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ON THE BOUNDEDNESS OF ASYMPTOTIC STABILITY REGIONS FOR THE STOCHASTIC THETA METHOD*

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Abstract.

The stochastic theta method gives a computational procedure for simulating ordinary stochastic differential equations. The method involves a free parameter, θ. Here, we characterise the precise value of θ beyond which the region of linear asymptotic stability of the method becomes unbounded. The cutoff point is seen to differ from that in the deterministic case. Computations that suggest further results are also given.

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1 Introduction.

We consider the numerical solution of autonomous scalar Itô stochastic differential equations (SDEs)

$$\text{(1.1)} \qquad \mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), \quad t > 0, \quad X(0) = X_0,$$

driven by the standard Wiener process $W(t)$ [4, 5]. In particular, we study the stochastic theta method (STM) class, which is also known as the family of implicit Euler methods [3, 4, 6]. Applying the STM with fixed stepsize $\Delta t > 0$ produces approximations $X_n \approx X(t_n)$, with $t_n = n\Delta t$, of the form

$$\text{(1.2)} \qquad X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + \Delta t^{\frac{1}{2}} g(X_n) V_n.$$

Here each V_n is an independent Normal$(0,1)$ random variable, so that $\Delta t^{\frac{1}{2}} V_n$ represents the Brownian path increment $W(t_{n+1}) - W(t_n)$. Specifying a value of θ determines a particular STM. We restrict ourselves to the standard range $\theta \in [0,1]$. For $\theta = 0$ we recover the explicit Euler–Maruyama method and for $\theta > 0$ the method is implicit.

We focus on the stability of the method (1.2) on the linear, multiplicative noise, test equation

$$\text{(1.3)} \qquad \mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad t > 0, \quad X(0) = X_0,$$

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where $\lambda, \mu \in \mathbb{R}$ are constants and we assume that $X_0 \neq 0$ with probability one. For our stability analysis, we may also assume without loss of generality that $\mu \geq 0$. The SDE (1.3) may be said to be *asymptotically stable* if $\lim_{t\to\infty} X(t) = 0$ with probability one. It is known [5] that asymptotic stability for (1.3) is equivalent to

$$\lambda - \tfrac{1}{2}\mu^2 < 0. \tag{1.4}$$

Analogously, the numerical solution X_n arising when the STM (1.2) is applied to (1.3) may be said to be asymptotically stable if $\lim_{n\to\infty} X_n = 0$ with probability one, [3].

Applying the method (1.2) to the test problem (1.3) produces the recurrence

$$X_{n+1} = (a + bV_n)X_n, \tag{1.5}$$

with

$$a := \frac{1 + (1-\theta)x}{1 - \theta x}, \qquad b := \frac{\sqrt{y}}{1 - \theta x}, \tag{1.6}$$

where we have defined $x := \Delta t \lambda$ and $y := \Delta t \mu^2$, and we have assumed that $1 - \theta x \neq 0$. We will define the stability region $\mathrm{S}^\theta \subset \mathbb{R}^2$ by

$$\mathrm{S}^\theta := \{(x, y) : x \neq 1/\theta, y \geq 0 \text{ and STM is asymptotically stable}\}.$$

Note that in the (x, y) variables the SDE asymptotic stability condition (1.4) becomes $y > 2x$.

From a numerical analysis perspective, it is of interest to compare the asymptotic stability properties of the SDE and numerical solution. So far, mean-square stability studies have proved more popular [2, 4, 6], largely because the analysis is more tractable. However, asymptotic stability is at least as relevant as mean-square stability in many modelling contexts [5]. Higham [3] recently gave a characterisation of the asymptotic stability region S^θ and our main purpose here is to show how that can be used to determine precisely when S^θ is bounded. We also present some numerical computations that suggest further results.

2 Stability result.

The following characterisation of S^θ comes directly from [3, Lemma 5.1].

LEMMA 2.1.

$$(x, y) \in \mathrm{S}^\theta \quad \Leftrightarrow \quad \mathbb{E}\left(\log|a + bV_n|\right) < 0, \tag{2.1}$$

where $\mathbb{E}(\cdot)$ *denotes the expected value.*

We note that for $a \neq 0$, the inequality in (2.1) may be written

$$\log|a| + \gamma(c) < 0, \tag{2.2}$$

where $c := b/a$ and $\gamma(c) := \mathbb{E}\left(\log|1 + cV_n|\right)$, so that

$$\gamma(c) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \log|1 + cs| e^{-\frac{1}{2}s^2} \, ds. \tag{2.3}$$

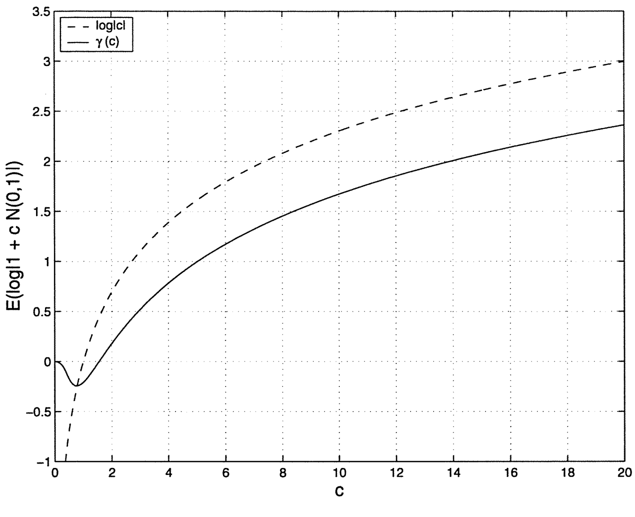

Figure 2.1: Plot of $\gamma(c) := \mathbb{E}(\log|1+cV_n|)$ (solid) and $\log|c|$ (dashed) for $c \geq 0$.

We also note that $\gamma(c) = \gamma(-c)$. A plot of $\gamma(c)$ for $c \geq 0$ is given in solid linetype in Figure 2.1. We have added a plot of $\log|c|$ to the picture, since the following asymptotic result is used in our analysis.

LEMMA 2.2. *As* $|c| \to \infty$, $\gamma(c) - \log|c|$ *tends to the constant*

$$R := \sqrt{\frac{2}{\pi}} \int_0^\infty s^2(\log s - 1)e^{-\frac{1}{2}s^2}\,ds \approx -0.635. \tag{2.4}$$

PROOF. We have

$$\gamma(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\log|c| + \log|s + \frac{1}{c}|)e^{-\frac{1}{2}s^2}\,ds = \log|c| + R(c),$$

where

$$\begin{aligned} R(c) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\log|s + \frac{1}{c}|)e^{-\frac{1}{2}s^2}\,ds \\ &= \frac{1}{\sqrt{2\pi}} \Big(\int_{-1/c}^{\infty} \log(s + \frac{1}{c})e^{-\frac{1}{2}s^2}\,ds + \int_{1/c}^{\infty} \log(s - \frac{1}{c})e^{-\frac{1}{2}s^2}\,ds \Big). \end{aligned}$$

Hence,

$$\lim_{c\to\infty} R(c) = \frac{2}{\sqrt{2\pi}} \int_0^\infty \log s\, e^{-\frac{1}{2}s^2}\,ds = \sqrt{\frac{2}{\pi}} \int_0^\infty s(s\log s - s)\, e^{-\frac{1}{2}s^2}\,ds.$$

□

Our concern here is to determine when the STM has an unbounded asymptotic stability region, S^θ. (We say that S^θ is unbounded if, given any $\rho > 0$, there exist (x, y) such that $(x, y) \in S^\theta$ with $|x| + |y| > \rho$.) First, we note that taking $y = 0$ (that is, $\mu = 0$ in (1.3)), we recover the traditional linear stability interval of the theta method for ordinary differential equations (ODEs), [1]. Since this interval is infinite for $\theta \geq \frac{1}{2}$, we immediately conclude that S^θ is unbounded for $\theta \geq \frac{1}{2}$. However, the ODE theory does not tell us whether S^θ is unbounded for $\theta < \frac{1}{2}$. The following theorem fully resolves the boundedness question and shows that the SDE case is genuinely different.

THEOREM 2.3. S^θ *is unbounded* $\Leftrightarrow \theta \geq \frac{1}{1+e^D}$, *where* $D := -\min_{c\in\mathbb{R}} \gamma(c)$.

PROOF. We first consider the case $\theta \geq \frac{1}{1+e^D}$. Let $c^\star > 0$ be the point where $\gamma(c^\star) = \gamma(-c^\star) = -D$. Let $\sqrt{y} = -c^\star(1 + (1-\theta)x)$ and $x \ll 0$. The stability condition

$$\log\left|\frac{1+(1-\theta)x}{1-\theta x}\right| - D < 0 \tag{2.5}$$

from (2.2) then reduces to

$$x(\theta(1+e^D) - 1) < 1 + e^D.$$

By inspecting the signs of the individual factors, we see that this inequality holds for all $x \ll 0$. Hence, we have constructed arbitrarily large (x, y) in S^θ.

Now we consider the case $0 < \theta < \frac{1}{1+e^D}$. First, we suppose that $x \neq -1/(1-\theta)$, so $a \neq 0$ in (1.6). Since $\gamma(c) \geq -D$ for all $c \in \mathbb{R}$, the condition (2.5) is necessary for stability. For $1 - \theta x > 0$, this necessary condition implies that $x > -(1+e^D)/(1-\theta(1+e^D))$. On the other hand, for $1 - \theta x < 0$ the necessary condition implies that $x < -(1+e^D)/(1-\theta(1+e^D))$, which contradicts $1-\theta x < 0$. Overall, we find that

$$-\frac{e^D+1}{1-\theta(e^D+1)} < x < \frac{1}{\theta} \tag{2.6}$$

is necessary for stability, and hence x must be bounded. Now, from Lemma 2.2 we see that for large y

$$\log\left|\frac{1+(1-\theta)x}{1-\theta x}\right| + \gamma\left(\frac{\sqrt{y}}{1+(1-\theta)x}\right) \approx \log\left|\frac{\sqrt{y}}{1-\theta x}\right| + R.$$

Hence, in order for the stability condition (2.2) to hold for large y we must have

$$\log\left|\frac{\sqrt{y}}{1-\theta x}\right| + R < 1,$$

say, which implies

$$\sqrt{y} < |1-\theta x| e^{1-R}. \tag{2.7}$$

It follows that stability cannot hold for arbitrarily large y. The case where $x = -1/(1-\theta)$ can also be dealt with by (2.7).

Finally, for $\theta = 0$, boundedness of x follows directly from (2.2). Then, from (2.7), stability for arbitrarily large y leads to the contradictory inequality $\sqrt{y} < e^{1-R}$. □

We found numerically that $D \approx 0.2454$, $c^\star \approx 0.7695$ and the cutoff value for θ is $\frac{1}{1+e^D} \approx 0.4390$. The left-hand picture in Figure 2.2 gives the boundary of S^θ for $\theta = 0, 0.15, 0.25, 0.3$. These boundaries were computed via a contour plotting routine by regarding them as zero-level curves of the function $\log|a| + \gamma(c)$ in (2.2), with $\gamma(c)$ approximated using quadrature. The boundary $y = 2x$ of the SDE stability region is also shown. We note that in these pictures the S^θ regions increase monotonically with θ, and all are strictly contained in the SDE stability region. The right-hand picture in Figure 2.2 shows S^θ for $\theta = 0.45$, a value between $\frac{1}{1+e^D}$ and $\frac{1}{2}$. Note the axis scaling has changed. In this case the intersection of S^θ with the x axis is finite, but, from Theorem 2.3, the region is unbounded. The proof of Theorem 2.3 showed that the curve $\sqrt{y} = -c^\star(1 + (1-\theta)x)$ lies in S^θ for $x \ll 0$. This curve is plotted in the picture as a dash-dotted line in order to confirm this behaviour. By mimicking the analysis used in the proof of Theorem 2.3 it can also be shown in this case that the curve $\sqrt{y} = c^\star(1 + (1-\theta)x)$ lies in S^θ for $x \gg 0$. This curve is plotted in the picture as a dashed line, and we see that it is indeed contained in S^θ for large x.

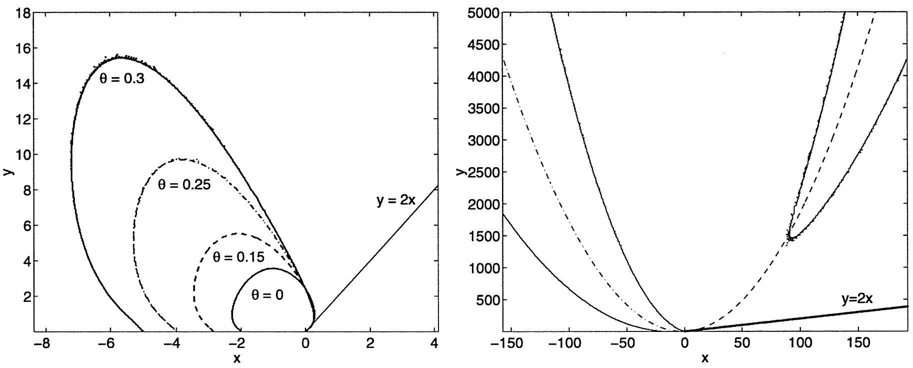

Figure 2.2: Left: boundary of S^θ for $\theta = 0, 0.15, 0.25, 0.3$. Right: portion of boundary of S^θ for $\theta = 0.45$.

3 Milstein version.

The strong order of the STM can be increased from $\frac{1}{2}$ to 1 by adding Milstein's correction to the stochastic increment [4]. Applying the corresponding method to the test equation (1.3) leads to the recurrence $X_{n+1} = (\widehat{a} + \widehat{b}V_n + \widehat{c}V_n^2)X_n$, where

$$\widehat{a} := \frac{1 + (1-\theta)x - y/2}{1 - \theta x}, \quad \widehat{b} := \frac{\sqrt{y}}{1-\theta x} \quad \text{and} \quad \widehat{c} := \frac{y/2}{1-\theta x}. \tag{3.1}$$

Mean-square stability properties of this recurrence were studied in [2]. From [3, Lemma 5.1], asymptotic stability is characterised by

$$\mathbb{E}(\log|\widehat{a} + \widehat{b}V_n + \widehat{c}V_n^2|) < 0.$$

Since the Milstein version is higher order than the basic STM, we may expect the resulting stability region to be smaller. However, Figure 3.1, which was computed in a similar way to Figure 2.2, shows that this is not the case. For the explicit Euler versions, where $\theta = 0$, the left-hand picture shows that the Milstein method asymptotic stability region is not strictly contained in the Euler–Maruyama region, and seems to be drawn towards the underlying SDE stability boundary $y = 2x$. The right-hand picture in Figure 3.1 shows further asymptotic stability boundaries for the Milstein version with $\theta = 0, 0.15.0.25, 0.3$. We see that monotonicity with respect to θ no longer holds.

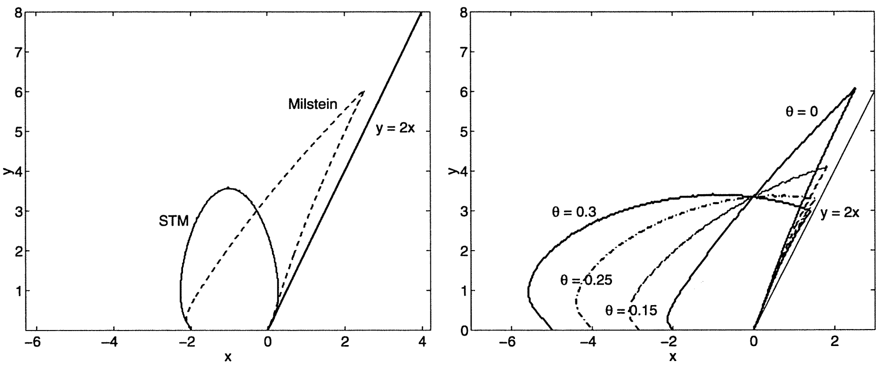

Figure 3.1: Left: $\theta = 0$ versions of STM and Milstein. Right: Milstein for $\theta = 0, 0.15.0.25, 0.3$.

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REFERENCES

1. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II, Stiff and Differential-Algebraic Problems*, Springer Verlag, Berlin, 2nd ed., 1996.
2. D. J. Higham, *A-stability and stochastic mean-square stability*, BIT, 40 (2000), pp. 404–409.
3. D. J. Higham, *Mean-square and asymptotic stability of numerical methods for stochastic ordinary differential equations*, SIAM J. Numer. Anal., 38 (2000), pp. 753–769.
4. P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1992.
5. X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, 1997.
6. Y. Saito and T. Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33 (1996), pp. 2254–2267.