INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 3, Number 2, Pages 125–140

CONVERGENCE AND STABILITY OF IMPLICIT METHODS FOR JUMP-DIFFUSION SYSTEMS

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Abstract. A class of implicit methods is introduced for Ito stochastic differential equations with Poisson-driven jumps. A convergence proof shows that these implicit methods share the same finite time strong convergence rate as the explicit Euler–Maruyama scheme. A mean-square linear stability analysis shows that implicitness offers benefits, and a natural analogue of mean-square A-stability is studied. Weak variants are also considered and their stability analyzed.

Key Words. A-stability, backward Euler, Euler–Maruyama, linear stability, Poisson process, stochastic differential equation, strong convergence, theta method, trapezoidal rule.

1. Introduction

Applications in economics, finance, and several areas of science and engineering, give rise to jump-diffusion Ito stochastic differential equations [2, 4, 24] of the form

$$dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))dN(t), \quad t > 0, \tag{1}$$

with $X(0)$ given, where $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$. Here, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the drift coefficient, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the diffusion coefficient and $W(t)$ is an m-dimensional Brownian motion. We assume that $N(t)$ is a scalar Poisson process with intensity λ, and hence the jump coefficient has the form $h : \mathbb{R}^n \to \mathbb{R}^n$. Extension of our work to vector-valued jumps with independent entries is straightforward. Conditions on the coefficients and initial data that guarantee a unique solution will be introduced in section 2.

We consider a class of theta methods for (1). For a constant stepsize $\Delta t > 0$ and a particular choice of $\theta \in [0, 1]$, the theta method is defined by $Y_0 = X(0)$ and

$$Y_{n+1} = Y_n + (1-\theta)f(Y_n)\Delta t + \theta f(Y_{n+1})\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n. \tag{2}$$

Here, Y_n is the approximation to $X(t_n)$, for $t_n = n\Delta t$, with $\Delta W_n := W(t_{n+1}) - W(t_n)$ and $\Delta N_n := N(t_{n+1}) - N(t_n)$ denoting the increments of the Brownian motion and the Poisson process, respectively.

We refer to (2) as a class of theta methods because in the deterministic ordinary differential equation (ODE) case, where $g(\cdot) \equiv h(\cdot) \equiv 0$ and $X(0)$ is constant, (2) reduces to the well-known class with this name. For Ito stochastic differential equations (SDEs), where $h(\cdot) \equiv 0$, the class has been referred to as the semi-implicit Euler method [11, 23] and the stochastic theta method [9]. Our motivation

Received by the editors September, 2004 and, in revised form, November, 2004.
2000 *Mathematics Subject Classification.* 65C30, 65L20, 60H10.
This work was supported by Engineering and Physical Sciences Research Council grant GR/T19100. DJH was also supported by a Research Fellowship from The Royal Society of Edinburgh/Scottish Executive Education and Lifelong Learning Department.

for introducing and studying (2) is that for the ODE and SDE cases, it has been found that the class offers good linear stability properties [8, 9, 23] and excellent potential for capturing long time dynamics [20, 25]. Our aim here is to show that the theta method offers a means to define useful implicit integrators in the presence of jumps. Section 2 justifies the methodology by giving a finite time strong convergence proof. Section 3 analyzes mean-square stability and quantifies precisely what may be gained by moving away from the Euler–Maruyama ($\theta = 0$) case. Stability for a weak version of the theta method is studied in section 4.

Previous work on numerical methods for jump-diffusion problems includes [3, 6, 7, 12, 14, 15, 16, 17, 21, 22]. The references [6, 7, 12, 14, 21, 22] deal with weak convergence. In [6, 7, 12, 21] 'jump-adapted' explicit methods that directly incorporate the jump points are studied, whereas [14, 22] use a fixed Δt. Glasserman [5, page 364] points out that jump-adaption may be expensive when the jump intensity λ is large. Strong convergence for fixed stepsize explicit methods is studied in [3, 15, 16, 17]. Our work differs from these references in that (a) implicit methods are considered, and (b) in addition to finite time strong convergence, mean-square stability properties are analyzed.

2. Strong Convergence Proof

In this section we suppose that the problem (1) is to be solved over a finite time interval, $[0, T]$, where T is a constant. We study classical strong convergence, and hence we are concerned with the regime where $\Delta t \to 0$ with T fixed. The reference [1] mentions a number of applications where this type of convergence is required, the most relevant for our work being mathematical finance. The initial steps of the proof follow the ideas in [10, Appendix A], where a strong convergence result for the theta method on a non-jump SDE is given. Our proof is more general in that it deals with the jump term and also places the supremum over time inside the expectation operator (see Theorem 2.4 below).

Letting $|\cdot|$ denote both the Euclidean vector norm and the Frobenius matrix norm, we assume that f, g, h satisfy the global Lipschitz condition:

$$|a(x) - a(y)|^2 \le K|x - y|^2, \qquad \text{for } a \equiv f, g, \text{ or } h, \tag{3}$$

where K is a constant independent of x and y, and we note that this implies the linear growth bound

$$|a(x)|^2 \le L(1 + |x|^2), \qquad \text{for } a \equiv f, g, \text{ or } h, \tag{4}$$

where L is a constant independent of x and y. Our assumption on the initial data is that $\mathbb{E}|X(0)|^2$ is finite and $X(0)$ is independent of $W(t)$ and $N(t)$ for all $t \ge 0$. We note that these conditions imply the existence of a unique solution for (1), see, for example, [4, 24].

For convenience, we will extend the discrete numerical solution to continuous time. We first define the 'step functions'

$$Z_1(t) = \sum_k Y_k \mathbf{1}_{[k\Delta t, (k+1)\Delta t)}(t), \quad Z_2(t) = \sum_k Y_{k+1} \mathbf{1}_{[k\Delta t, (k+1)\Delta t)}(t), \tag{5}$$

where $\mathbf{1}_G$ is the indicator function for the set G. Then we define

$$\begin{aligned} Y(t) \;=\;& Y_0 + \int_0^t (1-\theta) f(Z_1(s))\, ds + \int_0^t \theta f(Z_2(s))\, ds + \int_0^t g(Z_1(s))\, dW(s) \\ & + \int_0^t h(Z_1(s))\, dN(s). \end{aligned} \tag{6}$$

(Note that by construction $Z_1(s^-) = Z_1(s)$ and $Z_2(s^-) = Z_2(s)$ for $s \neq k\Delta t$.) It is straightforward to check that $Z_1(t_k) = Z_2(t_{k-1}) = Y(t_k) = Y_k$. Our plan is to prove a convergence result for $Y(t)$. Because $Y(t)$ interpolates the discrete numerical solution, this will immediately give a convergence result for $\{Y_k\}$.

To begin, we note that for $\theta \neq 0$ the numerical solution in (2) is defined by an implicit equation. Under our global Lipschitz assumption on f, it follows from the Banach contraction mapping theorem that a unique solution Y_{k+1} exists, with probability 1, for all $\sqrt{K}\Delta t < 1$; see, for example, [10, Lemma A.1]. Hence, we assume that this inequality holds so that there is a well defined numerical solution.

Throughout the following analysis we use $C_1, C_2, \ldots$ to denote generic constants that depend upon K, L and T, but not upon Δt. The precise values of these constants may be determined via the proofs.

Our first lemma shows that the discrete numerical solution has bounded second moments.

Lemma 2.1. *Under the assumptions above, there exists $\Delta t^\star > 0$ such that for all $0 < \Delta t \leq \Delta t^\star$,*

$$\mathbb{E}|Y_k|^2 \leq C_1(1 + \mathbb{E}|X(0)|^2), \text{ whenever } k\Delta t \leq T.$$

Proof. By construction, we have

$$\begin{aligned} Y_{k+1} &= Y_0 + \int_0^{(k+1)\Delta t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\, ds + \int_0^{(k+1)\Delta t} g(Z_1(s))\, dW(s) \\ &\quad + \int_0^{(k+1)\Delta t} h(Z_1(s))\, dN(s). \end{aligned}$$

So, for $(k+1)\Delta t \leq T$,

$$\begin{aligned} \mathbb{E}|Y_{k+1}|^2 &\leq 4\mathbb{E}|Y_0|^2 + 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} (1-\theta) f(Z_1(s)) + \theta f(Z_2(s))\, ds\right|^2 \\ &\quad + 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} g(Z_1(s))\, dW(s)\right|^2 + 4\mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s))\, dN(s)\right|^2. \end{aligned} \tag{7}$$

Now, using the Cauchy-Schwarz inequality, the linear growth bound (4) and Fubini's Theorem,

$$\begin{aligned} \mathbb{E}\left|\int_0^{(k+1)\Delta t} (1-\theta) f(Z_1(s)) \right. &+ \left. \theta f(Z_2(s))\, ds\right|^2 \\ &\leq T\mathbb{E}\int_0^{(k+1)\Delta t} |(1-\theta) f(Z_1(s)) + \theta f(Z_2(s))|^2\, ds \\ &\leq 2T\mathbb{E}\int_0^{(k+1)\Delta t} |f(Z_1(s))|^2 + |f(Z_2(s))|^2\, ds \\ &\leq 2TL\mathbb{E}\int_0^{(k+1)\Delta t} 2 + |Z_1(s)|^2 + |Z_2(s)|^2\, ds \\ &\leq 4T^2L + 2TL\int_0^{(k+1)\Delta t} \mathbb{E}|Z_1(s)|^2 + \mathbb{E}|Z_2(s)|^2\, ds \\ &\leq 4T^2L + 4TL\Delta t \sum_{i=0}^{k} \mathbb{E}|Y_i|^2 + 2TL\Delta t\mathbb{E}|Y_{k+1}|^2. \end{aligned} \tag{8}$$

Using the Ito isometry and the linear growth bound (4), we have

$$\begin{aligned}
\mathbb{E}\left|\int_0^{(k+1)\Delta t} g(Z_1(s))\, dW(s)\right|^2 &= \int_0^{(k+1)\Delta t} \mathbb{E}|g(Z_1(s))|^2\, ds \\
&= \Delta t \sum_{j=0}^{k} \mathbb{E}|g(Y_j)|^2 \\
&\le \Delta t L \sum_{j=0}^{k} (1 + \mathbb{E}|Y_j|^2) \\
&\le LT + \Delta t L \sum_{j=0}^{k} \mathbb{E}|Y_j|^2. \qquad (9)
\end{aligned}$$

For the jump integral, we convert to the compensated Poisson process $\widetilde{N}(t) := N(t) - \lambda t$, which is a martingale, and use the isometry

$$\mathbb{E}\left|\int_{t_1}^{t_2} h(Z_1(s))\, d\widetilde{N}(s)\right|^2 = \lambda \int_{t_1}^{t_2} \mathbb{E}|h(Z_1(s))|^2\, ds, \qquad (10)$$

see, for example, [3]. We then obtain

$$\begin{aligned}
\mathbb{E}\left| \int_0^{(k+1)\Delta t} h(Z_1(s))\, dN(s) \right|^2 & \\
&= \mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s))\, d\widetilde{N}(s) + \lambda \int_0^{(k+1)\Delta t} h(Z_1(s))\, ds\right|^2 \\
&\le 2\mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s))\, d\widetilde{N}(s)\right|^2 + 2\lambda^2 \mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s))\, ds\right|^2 \\
&\le 2\lambda \int_0^{(k+1)\Delta t} \mathbb{E}|h(Z_1(s))|^2\, ds + 2\lambda^2 T \int_0^{(k+1)\Delta t} \mathbb{E}|h(Z_1(s))|^2\, ds \\
&= (2\lambda + 2\lambda^2 T)\Delta t \sum_{j=0}^{k} \mathbb{E}|h(Y_j)|^2,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and Fubini's Theorem. The linear growth bound (4) then gives

$$\mathbb{E}\left|\int_0^{(k+1)\Delta t} h(Z_1(s))\, dN(s)\right|^2 \le 2\lambda T(1+\lambda T)L + 2\lambda \Delta t(1+\lambda T)L \sum_{j=0}^{k} \mathbb{E}|Y_j|^2. \qquad (11)$$

Inserting (8), (9) and (11) in (7) gives

$$\begin{aligned}
\mathbb{E}|Y_{k+1}|^2 \le\ & 4\left[\mathbb{E}|Y_0|^2 + 4T^2 L + LT + 2\lambda T(1+\lambda T)L\right] \\
& + 4\Delta t\left[4LT + L + 2\lambda(1+\lambda T)L\right] \sum_{j=0}^{k} \mathbb{E}|Y_j|^2 + 8TL\Delta t \mathbb{E}|Y_{k+1}|^2.
\end{aligned}$$

Choosing Δt sufficiently small for $1 - 8TL\Delta t \ge \frac{1}{2}$, we obtain

$$\begin{aligned}
\mathbb{E}|Y_{k+1}|^2 \le\ & 8\left[\mathbb{E}|Y_0|^2 + LT(4T + 1 + 2\lambda(1+\lambda T))\right] \\
& + 8\Delta t L\left[4T + 1 + 2\lambda(1+\lambda T)\right] \sum_{j=0}^{k} \mathbb{E}|Y_j|^2.
\end{aligned}$$

The result then follows from an application of the discrete Gronwall inequality, see for example, [18]. □

The next lemma shows that the continuous approximation has bounded second moments in a strong sense.

Lemma 2.2. *Under the assumptions above, there exists $\Delta t^\star > 0$ such that for all $0 < \Delta t \leq \Delta t^\star$,*

$$\mathbb{E} \sup_{t\in[0,T]} |Y(t)|^2 \leq C_2(1 + \mathbb{E}|X(0)|^2). \tag{12}$$

Proof. From (6),

$$\begin{aligned}
|Y(t)|^2 \quad &\leq \quad 4|Y_0|^2 + 4\left|\int_0^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\, ds\right|^2 \\
&\quad + 4|\int_0^t g(Z_1(s))\, dW(s)|^2 + 4\left|\int_0^t h(Z_1(s))\, dN(s)\right|^2.
\end{aligned}$$

Thus, using the Cauchy-Schwarz inequality and the definition of $\widetilde{N}$,

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} |Y(t)|^2 \quad &\leq \quad 4\mathbb{E}|Y_0|^2 + 4\mathbb{E} \sup_{t\in[0,T]}\left|\int_0^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\, ds\right|^2 \\
&\quad + 4\mathbb{E} \sup_{t\in[0,T]}\left|\int_0^t g(Z_1(s))\, dW(s)\right|^2 \\
&\quad + 4\mathbb{E} \sup_{t\in[0,T]}\left|\int_0^t h(Z_1(s))\, dN(s)\right|^2 \\
&\leq \quad 4\mathbb{E}|Y_0|^2 + 4\mathbb{E} \sup_{t\in[0,T]} \int_0^t 1^2\, ds \int_0^t 2|f(Z_1(s))|^2 + 2|f(Z_2(s))|^2\, ds \\
&\quad + 4\mathbb{E} \sup_{t\in[0,T]}\left|\int_0^t g(Z_1(s))\, dW(s)\right|^2 \\
&\quad + 8\mathbb{E} \sup_{t\in[0,T]}\left(\left|\int_0^t h(Z_1(s))\, d\widetilde{N}(s)\right|^2 + \left|\int_0^t h(Z_1(s))\lambda\, ds\right|^2\right).
\end{aligned}$$

Then, using the Doob inequality for the two martingale integrals and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} |Y(t)|^2 \quad &\leq \quad 4\mathbb{E}|Y_0|^2 + 8T\int_0^T \mathbb{E}\left(|f(Z_1(s))|^2 + |f(Z_2(s))|^2\right)\, ds \\
&\quad + 16\mathbb{E}\left|\int_0^T g(Z_1(s))\, dW(s)\right|^2 \\
&\quad + 32\mathbb{E}\left|\int_0^T h(Z_1(s))\, d\widetilde{N}(s)\right|^2 + 8T\lambda^2 \int_0^T \mathbb{E}|h(Z_1(s))|^2\, ds.
\end{aligned}$$

Finally, using the Ito isometry and the martingale isometry (10), along with the linear growth bounds (4), we have

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}|Y(t)|^2 &\leq 4\mathbb{E}|Y_0|^2 + 8LT\int_0^T 2+\mathbb{E}\left(|Z_1(s)|^2+|Z_2(s)|^2\right)\,ds\\
&\quad + 16L\int_0^T 1+\mathbb{E}|Z_1(s)|^2\,ds\\
&\quad + 32\lambda L\int_0^T 1+\mathbb{E}|Z_1(s)|^2\,ds + 8LT\lambda^2\int_0^T 1+\mathbb{E}|Z_1(s)|^2\,ds\\
&\leq 4\mathbb{E}|Y_0|^2 + 16LT^2 + 16LT + 32\lambda LT + 8L\lambda^2T^2\\
&\quad + [8LT+16L+32\lambda L+8LT\lambda^2]\int_0^T \mathbb{E}|Z_1(s)|^2\,ds\\
&\quad + 8LT\int_0^T \mathbb{E}|Z_2(s)|^2\,ds.
\end{aligned}$$

Applying Lemma 2.1 over the interval $[0, T+1]$ (since some $Z_2(t)$ terms may extend beyond T), we obtain the result (12). $\square$

Next, we show that the continuous-time approximation remains close to the step functions $Z_1(t)$ and $Z_2(t)$ in a strong sense.

Lemma 2.3. *Under the assumptions above, there exists $\Delta t^\star > 0$ such that for all $0 < \Delta t \leq \Delta t^\star$,*

$$\mathbb{E}\sup_{t\in[0,T]}|Y(t)-Z_1(t)|^2 \leq C_3\Delta t(1+\mathbb{E}|X(0)|^2) \tag{13}$$

and

$$\mathbb{E}\sup_{t\in[0,T]}|Y(t)-Z_2(t)|^2 \leq C_4\Delta t(1+\mathbb{E}|X(0)|^2). \tag{14}$$

Proof. Consider $t \in [k\Delta t, (k+1)\Delta t] \subseteq [0, T]$. We have

$$\begin{aligned}
Y(t)-Z_1(t) = Y(t)-Y_k &= \int_{k\Delta t}^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\,ds\\
&\quad + \int_{k\Delta t}^t g(Z_1(s))\,dW(s) + \int_{k\Delta t}^t h(Z_1(s))\,dN(s).
\end{aligned}$$

Thus,

$$\begin{aligned}
|Y(t)-Z_1(t)|^2 &\leq 3\left|\int_{k\Delta t}^t (1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\,ds\right|^2\\
&\quad + 3\left|\int_{k\Delta t}^t g(Z_1(s))\,dW(s)\right|^2 + 3\left|\int_{k\Delta t}^t h(Z_1(s))\,dN(s)\right|^2,
\end{aligned}$$

for each $t \in [k\Delta t, (k+1)\Delta t]$. Thus

$$\begin{aligned}\sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \quad \leq \quad & \max_{k=0,1,\ldots,T/\Delta t-1} \sup_{\tau\in[k\Delta t,(k+1)\Delta t]} \Big\{ \\ & 3\left|\int_{k\Delta t}^{\tau} [(1-\theta)f(Z_1(s)) + \theta f(Z_2(s))\, ds\right|^2 \\ & + 3\left|\int_{k\Delta t}^{\tau} g(Z_1(s))\, dW(s)\right|^2 \\ & + 6\left|\int_{k\Delta t}^{\tau} h(Z_1(s))\, d\widetilde{N}(s)\right|^2 \\ & + 6\left|\int_{k\Delta t}^{\tau} h(Z_1(s))\lambda\, ds\right|^2 \Big\}.\end{aligned}$$

Now, $|\int_{k\Delta t}^{\tau} h(Z_1(s))\, ds|^2 \leq \int_{k\Delta t}^{\tau} 1^2\, ds \int_{k\Delta t}^{\tau} |h(Z_1(s))|^2\, ds \leq \Delta t \int_{k\Delta t}^{\tau} |h(Z_1(s))|^2\, ds$ by the Cauchy-Schwarz inequality, so, taking expectations and using the Doob inequality on the martingale integrals we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \quad \leq \quad & \max_{k=0,1,\ldots,T/\Delta t-1} \sup_{\tau\in[k\Delta t,(k+1)\Delta t]} \Big\{ \\ & 6\Delta t \mathbb{E} \int_{k\Delta t}^{(k+1)\Delta t} |f(Z_1(s))|^2 + |f(Z_2(s))|^2\, ds \\ & + 12 \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}|g(Z_1(s))|^2\, ds \\ & + 24 \int_{k\Delta t}^{(k+1)\Delta t} \mathbb{E}|h(Z_1(s))|^2 \lambda\, ds \\ & + 6\Delta t \lambda^2 \mathbb{E} \int_{k\Delta t}^{(k+1)\Delta t} |h(Z_1(s))|^2\, ds \Big\}.\end{aligned}$$

Applying the linear growth bounds (4), we find that

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \quad \leq \quad & \max_{k=0,1,\ldots,T/\Delta t-1} \Big\{ \\ & 6\Delta t L \int_{k\Delta t}^{(k+1)\Delta t} 2 + \mathbb{E}|Z_1(s)|^2 + \mathbb{E}|Z_2(s)|^2\, ds \\ & + 6L[2 + 4\lambda + \Delta t \lambda^2] \int_{k\Delta t}^{(k+1)\Delta t} 1 + \mathbb{E}|Z_1(s)|^2\, ds \Big\}.\end{aligned}$$

But $Z_1(s) \equiv Y_k$ and $Z_2(s) \equiv Y_{k+1}$ on $[k\Delta t, (k+1)\Delta t)$, and hence, from Lemma 2.1.

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |Y(t) - Z_1(t)|^2 \quad \leq \quad & 6\Delta t^2 L \left[2 + 2C_1(1 + \mathbb{E}|Y_0|^2)\right] \\ & + 6L[2 + 4\lambda + \Delta t \lambda^2]\Delta t \left(1 + C_1(1 + \mathbb{E}|Y_0|^2)\right),\end{aligned}$$

giving (13).

A similar analysis gives (14). □

We are now in a position to prove a strong convergence result.

Theorem 2.4. *Under the assumptions above, there exists $\Delta t^\star > 0$ such that for all $0 < \Delta t \leq \Delta t^\star$,*

$$\mathbb{E} \sup_{t\in[0,T]} |Y(t) - X(t)|^2 \leq C_5 \Delta t (1 + \mathbb{E}|X(0)|^2). \tag{15}$$

Proof. By construction,

$$
\begin{aligned}
Y(t) - X(t) &= \int_0^t (1-\theta)[f(Z_1(s)) - f(X(s^-))] + \theta[f(Z_2(s)) - f(X(s^-))]\,ds \\
&\quad + \int_0^{(t} g(Z_1(s)) - g(X(s^-))\,dW(s) + \int_0^t h(Z_1(s)) - h(X(s^-))\,dN(s).
\end{aligned}
$$

Hence, for any $0 \leq t_1 \leq T$,

$$
\begin{aligned}
\mathbb{E} \sup_{t\in[0,t_1]} |Y(t) - X(t)|^2 &\leq 3\mathbb{E} \sup_{t\in[0,t_1]} \Big| \int_0^t (1-\theta)[f(Z_1(s)) - f(X(s^-))] \\
&\quad + \theta[f(Z_2(s)) - f(X(s^-))]\ ds\Big|^2 \\
&\quad + 3\mathbb{E} \sup_{t\in[0,t_1]} \Big| \int_0^t g(Z_1(s)) - g(X(s^-))\ dW(s)\Big|^2 \\
&\quad + 3\mathbb{E} \sup_{t\in[0,t_1]} \Big| \int_0^t h(Z_1(s)) - h(X(s^-))\ dN(s)\Big|^2 \\
&\leq 6\mathbb{E} \sup_{t\in[0,t_1]} \int_0^t 1^2\,ds \int_0^{t_1} |f(Z_1(s)) - f(X(s^-))|^2 \\
&\quad + |f(Z_2(s)) - f(X(s^-))|^2\ ds \\
&\quad + 3\mathbb{E} \sup_{t\in[0,t_1]} \Big| \int_0^t g(Z_1(s)) - g(X(s^-))\ dW(s)\Big|^2 \\
&\quad + 6\mathbb{E} \sup_{t\in[0,t_1]} \Big| \int_0^t h(Z_1(s)) - h(X(s^-))\ d\widetilde{N}(s)\Big|^2 \\
&\quad + 6\mathbb{E} \sup_{t\in[0,t_1]} \int_0^t 1^2\,ds \int_0^t |h(Z_1(s)) - h(X(s^-))|^2 \lambda^2\,ds,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality and the definition of $\widetilde{N}$. Now, using the Doob inequality in the two martingale terms,

$$
\begin{aligned}
\mathbb{E} \sup_{t\in[0,t_1]} |Y(t) - X(t)|^2 &\leq 6t_1\mathbb{E} \int_0^{t_1} |f(Z_1(s)) - f(X(s^-))|^2 \\
&\quad + |f(Z_2(s)) - f(X(s^-))|^2\ ds \\
&\quad + 12\mathbb{E} \left| \int_0^{t_1} g(Z_1(s)) - g(X(s^-))\ dW(s)\right|^2 \\
&\quad + 24\mathbb{E} \left| \int_0^{t_1} h(Z_1(s)) - h(X(s^-))\ d\widetilde{N}(s)\right|^2 \\
&\quad + 6t_1\lambda^2\mathbb{E} \int_0^{t_1} |h(Z_1(s)) - h(X(s^-))|^2\,ds.
\end{aligned}
$$

The integral isometries and Fubini's Theorem then give

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,t_1]} |Y(t) - X(t)|^2 \quad &\leq \quad 6T \int_0^{t_1} \mathbb{E}|f(Z_1(s)) - f(X(s^-))|^2 \\
&\quad + \mathbb{E}|f(Z_2(s)) - f(X(s^-))|^2 \; ds \\
&\quad + 12 \int_0^{t_1} \mathbb{E}|g(Z_1(s)) - g(X(s^-))|^2 \, ds \\
&\quad + 24 \int_0^{t_1} \mathbb{E}|h(Z_1(s)) - h(X(s^-))|^2 \lambda \, ds \\
&\quad + 6T\lambda^2 \int_0^{t_1} \mathbb{E}|h(Z_1(s)) - h(X(s^-))|^2 \, ds.
\end{aligned}$$

Applying the Lipschitz conditions (3) we have

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,t_1]} |Y(t) - X(t)|^2 \quad &\leq \quad 6TK \int_0^{t_1} \mathbb{E}|Z_1(s) - X(s^-)|^2 + \mathbb{E}|Z_2(s) - X(s^-)|^2 \; ds \\
&\quad + 12K \int_0^{t_1} \mathbb{E}|Z_1(s) - X(s^-)|^2 \, ds \\
&\quad + 24\lambda K \int_0^{t_1} \mathbb{E}|Z_1(s) - X(s^-)|^2 \, ds \\
&\quad + 6T\lambda^2 K \int_0^{t_1} \mathbb{E}|Z_1(s) - X(s^-)|^2 \, ds \\
&= \quad 6K[T + 2 + 4\lambda + T\lambda^2] \int_0^{t_1} \mathbb{E}|Z_1(s) - X(s^-)|^2 \, ds \\
&\quad + 6TK \int_0^{t_1} \mathbb{E}|Z_2(s) - X(s^-)|^2 \, ds \\
&\leq \quad 12K[T + 2 + 4\lambda + T\lambda^2] \int_0^{t_1} \mathbb{E}|Z_1(s) - Y(s)|^2 \\
&\quad + \mathbb{E}|Y(s) - X(s^-)|^2 \; ds \\
&\quad + 12TK \int_0^{t_1} \mathbb{E}|Z_2(s) - Y(s)|^2 + \mathbb{E}|Y(s) - X(s^-)|^2 \; ds.
\end{aligned}$$

Applying Lemma 2.3 we obtain a bound of the form

$$\mathbb{E} \sup_{t\in[0,t_1]} |Y(t) - X(t)|^2 \leq C_6 \Delta t (1 + \mathbb{E}|Y(0)|^2) + C_7 \int_0^{t_1} \mathbb{E} \sup_{t\in[0,s]} |Y(t) - X(t^-)|^2 \, ds.$$

The result (15) then follows from the continuous Gronwall inequality, see, for example, [19]. □

Theorem 2.4 shows that the theta method has strong convergence rate of at least $\frac{1}{2}$. This result is sharp in the sense that $\frac{1}{2}$ is the actual rate obtained for Euler–Maruyama; that is, for $\theta = 0$. It is of interest to note that the error bound (15) places the supremum inside the expectation, which shows that the error in the continuous-time version of the method is controlled uniformly across $[0, T]$, even though the discrete method does not use information about the precise location of the jumps. We also mention that the $\theta = 0$ case is covered in [3]. Having established that the theta method has acceptable finite time convergence, in the next section we consider long-time stability.

3. Mean-Square Stability

This section looks at stability in mean-square. Here, we are concerned with the regime where $t \to \infty$ with Δt fixed. Following the approach used in the deterministic case, we examine the behaviour of the method on a linear test equation and ask: *for what Δt does the method share the stability/instability of the test equation?*

3.1. Test Equation. We consider the case where $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ in (1) are scalar and multiplicatively linear, that is

$$dX(t) = \mu X(t^-)dt + \sigma X(t^-)dW(t) + \gamma X(t^-)dN(t), \tag{16}$$

where μ, σ and γ are real constants. We assume $X(0) \neq 0$ with probability one.

We note that (16) is a natural generalization of both the classical linear equation used to study stability of methods for deterministic ODEs, [8], and the multiplicative noise SDE that has been used to study linear stability of methods for SDEs, [1, 9, 23]. We also remark that (16) has been proposed as a model in mathematical finance, [5, 13].

The problem (16) has solution

$$X(t) = X(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}\,(1+\gamma)^{N(t)}\,, \tag{17}$$

see, for example, [5, 13]. For $\gamma \neq -1$, using $\mathbb{E}\left((1+\gamma)^{N(t)}\right) = e^{\lambda t\gamma(2+\gamma)}$, see, for example, [13, Ex. 39], we have

$$\begin{aligned}
\mathbb{E}X(t)^2 &= \mathbb{E}\left(X(0)^2\right)e^{2(\mu-\frac{1}{2}\sigma^2)t}\mathbb{E}\left(e^{2\sigma W(t)}\right)\mathbb{E}\left((1+\gamma)^{2N(t)}\right)\\
&= \mathbb{E}\left(X(0)^2\right)e^{\left(2\mu+\sigma^2+\lambda\gamma(2+\gamma)\right)t}.
\end{aligned}$$

Hence, mean-square stability (of the zero solution) for $\gamma \neq -1$ in (16) may be characterized by

$$\lim_{t\to\infty}\mathbb{E}X(t)^2 = 0 \quad \Leftrightarrow \quad 2\mu+\sigma^2+\lambda\gamma(2+\gamma) < 0, \tag{18}$$

and it is straightforward to check that (18) remains true when $\gamma = -1$.

A few comments are in order regarding the parameters in (18).

- The jump intensity $\lambda > 0$.
- The sign of the diffusion parameter does not matter, it will only appear in the form σ^2 when we look at mean-square stability.
- The jump parameter γ may be positive or negative. We see that γ in (18) appears through the factor $\gamma(2+\gamma)$, which is symmetric about $\gamma = -1$. This is intuitively reasonable, as a jump at time τ changes the solution from $X(\tau^-)$ to $X(\tau)$, where $X(\tau) - X(\tau^-) = \gamma X(\tau^-)$; that is, $X(\tau) = (1+\gamma)X(\tau^-)$. Only the absolute value of $1+\gamma$ matters for mean-square stability.
- The drift parameter μ may be positive or negative. It is interesting to note that in the non-jump case ($\gamma = 0$) we must have $\mu < 0$ in order for the problem to be mean-square stable. We still need $\mu < 0$ for mean-square stability when we introduce a positive jump coefficient $\gamma > 0$. However, with $\gamma < 0$, the amplification factor of the jump can be less than one in modulus and it is then possible to have mean-square stability when $\mu > 0$. (So the jump term may stabilize the problem.) For example, $\mu = 1$, $\lambda = 4$, $\sigma = 0$ gives stability for all γ between $-1-\sqrt{\frac{1}{2}}$ and $-1+\sqrt{\frac{1}{2}}$.

Our aim is now to analyze the corresponding mean-square stability property, $\lim_{n\to\infty}\mathbb{E}X_n^2 = 0$, for the theta method applied to (16). For brevity we will refer to "stability" rather than "mean-square stability (of the zero solution)" and we will make reference to the "deterministic case", which is $\sigma = \gamma = 0$ with $X(0) \in \mathbb{R}$, and the "non-jump" case, which is $\gamma = 0$.

3.2. Stability of the Theta Method. Applying the theta method (2) to (16) gives the recurrence

$$Y_{n+1} = Y_n + (1-\theta)\Delta t\mu Y_n + \theta\Delta t\mu Y_{n+1} + \sigma Y_n \Delta W_n + \gamma Y_n \Delta N_n. \tag{19}$$

For the implicit case, $\theta > 0$, we require $\mu\Delta t\theta \neq 1$ in order for the method to be well defined, as in the deterministic case. The Brownian increments satisfy $\mathbb{E}\Delta W_n = 0$ and $\mathbb{E}\Delta W_n^2 = \Delta t$, and the Poisson increments satisfy $\mathbb{E}\Delta N_n = \lambda\Delta t$ and $\mathbb{E}(\Delta N_n)^2 = \lambda\Delta t(1+\lambda\Delta t)$, see, for example, [11, 13]. Hence, using the independence of the increments, we find that

$$\begin{aligned}(1-\theta\Delta t\mu)^2\mathbb{E}(Y_{n+1}^2) &= \mathbb{E}(Y_n^2)\left(1 + \Delta t[2(1-\theta)\mu + \sigma^2 + \lambda\gamma(2+\gamma)]\right. \\ &\qquad \left. + \Delta t^2[(1-\theta)^2\mu^2 + 2(1-\theta)\mu\gamma\lambda + \lambda^2\gamma^2]\right).\end{aligned}$$

This leads to a simple stability characterization for the theta method:

$$\lim_{n\to\infty} \mathbb{E}\left(Y_n^2\right) = 0 \Leftrightarrow \Delta t\,(\mu + \lambda\gamma)\,(\mu(1-2\theta) + \lambda\gamma) < -\left(2\mu + \sigma^2 + \lambda\gamma(2+\gamma)\right). \tag{20}$$

Note that the right-hand side involves the mean-square stability term in (18) for the underlying problem.

3.3. Euler–Maruyama. Taking $\theta = 0$ in (19) gives the explicit Euler–Maruyama (EM) method, which has been studied by, for example, [3, 15]. We have the following result immediately.

Lemma 3.1. *For the EM method applied to (16),*

1: *problem stable $\Rightarrow$ EM stable for*

$$\Delta t < \frac{|2\mu + \sigma^2 + \lambda\gamma(2+\gamma)|}{(\mu + \gamma\lambda)^2},$$

2: *problem unstable $\Rightarrow$ EM unstable for all $\Delta t > 0$.*

Proof. Note that if the problem is stable then we cannot have $\mu = -\gamma\lambda$ (see the proof of part 2). Thus, for part 1., we may write the mean-square stability condition (20) as

$$\Delta t < -\frac{2\mu + \sigma^2 + \lambda\gamma(2+\gamma)}{(\mu + \gamma\lambda)^2}, \tag{21}$$

and the result follows. For $\mu \neq -\gamma\lambda$, the condition (21) continues to determine EM stability, and part 2. follows. In the remaining case where $\mu = -\gamma\lambda$, the SDE is unstable because the right-hand side of (18) involves $2\mu + \sigma^2 + \lambda\gamma(2+\gamma) = \sigma^2 + \lambda\gamma^2$. In this case the stability condition in (20) becomes $\Delta t \times 0 < -(\sigma^2 + \lambda\gamma^2)$, which never holds, confirming part 2. $\square$

Note that this result is a clean generalization of the deterministic and non-jump cases, see, for example, [9, Theorem 4.1]. The EM method has a bounded stability region that is strictly contained in that of the differential equation.

3.4. General θ. Theorem 4.1 of [9] showed that in the non-jump case there is a well-defined inclusion: for $\theta < \frac{1}{2}$, the method's stability region is strictly contained in that of the problem, for $\theta = \frac{1}{2}$, the two regions coincide, and for $\theta > \frac{1}{2}$, the method's stability region strictly contains that of the problem, The next theorem shows that these inclusions do not extend to the jump case. Also, for all $0 < \theta \leq 1$ the theta method stability region is unbounded. In particular, the theta method can be stable for all Δt in cases where the problem is stable, and we can find cases where the problem is unstable but the method is stable for all $\Delta t > \epsilon$.

Theorem 3.2. *For any* $0 < \theta \leq 1$

1: *there exist* $\{\mu, \sigma, \lambda, \gamma\}$ *for which the problem is stable and the theta method is stable for all* $\Delta t > 0$,

2: *given any* $\epsilon > 0$, *there exist* $\{\mu, \sigma, \lambda, \gamma\}$ *for which the problem is unstable, yet the theta method is stable for all* $\Delta t > \epsilon$.

Proof. Part 1. For $\frac{1}{2} \leq \theta$, the result follows from the standard deterministic stability theory. For $0 < \theta < \frac{1}{2}$ take

$$\mu = -1, \qquad \sigma = 0, \qquad \lambda = \frac{(1-\theta)^2}{\theta}, \qquad \gamma = \frac{\theta}{1-\theta}.$$

We have $2\mu + \sigma^2 + \lambda\gamma(2+\gamma) = -\theta$, so the problem is stable by (18). The stability condition in (20) reduces to $\Delta t(-\theta)(\theta) < \theta$, which holds for all $\Delta t > 0$.

Part 2. For $\frac{1}{2} < \theta$, the result follows from the standard deterministic stability theory. For $0 < \theta \leq \frac{1}{2}$ take

$$\mu = -1, \qquad \sigma = \sqrt{\theta + \epsilon\theta^2}, \qquad \lambda = \frac{(1-\theta)^2}{\theta}, \qquad \gamma = \frac{\theta}{1-\theta}.$$

We have $2\mu + \sigma^2 + \lambda\gamma(2+\gamma) = \epsilon\theta^2$, so the problem is unstable. Yet the stability condition in (20) reduces to $\Delta t(-\theta)(\theta) < -\epsilon\theta^2$, so the method is stable for all $\Delta t > \epsilon$. □

In the non-jump case, the theta method stability is monotonic with respect to θ, in the sense that (with $\gamma = 0$), given any $\{\mu, \sigma, \lambda\}$ and Δt, if the method is stable for $\theta = \theta^\star$ then the method is also stable for $\theta > \theta^\star$. Such a monotonicity property does not hold in the jump case. To see this, take $\mu = 1$, $\sigma = 0$, $\lambda = 4$ and $\gamma = -\frac{1}{2}$. The theta method stability constraint (20) is then $\Delta t < 1/(1+2\theta)$, which becomes more stringent as θ increases.

3.5. A-stability. The concept of A-stability for a numerical method may be summarized as "problem stable $\Rightarrow$ method stable for all Δt". In the deterministic and non-jump cases, the theta method is A-stable if and only if $\theta \geq \frac{1}{2}$, [9, Theorem 4.1]. The result extends to the jump case if we constrain the jump parameter to be non-negative.

Theorem 3.3. *Under the restriction* $\gamma \geq 0$, *the theta method is A-stable for* $\theta \geq \frac{1}{2}$. *In other words, given* $\{\mu, \sigma, \lambda, \gamma\}$ *with* $\gamma \geq 0$ *for which the stability condition in (18) holds, the theta method with* $\theta \geq \frac{1}{2}$ *is stable for all* $\Delta t > 0$.

Proof. Consider $\theta \geq \frac{1}{2}$. Suppose the stability condition in (18) holds and $\gamma \geq 0$. Then $\mu < 0$, $\mu + \lambda\gamma < 0$ and $\mu(1-2\theta) + \lambda\gamma \geq 0$. It follows that the stability condition in (20) holds for any $\Delta t > 0$. □

If we allow $\gamma < 0$, then the A-stability property no longer holds.

Theorem 3.4. *For $\gamma \in \mathbb{R}$, the theta method is* ***not*** *A-stable for any $\frac{1}{2} \leq \theta \leq 1$. In fact, there exist $\{\mu, \sigma, \lambda, \gamma\}$ satisfying the stability condition in (18) with the property that for any $\frac{1}{2} \leq \theta \leq 1$ there exists a finite Δt_θ such that the theta method is unstable for all $\Delta t > \Delta t_\theta$.*

Proof. The proof hinges on the fact that when γ is allowed to be negative, we can get stability of the problem when $\mu > 0$. For example, take $\mu = 1$, $\lambda = 4$, $\sigma = 0$ and $\gamma = -1$. In this case we have $\mu + \lambda\gamma < 0$ and $\mu(1-2\theta) + \lambda\gamma < 0$. It follows that the stability condition in (20) becomes $\Delta t < 2/(9 + 6\theta)$, and the result is proved. □

3.6. Jump Symmetry. We observed in section 3.1 that, from (18), the stability of (16) is symmetric about $\gamma = -1$; that is, changing γ to $-2 - \gamma$ does not affect the mean-square stability. It is interesting to note that numerical methods do not generally preserve this symmetry. To formalize this we make the following definition.

Definition 3.5. *A numerical method applied to (16) is said to be* jump symmetric *if, whenever stable (unstable) for $\{\mu, \sigma, \lambda, \gamma, \Delta t\}$ it is also stable (unstable) for $\{\mu, \sigma, \lambda, -2 - \gamma, \Delta t\}$.*

Lemma 3.6. *The theta method is not jump symmetric for any θ.*

Proof. The right-hand side of the stability condition in (20) is unchanged under $\gamma \mapsto -2 - \gamma$. However, it is readily confirmed that this is not the case for the product $(\mu + \lambda\gamma)(\mu(1 - 2\theta) + \lambda\gamma)$ on the left-hand side. □

4. Mean-Square Stability for the Weak Theta Method

An alternative version of the theta method that uses increments that are cheaper to simulate is given by

$$Y_{n+1} = Y_n + (1-\theta) f(Y_n)\Delta t + \theta f(Y_{n+1})\Delta t + g(Y_n)\widehat{\Delta W}_n + h(Y_n)\widehat{\Delta N}_n, \tag{22}$$

where

$$\mathbb{P}\left(\widehat{\Delta W}_n = \sqrt{\Delta t}\right) = \frac{1}{2} = \mathbb{P}\left(\widehat{\Delta W}_n = -\sqrt{\Delta t}\right), \tag{23}$$

and

$$\mathbb{P}\left(\widehat{\Delta N}_n = 0\right) = 1 - \lambda\Delta t, \qquad \mathbb{P}\left(\widehat{\Delta N}_n = 1\right) = \lambda\Delta t. \tag{24}$$

For $\theta = 0$ and $g(\cdot) = 0$ this is the classical weak Euler–Maruyama method, [11], and in the jump case with $\theta = 0$ this is the weak stochastic Taylor method of order $\frac{1}{2}$, called WST1 in [14, equation (2.2)] . We note that the probabilities in (24) must be non-negative, and hence the method is only defined for $\lambda\Delta t \leq 1$. This places a parameter-dependent constraint on the stepsize, and hence the method cannot be A-stable.

Changing from ΔW_n in the strong method (2) to $\widehat{\Delta W}_n$ in the weak method (22) does not affect the mean-square stability properties, since $\mathbb{E}(\widehat{\Delta W}_n) = 0$ and $\mathbb{E}(\widehat{\Delta W}_n^2) = \Delta t$, as before. (Similarly, matching higher moments of the Brownian increments, as with $\widehat{\Delta W}_n$ such that $\mathbb{P}(\widehat{\Delta N}_n = \sqrt{3\Delta t}) = 1/6 = \mathbb{P}(\widehat{\Delta N}_n = -\sqrt{3\Delta t})$ and $\mathbb{P}(\widehat{\Delta N}_n = 0) = 2/3$, has no effect on mean-square stability.) However, changing from ΔN_n to $\widehat{\Delta N}_n$ three times *does* make a difference, because $\mathbb{E}(\widehat{\Delta N}_n) = \lambda\Delta t$, as before, but $\mathbb{E}(\widehat{\Delta N}_n^2) = \lambda\Delta t$, which differs from the correct version by $O(\Delta t^2)$.

Since we are concerned with the "Δt fixed, $n \to \infty$" regime, this difference will be significant. Working through the algebra, we find that (20) changes to

$$\lim_{n\to\infty} \mathbb{E}\left(Y_n^2\right) = 0$$

$$\Leftrightarrow \quad \Delta t\left((\mu+\lambda\gamma)(\mu(1-2\theta)+\lambda\gamma)-\lambda^2\gamma^2\right) < -\left(2\mu+\sigma^2+\lambda\gamma(2+\gamma)\right). \tag{25}$$

The next result is immediate.

Theorem 4.1. *For any $0 \le \theta \le 1$ and any $\{\mu, \sigma, \gamma, \lambda, \Delta t\}$ (with $\lambda\Delta t \le 1$):*

1.: *strong theta method stable $\Rightarrow$ weak theta method stable, and hence*
2.: *weak theta method unstable $\Rightarrow$ strong theta method unstable.*

Proof. A proof follows directly from (20) and (25). □

Theorem 4.1 shows that (in the case where $\lambda\Delta t \le 1$) the weak method is "more stable" than the strong method. In particular, the "A-stability" result for $\frac{1}{2} \le \theta \le 1$ when $\gamma \ge 0$ in Theorem 3.3 that we derived for the strong method also holds for the weak method, except that we need $\lambda\Delta t \le 1$.

For $\gamma \in \mathbb{R}$ we have the following lemma, giving a positive result for the $\theta = 1$ (backward Euler) weak method that holds for all $\gamma \in \mathbb{R}$. This lemma gives "A-stability" except for the restriction $\lambda\Delta t \le 1$.

Lemma 4.2. *For $\theta = 1$, if $\lambda\Delta t \le 1$ then problem stable $\Rightarrow$ weak theta method stable.*

Proof. For $\theta = 1$, the weak theta method stability condition in (25) reduces to

$$-\Delta t\mu^2 < -\left(2\mu+\sigma^2+\lambda\gamma(2+\gamma)\right). \tag{26}$$

If the problem is stable then, from (18), the right-hand side of this inequality is positive; whence the inequality is satisfied for all $\Delta t > 0$. □

4.1. Jump Symmetry. It is clear from (26) that the $\theta = 1$ WTM has the jump symmetry property in Definition 3.5. This turns out to be the only case.

Lemma 4.3. *The weak theta method is jump symmetric if and only if $\theta = 1$.*

Proof. From (25), the weak theta method has jump symmetry if and only if the factor $(\mu+\lambda\gamma)(\mu(1-2\theta)+\lambda\gamma)-\lambda^2\gamma^2$ is unchanged under $\gamma \mapsto -2-\gamma$. It is readily confirmed that this reduces to $\theta = 1$. □

4.2. Plotting Stability Regions. Stability of the test equation (16) depends on three quantities: the drift factor, μ, the diffusion factor, σ, and the factor $\lambda\gamma(2+\gamma)$ that could be called the jump strength. Generally, for the methods considered here, because of the lack of jump symmetry, the numerical stability depends on the four parameters μ, σ, λ and γ, plus the stepsize Δt. For this reason, it does not seem worthwhile, in general, to attempt to visualize stability regions; that is, to display regions of the parameter space where stability occurs. However, for the jump symmetric $\theta = 1$ weak theta method, it is possible to get some insight this way, and the pictures form a natural generalization of that in [9, Figure 4.1].

Let $x = \Delta t\mu$, $y = \Delta t\sigma^2$ and $z = \Delta t\lambda\gamma(2+\gamma)$. Note that this forces $y \ge 0$. The stability condition in (18) may then be written

$$y < -2x - z \tag{27}$$

and the $\theta = 1$ weak theta method stability condition in (26) may be written

$$y < x^2 - 2x - z. \tag{28}$$

If we consider slices through (x, y, z) space with z fixed, then there are three distinct cases. Figure 1 illustrates the behaviour by showing finite portions of the regions. Here, horizontal hashing shows the region (27) where the problem is stable and vertical hashing shows the region (28) where the numerical method is stable. For each fixed z, the problem stability region (27) consists of a wedge and the numerical method stability region (28) consists of the region below a parabola. For $z > -1$, the parabola cuts the x axis at two distinct points, $x = 1 \pm \sqrt{1+z}$, and hence the numerical method stability region consists of two unconnected sets. The stability wedge (27) lies strictly below the left-hand set. For $z = -1$, the parabola touches the x axis at $x = 1$ and for $z < -1$, the parabola lies strictly above the x axis. In all cases the "A-stability" property is clear—the numerical method has a stability region that contains the wedge.

Note that for fixed values of μ, σ, and $\lambda\gamma(2+\gamma)$, varying the timestep Δt corresponds to moving along a ray through the origin in (x, y, z) space.

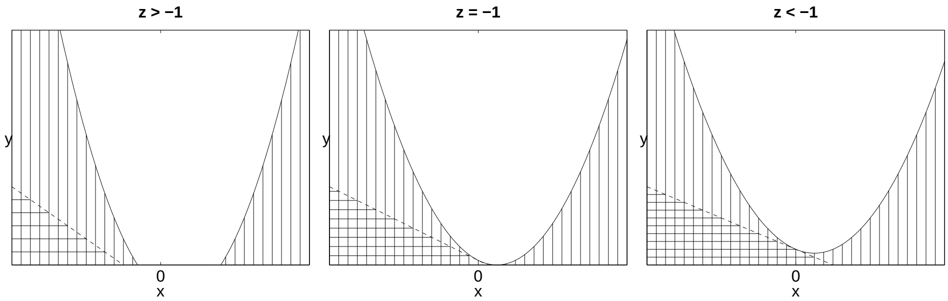

FIGURE 1. Mean-square stability regions for $z > -1$ (left), $z = -1$ (middle) and $z < -1$ (right). Vertical hashing is stability region (28) for weak theta method with $\theta = 1$. Horizontal hashing is stability region (27) for the SDE.

5. Conclusions

The main aim of this work is to show that it is feasible to use implicit methods for jump-diffusion problems, and that this leads to an improvement in mean-square stability properties. There are many directions in which this work could be extended, including (a) the derivation of higher order implicit methods, (b) the search for an implicit method that is A-stable for all $\gamma \in \mathbb{R}$ (c.f. Theorem 3.4) and (c) the analysis of long-time dynamics of nonlinear jump-diffusion simulations.

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