-hand side in (33) appears in the mean-square stability characterisation for the underlying SDE in (31). It follows immediately from (31) and (33) that (a) if the SDE is stable then the numerical method will also be stable for all sufficiently small Δt, and, conversely, (b) if the SDE is unstable then the numerical method will also be unstable for all sufficiently small Δt. However, by analysing the characterisations more carefully, we can be more precise.

4.2. **Euler-Maruyama.** Taking $\theta = 0$ in (32) gives the explicit Euler-Maruyama (EM) method, for which we have the following result.

Lemma 4.1. *Suppose $\mathbb{E}\left[(\mu+\lambda\gamma)^2\right] \neq 0$. Then for the Euler-Maruyama method applied to (30) we have*

1. *problem stable $\Rightarrow$ EM stable for*

$$\Delta t < \frac{\left|2\mu + \sigma^2 + 2\lambda\mathbb{E}[\gamma] + \lambda\mathbb{E}[\gamma^2]\right|}{\mathbb{E}\left[(\mu+\lambda\gamma)^2\right]},$$

2. *problem unstable $\Rightarrow$ EM unstable for all $\Delta t > 0$.*

Proof. Both results follow directly from (31) and the $\theta = 0$ case of (33). ∎

We note that this result gives a clean generalization of the stability behaviour for explicit Euler on determinstic ordinary differential equations and standard SDEs. However, as we show in the next subsection, moving to a random jump magintude generally degrades the stability of implicit methods.

4.3. **General θ.** In [14], which deals with the case of constant jump magnitude, γ, it was shown that jumps can affect stability in the sense that

1. there exist $\{\mu, \sigma, \lambda, \gamma\}$ for which the problem is stable and the theta method is stable for all $\Delta t > 0$, even when $0 < \theta < \frac{1}{2}$,
2. given any $\varepsilon > 0$, there exist $\{\mu, \sigma, \lambda, \gamma\}$ for which the problem is unstable, yet the theta method is stable for all $\Delta t > \varepsilon$.
3. if the method is stable for some $\theta^\star$, then it is not true in general that it is stable for the same parameter set and fixed Δt for $\theta > \theta^\star$.

However, one result that did carry through from the regular SDE case to the constant jump magnitude case was the A-stability property, for $\frac{1}{2} \leq \theta \leq 1$, when γ is positive. Here, A-stability means that "problem stable $\Rightarrow$ method stable for all Δt".

The following result shows that this property fails to hold, in general, when random jump magnitudes are considered.

Theorem 4.2. *The theta method is* **not** *A-stable for any $\theta \in \left[\frac{1}{2}, 1\right]$ for random jump magnitudes of positive mean. More precisely, there exists $\{\mu, \sigma, \lambda, \gamma\}$, with $\mathbb{E}[\gamma] > 0$, satisfying the stability condition (31) with the property that for any $\frac{1}{2} \leq \theta \leq 1$, there exists a finite Δt_θ such that the theta method is unstable for all $\Delta t > \Delta t_\theta$.*

Proof. From (31) and (33), it is sufficient to show that there exists some random variable γ such that

$$2\mu + \sigma^2 + 2\lambda\mathbb{E}[\gamma] + \lambda\mathbb{E}\left[\gamma^2\right] < 0$$

and

$$(1-2\theta)\mu^2 + 2(1-\theta)\mu\lambda\mathbb{E}[\gamma] + \lambda^2\mathbb{E}\left[\gamma^2\right] > 0$$

for all $\theta \in \left[\frac{1}{2}, 1\right]$.

Take $\mu = -0.7$, $\sigma = 0.1$, $\lambda = 2$ and γ to be a random variable such that $\mathbb{E}[\gamma] = 0.2$, $\mathbb{E}\left[\gamma^2\right] = 0.25$. In this case, it is easy to see that the problem is stable by (31) and it then follows that the stability condition in (33) becomes $\Delta t < 9/(93 - 42\theta)$. □

5. **Concluding Remarks.** Implicit methods are useful both for (a) establishing existence and convergence on nonlinear problems [16] and (b) obtaining good long term stability properties [13]. In this work, we have extended previous results on strong convergence and mean-square stability [14] to the case of jump-SDEs where the jump magnitude is a random variable. This problem class is now widely used in mathematical finance.

We showed that under appropriate moment bounds on the jump magnitude, an implicit theta method gives strong convergence rate arbitrarily close to order $\frac{1}{2}$, and our numerical results supported the analysis. On a linear test problem, we characterised mean-square stability of these methods and, in particular, showed that an A-stability result for constant jump magnitude is lost in this more general setting.

There is much scope for further work in the context of random jump magnitudes. For example, it would clearly be of interest to extend the strong convergence theory to the case where coefficients are not globally Lipschitz, and to develop and analyse new methods that maintain the A-stability of the underlying deterministic methods.

REFERENCES

[1] J. Appleby, A. Rodkina and H. Schurz, *Pathwise non-exponential decay rates of solutions of scalar nonlinear stochastic differential equations*, Discrete Contin. Dyn. Syst. Ser. B, **6** (2006), 667–696.

[2] N. Bruti-Liberati and E. Platen, *On the strong approximation of jump-diffusion processes*, Tech. Rep. QFRC Research Paper 157, University of Technology, Sydney, (2005).

[3] R. Cont and P. Tankov, "Financial Modelling with Jump Processes," Chapman & Hall/CRC, Florida, 2004.

[4] A. Gardoń, *The order of approximation for solutions of Itô-type stochastic differential equations with jumps*, Stoch. Anal. Appl., **22** (2004), 679–699.

[5] M. Giles, *Multi-level Monte Carlo path simulation*, Tech. Rep. NA-06/03, University of Oxford Computing Laboratory, (2006).

[6] P. Glasserman, "Monte Carlo Methods in Financial Engineering," Springer-Verlag, Berlin, 2004.

[7] P. Glasserman and N. Merener, *Numerical solution of jump-diffusion LIBOR market models*, Finance Stoch., **7** (2003), 1–27.

[8] P. Glasserman and N. Merener, *Convergence of a discretization scheme for jump-diffusion processes with state-dependent intensities*, Proc. R. Soc. A, **460** (2004), 111–127.

[9] F. Hanson and J. Westman, *Optimal consumption and portfolio control for jump-diffusion stock process with log-normal jumps*, in "Proceedings of the 2002 American Control Conference", Piscataway, NJ: IEEE, (2002), 4256–4261.

[10] F. Hanson and G. Yan, *Option pricing for a stochastic-volatility jump-diffusion model with log-uniform jump amplitudes*, in "Proceedings of the 2006 American Control Conference", Evanston, IL: American Automatic Control Council, (2006), 1–7.

[11] F. Hanson and Z. Zhu, *Log-double-uniform jump-diffusion model*, Unpublished working paper, `http://www.math.uic.edu/~hanson/pub/ZZhu/dbuniformfree2.pdf`, 2004.

[12] D. Higham, *A-stability and stochastic mean-square stability*, BIT, **40** (2000), 404–409.

[13] D. Higham, *Mean-square and asymptotic stability of the stochastic theta method* SIAM J. Numer. Anal., **38** (2000), 753–769.

[14] D. Higham and P. Kloeden, *Convergence and stability of implicit methods for jump-diffusion systems*, Int. J. Numer. Anal. Model., **3** (2006), 125–140.

[15] D. Higham, X. Mao and A. Stuart, *Exponential mean-square stability of numerical solutions to stochastic differential equations*, LMS J. Comput. Math., **6** (2003), 297–313.

[16] D. Higham, X. Mao and A. Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations*, SIAM J. Numer. Anal., **40** (2002), 1041–1063.

[17] S. Kou, *A jump-diffusion model for option pricing*, Management Science, **48** (2002), 1086–1101.

[18] Y. Maghsoodi, *Mean square efficient numerical solution of jump-diffusion stochastic differential equations*, Sankhyā Indian J. Stat., **58** (1996), 25–47.

[19] Y. Maghsoodi, *Exact solutions and doubly efficient approximations and simulation of jump-diffusion Itô equations*, Stoch. Anal. Appl., **16** (1998), 1049–1072.

[20] Y. Maghsoodi and C. Harris, *In-probability approximation and simulation of non-linear jump-diffusion stochastic differential equations*, IMA J. Math. Control Inform., **4** (1987), 65–92.

[21] R. Merton, *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics, **3** (1976), 125–144.

[22] R. Milkulevičius and E. Platen, *Exact solutions and doubly efficient approximations and simulation of jump-diffusion Itô equations*, Stoch. Anal. Appl., **16** (1998), 1049–1072.

[23] R. Schilling, "Measures, Integrals and Martingales," Cambridge University Press, Cambridge, 2005.

[24] K. Sobczyk, "Stochastic Differential Equations. With Applications to Physics and Engineering," Kluwer Academic, Dordrecht, 1991.

[25] A. Svishchuk and A. Kalemanova, *The stochastic stability of interest rates with jump changes*, Theory Probab. Math. Statist., **61** (2000), 161–172.

[26] Z. Zhu and F. Hanson, *A Monte Carlo option-pricing algorithm for log-uniform jump-diffusion model*, in "Proceedings of the 44th IEEE Conference on Decision and Control, and European Control Conference", Piscataway, NJ: IEEE, (2005), 5221–5226.

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