# Elliptic Springer Theory And Singularities 

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#### Abstract

We construct an elliptic Grothendieck-Springer resolution as a simultaneous log resolution of algebraic stacks. Our construction extends earlier work from the stack of semistable principal bundles on an elliptic curve to the stack of all principal bundles. We use elliptic analogues of transversal slices to study the geometry of the unstable part of our resolution in codimension $\leq 2$, and give detailed case by case calculations of the corresponding surfaces in all types.


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## Chapter 1

## Introduction

Let $G$ be a simply connected simple algebraic group over an algebraically closed field $k$. Classically, the Springer theory of $G$ is the study of various features of the additive and multiplicative adjoint quotient maps

$$
\begin{equation*}
\chi^{\text {add }}: \mathfrak{g} \longrightarrow \mathfrak{g} / / G=\operatorname{Spec} k[\mathfrak{g}]^{G} \quad \text { and } \quad \chi^{m u l}: G \longrightarrow G / / G=\operatorname{Spec} k[G]^{G}, \tag{1.0.1}
\end{equation*}
$$

where $\mathfrak{g}=\operatorname{Lie}(G)$ is the Lie algebra of $G$ and $G$ acts on $\mathfrak{g}$ (resp., $G$ ) via the adjoint representation (resp., by conjugation).

One of the most important of these features is that both $\chi^{\text {add }}$ and $\chi^{m u l}$ are flat with some singular fibres, and admit simultaneous resolutions of singularities after pulling back along finite coverings of the targets. More precisely, there are isomorphisms $\mathfrak{g} / / G \cong \mathfrak{t} / / W$ and $G / / G \cong T / / W$, due to Chevalley, and commutative diagrams

where $T \subseteq B \subseteq G$ are a maximal torus and Borel subgroup respectively, $\mathfrak{t}=\operatorname{Lie}(T)$ and $\mathfrak{b}=\operatorname{Lie}(B)$ their Lie algebras, and $W=N_{G}(T) / T$ is the Weyl group. The diagrams (1.0.2) are called the additive and multiplicative Grothendieck-Springer resolutions. Assuming that $k$ has characteristic 0 in the additive case, they are simultaneous resolutions in the sense that $\tilde{\chi}^{a d d}$ and $\tilde{\chi}^{m u l}$ are smooth, $\psi^{a d d}$ and $\psi^{m u l}$ are proper, and for all $t \in \mathfrak{t}($ resp., $T)$, the morphism $\left(\tilde{\chi}^{\text {add }}\right)^{-1}(t) \rightarrow\left(\chi^{\text {add }}\right)^{-1}(t W)$ (resp., $\left.\left(\tilde{\chi}^{m u l}\right)^{-1}(t) \rightarrow\left(\chi^{m u l}\right)^{-1}(t W)\right)$ is a resolution of singularites.

The adjoint quotient maps (1.0.1) and their Grothendieck-Springer resolutions (1.0.2) are rich and interesting objects, with connections to many areas of mathematics. For example, in representation theory, the cohomology of the fibres of $\psi^{\text {add }}$ give natural representations of the Weyl group $W$ [S3], [S4], [S1], the multiplicative Grothendieck-Springer map $\psi^{m u l}$ plays an essential role in Lusztig's theory of character sheaves for representations of finite groups of Lie type [L3] (see also [BZN2, §1.3.1]), and the Belinson-Bernstein localisation theorem [BB] relating representations of $\mathfrak{g}$ to twisted $\mathcal{D}$-modules on the flag variety $G / B$ can be interpreted in terms of a quantisation of the additive Grothendieck-Springer resolution [BZN1] [MN]. The Grothendieck-Springer resolutions are also of interest in algebraic geometry, as they give a direct link between algebraic groups and du Val singularities of algebraic surfaces as follows.

For simplicity, assume that $k=\mathbb{C}$ and restrict attention to the additive case. Then [S1, §2.4] there exist closed subvarieties $Z \subseteq \mathfrak{g}$ of dimension $l+2$, where $l=\operatorname{dim}(T)$, such that $Z$ is transverse to every $G$-orbit in $\mathfrak{g}$, and contains a single "subregular" element with stabiliser group of dimension $l+2$. Given such a subregular transversal slice $Z$, the additive

Grothendieck-Springer resolution pulls back to a simultaneous resolution

which is smoothly equivalent to an open set in (1.0.2) in the sense that they have a common smooth cover, where $\chi_{Z}$ is now a flat family of affine surfaces. It was shown by E. Brieskorn [B2] that when $G$ is of type $A, D$ or $E, \chi_{Z}^{-1}(0)$ has a du Val singularity of the same type, the family $\chi_{Z}$ is a miniversal deformation of this singularity, and $\tilde{\chi}_{Z}$ is a family of minimal resolutions of the fibres of $\chi_{z}$. This was extended to types $B, C, F$ and $G$ by P. Slodowy [S2], who showed that in these cases the singularity is again du Val, of type dual to the unfolding of $G$, that (1.0.3) admits an action of the discrete folding group of order 2 or 3, and that the deformation is miniversal among deformations respecting this symmetry.

More recently, it has been understood that many constructions from additive and multiplicative Springer theory also have "elliptic" analogues. To motivate this, note that the adjoint quotients $\mathfrak{g} / / G$ and $G / / G$ are coarse moduli spaces (in an appropriate sense) for the stack quotients $\mathfrak{g} / G$ and $G / G$. The central idea of elliptic Springer theory is to replace these stacks with the stack $\operatorname{Bun}_{G}(E)$ of principal $G$-bundles on a smooth elliptic curve $E$ over $k$.

At a basic level, this substitution is not unreasonable: for example, if we allow the elliptic curve $E$ to degenerate to a curve with a cusp (resp., a node), then the additive stack $\mathfrak{g} / G$ (resp., the multiplicative stack $G / G$ ) is naturally identified with the open substack of $G$-bundles whose pullbacks to the normalisation are trivial. From a slightly different perspective, the passage to elliptic Springer theory can be viewed as a passage from finite dimensional groups to loop groups: if we fix a complex number $q \in \mathbb{C}^{\times}$with $0<|q|<1$, then $\mathbb{C}^{\times} / q^{\mathbb{Z}}$ is the analytification of an elliptic curve and, by an unpublished observation of E. Looijenga, there is an isomorphism of complex analytic stacks

$$
\begin{align*}
\mathcal{L} G / q \mathcal{L} G & \xrightarrow{\sim} \operatorname{Bun}_{G}^{a n}\left(\mathbb{C}^{\times} / q^{\mathbb{Z}}\right)  \tag{1.0.4}\\
\varphi & \longmapsto \frac{\mathbb{C}^{\times} \times G}{(q z, g) \sim(z, \varphi(z) g)}
\end{align*}
$$

where $\mathcal{L} G$ is the group of holomorphic maps $\varphi: \mathbb{C}^{\times} \rightarrow G$ acting on itself by $q$-twisted conjugation

$$
(\theta \cdot \varphi)(z)=\theta(z) \varphi(z) \theta(q z)^{-1}
$$

It was observed by I. Grojnowski and N. Shepherd-Barron in [GSB, $\S 3]$ that the restriction $G / G \rightarrow \operatorname{Bun}_{G}^{a n}\left(\mathbb{C}^{\times} / q^{\mathbb{Z}}\right)$ of (1.0.4) to the constant loops $G \subseteq \mathcal{L} G$ is in fact étale in a neighbourhood of the identity. Since $\operatorname{Bun}_{G}^{a n}\left(\mathbb{C}^{\times} / q^{\mathbb{Z}}\right)$ is the analytification of $\operatorname{Bun}_{G}\left(\mathbb{C}^{\times} / q^{\mathbb{Z}}\right)$ (by GAGA) and since the image of $G / G$ consists entirely of semistable bundles, this identifies an analytic neighbourhood for the identity in $G / G$ with an analytic (étale) neighbourhood for the trivial bundle in $\operatorname{Bun}_{G}^{s s}\left(\mathbb{C}^{\times} / q^{\mathbb{Z}}\right)$.

There are, however, some qualitative differences between the stacks $\mathfrak{g} / G$ and $G / G$ and the stack $\operatorname{Bun}_{G}(E)$. For instance, $\operatorname{Bun}_{G}(E)$ is only locally of finite type, and only admits a well-behaved coarse moduli space after restricting to the finite type open substack $\operatorname{Bun}_{G}^{s s}(E)$ of semistable bundles. This coarse moduli space was studied by R. Friedman and J. Morgan in [FM1], who identified it with the quotient $Y / / W$, where $Y=\operatorname{Hom}\left(\mathbb{X}^{*}(T), \operatorname{Pic}^{0}(E)\right) \cong$ $\operatorname{Pic}^{0}(E)^{l}$ is the abelian variety parametrising degree $0 T$-bundles on $E$. The semistable
coarse moduli space map $\operatorname{Bun}_{G}^{s s}(E) \rightarrow Y / / W$ fits into a commutative diagram


The diagram (1.0.5) has been studied by D. Ben-Zvi and D. Nadler in [BZN2] from the perspective of character sheaves, who showed that it shares many properties with (1.0.2). It was also shown in [GSB, Theorem 3.11] that, over $\mathbb{C}$, the analytic morphism $G / G \rightarrow \operatorname{Bun}_{G}^{a n}(E)$ extends to a morphism (of diagrams) from the multiplicative Grothendieck-Springer resolution to (1.0.5) that is smooth in a neighbourhood of the identity in $G$.

The guiding principle behind this thesis is that the semistable elliptic Springer theory described above should extend in an interesting way to the whole of $\operatorname{Bun}_{G}(E)$. Our first main result gives a precise incarnation of this principle.

Theorem 1.0.1 (Corollaries 4.5.2 and 5.5.7 and Proposition 4.5.4). There exists an ample $W$-linearised line bundle $\Theta_{Y}$ on $Y$, with inverse $\Theta_{Y}^{-1}$, and a commutative diagram

which is a simultaneous log resolution with respect to the zero section of $\Theta_{Y}^{-1} / \mathbb{G}_{m}$ in the sense of Definition 1.0.2 below, where $\widehat{Y}$ is the affine cone over $Y$ obtained by contracting the zero section of $\Theta_{Y}^{-1}$ to a point. The preimage of the cone point under $\chi$ is precisely the locus of unstable bundles in $\operatorname{Bun}_{G}(E)$.

We will call the diagram (1.0.6) the elliptic Grothendieck-Springer resolution. Unlike the additive and multiplicative Grothendieck-Springer resolutions, the elliptic GrothendieckSpringer resolution is not quite a simultaneous resolution, as the morphism $\tilde{\chi}$ fails to be smooth over the zero section of $\Theta_{Y}^{-1}$. It does, however, satisfy the following weaker property.

Definition 1.0.2. Let

be a commutative diagram of algebraic stacks and let $D \subseteq \tilde{S}$ be a divisor. We say that (1.0.7) is a simultaneous log resolution with respect to $D$ if the following conditions are satisfied.
(1) The morphisms $f$ and $\tilde{f}$ are flat, $q$ is representable, proper, surjective and generically finite, and $\pi$ is proper with finite diagonal.
(2) For any point $s: \operatorname{Spec} k \rightarrow \tilde{S}$, the morphism $\tilde{f}^{-1}(s) \rightarrow f^{-1}(q(s))$ is an isomorphism over a dense open substack of $f^{-1}(q(s))$.
(3) The stack $\tilde{X}$ is regular, the morphism $\tilde{f}$ is smooth away from $D$, and $\tilde{f}^{-1}(D)$ is a (possibly non-reduced) divisor with normal crossings.

Remark 1.0.3. Definition 1.0 .2 is weaker than the definition of simultaneous $\log$ resolution given in [GSB, Definition 1.1] in several important ways. First, we do not require that the $\operatorname{map} \pi: \tilde{X} \rightarrow X$ be representable, but impose only the weaker condition that it have finite diagonal. Since the diagonal is proper (by properness of $\pi$ ) and the fibres of the diagonal are the stabilisers (or automorphism groups) of points in the fibres of $\pi$, this is equivalent to requiring that the fibres of $\pi$ have only finite stabilisers, which in characteristic 0 is the same thing as being a Deligne-Mumford stack. Second, we allow the singular fibres of $\tilde{f}$ to have non-reduced irreducible components, and for these irreducible components to have self-intersections. Finally, $[\mathrm{GSB}]$ require that the relative canonical bundle $K_{\tilde{X} / \tilde{S}}$ be the pullback of $K_{X / S}$, which we do not. We have chosen to make these modifications in order for Theorem 1.0.1 to be true. It follows from Corollary 4.5.9, Theorem 4.6.1 and Remark 6.1.12 that all of these modifications are necessary.

Theorem 1.0.1 builds on the work of S. Helmke and Slodowy [HS2] and of Grojnowski and Shepherd-Barron [GSB]. First, a version of the extended coarse moduli space map $\chi$ was constructed in [HS2] in terms of the isomorphism (1.0.4) as follows. By a theorem of Looijenga [L2], the affine variety $\widehat{Y} / / W$ is isomorphic to an affine space $\mathbb{A}^{l+1}$. From Looijenga's explicit isomorphism, the ring of functions on $\mathbb{A}^{l+1}=\widehat{Y} / / W$ can be identified with a ring of characters of irreducible representations of $\widehat{\mathcal{L}} G=\tilde{\mathcal{L}} G \rtimes \mathbb{C}^{\times}$, where $\tilde{\mathcal{L}} G$ is the universal central extension $1 \rightarrow \mathbb{C}^{\times} \rightarrow \tilde{\mathcal{L}} G \rightarrow \mathcal{L} G \rightarrow 1$, and $q \in \mathbb{C}^{\times}$acts on $\tilde{\mathcal{L}} G$ by

$$
(q \cdot \varphi)(z)=\varphi(q z)
$$

So there is a map

$$
\begin{equation*}
\tilde{\mathcal{L}} G /{ }_{q} \mathcal{L} G=(\tilde{\mathcal{L}} G \times\{q\}) / \mathcal{L} G \longrightarrow\left(\mathbb{A}^{l+1}\right)^{a n}=(\widehat{Y} / / W)^{a n} . \tag{1.0.8}
\end{equation*}
$$

Although our actual construction of $\chi$ will be given without reference to loop groups, one could also obtain its analytification over $\mathbb{C}$ by taking the quotient of (1.0.8) by the centre $\mathbb{C}^{\times}=\mathbb{G}_{m}^{a n} \subseteq \tilde{\mathcal{L}} G$. Second, the stack $\widetilde{\operatorname{Bun}}_{G}(E)$ appearing in (1.0.6) is the "Kontsevich-Mori compactification" of $\operatorname{Bun}_{B}^{0}(E)$ defined in [GSB], which was used in a slightly ad hoc way to construct a sliced version [GSB, Theorem 1.2] of (1.0.6), analogous to (1.0.3), for groups of type $D_{5}, E_{6}, E_{7}$ and $E_{8}$. Theorem 1.0.1 extends Helmke and Slodowy's work by constructing $\chi$ as an algebraic (rather than analytic) morphism, and Grojnowski and Shepherd-Barron's work by extending the definitions of $\chi$ and $\tilde{\chi}$ to all simply connected simple groups and to the whole of $\operatorname{Bun}_{G}(E)$ and $\widetilde{\operatorname{Bun}}_{G}(E)$.

The proof of Theorem 1.0.1 is divided into two parts: the construction of the diagram (1.0.6) (Corollary 4.5.2) and the proof that it is a simultaneous log resolution (Corollary 5.5.7). The main ingredient in the construction is an elliptic version of Chevalley's isomorphisms $\mathfrak{g} / / G \cong \mathfrak{t} / / W$ and $G / / G \cong T / / W$, which refines Friedman and Morgan's identification [FM1, Corollary 5.12] of the coarse moduli space of $\operatorname{Bun}_{G}^{s s}(E)$ with $Y / / W$.

Theorem 1.0.4 (Theorem 4.3.4). There is a certain subgroup $\operatorname{Pic}^{W}(Y)_{\text {good }} \subseteq \operatorname{Pic}^{W}(Y)$ of the group of $W$-linearised line bundles on $Y$ and an isomorphism

$$
\begin{equation*}
\operatorname{Pic}^{W}(Y)_{\text {good }} \xrightarrow{\sim} \operatorname{Pic}\left(\operatorname{Bun}_{G}(E)\right) . \tag{1.0.9}
\end{equation*}
$$

Moreover, if $L_{\mathrm{Bun}_{G}}$ is the image of $L_{Y} \in \operatorname{Pic}^{W}(Y)_{\text {good }}$ under (1.0.9), then there is a canonical isomorphism

$$
H^{0}\left(Y, L_{Y}\right)^{W} \xrightarrow{\sim} H^{0}\left(\operatorname{Bun}_{G}(E), L_{\mathrm{Bun}_{G}}\right) .
$$

We give the proof of Theorem 1.0.4 (as Theorem 4.3.4) in §4.3.
The fact that (1.0.6) is a simultaneous log resolution is proved in $\S 5.5$ as a fairly straightforward consequence of the following analogue of the Kostant and Steinberg section theorems.

Theorem 1.0.5 (Theorem 5.4.6 and Proposition 5.4.13). There exists a morphism $Z \rightarrow$ $\operatorname{Bun}_{G}(E)$ from an affine space $Z$ such that the composition $Z \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ with $\chi$ factors through an isomorphism $Z \cong \widehat{Y} / / W$. Moreover, writing

$$
\tilde{Z}=\widetilde{\operatorname{Bun}}_{G}(E) \times_{\operatorname{Bun}_{G}(E)} Z,
$$

the morphism $\tilde{Z} \rightarrow \Theta_{Y}^{-1} / \mathbb{G}_{m}$ induced by $\tilde{\chi}$ also factors through an isomorphism $\tilde{Z} \cong \Theta_{Y}^{-1}$.
Theorem 1.0.5 is a mild refinement of another theorem of Friedman and Morgan [FM2, Theorem 5.1.1], so we call it the Friedman-Morgan section theorem. The new observations here are that Friedman and Morgan's parabolic induction construction for the map $Z \rightarrow$ $\operatorname{Bun}_{G}(E)$ can be made to give a natural lift $Z \rightarrow \widehat{Y} / / W$ (Proposition 5.2.10), and that Theorem 1.0.5 can be proved by computing a small part of the elliptic Grothendieck-Springer resolution (§5.4).

Theorem 1.0.1 justifies our guiding principle that elliptic Springer theory extends to unstable $G$-bundles. In Chapter 6, we also give some evidence for the assertion that this extension is geometrically interesting. We prove that, with the exception of $G=S L_{2}$, there always exist slices $Z \rightarrow \operatorname{Bun}_{G}(E)$ through subregular unstable bundles with very nice properties (Theorem 6.1.5), which are analogous to the slices appearing in Brieskorn and Slodowy's work on du Val singularities. For each of these slices, the elliptic GrothendieckSpringer resolution pulls back to a simultaneous $\log$ resolution


Our main results about these slices (Theorems 6.1.9 and 6.6.1) are identifications of the fibres of $\tilde{\chi}_{Z}$ over the zero section of $\Theta_{Y}^{-1}$ as explicit surfaces built from blowups of line bundles over $E$, Hirzebruch surfaces and projective spaces at points along a embedded copies of the elliptic curve $E$. Our computations recover as a special case the computation [GSB, Theorem 6.7] of Grojnowski and Shepherd-Barron in type $E$, and can also be used (Theorem 6.7.3) to extend Helmke and Slodowy's description [HS2] of the codimension 2 singularities of $\chi^{-1}(0)$ to all simply connected groups $G$.

Remark 1.0.6. For technical reasons, we prove many of our main results for the rigidified stack $\operatorname{Bun}_{G}(E)_{\text {rig }}$ obtained from $\operatorname{Bun}_{G}(E)$ by taking the quotient of all automorphism groups by the centre of $G$, rather than for $\operatorname{Bun}_{G}(E)$ itself. (For a more precise explanation of what this means, see Definition 2.2.6.) The advantages of $\operatorname{Bun}_{G}(E)_{\text {rig }}$ over $\operatorname{Bun}_{G}(E)$ are that various automorphism groups (coming from centres of Levi subgroups) that are disconnected in $\operatorname{Bun}_{G}(E)$ become connected in $\operatorname{Bun}_{G}(E)_{\text {rig }}$, and that it is easier in practice to construct morphisms $Z \rightarrow \operatorname{Bun}_{G}(E)_{\text {rig }}$. For example, the Friedman-Morgan map $\widehat{Y} / / W \rightarrow \operatorname{Bun}_{G}(E)$ does not factor through a section $(\widehat{Y} / / W) / \mathbb{G}_{m} \rightarrow \operatorname{Bun}_{G}(E)$ of the coarse quotient map, but it does factor through a section $(\widehat{Y} / / W) / \mathbb{G}_{m} \rightarrow \operatorname{Bun}_{G}(E)_{\text {rig }}$. Similarly, in Chapter 6, we will actually work with slices $Z \rightarrow \operatorname{Bun}_{G}(E)_{r i g}$, some of which cannot be lifted to maps to $\operatorname{Bun}_{G}(E)$ without first passing to a gerbe over $Z$.

Remark 1.0.7. Throughout the body of this thesis, we will work in a somewhat more general context than in this introduction. Instead of working with a single elliptic curve $E$ defined over an algebraically closed field $k$, we will allow arbitrary families $E \rightarrow S$ of smooth curves of genus 1 over a regular stack $S$ (and work with a split simply connected simple group scheme $G$ over $\operatorname{Spec} \mathbb{Z}$ ), subject only sometimes to the restriction that $E \rightarrow S$ have a section. The key examples that should be kept in mind are:
(1) $S=\operatorname{Spec} k$ for $k$ a field, and $E$ an elliptic curve over $k$,
(2) $S=\mathbb{B} E^{\prime}$ and $E=\operatorname{Spec} k$, where $\mathbb{B} E^{\prime}$ is the classifying stack of an elliptic curve $E^{\prime}$ over $k$ (this amounts to working with $G$-bundles on $E^{\prime}$ up to translation), and
(3) $S=M_{1,1}$ the stack of elliptic curves over $\operatorname{Spec} \mathbb{Z}$ (or over some field) and $E \rightarrow S$ the universal elliptic curve.

It should be emphasised that very little will be lost to the reader who wishes to assume that we are in case (1) throughout.

### 1.1 Acknowledgments

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## Chapter 2

## Principal bundles on curves

In this chapter, we review some of the basic theory of stacks of principal bundles on curves. We begin by recalling some of the abstract definitions and properties of Artin stacks in §2.1 before introducing the stack of principal bundles on a curve in $\S 2.3$. In $\S 2.2$ we discuss gerbes under commutative group schemes and the process of rigidifying an Artin stack with respect to a group of automorphisms. In $\S 2.4$, we give some more concrete descriptions of stacks of principal bundles under certain unipotent groups, and in $\S 2.5$ we discuss some general features of the stack of principal bundles under a reductive group, centering around the notion of semistability. Finally, in $\S 2.6$ we describe some simplifications of the general theory for curves of genus $\leq 1$.

To fix ideas, unless otherwise specified, all schemes will be locally Noetherian, and all group schemes will be flat, affine, and of finite type.

All the results stated in this chapter are either well known or folklore, with the possible exception of Proposition 2.6.8.

### 2.1 Recollections on deformation theory and Artin stacks

In this section, we recall and fix terminology for some of the basic notions from deformation theory and the theory of Artin stacks. We stress that this is not by any means a selfcontained introduction to the subject, for which we direct the reader to one of the standard references such as [LMB] or [O2].

For the purposes of this thesis, by an Artin stack (or algebraic stack) $X$, we mean a functor (i.e., a lax 2-functor)

$$
X: \text { Sch }^{o p} \longrightarrow \text { Grpd, }
$$

where Sch is the category of locally Noetherian schemes and Grpd is the 2-category of groupoids, such that
(1) $X$ satisfies descent for the étale (equivalently, the fppf) topology,
(2) the diagonal $\Delta: X \rightarrow X \times X$ is representable (by algebraic spaces), and
(3) there exists a (locally Noetherian) scheme $U$ and a smooth surjective morphism $U \rightarrow X$.

For the sake of brevity, we will often drop the adjective "Artin" or "algebraic" and speak simply of stacks.

Remark 2.1.1. Artin stacks naturally form a 2 -category, rather than an ordinary category. When we speak of a commutative diagram, fibre product, etc., of Artin stacks, we will always mean a 2 -commutative diagram, 2 -fibre product, etc.

Definition 2.1.2. If $X$ is an Artin stack, then a quasi-coherent (resp., coherent) sheaf on $X$ is a sheaf $F$ of $\mathcal{O}$-modules on the site $\mathrm{Sch}_{/ X}$ of locally Noetherian schemes over $X$ (say, with the étale topology) such that
(1) if $U \in \operatorname{Sch}_{/ X}$, then the restriction $F_{U}$ of $F$ to the subcategory of Zariski open sets of $U$ is a quasi-coherent (resp., coherent) sheaf on $U$, and
(2) if $f: U \rightarrow V$ is a morphism in $\mathrm{Sch}_{/ X}$, then the induced morphism $f^{*} F_{V} \rightarrow F_{U}$ is an isomorphism.

This defines full subcategories $\operatorname{Coh}(X) \subseteq \mathrm{QCoh}(X) \subseteq \mathcal{O}_{\operatorname{Sch}_{/ X}}-\bmod$ of coherent and quasicoherent sheaves respectively.

Remark 2.1.3. Using the fact that the categories of coherent and quasi-coherent sheaves satisfy fpqc descent, one can show [O1, §10] that we obtain equivalent categories $\operatorname{Coh}(X)$ and $\mathrm{QCoh}(X)$ if we replace $\operatorname{Sch}_{/ X}$ with the lisse-étale site of $X$ or the étale topology on $\operatorname{Sch}_{/ X}$ with the fppf or fpqc topologies.

Similarly, to any Artin stack $X$ one can associate a derived category $D(X)$ of complexes quasi-coherent sheaves. There are multiple definitions of this available in the literature: to fix ideas, we will define $D(X)$ to be the category $D_{q c o h}^{\prime}(X)$ in the notation of $[\mathrm{O} 1, \S 7]$. (Another good option would be to follow [GR, Chapter $3, \S 1$, which gives the more refined structure of a stable $\infty$-category $D(X)$, denoted there by $\mathrm{QCoh}(X)$, rather than a triangulated category.)

If $f: X \rightarrow S$ is a morphism of Artin stacks, then there is an associated complex $\mathbb{L}_{X / S} \in$ $D(X)$ controlling the deformation theory of morphisms from $S$-schemes to $X$, called the cotangent complex of $X$ over $S[\mathrm{O}, \S 8]$. The tangent complex of $X$ over $S$ is the derived dual $\mathbb{T}_{X / S}=\left(\mathbb{L}_{X / S}\right)^{\vee}=\mathbb{R} \underline{\operatorname{Hom}}\left(\mathbb{L}_{X / S}, \mathcal{O}_{X}\right)$.

The cotangent complex has the following basic functoriality properties.
Theorem 2.1.4 ([O1, Theorem 8.1]). Let $f: X \rightarrow S$ be a morphism of stacks.
(1) If

is a commutative diagram of stacks, then there is a natural functoriality morphism

$$
\mathbb{L} g^{*} \mathbb{L}_{X / S} \longrightarrow \mathbb{L}_{X^{\prime} / S^{\prime}},
$$

which is an isomorphism if (2.1.1) is Cartesian and either $f$ or $h$ is flat.
(2) If $g: U \rightarrow X$ is another morphism of stacks, then there is a natural exact triangle

$$
\mathbb{L} g^{*} \mathbb{L}_{X / S} \longrightarrow \mathbb{L}_{U / S} \longrightarrow \mathbb{L}_{U / X} \longrightarrow \mathbb{L}_{X / S}[1]
$$

For well behaved representable morphisms, the cotangent complex can be computed easily from more classical objects.

Proposition 2.1.5. Suppose that $f: X \rightarrow S$ is smooth and representable and that $i: Y \rightarrow X$ is a regular embedding with ideal sheaf $I$. Then the $\mathbb{L}_{X / S} \cong \Omega_{X / S}^{1}$ is the sheaf of relative Kähler differentials, and $\mathbb{L}_{Y / S}$ is given by the complex

$$
\mathbb{L}_{Y / S}=\left[I / I^{2} \xrightarrow{d} i^{*} \Omega_{X / S}^{1}\right]
$$

concentrated in degree -1 and 0 .

At a basic level, the connection between cotangent complexes and deformation theory can be understood as follows. Given a point $x: \operatorname{Spec} k \rightarrow X$ (with $k$ a field) over the point $s=f(x):$ Spec $k \rightarrow S$, we can define deformation functors

$$
X_{l o c}: \operatorname{Art}_{k} \longrightarrow \operatorname{Grpd} \quad \text { and } \quad S_{l o c}: \operatorname{Art}_{k} \longrightarrow \text { Grpd, }
$$

by setting

$$
X_{l o c}(A)=\operatorname{Hom}(\operatorname{Spec} A, X) \times_{\operatorname{Hom}(\operatorname{Spec} k, X)}\{x\}
$$

and

$$
S_{l o c}(A)=\operatorname{Hom}(\operatorname{Spec} A, S) \times_{\operatorname{Hom}(\operatorname{Spec} k, S)}\{s\}
$$

for $A \in \operatorname{Art}_{k}$, where $\mathrm{Art}_{k}$ is the category of local Artinian rings with residue field $k$. In the following proposition, we write $\mathbb{L}_{X / S, x}=\mathbb{L} x^{*} \mathbb{L}_{X / S}$ for the derived pullback of $\mathbb{L}_{X / S}$ to Spec $k$.

Proposition 2.1.6. We have the following.
(1) If $V$ is a finite dimensional $k$-vector space, then $\operatorname{Ext}^{0}\left(\mathbb{L}_{X / S, x}, V\right)$ and $\operatorname{Ext}^{-1}\left(\mathbb{L}_{X / S, x}, V\right)$ are canonically isomorphic respectively to the set of isomorphism classes and the automorphism group of any object in the groupoid

$$
X_{l o c}(k \oplus V) \times_{S_{l o c}(k \oplus V)}\{s\},
$$

where the product on $k \oplus V$ is defined by $(a, u)(b, v)=(a b, a v+b u)$ and we also write $s$ for the image of $s \in S_{l o c}(k)$ under the morphism $S_{l o c}(k) \rightarrow S_{l o c}(k \oplus V)$ given by $a \mapsto(a, 0)$.
(2) If $B \rightarrow A$ is a surjection in Art $_{k}$ with kernel $V \subseteq B$ satisfying $\mathfrak{m}_{B} V=0\left(\mathfrak{m}_{B}\right.$ the maximal ideal of $B$ ), then there is a canonical "obstruction" function

$$
\mathrm{ob}: X_{l o c}(A) \times_{S_{l o c}(A)} S_{l o c}(B) \longrightarrow \operatorname{Ext}^{1}\left(\mathbb{L}_{X / S, x}, V\right)
$$

such that $\xi \in X_{l o c}(A) \times{ }_{S_{l o c}(A)} S_{l o c}(B)$ is in the image of the natural functor from $X_{l o c}(B)$ if and only if $\operatorname{ob}(\xi)=0$.

Remark 2.1.7. Note that if $\mathbb{L}_{X / S}$ is perfect, then $\operatorname{Ext}^{i}\left(\mathbb{L}_{X / S, x}, V\right)=H^{i}\left(\mathbb{T}_{X / S, x}\right) \otimes V$.
Remark 2.1.8. Proposition 2.1.6 shows that we can identify $\operatorname{Ext}^{-1}\left(\mathbb{L}_{X / S, x}, k\right)=H^{-1}\left(\mathbb{T}_{X / S, x}\right)$ with the Lie algebra of the kernel of the homomorphism $\operatorname{Aut}_{X}(x) \rightarrow \operatorname{Aut}_{S}(s)$, where we write $\operatorname{Aut}_{Y}(y)$ for the automorphism $k$-group scheme of a $k$-point $y$ in a stack $Y$.

Remark 2.1.9. The connection between the cotangent complex and deformation theory outlined above can be made much sharper in the framework of derived deformation theory, where the cotangent complex can be characterised as the unique complex corepresenting some functor. The statements of Proposition 2.1.6 are the most straightforward consequences of this sharper statement that can be seen in the underived world.

For many stacks of interest, the cotangent and tangent complexes are very difficult to compute. However, in these cases there is often a much simpler and more natural complex approximating the tangent complex closely enough to retain the deformation theoretic properties of Proposition 2.1.6.

Definition 2.1.10. If $f: X \rightarrow S$ is a morphism of stacks, a tangent-obstruction complex for $X$ over $S$ is a complex $\mathbb{T} \in D(X)$ together with a morphism $\left(\mathbb{L}_{X / S}\right)^{\vee} \rightarrow \mathbb{T}$ such that the (derived) cokernel has vanishing cohomology in degrees $\leq 0$.

We can also speak of tangent-obstruction complexes $\mathbb{T}_{x} \in D(\operatorname{Spec} k)$ at $x: \operatorname{Spec} k \rightarrow X$ by replacing $\mathbb{L}_{X / S}$ with $\mathbb{L}_{X / S, x}$ in Definition 2.1.10.

If $\mathbb{T}$ is a tangent-obstruction complex for $X$ over $S$, then the vanishing condition implies that for any point $x$ : Spec $k \rightarrow X$ and any $k$-vector space $V$, the induced morphism $\operatorname{Ext}^{i}\left(\mathbb{L}_{X / S, x}, V\right) \rightarrow H^{i}\left(\mathbb{T}_{x}\right) \otimes V$ is an isomorphism for $i=0,-1$ and injective for $i=1$. So Proposition 2.1.6 holds with $H^{i}\left(\mathbb{T}_{x}\right) \otimes V$ in place of $\operatorname{Ext}^{i}\left(\mathbb{L}_{X / S, x}, V\right)$.

The following proposition follows easily from the above discussion and the fact that the tangent complex of a smooth morphism of stacks is perfect of amplitude contained in $[-1,0]$.

Proposition 2.1.11. Assume that $f: X \rightarrow S$ is locally of finite presentation. Then $f$ is smooth at $x$ : Spec $k \rightarrow X$ if and only if $f$ has a tangent-obstruction complex $\mathbb{T}_{x}$ at $x$ with $H^{i}\left(\mathbb{T}_{x}\right)=0$ for $i \neq 0,-1$. Moreover, for any such tangent-obstruction complex, the morphism $\mathbb{T}_{X / S, x} \rightarrow \mathbb{T}_{x}$ must be an isomorphism in $D(\operatorname{Spec} k)$.

Remark 2.1.12. While tangent-obstruction complexes appear somewhat unnatural at first sight, they have a natural interpretation in the context of derived algebraic geometry: in practice, interesting tangent-obstruction complexes for a stack $X$ are almost always the tangent complexes of some non-trivial (but natural) derived thickenings of $X$.

### 2.2 Gerbes and rigidification

At many points throughout this thesis, we will encounter stacks in which all automorphism groups naturally contain a common commutative subgroup, which we will either wish to remove or use in subsequent constructions. In this section, we review some useful theory for working with these structures.

Recall that if $X \rightarrow S$ is a morphism of stacks and $G \rightarrow S$ is a group scheme over $S$, then a $G$-torsor or principal $G$-bundle on $X$ is a morphism $\xi \rightarrow X$ equipped with a right $G$-action on $\xi$ over $X$ such that there exists an fppf surjection $U \rightarrow X$ and a $G$-equivariant isomorphism $U \times_{S} \xi \cong U \times_{S} G$. The classifying stack of $G$ is the algebraic stack $\mathbb{B} G=\mathbb{B}_{S} G$ representing the functor

$$
\begin{aligned}
\mathrm{Stk}_{/ S}^{o p} & \longrightarrow \mathrm{Grpd} \\
X & \longmapsto\{G \text {-torsors on } X\} .
\end{aligned}
$$

It is easy to see that giving a morphism $\mathbb{B}_{S} G \rightarrow X$ is equivalent to giving a morphism $x: S \rightarrow X$ (the image of the trivial torsor $S \times G$ ) equipped with a homomorphism $G=$ $\underline{\text { Aut }}_{\mathbb{B}_{S} G}(S \times G) \rightarrow{\underline{\text { Aut }_{X}}}_{X}(x)$ of automorphism group schemes over $S$.

If $G \rightarrow S$ is a commutative group scheme, then the classifying stack $\mathbb{B} G$ is a (commutative) group stack over $S$ with identity given by the map $S \rightarrow \mathbb{B} G$ classifying the trivial torsor, and group operation given by

$$
\begin{aligned}
m: \mathbb{B} G \times_{S} \mathbb{B} G & \longrightarrow \mathbb{B} G \\
(\xi, \eta) & \longmapsto \xi \otimes \eta
\end{aligned}
$$

where, for $\xi$ and $\eta G$-torsors over some $S$-stack $U, \xi \otimes \eta$ is the $G$-torsor given by

$$
\xi \otimes \eta=\xi \times{ }^{G} \eta=\left(\xi \times_{U} \eta\right) / G
$$

where $G$ acts on $\xi \times_{U} \eta$ by the formula $(x, y) \cdot g=\left(x g, y g^{-1}\right)$, and the action of $G$ on $\xi \otimes \eta$ is induced by the action on either factor $\xi$ or $\eta$ in the product. Using the general theory of group objects in a 2-category, one can therefore define to notion of an action of
$\mathbb{B} G$ on an $S$-stack $X$. While the general definition of an action of a group stack is somewhat complicated, in the special case of $\mathbb{B} G$ it reduces to the following simple structure.

Definition 2.2.1. Let $X \rightarrow S$ be a morphism of stacks, and let $G \rightarrow S$ be a flat commutative group scheme of finite type. An action of $\mathbb{B} G$ on $X$ over $S$ is a morphism $a: X \times_{S} \mathbb{B} G \rightarrow X$ of stacks over $S$, equipped with a 2 -isomorphism making the diagram of $S$-stacks

commute, where the top morphism is the canonical quotient map classifying the trivial $G$-torsor on $X$.

Remark 2.2.2. If $s: U \rightarrow S$ is an $S$-scheme, a morphism $U \times_{S} \mathbb{B} G \rightarrow X$ over $S$ is the same thing as a point $x \in X(U)$ over $s \in S(U)$, together with a homomorphism $G_{U} \rightarrow$
 fact, an action of $\mathbb{B} G$ on $X$ is the same thing as a collection of homomorphisms $G_{U} \rightarrow$ $\operatorname{ker}\left(\underline{\operatorname{Aut}}_{X}(x) \rightarrow \underline{\operatorname{Aut}}_{S}(s)\right)$ for every $X$-scheme $x: U \rightarrow X$ compatible with base change.

More generally, an action of a group stack $H$ on $X$ consists of an action in the sense of Definition 2.2.1, together with a choice of 2 -isomorphism making the diagram

commute, which is required to be compatible with various other 2-isomorphisms in a precise way. For $H=\mathbb{B} G$, however, there is a unique such 2 -isomorphism, automatically compatible, given by the obvious identification of the two homomorphisms $G_{U} \times_{U} G_{U} \rightarrow \operatorname{ker}\left(\underline{\text { Aut }}_{X}(x) \rightarrow\right.$ $\left.\underline{\operatorname{Aut}}_{S}(s)\right)$ for every $x: U \rightarrow X$ as in Remark 2.2.2.

Given the notion of an action, one can define torsors under group stacks just as for group schemes. If $G$ is a commutative group scheme, then torsors under the group stack $\mathbb{B} G$ have a special name.

Definition 2.2.3. If $G \rightarrow S$ is a flat commutative group scheme of finite type and $X \rightarrow S$ is any stack over $S$, then a $G$-gerbe on $X$ is a morphism of stacks $\xi \rightarrow X$ equipped with an action of $\mathbb{B} G$ on $\xi$ over $X$, such that there exists a smooth (equivalently, fppf) surjection $U \rightarrow Y$ such that $U \times_{Y} \xi$ is isomorphic to $U \times_{S} \mathbb{B} G$ as stacks over $U$ equipped with a $\mathbb{B} G$-action. Here $\mathbb{B} G$ acts on $U \times{ }_{S} \mathbb{B} G$ via

$$
U \times_{S} \mathbb{B} G \times_{S} \mathbb{B} G \xrightarrow{\mathrm{id} \times m} U \times_{S} \mathbb{B} G .
$$

Remark 2.2.4. There is another notion of gerbe defined, for example, in [LMB]. These weaker objects are simply surjective morphisms $U \rightarrow X$ such that the diagonal $U \rightarrow U \times_{X}$ $U$ is also surjective, which in particular implies that all geometric fibres of $U \rightarrow X$ are classifying stacks of groups. It is easy to see that any $G$-gerbe in our sense is a gerbe in this weaker sense, and has the stronger property that all the automorphism groups appearing in fibres of $U \rightarrow X$ are identified with the corresponding fibres of $G \rightarrow S$.

Just as for actions of group schemes, there is a good theory of quotients for nice enough actions of $\mathbb{B} G$.

Proposition 2.2.5. Assume that $G \rightarrow S$ is a flat commutative group scheme of finite type and that we are given an action $a: X \times_{S} \mathbb{B} G \rightarrow X$ on an $S$-stack $X$ such that for every morphism $x: U \rightarrow X$ with $U$ a scheme, the induced homomorphism $G_{U} \rightarrow \underline{\text { Aut }_{X}(x) \text { is a }}$ closed immersion. Then there exists a unique stack $X_{\text {rig }}$ over $S$ equipped with a morphism $X \rightarrow X_{\text {rig }}$ such that the $\mathbb{B} G$-action makes $X$ into a $G$-gerbe over $X_{\text {rig }}$. Moreover, if $X$ has affine diagonal over $S$ and $G$ is an extension of a finite group scheme by a torus, then $X_{\text {rig }}$ also has affine diagonal over $S$.

Proof. Existence and uniqueness of the stack $X_{\text {rig }}$ is proved in [ACV, Theorem 5.1.5]. It remains to check that $X_{\text {rig }} \rightarrow S$ has affine diagonal when $X$ does and $G$ is an extension of a finite group scheme by a torus. To see this, observe that since $X \rightarrow X_{\text {rig }}$ is a $G$-gerbe, there is a pullback square

where the top horizontal morphism is given by natural projection on the first factor and the $\mathbb{B} G$ action on the second. Since $X \rightarrow X_{\text {rig }}$ is faithfully flat, it suffices to show that this top morphism is affine. If $U$ is an affine scheme and $U \rightarrow X \times_{S} X$ is a morphism classifying a pair $(x, y) \in X(U) \times{ }_{X(S)} X(U)$ over $s \in S(U)$, then there is a pullback

where $\underline{\operatorname{Isom}}_{s}(x, y) \rightarrow U$ is the scheme of isomorphisms $x \xrightarrow{\sim} y$ covering id: $s \xrightarrow{\sim} s$. But $G$ acts freely on the affine scheme $\underline{\operatorname{Isom}}_{s}(x, y)$, so $\underline{\operatorname{Isom}}(x, y) / G$ is itself affine over $U$ since $G$ is an extension of a finite group scheme by a torus. This completes the proof.

Definition 2.2.6. The stack $X_{\text {rig }}$ is called the rigidification of $X$ with respect to the group $G$.

### 2.3 Stacks of principal bundles and basic properties

In this thesis, the most important example of an Artin stack is, of course, the stack of principal bundles on a curve, or family of curves. We define this stack in this section, and discuss some of its most elementary properties.

Let $X \rightarrow S$ be a morphism of Artin stacks, and suppose that $G \rightarrow X$ is a group scheme, which we will assume to be flat, affine and of finite presentation over $X$. Consider the functor

$$
\begin{equation*}
\underline{\operatorname{Bun}}_{G / S}(X):\left(\mathrm{Stk}_{/ S}\right)^{o p} \longrightarrow \operatorname{Grpd} \tag{2.3.1}
\end{equation*}
$$

sending an $S$-stack $U$ to the groupoid of $G$-torsors on $U \times{ }_{S} X$.
Proposition 2.3.1. Assume that $X$ is flat, proper and representable over $S$. Then the functor (2.3.1) is representable by an Artin stack $\operatorname{Bun}_{G / S}(X)$ locally of finite presentation with affine diagonal over $S$.

Remark 2.3.2. In more prosaic terms, $\operatorname{Bun}_{G / S}(X)$ is the stack of pairs $\left(s, \xi_{G}\right)$, where $s \in S$ and $\xi_{G} \rightarrow X_{s}$ is a $G_{s}$-bundle over the fibre $X_{s}$ of $X$ over $s$.

Remark 2.3.3. If $X \rightarrow S$ is as above, $X \rightarrow X^{\prime}$ is any morphism of stacks, and $G \rightarrow X^{\prime}$ is a group scheme over $X^{\prime}$, then $G_{X}=X \times_{X^{\prime}} G$ is a group scheme over $X$, and we will write $\operatorname{Bun}_{G / S}(X)$ for the stack $\operatorname{Bun}_{G_{X} / S}(X)$. Note that, for any $S$-stack $U \rightarrow S$, a principal $G_{X}$-bundle on $U \times_{S} X$ (viewing $U \times_{S} X$ as a stack over $X$ ) is by definition the same thing as a principal $G$-bundle over $U \times_{S} X$ (viewing $U \times_{S} X$ as a stack over $X^{\prime}$ ).

Remark 2.3.4. If $S=\operatorname{Spec} k$ for some field $k$, we will often write $\operatorname{Bun}_{G}(X)=\operatorname{Bun}_{G / S}(X)$.
Assume that $G \rightarrow X^{\prime}$ is a group scheme over $X^{\prime}$ and that $Y \rightarrow X^{\prime}$ is a stack over $X^{\prime}$ equipped with a left $G$-action. If $X$ is a stack over $X^{\prime}$ and $\xi_{G} \rightarrow X$ is a principal $G$-bundle, then we set

$$
\xi_{G} \times{ }^{G} Y=\left(\xi_{G} \times X_{X^{\prime}} Y\right) / G \longrightarrow X
$$

on $X$, where $G$ acts on $\xi_{G} \times_{X^{\prime}} Y$ by $(x, v) g=\left(x g, g^{-1} y\right)$ for $x \in \xi_{G}, g \in G$ and $y \in Y$.
If $G \rightarrow H$ is a homomorphism of group schemes over $X^{\prime}$ then $\xi_{G} \times{ }^{G} H$ is naturally an $H$-torsor, where $G$ acts on $H$ by multiplication on the left and $H$ acts on $\xi_{G} \times{ }^{G} H$ by multiplication on the right. This construction defines a morphism

$$
\begin{equation*}
\operatorname{Bun}_{G / S}(X) \longrightarrow \operatorname{Bun}_{H / S}(X) \tag{2.3.2}
\end{equation*}
$$

Definition 2.3.5. In the setup above, if $\xi_{H} \rightarrow X$ is an $H$-bundle, a reduction of the structure group of $\xi_{H}$ to $G$ is a $G$-bundle $\xi_{G} \rightarrow X$ and an isomorphism $\xi_{G} \times{ }^{G} H \cong \xi_{H}$ (i.e., a preimage of $\xi_{H}$ under (2.3.2)).

If $X \rightarrow Y$ is a morphism of stacks over another stack $S$, we write $\Gamma_{S}(Y, X)$ for the stack whose functor of points sends an $S$-scheme $U$ to the groupoid of sections over $U$ of the map $U \times_{S} X \rightarrow U \times_{S} Y$; if $Y \rightarrow S$ is proper, then this is algebraic.

The following proposition is elementary and well-known.
Proposition 2.3.6. If $G \rightarrow H$ is a homomorphism of group schemes, then there is an isomorphism

$$
\operatorname{Bun}_{G / S}(X) \cong \Gamma_{\operatorname{Bun}_{H / S}(X)}\left(\operatorname{Bun}_{H / S}(X) \times_{S} X, \xi_{H}^{u n i} / G\right),
$$

where $\xi_{H}^{u n i}$ is the universal $H$-bundle on $\operatorname{Bun}_{H / S}(X) \times_{S} X$.
Remark 2.3.7. In more down to earth terms, Proposition 2.3 .6 can be interpreted as says that a reduction of the structure group of an $H$-bundle $\xi_{H} \rightarrow X$ to $G$ is the same thing as a section of the map $\xi_{H} / G=\xi_{H} \times{ }^{H} H / G \rightarrow X$.

Now suppose that $V$ is a representation of $G$, i.e., a vector bundle equipped with a linear $G$-action. Then $\xi_{G} \times{ }^{G} V$ is a vector bundle on $X$, called the associated vector bundle.

Proposition 2.3.8. Under the assumptions of Proposition 2.3.1, assume in addition that $G \rightarrow X$ is smooth. Then $\operatorname{Bun}_{G / S}(X) \rightarrow S$ has a tangent-obstruction complex given by

$$
\mathbb{T}=\mathbb{R} \pi_{*}\left(\xi_{G}^{u n i} \times^{G} \mathfrak{g}\right)[1]
$$

where $\mathfrak{g}$ is the Lie algebra of $G$ with $G$ acting via the adjoint representation, $\xi_{G}^{u n i}$ is the universal $G$-bundle over $\operatorname{Bun}_{G / S}(X) \times_{S} X$, and $\pi: \operatorname{Bun}_{G / S}(X) \times_{S} X \rightarrow \operatorname{Bun}_{G / S}(X)$ is the natural projection.

Corollary 2.3.9. If $G \rightarrow X$ is a smooth affine group scheme and $X \rightarrow S$ is a proper curve, then $\operatorname{Bun}_{G / S}(X) \rightarrow S$ is smooth and the tangent complex $\mathbb{T}_{\operatorname{Bun}_{G / S}(X) / S}$ is equal to the tangent-obstruction complex of Proposition 2.3.8.

Proof. Fix a point $s: \operatorname{Spec} k \rightarrow S$ and a $G$-bundle $\xi_{G} \rightarrow X_{s}$ corresponding to a point in $\operatorname{Bun}_{G / S}(X)$. Then Proposition 2.3.8 gives a tangent-obstruction complex

$$
\mathbb{T}_{\left(s, \xi_{G}\right)}=\mathbb{R} \Gamma\left(X_{s}, \xi_{G} \times{ }^{G} \mathfrak{g}\right)[1]
$$

for $\operatorname{Bun}_{G / S}(X)$ at $\left(s, \xi_{G}\right)$. Since $X_{s}$ is a curve, $H^{i}\left(X_{s}, \xi_{G} \times{ }^{G} \mathfrak{g}\right)=0$ for $i>1$, so Proposition 2.1.11 implies the claim.

It will often be convenient to simplify $\operatorname{Bun}_{G / S}(X)$ using the rigidification construction described in $\S 2.2$. Let $G \rightarrow X$ be a group scheme, and choose a closed subgroup $H \subseteq$ $\Gamma_{S}(X, Z(G))$ that is flat over $S$. Then we have a natural action

$$
\begin{aligned}
\operatorname{Bun}_{G / S}(X) \times_{S} \mathbb{B} H & \longrightarrow \operatorname{Bun}_{G / S}(X) \\
\left(\xi_{G}, \eta_{H}\right) & \longmapsto \xi_{G} \otimes \eta_{H}
\end{aligned}
$$

where, for $U$ an $S$-scheme, $\xi_{G} \rightarrow U \times_{S} X$ a $G$-bundle and $\eta_{H} \rightarrow U$ an $H$-bundle, we set

$$
\xi_{G} \otimes \eta_{H}=\left(\xi_{G} \times_{U} \eta_{H}\right) / H,
$$

where $H$ acts on $\xi_{G} \times_{U} \eta_{H}$ by the formula $(x, y) \cdot h=\left(x h, y h^{-1}\right)$. This action corresponds to the collection of homomorphisms $H_{U} \rightarrow \underline{\operatorname{Aut}}\left(\xi_{G}\right)$ defined by the action

$$
\begin{aligned}
H_{U} \times_{U} \xi_{G} & \longrightarrow \xi_{G} \\
(h, x) & \longmapsto x h .
\end{aligned}
$$

Since these homomorphisms are closed immersions, we have a rigidification $\operatorname{Bun}_{G / S}(X) \rightarrow$ $\operatorname{Bun}_{G / S}(X)_{\text {rig }}$ with respect to $H$.

Remark 2.3.10. In the special case where $H$ is the centre of some reductive group, $H$ is an extension of a torus by a finite commutative group scheme, so $\operatorname{Bun}_{G / S}(X)_{\text {rig }}$ has affine diagonal by Proposition 2.2.5.

Remark 2.3.11. If $G \rightarrow S$ is a group scheme over $S$, then we have that $Z(G) \subseteq \Gamma_{S}\left(X, X \times{ }_{S}\right.$ $Z(G))$ is a closed subgroup, so we can rigidify $\operatorname{Bun}_{G / S}(X)=\operatorname{Bun}_{X \times_{S} G / S}(X)$ with respect to closed subgroups of $Z(G)$.

### 2.4 Principal bundles under unipotent groups

In this section, we describe methods for studying the geometry of $\operatorname{Bun}_{G}(X)$ when the group $G$ is built from additive groups and $X$ is a curve.

We first recall the relationship between sheaf cohomology and principal bundles under a vector bundle (viewed as a group scheme under addition).

Proposition 2.4.1. Let $\pi_{X}: X \rightarrow S$ be a proper curve over $S$ and $V$ a vector bundle on $X$. Assume that the coherent sheaves $\mathbb{R}^{i} \pi_{X *}(V)$ are vector bundles on $S$ for $i=0,1$. Then the rigidified stack $\operatorname{Bun}_{V / S}(X)_{\text {rig }}$ with respect to $H=\Gamma_{S}(X, V)=\pi_{X *}(V)$ is isomorphic to the total space of the vector bundle $\mathbb{R}^{1} \pi_{X *}(V)$ on $S$.

The next proposition describes how principal bundles behave under extensions of group schemes.

Proposition 2.4.2. Let $: X \rightarrow S$ be a proper curve over $S, X \rightarrow X^{\prime}$ a morphism of stacks and let

$$
\begin{equation*}
1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1 \tag{2.4.1}
\end{equation*}
$$

be a short exact sequence of flat affine group schemes over $X^{\prime}$.
(1) If the sequence (2.4.1) is split, so that $G \cong K \rtimes H$, then

$$
\begin{equation*}
\operatorname{Bun}_{G / S}(X) \cong \operatorname{Bun}_{\mathcal{K} / M}\left(M \times_{S} X\right) \tag{2.4.2}
\end{equation*}
$$

where $M=\operatorname{Bun}_{H / S}(X)$ and $\mathcal{K} \rightarrow M \times_{S} X$ is the group scheme $\mathcal{K}=\xi_{H}^{u n i} \times{ }^{H} K$, where $\xi_{H}^{u n i} \rightarrow M \times{ }_{S} X$ is the universal $H$-bundle and $H$ acts on $K$ by conjugation.
(2) If $K$ is a vector bundle contained in the centre of $G$ then $\operatorname{Bun}_{K / S}(X)$ is naturally a group stack, and $\operatorname{Bun}_{G / S}(X)$ is a $\operatorname{Bun}_{K / S}(X)$-torsor over $\operatorname{Bun}_{H / S}(X)$.

Proof. In (1), if we view the right hand side of the isomorphism (2.4.2) as the stack of pairs $\left(\xi_{H}, \xi_{\mathcal{K}}\right)$, where $\xi_{H}$ is an $H$-torsor and $\xi_{\mathcal{K}}$ a torsor for the corresponding fibre of $\mathcal{K}$, the isomorphism sends $\xi_{G} \in \operatorname{Bun}_{G / S}(X)$ to the pair $\left(\xi_{H}=\xi_{G} \times{ }^{G} H, \xi_{G} / H\right)$. (Note that the action of $K$ on $\xi_{G}$ on the right determines an action of $\mathcal{K}=\xi_{G} \times{ }^{G} K$ on $\xi_{G} / H$.) It is easy to check that this is indeed an isomorphism of stacks.

To prove (2), observe that the action of $K$ on $G$ by multiplication on the right induces an action of $\operatorname{Bun}_{K / S}(X)$ on $\operatorname{Bun}_{G / S}(X)$ over $\operatorname{Bun}_{H / S}(X)$ such that the induced morphism $\operatorname{Bun}_{G / S}(X) \times \operatorname{Bun}_{K / S}(X) \rightarrow \operatorname{Bun}_{G / S}(X) \times_{\operatorname{Bun}_{H / S}(X)} \operatorname{Bun}_{G / S}(X)$ is an isomorphism. Hence, to show that $\operatorname{Bun}_{G / S}(X) \rightarrow \operatorname{Bun}_{H / S}(X)$ is a $\operatorname{Bun}_{K / S}(X)$-torsor, it suffices to show that $\operatorname{Bun}_{G / S}(X) \rightarrow \operatorname{Bun}_{H / S}(X)$ is surjective. To see this, observe that if $\xi_{H} \rightarrow X_{s}$ is an $H$ bundle on a geometric fibre of $X \rightarrow S$, then the stack quotient $\xi_{H} / G$ is naturally a $K$ gerbe over $X_{s}$. Since $H^{2}\left(X_{s}, K\right)=0$, as $K$ is a vector bundle and $X_{s}$ is a curve, all $K$-gerbes over $X_{s}$ are trivial, so $\xi_{H} / G$ admits a section, and hence $\xi_{H}$ is in the image of $\operatorname{Bun}_{G / S}(X) \rightarrow \operatorname{Bun}_{H / S}(X)$.

Using Proposition 2.4.1 and Proposition 2.4.2, we have the following two extreme cases for the geometry of $\operatorname{Bun}_{\mathcal{U} / S}(X)$ with $\mathcal{U} \rightarrow X$ a connected unipotent group scheme.

Corollary 2.4.3. Let $\pi_{X}: X \rightarrow S$ be a proper curve and $\mathcal{U} \rightarrow X$ a connected unipotent group scheme such that $\Gamma_{S}(X, \mathcal{U})=\{1\}$. Then $\operatorname{Bun}_{\mathcal{U} / S}(X)$ is an affine space bundle over $S$.

Proof. Fix a central series

$$
\{1\} \subseteq \mathcal{U}_{n} \subseteq \mathcal{U}_{n-1} \subseteq \cdots \subseteq \mathcal{U}_{1}=\mathcal{U}
$$

for $\mathcal{U}$. We prove by the corollary by induction on the length $n$.
When $n=1$, this reduces to a special case of Proposition 2.4.1, so suppose that $n>1$. Then the assumptions on $\mathcal{U}$ imply that $\mathcal{U}_{n}$ is a vector bundle on $X$ such that $H^{0}\left(X_{s}, \mathcal{U}_{n}\right)=0$ for every $s$ : Spec $k \rightarrow S$. So Proposition 2.4.1 implies that $\operatorname{Bun}_{\mathcal{U}_{n} / S}(X)=\mathbb{R}^{1} \pi_{X *} \mathcal{U}_{n}$ is a vector bundle over $S$. We have a central extension

$$
1 \longrightarrow \mathcal{U}_{n} \longrightarrow \mathcal{U} \longrightarrow \mathcal{U} / \mathcal{U}_{n} \longrightarrow 1
$$

so $\operatorname{Bun}_{\mathcal{U} / S}(X)$ is a $\mathbb{R}^{1} \pi_{X *} \mathcal{U}_{n}$-torsor over $\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{n}\right) / S}(X)$. Since $\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{n}\right) / S}(X)$ is an affine space bundle over $S$ by induction, so is $\operatorname{Bun}_{\mathcal{U} / S}(X)$.

Corollary 2.4.4. Let $\pi_{X}: X \rightarrow S$ be a proper curve and $\mathcal{U} \rightarrow X$ a connected unipotent group scheme admitting a central series

$$
\{1\}=\mathcal{U}_{n+1} \subseteq \mathcal{U}_{n} \subseteq \mathcal{U}_{n-1} \subseteq \cdots \subseteq \mathcal{U}_{1}=\mathcal{U}
$$

in which each quotient $\mathcal{U}_{i} / \mathcal{U}_{i+1}$ is a vector bundle with $\mathbb{R}^{1} \pi_{X *} \mathcal{U}_{i} / \mathcal{U}_{i+1}=0$. Then the map $\mathbb{B} \Gamma_{S}(X, \mathcal{U}) \rightarrow \operatorname{Bun}_{\mathcal{U} / S}(X)$ classifying the trivial bundle is an isomorphism.

Proof. For $n=1$, the statement is immediate from Proposition 2.4.1. For $n>1$, we have a central extension

$$
1 \longrightarrow \mathcal{U}_{n} \longrightarrow \mathcal{U} \longrightarrow \mathcal{U} / \mathcal{U}_{n} \longrightarrow 1
$$

from which it follows by Proposition 2.4.1 and induction that $\mathbb{B} \Gamma_{S}(X, \mathcal{U}) \rightarrow \operatorname{Bun}_{\mathcal{U} / S}(X)$ is a morphism of $\mathbb{B} \Gamma_{S}\left(X, \mathcal{U}_{n}\right)$-gerbes over $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}_{n}\right)=\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{n}\right) / S}(X)$, hence an isomorphism as claimed.

The stack of bundles under a connected unipotent group scheme has the following convenient properties.

Corollary 2.4.5. Let $X \rightarrow \operatorname{Spec} k$ be a proper curve over an algebraically closed field $k$ and let $\mathcal{U} \rightarrow X$ be a connected unipotent group scheme. Then $\operatorname{Bun}_{\mathcal{U}}(X)$ is connected.

Proof. By induction on the length of a central series for $\mathcal{U}$, we can reduce by Proposition 2.4.2 to the case where $\mathcal{U}$ is a vector bundle on $X$. The statement in this case follows immediately from Proposition 2.4.1.

Proposition 2.4.6. Let $X \rightarrow$ Spec $k$ be a proper curve over an algebraically closed field $k$, and let $\mathcal{U} \rightarrow X$ be a connected unipotent group scheme. If $\xi_{\mathcal{U}} \rightarrow X$ is $\mathcal{U}$-bundle, then the canonical morphism $\mathbb{B} \operatorname{Aut}\left(\xi_{\mathcal{U}}\right) \rightarrow \operatorname{Bun}_{\mathcal{U}}(X)$ is a closed immersion.

Proof. Fix a central series

$$
\{1\} \subseteq \mathcal{U}_{n} \subseteq \mathcal{U}_{n-1} \subseteq \cdots \subseteq \mathcal{U}_{1}=\mathcal{U}
$$

for $\mathcal{U}$. We prove by the proposition by induction on the length $n$.
For $n=0$ the claim is trivial. For $n>0$, we have a short exact sequence

$$
1 \longrightarrow \mathcal{U}_{n} \longrightarrow \mathcal{U} \longrightarrow \mathcal{U} / \mathcal{U}_{n} \longrightarrow 1
$$

where $\mathcal{U}_{n}$ is a vector bundle on $X$ and the claim holds for $\mathcal{U} / \mathcal{U}_{n}$. So Proposition 2.4.2 implies that the fibre product

is a trivial $\operatorname{Bun}_{\mathcal{U}_{n}}(X)$-torsor over $\mathbb{B} \operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right)$ and hence isomorphic to $\operatorname{Bun}_{\mathcal{U}_{n}}(X) / \operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right)$, where $\xi_{\mathcal{U} / \mathcal{U}_{n}}=\xi_{\mathcal{U}} \times^{\mathcal{U}} \mathcal{U} / \mathcal{U}_{n}$. Since the horizontal morphisms in (2.4.3) are closed immersions, we just have to show that the morphism

$$
\mathbb{B A u t}\left(\xi_{\mathcal{U}}\right) \longrightarrow \operatorname{Bun}_{\mathcal{U}_{n}}(X) / \operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right)
$$

is a closed immersion, which reduces by Proposition 2.4.1 to showing that the orbits of $\operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right)$ on the variety $H^{1}\left(X, \mathcal{U}_{n}\right)$ are closed. But this is immediate since $\operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right)$ acts through some algebraic group homomorphism $\operatorname{Aut}\left(\xi_{\mathcal{U} / \mathcal{U}_{n}}\right) \rightarrow \operatorname{Bun}_{\mathcal{U}_{n}}(X) \rightarrow H^{1}\left(X, \mathcal{U}_{n}\right)$, which must have closed image.

One can view (2) of Proposition 2.4.2 as a part of the long exact sequence in nonabelian cohomology applied to short exact sequences in which every term is a flat group scheme. However, it is also useful to consider nonabelian analogues of short exact sequences

$$
0 \longrightarrow U \longrightarrow V \longrightarrow F \longrightarrow 0,
$$

where $U$ and $V$ are vector bundles and $F$ is a torsion coherent sheaf. The following proposition gives one such analogue.

Proposition 2.4.7. Let $X \rightarrow S$ be a proper curve and let $\mathcal{U}^{\prime} \rightarrow \mathcal{U}$ be a homomorphism of connected unipotent group schemes over $X$ such that
(1) $\mathcal{U} \rightarrow \mathcal{U}^{\prime}$ is an isomorphism over an open subset of $X$ that is dense in every geometric fibre of $X \rightarrow S$, and
(2) for every morphism $S^{\prime} \rightarrow S$ with $S^{\prime}$ an affine scheme and every affine open $V \subseteq S^{\prime} \times{ }_{S} X$, $\Gamma\left(V, \mathcal{U}^{\prime}\right)$ is a normal subgroup of $\Gamma(V, \mathcal{U})$.

Then the stack $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}^{\prime}\right)$ is a group scheme over $S$, and $\operatorname{Bun}_{\mathcal{U}^{\prime} / S}(X) \rightarrow \operatorname{Bun}_{\mathcal{U} / S}(X)$ is naturally a $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}^{\prime}\right)$-torsor.

Proof. To show that the stack $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}^{\prime}\right)$ is a group scheme, consider its functor of points

$$
\begin{aligned}
F_{0}: \mathrm{Stk}_{/ S}^{o p} & \longrightarrow \mathrm{Grpd} \\
S^{\prime} & \longrightarrow \Gamma\left(X^{\prime}, \mathcal{U} / \mathcal{U}^{\prime}\right),
\end{aligned}
$$

where we write $X^{\prime}=S^{\prime} \times_{S} X$. Since $F_{0}$ satisfies fppf descent, it is determined by its restriction to the full subcategory $\mathcal{C}_{0} \subseteq \mathrm{Stk}_{/ S}$ spanned by the affine schemes $S^{\prime}$ over $S$ such that $X^{\prime}$ is itself a scheme. Moreover, $F_{0}$ extends to a functor

$$
\begin{aligned}
F_{0, \mathrm{Zar}}: \mathcal{C}_{0, \mathrm{Zar}}^{o p} & \longrightarrow \mathrm{Grpd} \\
\left(S^{\prime}, V \subseteq X^{\prime}\right) & \longmapsto \Gamma\left(V, \mathcal{U} / \mathcal{U}^{\prime}\right),
\end{aligned}
$$

where $\mathcal{C}_{0, \text { Zar }}$ is the category of pairs $\left(S^{\prime}, V \subseteq X^{\prime}\right)$ where $S^{\prime} \in \mathcal{C}_{0}$ and $V \subseteq X^{\prime}$ is a Zariski open subset. Since $\mathcal{U}^{\prime}$ is unipotent, every $\mathcal{U}^{\prime}$-torsor on an affine scheme is trivial, so whenever $V$ is affine we have that

$$
\Gamma\left(V, \mathcal{U} / \mathcal{U}^{\prime}\right)=\Gamma(V, \mathcal{U}) / \Gamma\left(V, \mathcal{U}^{\prime}\right)
$$

is the quotient of a group by a normal subgroup, and is hence a group. So by Zariski descent, $F_{0, \mathrm{Zar}}$ and hence $F_{0}$ is valued in groups, so $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}^{\prime}\right)$ is a group scheme over $S$ as claimed.

Similarly, to show that $\operatorname{Bun}_{\mathcal{U}^{\prime} / S}(X)$ is a $\Gamma_{S}\left(X, \mathcal{U} / \mathcal{U}^{\prime}\right)$-torsor, notice that by Proposition 2.3.6, $\operatorname{Bun}_{\mathcal{U}^{\prime} / S}(X)$ has functor of points

$$
\begin{aligned}
F: \operatorname{Stk}_{/ \operatorname{Bun}_{\mathcal{U} / S}(X)}^{o p} & \longrightarrow \operatorname{Grpd} \\
\left(S^{\prime}, \xi_{\mathcal{U}}\right) & \longmapsto \Gamma\left(X^{\prime}, \xi_{\mathcal{U}} / \mathcal{U}^{\prime}\right)
\end{aligned}
$$

which is again determined by the functor

$$
\begin{aligned}
F_{\text {Zar }}: \mathcal{C}_{\text {Zar }}^{o p} & \longrightarrow \mathrm{Grpd} \\
\left(S^{\prime}, \xi_{\mathcal{U}}, V \subseteq X^{\prime}\right) & \longmapsto \Gamma\left(V, \xi_{\mathcal{U}} / \mathcal{U}^{\prime}\right)=\Gamma\left(V, \xi_{\mathcal{U}}\right) / \Gamma\left(V, \mathcal{U}^{\prime}\right),
\end{aligned}
$$

where $\mathcal{C}_{\text {Zar }}$ is the category of triples $\left(S^{\prime}, \xi_{\mathcal{U}}, V \subseteq X^{\prime}\right)$ where $S^{\prime}$ is an affine scheme over $S$ with $X^{\prime}=S^{\prime} \times{ }_{S} X$ also a scheme, $\xi_{\mathcal{U}} \rightarrow X^{\prime}$ is a $\mathcal{U}$-torsor and $V \subseteq X^{\prime}$ is an open subset. It is clear that $F_{0, \mathrm{Zar}}\left(S^{\prime}, V\right)$ acts naturally on $F_{\mathrm{Zar}}\left(S^{\prime}, \xi_{\mathcal{U}}, V\right)$. Since $\xi_{\mathcal{U}}$ must be trivial restricted to $V$ when $V$ is affine, it follows that $F_{\mathrm{Zar}}\left(S^{\prime}, \xi_{\mathcal{U}},-\right)$ defines an $F_{0, \mathrm{Zar}}\left(S^{\prime},-\right)$-torsor on $X^{\prime}$ for fixed $S^{\prime}$. Finally, since $\mathcal{U}^{\prime} \rightarrow \mathcal{U}$ is generically an isomorphism on every fibre of $X \rightarrow S$, there exists an affine open $V \subseteq X^{\prime}$ and a subset $Z \subseteq V$ closed in $X^{\prime}$ such that $F_{0, \mathrm{Zar}}\left(S^{\prime}, \xi_{\mathcal{U}},-\right)$ is trivial on any open subset of $X^{\prime} \backslash Z$. So any trivialisation of $F_{\text {Zar }}$ over $V$ extends uniquely to a trivialisation on all of $X^{\prime}$, which implies that $\operatorname{Bun}_{\mathcal{U}^{\prime} / S}(X) \rightarrow \operatorname{Bun}_{\mathcal{U} / S}(X)$ is a $\Gamma_{S}\left(X, \mathcal{U}^{\prime} / \mathcal{U}\right)$ torsor as claimed.

### 2.5 Principal bundles under reductive groups and stability

We now turn to the much more subtle problem of describing the stack $\operatorname{Bun}_{G / S}(X)$ when $G$ is a reductive group, which in some sense will take up the bulk of this thesis. In this section, we recall some of the basic theory of reductive groups, establish some notation, and discuss the important notion of (semi)stability for principal bundles under a reductive group.

Unless otherwise specified, by a reductive group, we will always mean a split connected reductive group scheme over $\operatorname{Spec} \mathbb{Z}$. We will fix throughout this section a proper curve $X \rightarrow S$ over a stack $S$.

In the case of the simplest reductive groups, tori, the structure of $\operatorname{Bun}_{G / S}(X)$ is captured by the classical theory of Picard schemes.

Proposition 2.5.1. Let $T$ be a split torus over $\operatorname{Spec} \mathbb{Z}$ with character group $\mathbb{X}^{*}(T)=$ $\operatorname{Hom}\left(T, \mathbb{G}_{m}\right) \cong \mathbb{Z}^{l}$ and $X \rightarrow S$ a proper curve with reduced and irreducible geometric fibres. Then, with respect to the subgroup $H=T=Z(T)$, the rigidified stack of T-bundles is given by

$$
\operatorname{Bun}_{T / S}(X)_{r i g}=\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{X}^{*}(T), \operatorname{Pic}_{S}(X)\right) \cong \operatorname{Pic}_{S}(X)^{l},
$$

where $\operatorname{Pic}_{S}(X)$ is the relative Picard scheme of $X$ over $S$. Moreover, if we fix a section $x: S \rightarrow X$, then there is a unique section $\operatorname{Bun}_{T / S}(X)_{\text {rig }} \rightarrow \operatorname{Bun}_{T / S}(X)$ such that the pullback of the universal T-bundle along the map

$$
\operatorname{Bun}_{T / S}(X)_{r i g} \xrightarrow{x} \operatorname{Bun}_{T / S}(X)_{r i g} \times_{S} X \longrightarrow \operatorname{Bun}_{T / S}(X) \times_{S} X
$$

is trivial.
Remark 2.5.2. Since $\operatorname{Bun}_{T / S}(X) \rightarrow \operatorname{Bun}_{T / S}(X)_{\text {rig }}$ is a $T$-gerbe, the section of Proposition 2.5.1 determines an isomorphism $\operatorname{Bun}_{T / S}(X) \cong \operatorname{Bun}_{T / S}(X)_{\text {rig }} \times \mathbb{B} T$.

In the setup of Proposition 2.5.1, the degree map $\operatorname{Pic}_{S}(X) \rightarrow \mathbb{Z}$ determines a degree function

$$
\operatorname{deg}: \operatorname{Bun}_{T / S}(X) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{X}^{*}(T), \operatorname{Pic}_{S}(X)\right) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{X}^{*}(T), \mathbb{Z}\right)=\mathbb{X}_{*}(T)
$$

where $\mathbb{X}_{*}(T)=\operatorname{Hom}\left(\mathbb{G}_{m}, T\right)$ is the group of cocharacters of $T$. More explicitly, if $\xi_{T} \rightarrow X_{s}$ is a $T$-bundle over a geometric fibre of $X \rightarrow S$, then $\operatorname{deg}\left(\xi_{T}\right) \in \mathbb{X}_{*}(T)$ is the unique cocharacter satisfying

$$
\left\langle\lambda, \operatorname{deg}\left(\xi_{T}\right)\right\rangle=\operatorname{deg} \lambda\left(\xi_{T}\right)
$$

for all $\lambda \in \mathbb{X}^{*}(T)$, where

$$
\begin{aligned}
\langle-,-\rangle: \mathbb{X}^{*}(T) \times \mathbb{X}_{*}(T) & \longrightarrow \mathbb{Z} \\
(\lambda, \mu) & \longmapsto \lambda \circ \mu \in \operatorname{Hom}\left(\mathbb{G}_{m}, \mathbb{G}_{m}\right)=\mathbb{Z}
\end{aligned}
$$

is the natural pairing and we write $\lambda\left(\xi_{T}\right)=\xi_{T} \times{ }^{T} \mathbb{Z}_{\lambda}$ for the line bundle on $X_{s}$ associated to $\xi_{T}$ and the 1-dimensional representation $\mathbb{Z}_{\lambda}$ on which $T$ acts with weight $\lambda$.

Definition 2.5.3. If $\lambda \in \mathbb{X}_{*}(T)$, we write $\operatorname{Bun}_{T / S}^{\lambda}(X) \subseteq \operatorname{Bun}_{T / S}(X)$ for the open and closed substack of $T$-bundles of degree $\lambda$.

For $G$ an arbitrary reductive group, we can use the theory for tori to define numerical invariants of $G$-bundles. Observe that the reduced identity component $Z(G)^{\circ}$ of $Z(G)$ and the abelianisation $G /[G, G]$ of $G$ are tori, and that the morphism $Z(G)^{\circ} \rightarrow G /[G, G]$ is an isogeny.

Definition 2.5.4. Let $\xi_{G} \rightarrow X_{s}$ be a principal $G$-bundle on a geometric fibre of $X \rightarrow S$. The degree of $\xi_{G}$ is the degree of the associated $G /[G, G]$-bundle. The slope of $\xi_{G}$ is the image $\mu\left(\xi_{G}\right) \in \mathbb{X}_{*}\left(Z(G)^{\circ}\right)_{\mathbb{Q}}$ of $\operatorname{deg} \xi_{G}$ under the natural map

$$
\mathbb{X}_{*}(G /[G, G]) \longleftrightarrow \mathbb{X}_{*}(G /[G, G])_{\mathbb{Q}} \stackrel{\sim}{\leftarrow} \mathbb{X}_{*}\left(Z(G)^{\circ}\right)_{\mathbb{Q}}
$$

More generally, if $H$ is any group scheme such that $H / R_{u}(H)$ is split reductive, we will define the degree (resp., slope) of an $H$-bundle $\xi_{H} \rightarrow X_{s}$ to be the degree (resp., slope) of $\xi_{H} \times{ }^{H} H / R_{u}(H)$, and, given $d \in \mathbb{X}_{*}\left(H /\left(R_{u}(H) \cdot[H, H]\right)\right)\left(\right.$ resp., $\left.\mu \in \mathbb{X}_{*}\left(Z\left(H / R_{u}(H)\right)^{\circ}\right)_{\mathbb{Q}}\right)$ we will write $\operatorname{Bun}_{H / S}^{d}(X)$ (resp., $\left.\operatorname{Bun}_{H / S}^{\mu}(X)\right)$ for the open and closed substack of $\operatorname{Bun}_{H / S}(X)$ of $H$-bundles of degree $d$ (resp., slope $\mu$ ).

Remark 2.5.5. The terminology "degree" and "slope" can be justified as follows: for $G=$ $G L_{n}$, we have $G L_{n} /\left[G L_{n}, G L_{n}\right] \cong \mathbb{G}_{m}$ and $Z\left(G L_{n}\right)^{\circ} \cong \mathbb{G}_{m}$ and hence natural identifications $\mathbb{X}_{*}\left(G L_{n} /\left[G L_{n}, G L_{n}\right]\right) \cong \mathbb{Z}$ and $\mathbb{X}_{*}\left(Z\left(G L_{n}\right)^{\circ}\right)_{\mathbb{Q}} \cong \mathbb{Q}$ such that the degree (resp., slope) of a $G L_{n}$-bundle $\xi_{G L_{n}}$ is identified with the degree $\operatorname{deg} V$ (resp., slope $\mu(V)=\operatorname{deg} V / \operatorname{rank} V$ ) of the vector bundle $V$ associated to the standard representation. More generally, if $G$ is an arbitrary reductive group, $\xi_{G} \rightarrow X_{s}$ is a $G$-bundle of slope $\mu$ and $V$ is any $G$-representation on which $Z(G)^{\circ}$ acts with weight $\lambda$, then the associated vector bundle $\xi_{G} \times{ }^{G} V \rightarrow X_{s}$ has slope $\langle\lambda, \mu\rangle \in \mathbb{Q}$.

Before we can go further, we need to recall some of the basic structure theory of a reductive group $G$.

Definition 2.5.6. Let $G$ be a reductive group. If $U$ is any scheme, a Borel subgroup of $G_{U}=G \times U$ is a closed solvable subgroup $B \subseteq G_{U}$, flat over $U$ with connected fibres, such that for every geometric point $u$ : Spec $k \rightarrow U, B_{u} \subseteq G_{k}$ is maximal among closed connected solvable subgroups of $G_{k}$. The flag variety of $G$ is the $\mathbb{Z}$-scheme $F$ representing the functor Sch $^{o p} \rightarrow$ Set sending a scheme $U$ to the set of Borel subgroups of $G_{U}$.

The flag variety of a reductive group $G$ is always a connected projective $\mathbb{Z}$-scheme with a transitive action of $G$ by conjugation of subgroups, such that the universal Borel subgroup $\mathcal{B} \subseteq G \times F$ is given by

$$
\mathcal{B}=\{(g, x) \in G \times F \mid g x=x\}
$$

Since we are assuming our reductive groups are split, $F \rightarrow \mathrm{Spec} \mathbb{Z}$ has a section, defining a Borel subgroup $B \subseteq G$ and an isomorphism

$$
\begin{aligned}
G / B & \sim \\
g B & \longmapsto g B g^{-1}
\end{aligned}
$$

In view of this isomorphism and Proposition 2.3.6, the flag variety plays an important role in studying reductions of $G$-bundles to $B$.

Again because $G$ is split, there is a unique split torus $T$ over $\operatorname{Spec} \mathbb{Z}$, called the abstract Cartan subgroup, equipped with an isomorphism of group schemes over $F$

$$
\begin{equation*}
\mathcal{B} / R_{u}(\mathcal{B})=\mathcal{B} /[\mathcal{B}, \mathcal{B}] \cong T \times F \tag{2.5.1}
\end{equation*}
$$

where $R_{u}(H)$ denotes the unipotent radical of a group scheme $H$ and $[H, H]$ the commutator subgroup (the normal subgroup generated by commutators $g h g^{-1} h^{-1}$ for $g, h \in H$ ). For any choice of Borel subgroup $B \subseteq G$, there is a canonical isomorphism $B / R_{u}(B) \cong T$ given by pulling back (2.5.1) along the section $\operatorname{Spec} \mathbb{Z} \rightarrow F$ classifying $B$. If we choose a (split)
maximal torus $T^{\prime} \subseteq B$, which exists since $G$ is split, then we therefore get an isomorphism $T^{\prime} \cong B / R_{u}(B) \cong T$. Note that this last isomorphism depends on the choice of Borel subgroup $B$ containing $T^{\prime}$.

The reductive group $G$ is completely classified by its root datum $\left(\mathbb{X}^{*}(T), \Phi, \mathbb{X}_{*}(T), \Phi^{\vee}\right)$, where $\Phi \subseteq \mathbb{X}^{*}(T)$ is the set of roots and $\Phi^{\vee} \subseteq \mathbb{X}_{*}(T)$ the set of coroots, and implicitly we are given a bijection $\Phi \rightarrow \Phi^{\vee}, \alpha \mapsto \alpha^{\vee}$. If we choose a Borel subgroup and maximal torus $T \subseteq B \subseteq G$ (where we implicitly identify the maximal torus with the abstract Cartan subgroup via (2.5.1)), the set of roots $\Phi$ is simply the set of weights of $T$ acting on the Lie algebra $\mathfrak{g}$ of $G$. If $\alpha \in \Phi$ is a root, then the corresponding coroot $\alpha^{\vee}$ is defined by choosing homomorphism $\rho_{\alpha}: S L_{2} \rightarrow G$ whose derivative sends the strictly upper triangular matrices isomorphically onto the $\alpha$-weight space $\mathfrak{g}_{\alpha}$ and the strictly lower triangular matrices isomorphically onto the $(-\alpha)$-weight space $\mathfrak{g}_{-\alpha}$ and setting

$$
\alpha^{\vee}(t)=\rho_{\alpha}\left(\begin{array}{cc}
t & 0 \\
0 & t^{-1}
\end{array}\right) \quad \text { for } \quad t \in \mathbb{G}_{m} .
$$

We will adopt the convention that the set $\Phi_{-} \subseteq \Phi$ of negative roots is the set of nonzero weights of $T$ acting on $\operatorname{Lie}(B)$, and let $\Phi_{+}=-\Phi_{-}$be the corresponding set of positive roots. The root datum $\left(\mathbb{X}^{*}(T), \Phi, \mathbb{X}_{*}(T), \Phi^{\vee}\right)$ and the sets $\Phi_{+}$and $\Phi_{-}$are independent of the choice of $B$ and embedding $T \rightarrow B$.

We will write $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{l}\right\} \subseteq \Phi_{+}$and $\Delta^{\vee}=\left\{\alpha_{1}^{\vee}, \ldots, \alpha_{l}^{\vee}\right\} \subseteq \Phi_{+}^{\vee}$ for the sets of positive simple roots and coroots respectively. We also write $\left\{\varpi_{1}, \ldots, \varpi_{l}\right\}$ and $\left\{\varpi_{1}^{\vee}, \ldots, \varpi_{l}^{\vee}\right\}$ for the bases of $\left(\mathbb{Z} \Phi^{\vee}\right)^{\vee}$ and $(\mathbb{Z} \Phi)^{\vee}$ dual to $\Delta$ and $\Delta^{\vee}$ respectively. Note that $\mathbb{Z} \Phi=\mathbb{X}^{*}(T)$ if and only if the centre $Z(G)$ is trivial, and $\mathbb{Z} \Phi^{\vee}=\mathbb{X}_{*}(T)$ if and only if $G$ is simply connected and semisimple.

Definition 2.5.7. Let $P \subseteq G$ be a parabolic subgroup (i.e., a closed subgroup containing some Borel). The type of $P$ is the set $t(P) \subseteq \Delta$ of simple roots that are not weights of $T$ acting on $\operatorname{Lie}(P) \subseteq \mathfrak{g}$ for some (hence any) choice $T \subseteq B \subseteq P \subseteq G$ of Borel subgroup and maximal torus contained in $P$.

If $P \subseteq G$ is a parabolic subgroup, we write $T_{P}=P /[P, P]$. Note that $T_{P}$ is a torus, and is the unique quotient of $T$ with character group

$$
\mathbb{X}^{*}\left(T_{P}\right)=\left\{\lambda \in \mathbb{X}^{*}(T) \mid\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle=0 \text { for } \alpha_{i} \in \Delta \backslash t(P)\right\}
$$

In particular, $T_{P}$ depends only on the type $t(P)$.
Definition 2.5.8. If $P \subseteq G$ is a parabolic subgroup, we say that $\lambda \in \mathbb{X}^{*}\left(T_{P}\right)=\operatorname{Hom}\left(P, \mathbb{G}_{m}\right)$ is dominant if $\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle \geq 0$ for all $\alpha_{i} \in t(P)$.

Remark 2.5.9. For any $t \subseteq \Delta$, there is a $G$-homogeneous projective $\mathbb{Z}$-scheme $F_{t}$ parametrising parabolic subgroups $P$ of type $t(P)=t$, and fixing any such $P \subseteq G$ we have an isomorphism $F_{t} \cong G / P$. There is a canonical isomorphism

$$
\begin{aligned}
\mathbb{X}^{*}\left(T_{P}\right) & \xrightarrow{\sim} \operatorname{Pic}^{G}(G / P)=\operatorname{Pic}^{G}\left(F_{t}\right) \\
\lambda & \longmapsto \mathcal{L}_{\lambda}=G \times{ }^{P} \mathbb{Z}_{\lambda},
\end{aligned}
$$

where $\mathrm{Pic}^{G}$ denotes the group of $G$-linearised line bundles, which is independent of the choice of parabolic subgroup $P$ of type $t$. Our conventions are chosen so that $\lambda \in \mathbb{X}^{*}\left(T_{P}\right)$ is dominant if and only if the line bundle $\mathcal{L}_{\lambda}$ on $G / P$ is nef.

Returning to $G$-bundles, assume for the rest of this section that the curve $X \rightarrow S$ is smooth.

Definition 2.5.10. Fix a geometric point $s$ : Spec $k \rightarrow S$ and a principal $G$-bundle $\xi_{G} \rightarrow$ $X_{s}$. We say that $\xi_{G}$ is stable (resp., semistable) if for every reduction $\xi_{P}$ of $\xi_{G}$ to a parabolic subgroup $P \subseteq G$ and every dominant character $\lambda: P \rightarrow \mathbb{G}_{m}$ of $P$ that vanishes on the reduced identity component $Z(G)^{\circ}$ of the centre $Z(G)$, we have

$$
\left.\operatorname{deg}\left(\xi_{P} \times^{P} \mathbb{Z}_{\lambda}\right)>0 \quad \text { (resp., } \quad \operatorname{deg}\left(\xi_{P} \times{ }^{P} \mathbb{Z}_{\lambda}\right) \geq 0\right)
$$

We say that $\xi_{G}$ is unstable if it is not semistable.
Remark 2.5.11. By Proposition 2.3.6, the datum of a reduction $\xi_{P}$ of $\xi_{G}$ to $P$ is equivalent to a section of the associated flag variety bundle $\sigma: X_{s} \rightarrow \xi_{G} / P=\xi_{G} \times{ }^{G} F_{t(P)}$, and $\xi_{P} \times{ }^{P} \mathbb{Z}_{\lambda}=\sigma^{*} \mathcal{L}_{\lambda}^{\xi_{G}}$, where $\mathcal{L}_{\lambda}^{\xi_{G}}=\xi_{G} \times{ }^{P} \mathbb{Z}_{\lambda}=\xi_{G} \times{ }^{G} \mathcal{L}_{\lambda}$ is the natural line bundle on $\xi_{G} / P$ associated to $\mathcal{L}_{\lambda}$.

In the special case when $G=G L_{n}$, the associated vector bundle of the standard representation gives an identification of $\operatorname{Bun}_{G L_{n} / S}(X)$ with the relative stack of rank $n$ vector bundles on $X$. We will say that a vector bundle is stable (resp., semistable, unstable) if the corresponding principal $G L_{n}$-bundle is. We can also understand this notion more directly in terms of vector bundles as follows.

If $K$ is a nonzero coherent sheaf on a geometric fibre $X_{s}$ of $X \rightarrow S$, then the slope of $K$ is

$$
\mu(F)=\frac{\operatorname{deg} K}{\operatorname{rank} K},
$$

where we adopt the convention that $\mu(K)=+\infty$ if rank $K=0$.
Proposition 2.5.12. Let $V$ be a vector bundle on a geometric fibre $X_{s}$ of $X \rightarrow S$. The following are equivalent.
(1) The associated $G L_{n}$-bundle is stable (resp., semistable).
(2) If $K \hookrightarrow V$ is a nonzero subsheaf, then $\mu(F)<\mu(V)$ (resp., $\mu(K) \leq \mu(V)$ ).
(3) If $V \rightarrow Q$ is a nonzero quotient sheaf, then $\mu(Q)>\mu(V)$ (resp., $\mu(Q) \geq \mu(V))$.
(4) If $U \hookrightarrow V$ is a nonzero vector subbundle, then $\mu(U)<\mu(V)$ (resp., $\mu(U) \leq \mu(V)$ ).
(5) If $V \rightarrow W$ is a nonzero quotient bundle, then $\mu(W)>\mu(V)$ (resp., $\mu(W) \geq \mu(V)$ ).

Proposition 2.5.13. Let $X \rightarrow S$ be a smooth proper curve. Then the substack

$$
\operatorname{Bun}_{G / S}^{s s}(X) \subseteq \operatorname{Bun}_{G / S}(X)
$$

of semistable bundles $\xi_{G} \rightarrow X_{s}$ is open.
Proposition 2.5.13 is well-known. We will give a proof in $\S 3.6$ as an easy application of the theory of Kontesevich-Mori compactifications.

In characteristic 0 , semistability has very good functoriality properties under extension of structure group and pullback along morphisms between curves. In positive characteristic, many of these properties require a slightly stronger condition than semistability.

Definition 2.5.14. Let $s$ : Spec $k \rightarrow X$ be a geometric point and let $\xi_{G} \rightarrow X_{s}$ be a principal $G$-bundle. We say that $\xi_{G}$ is strongly semistable if either $k$ has characteristic 0 , or $k$ has characteristic $p>0$ and the Frobenius twists $\xi_{G} \times{ }^{G} G_{k}^{\left(p^{n}\right)}$ are semistable $G_{k}^{\left(p^{n}\right)}$-bundles for all $n \geq 0$, where $G_{k} \rightarrow G_{k}^{\left(p^{n}\right)}$ is the relative $p^{n}$-Frobenius for $G_{k}=G \times \operatorname{Spec} k$.

The class of strongly semistable $G$-bundles behaves well under extension of structure group.

Theorem 2.5.15 ([RR, Theorems 3.19 and 3.23]). Let $f: G \rightarrow H$ be a homomorphism of reductive groups such that $f\left(Z(G)^{\circ}\right) \subseteq Z(H)^{\circ}$, and let $\xi_{G} \rightarrow X_{s}$ be a strongly semistable $G$-bundle. Then the $H$-bundle $\xi_{G} \times{ }^{G} H$ is strongly semistable.

In particular, we have the following result concerning semistability of associated vector bundles.

Corollary 2.5.16. Let $\xi_{G} \rightarrow X_{s}$ be a strongly semistable $G$-bundle, and let $V$ be a representation of $G$ on which $Z(G)^{\circ}$ acts with a single weight. Then the associated vector bundle $\xi_{G} \times{ }^{G} V$ on $X_{s}$ is semistable.

Let $\xi_{G} \rightarrow X_{s}$ be an unstable $G$-bundle. Then by definition, there exists a parabolic subgroup $P \subseteq G$, a reduction $\xi_{P}$ of $\xi_{G}$ to $P$ and a dominant character $\lambda: P \rightarrow \mathbb{G}_{m}$ such that the line bundle $\xi_{P} \times^{P} \mathbb{Z}_{\lambda}$ has strictly negative degree. In fact, there is a canonical choice of such a reduction, which is in some sense as destabilising as possible.

Definition 2.5.17. Let $P \subseteq G$ be a parabolic subgroup with Levi factor $L$. We say that $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ is a Harder-Narasimhan vector for $P$ if

$$
\mathfrak{p}=\operatorname{Lie}(P)=\bigoplus_{\left\langle\lambda, \mu\left(\xi_{L}\right)\right\rangle \geq 0} \mathfrak{g}_{\lambda},
$$

where

$$
\mathfrak{g}=\bigoplus_{\lambda \in \mathbb{X}^{*}\left(Z(L)^{\circ}\right)} \mathfrak{g}_{\lambda}
$$

is the weight space decomposition under the action of the torus $Z(L)^{\circ}$.
Definition 2.5.18. Let $\xi_{G} \rightarrow X_{s}$ be a principal $G$-bundle, and $\xi_{P}$ a reduction of $\xi_{G}$ to a parabolic subgroup $P \subseteq G$ with Levi factor $L \cong P / R_{u}(P)$, then we say that the reduction $\xi_{P}$ is canonical, or Harder-Narasimhan, if the induced $L$-bundle $\xi_{L}$ is semistable and $\mu\left(\xi_{L}\right)$ is a Harder-Narasimhan vector for $P$.

Theorem 2.5.19 ([B1, Theorem 7.3]). Given a $G$-bundle $\xi_{G} \rightarrow X_{s}$, there exists a parabolic subgroup $P \subseteq G$, unique up to conjugation, and a unique Harder-Narasimhan reduction of $\xi_{G}$ to $P$.

Remark 2.5.20. When $G=G L_{n}$, Theorem 2.5.19 reduces to the statement that any vector bundle $V \rightarrow X_{s}$ of rank $n$ has a unique filtration

$$
0=V_{0} \subseteq V_{1} \subseteq \cdots \subseteq V_{m}=V
$$

such that $V_{i} / V_{i-1}$ is semistable for each $i$ and

$$
\mu\left(V_{1} / V_{0}\right)>\mu\left(V_{2} / V_{1}\right)>\cdots>\mu\left(V_{m} / V_{m-1}\right) .
$$

This filtration is called the Harder-Narasimhan filtration on $V$.

### 2.6 Principal bundles on curves of genus 0 and 1

In this section, we highlight some of the special features of the theory of principal bundles on a curve of low genus. As in $\S 2.5$, we fix a family of smooth curves $X \rightarrow S$, now assumed to be of genus $\leq 1$, and a reductive group $G$.

The first simplification of the general theory in this case is that semistability and strong semistability coincide.

Theorem 2.6.1 (E.g., [L1, Corollary 6.4]). Assume that $X_{s}$ has genus $\leq 1$. Then every semistable $G$-bundle on $X_{s}$ is strongly semistable.

As is well known, in this case the Harder-Narasimhan reduction of a $G$-bundle on $X_{s}$ can be reduced further to a Levi subgroup of $G$.

Proposition 2.6.2. Assume that $X_{s}$ is a curve of genus $g\left(X_{s}\right) \leq 1$, let $\xi_{G} \rightarrow X_{s}$ be an unstable principal bundle, and let $\xi_{P} \rightarrow X_{s}$ be its Harder-Narasimhan reduction. Fix a Levi subgroup $L \subseteq P$ (so that $L \rightarrow P / R_{u}(P)$ is an isomorphism) and set $\xi_{L}=\xi_{P} \times{ }^{P} L$. Then there is an isomorphism of $P$-bundles

$$
\xi_{P} \cong \xi_{L} \times{ }^{L} P
$$

In particular, the unstable $G$-bundle $\xi_{G}$ has a reduction to a semistable bundle for the Levi subgroup $L$.

Proof. Consider the fibre $\operatorname{Bun}_{P}\left(X_{s}\right)_{\xi_{L}}$ over $\xi_{L}$ of the morphism $\operatorname{Bun}_{P}\left(X_{s}\right) \rightarrow \operatorname{Bun}_{L}\left(X_{s}\right)$ induced by the homomorphism $P \rightarrow L$. By Proposition 2.4.2, we have a canonical isomorphism of stacks

$$
\operatorname{Bun}_{P}\left(X_{s}\right)_{\xi_{L}} \cong \operatorname{Bun}_{\mathcal{U}}\left(X_{s}\right)
$$

where $\mathcal{U} \rightarrow X_{s}$ is the group scheme $\mathcal{U}=\xi_{L} \times{ }^{L} R_{u}(P)$, and $L$ acts on the unipotent radical $R_{u}(P) \subseteq P$ by conjugation. To show that $\xi_{P} \cong \xi_{L} \times{ }^{L} P$, we need to show that the corresponding $\mathcal{U}$-bundle is trivial. By Corollary 2.4.4, it therefore suffices to give a central series

$$
\{1\}=\mathcal{U}_{n+1} \subseteq \mathcal{U}_{n} \subseteq \mathcal{U}_{n-1} \subseteq \cdots \subseteq \mathcal{U}_{1}=\mathcal{U}
$$

in which each quotient $\mathcal{U}_{i} / \mathcal{U}_{i+1}$ is a vector bundle with $H^{1}\left(X_{s}, \mathcal{U}_{i} / \mathcal{U}_{i+1}\right)=0$. Let $0<\mu_{1}<$ $\mu_{2}<\cdots<\mu_{n}$ be all the possible positive values of $\left\langle\alpha, \mu\left(\xi_{L}\right)\right\rangle$ for $\alpha \in \Phi$, let

$$
\{1\}=U_{n+1} \subseteq U_{n} \subseteq \cdots \subseteq U_{1}=R_{u}(P)
$$

be the $L$-invariant filtration defined by

$$
U_{i}=\prod_{\substack{\alpha \in \Phi \\\left\langle\alpha, \mu\left(\xi_{L}\right)\right\rangle \geq \mu_{i}}} U_{\alpha}
$$

for $1 \leq i \leq n+1$, where $U_{\alpha} \cong \mathbb{G}_{a}$ is the root subgroup corresponding to $\alpha$, and define $\mathcal{U}_{i}=\xi_{L} \times{ }^{L} U_{i}$. Since $X_{s}$ has genus $g\left(X_{s}\right) \leq 1, \xi_{L}$ is strongly semistable by Theorem 2.6.1, so for each weight $\lambda$ of $Z(L)^{\circ}$ acting on $U_{i} / U_{i+1}$, the vector bundle $\xi_{L} \times{ }^{L}\left(U_{i} / U_{i+1}\right)_{\lambda}$ associated to the $\lambda$-weight space is semistable of slope $\left\langle\lambda, \mu\left(\xi_{L}\right)\right\rangle=\mu_{i}>0 \geq 2 g\left(X_{s}\right)-2$ by Theorem 2.5.15. So Lemma 2.6.3 below implies that $H^{1}\left(X_{s}, \mathcal{U}_{i} / \mathcal{U}_{i+1}\right)=0$ as required.

Lemma 2.6.3. Let $V$ be a semistable vector bundle on a curve $X_{s}$ of genus $g$ such that $\mu(V)>2 g-2$. Then $H^{1}\left(X_{s}, V\right)=0$.

Proof. Assume for a contradiction that $H^{1}\left(X_{s}, V\right) \neq 0$. Then by Serre duality, $H^{0}\left(X_{s}, V^{\vee} \otimes\right.$ $\left.K_{X_{s}}\right) \neq 0$, so there exists a nonzero morphism $V \rightarrow K_{X_{s}}$. Let $M \subseteq K_{X_{s}}$ be the sheaftheoretic image of this morphism. Then $M$ is a quotient line bundle of $V$, so by semistability we have

$$
\operatorname{deg} M=\mu(M) \geq \mu(V)>2 g-2=\operatorname{deg} K_{X_{s}}
$$

which is a contradiction.

Remark 2.6.4. Proposition 2.6.2 implies that when $X_{s}$ has genus $g \leq 1$, the HarderNarasimhan filtration of any vector bundle $V$ splits as $V=\bigoplus_{i} U_{i}$, where the $U_{i}=V_{i} / V_{i-1}$ are semistable vector bundles of distinct slopes. We call such a decomposition a HarderNarasimhan decomposition of $V$. Note that the terms $U_{i}$ are unique, but the particular decomposition of $V$ is not.

Curves of genus $\leq 1$ also have the following convenient property, which is shared by curves of higher genus only in characteristic 0 .

Proposition 2.6.5. Fix a parabolic subgroup $P \subseteq G$ with Levi factor $L$ and $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ a Harder-Narasimhan vector for $P$. Then the morphism

$$
\begin{equation*}
\operatorname{Bun}_{P / S}^{s s, \mu}(X) \longrightarrow \operatorname{Bun}_{G / S}(X) \tag{2.6.1}
\end{equation*}
$$

is a locally closed immersion, where $\operatorname{Bun}_{P / S}^{s s, \mu}(X)$ is the open substack of $P$-bundles such that the induced L-bundle $\xi_{L}$ is semistable with slope $\mu\left(\xi_{L}\right)=\mu$.

Proof. By Theorem 2.5.19, (2.6.1) is injective on points, so it suffices to show that it is unramified, i.e., that the relative tangent complex

$$
\mathbb{T}=\mathbb{R} \pi_{*}\left(\xi_{P}^{u n i} \times^{P} \mathfrak{g} / \mathfrak{p}\right)
$$

is a vector bundle concentrated in degree 1, where $\pi: \operatorname{Bun}_{P / S}^{s s, \mu}(X) \times{ }_{S} X \rightarrow \operatorname{Bun}_{P / S}^{s s, \mu}(X)$ is the projection onto the first factor, and $\xi_{P}^{u n i}$ is the universal $P$-bundle. The restriction of $\xi_{P}^{u n i} \times{ }^{P} \mathfrak{g} / \mathfrak{p}$ to each fibre of $\pi$ has a filtration by semistable vector bundles of negative slopes, so $H^{0}(\mathbb{T})=0$. So $\mathbb{T}=H^{1}(\mathbb{T})[-1]$ is indeed a vector bundle in degree 1 as claimed.

Definition 2.6.6. If $X \rightarrow S$ is a family of curves of genus $\leq 1$, and $\xi_{G} \rightarrow X_{s}$ is an unstable $G$-bundle on a geometric fibre of $X \rightarrow S$ with a Harder-Narasimhan reduction to $P \subseteq G$ with slope $\mu$, then the Harder-Narasimhan locus of $\xi_{G}$ is the locally closed substack

$$
\operatorname{Bun}_{P / S}^{s s, \mu}(X) \longleftrightarrow \operatorname{Bun}_{G / S}(X)
$$

In the case of a curve of genus 1 , we can also compute the codimension of the HarderNarasimhan loci. Note that if $L \subseteq P$ is a Levi factor of a parabolic subgroup $P \subseteq G$, then choosing a maximal torus and Borel $T \subseteq B \subseteq G$ such that $B \subseteq P$ and $T \subseteq L$ we have an inclusion $Z(L) \subseteq T$ and hence a homomorphism $\mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}} \rightarrow \mathbb{X}_{*}(T)_{\mathbb{Q}}$ and a pairing

$$
\langle-,-\rangle: \mathbb{X}^{*}(T) \times \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}} \longrightarrow \mathbb{Q}
$$

Proposition 2.6.7. In the situation of Proposition 2.6.5 if in addition $X \rightarrow S$ is a family of curves of genus 1 , then (2.6.1) has codimension $-\langle 2 \rho, \mu\rangle$, where $2 \rho \in \mathbb{X}^{*}(T)$ is the sum of the positive roots.

Proof. From the proof of Proposition 2.6.5, is suffices to show that the vector bundle $H^{1}(\mathbb{T})$ has rank $-\langle 2 \rho, \mu\rangle$. But by Riemann-Roch,

$$
\operatorname{rank} H^{1}(\mathbb{T})=-\operatorname{deg}\left(\xi_{P} \times^{P} \mathfrak{g} / \mathfrak{p}\right)=-\sum_{\substack{\alpha \in \Phi \\\langle\alpha, \mu\rangle<0}}\langle\alpha, \mu\rangle=-\sum_{\alpha \in \Phi_{+}}\langle\alpha, \mu\rangle=-\langle 2 \rho, \mu\rangle
$$

as claimed.
Proposition 2.6.8. Let $X \rightarrow S$ be a family of smooth curves of genus 1 , and let $G$ be simply connected and simple of rank $l \in \mathbb{Z}_{>0}$. Then the locus of unstable bundles has codimension $l+1$ in $\operatorname{Bun}_{G / S}(E)$.

Proof. Since the locus of unstable bundles in $\operatorname{Bun}_{G / S}(X)$ is the union of the images of $\operatorname{Bun}_{P / S}^{s s, \mu}(X) \rightarrow \operatorname{Bun}_{G / S}(X)$ where $P$ ranges over all parabolic subgroups containing some fixed Borel subgroup $B$ and $\mu$ ranges over all Harder-Narasimhan vectors for $P$, by Proposition 2.6.7, it suffices to prove that

$$
-\langle 2 \rho, \mu\rangle \geq l+1
$$

for all such $P$ and $\mu$ for which $\operatorname{Bun}_{P / S}^{s s, \mu}(X)$ is nonempty, with equality for some such choice of $P$ and $\mu$. Note that $\operatorname{Bun}_{P / S}^{s s, \mu}(X)$ is nonempty if and only if $\left\langle\varpi_{i}, \mu\right\rangle \in \mathbb{Z}$ for all $\alpha_{i} \in t(P)$.

Consider the case where $P$ is a maximal parabolic of type $t(P)=\left\{\alpha_{i}\right\}$. Then the conditions on $\mu$ are equivalent to

$$
\mu \in \mathbb{Z}_{>0-\text {-span }}\left\{-\frac{\varpi_{i}^{\vee}}{\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle}\right\} .
$$

So

$$
-\langle 2 \rho, \mu\rangle \geq \frac{\left\langle 2 \rho, \varpi_{i}^{\vee}\right\rangle}{\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle},
$$

which by [FM2, Lemma 3.3.2] is always $\geq l+1$, with equality achieved for some choice of $\alpha_{i}$.

More generally, suppose that $P \subseteq G$ is an arbitrary parabolic, choose $\alpha_{i} \in t(P)$, and let $L_{i} \supseteq L$ be the Levi factor of the unique maximal parabolic of type $\left\{\alpha_{i}\right\}$ containing $P$. Let $\tilde{\mu} \in \mathbb{X}_{*}\left(Z\left(L_{i}\right)^{\circ}\right)_{\mathbb{Q}}$ be the unique element such that $\left\langle\varpi_{i}, \tilde{\mu}\right\rangle=\left\langle\varpi_{i}, \mu\right\rangle$. Then

$$
\tilde{\mu} \in \mathbb{Z}_{>0} \text {-span }\left\{-\frac{\varpi_{i}^{\vee}}{\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle}\right\},
$$

so $-\langle 2 \rho, \tilde{\mu}\rangle \geq l+1$ as shown above. But

$$
-\langle 2 \rho, \tilde{\mu}\rangle=-\sum_{\substack{\alpha \in \Phi_{+} \\\left\langle\alpha, \varpi_{i}^{V}\right\rangle>0}}\langle\alpha, \tilde{\mu}\rangle=-\sum_{\substack{\alpha \in \Phi_{+} \\\left\langle\alpha, \varpi_{i}^{V}\right\rangle>0}}\langle\alpha, \mu\rangle \leq-\langle 2 \rho, \mu\rangle
$$

since

$$
\sum_{\substack{\alpha \in \Phi_{+} \\\left\langle\alpha, \varpi_{i}^{v}\right\rangle>0}} \alpha \in \mathbb{Z}_{\geq 0}-\operatorname{span}\left\{\varpi_{i}\right\},
$$

so we are done.

## Chapter 3

## Stable maps and Kontsevich-Mori compactifications

In $\S 2.5$, we saw that reductions to parabolic subgroups $P \subseteq G$ provide a useful tool in the study of principal $G$-bundles for $G$ a reductive group. In this chapter, we will discuss how these methods can be refined by allowing such reductions to degenerate. That is, we will construct a relative compactification, called the Kontsevich-Mori compactification, of the stack of $P$-bundles over the stack of $G$-bundles.

Degenerations of this kind play an especially vital role in elliptic Springer theory. If $E$ is an elliptic curve and $B \subseteq G$ a Borel subgroup, we will see that there is a proper morphism $\operatorname{Bun}_{B}^{0}(E) \rightarrow \operatorname{Bun}_{G}^{s s}(E)$ with many of the good properties of the classical Springer resolution $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. To produce a good theory for unstable bundles, this morphism must be extended to a proper morphism to $\operatorname{Bun}_{G}(E)$ using some relative compactification.

The Kontsevich-Mori compactification $\mathrm{KM}_{P, G}(X)$ is constructed by thinking of $\operatorname{Bun}_{P}(X)$ as the stack of pairs $\left(\xi_{G}, \sigma\right)$, where $\xi_{G} \rightarrow X$ is a $G$-bundle, and $\sigma: X \rightarrow \xi_{G} / P$ is a section of the associated bundle of partial flag varieties. One then allows the section $\sigma$ to degenerate using Kontsevich's theory of stable maps. The Kontsevich-Mori compactification has been studied in the context of elliptic Springer theory in [GSB] and for curves of arbitrary genus in [C]. It has the convenient properties that the total space $\mathrm{KM}_{P, G}(X)$ is smooth, the complement of $\operatorname{Bun}_{P}(X)$ is a divisor with normal crossings, and the morphism $\mathrm{KM}_{P, G}(X) \rightarrow \operatorname{Bun}_{G}(X)$ is proper with finite relative stabilisers (and is even representable in low codimension).

We remark that there is another relative compactification $\operatorname{Drin}_{P, G}(X)$ of $\operatorname{Bun}_{P}(X)$ over $\operatorname{Bun}_{G}(X)$, called the Drinfeld compactification, which is popular in the literature. It was introduced in [BG], and is constructed by thinking of a reduction of $\xi_{G}$ to $P$ as a system of subbundles inside vector bundles associated to representations of $G$ and allowing these subbundles to degenerate to subsheaves. The Drinfeld compactification has the advantage over the Kontsevich-Mori that the morphism $\operatorname{Drin}_{P, G}(X) \rightarrow \operatorname{Bun}_{G}(X)$ is representable, but the disadvantage that the total space $\operatorname{Drin}_{P, G}(X)$ is singular.

For many of the degeneration arguments we will use, either compactification would serve equally well. However, we have chosen to work with the Kontsevich-Mori since, for the applications to elliptic Springer theory, smoothness is much more useful for our purposes than representability.

### 3.1 Stable and prestable maps

In this section, we review the basic ideas of Kontsevich's theory of stable maps. We begin by recalling some definitions.

Definition 3.1.1. Let $S$ be a stack and let $g, n \in \mathbb{Z}_{\geq 0}$. An $n$-pointed prestable curve of genus $g$ over $S$ is a tuple $\left(\pi: C \rightarrow S, x_{1}, \ldots, x_{n}\right)$, where $\pi: C \rightarrow S$ is proper, flat and representable, and $x_{i}: S \rightarrow C$ is a section of $\pi$ such that for every geometric point $s: \operatorname{Spec} k \rightarrow S$, the
fibre $C_{s}$ over $s$ is a reduced connected curve of arithmetic genus $g$, with at worst nodal singularities, and $x_{1}(s), \ldots, x_{n}(s) \in C_{s}$ are distinct smooth points of $C_{s}$.

Definition 3.1.2. Let $\pi: X \rightarrow S$ be a proper representable morphism of stacks and let $g, n \in \mathbb{Z}_{\geq 0}$. An n-pointed prestable map to $X$ over $S$ of genus $g$ is a tuple $(f: C \rightarrow$ $\left.X, x_{1}, \ldots, x_{n}\right)$ where $\left(\pi \circ f: C \rightarrow S, x_{1}, \ldots, x_{n}\right)$ is a prestable curve over $S$ of genus $g$.

There are obvious notions of isomorphism of prestable curves and maps, and of pullback of prestable curves and maps along a morphism $S^{\prime} \rightarrow S$. So given any stack $S$ and a proper representable morphism $X \rightarrow S$, there are functors

$$
\underline{\mathfrak{M}}_{g, n, S} \text { and } \underline{M}_{g, n, S}(X):(\mathrm{Stk} / S)^{o p} \longrightarrow \mathrm{Grpd}
$$

sending a stack $S^{\prime} \rightarrow S$ over $S$ to the groupoids of $n$-pointed prestable curves of genus $g$ over $S^{\prime}$ and $n$-pointed prestable maps to $X \times{ }_{S} S^{\prime}$ over $S^{\prime}$ of genus $g$ respectively.

Theorem 3.1.3. The functors $\mathfrak{M}_{g, n, S}$ and $\mathfrak{M}_{g, n, S}(X)$ are representable by Artin stacks $\mathfrak{M}_{g, n, S}$ and $\mathfrak{M}_{g, n, S}(X)$ locally of finite type over $S$.

If $S=\operatorname{Spec} k$ for some field $k$, we will sometimes write $\mathfrak{M}_{g, n}=\mathfrak{M}_{g, n, S}$ and $\mathfrak{M}_{g, n}(X)=$ $\mathfrak{M}_{g, n, S}(X)$.

Proposition 3.1.4. The stack $\mathfrak{M}_{g, n, S}$ is smooth over $S$ of relative dimension $3 g-3+n$.
Proof. Let $p: \operatorname{Spec} k \rightarrow \mathfrak{M}_{g, n, S}$ be a geometric point over $s: \operatorname{Spec} k \rightarrow S$ and let $\left(C, x_{1}, \ldots, x_{n}\right)$ be the corresponding prestable curve over Spec $k$. There is a tangent-obstruction complex for $\mathfrak{M}_{g, n, S} \rightarrow S$ at $p$ given by

$$
\mathbb{T}=\mathbb{R} \Gamma\left(C, \mathbb{T}_{C / k}\left(-x_{1}-\cdots-x_{n}\right)\right)[1]
$$

But Proposition 2.1.5 implies that $\mathbb{T}_{C / k}$ is a complex supported in degrees 0 and 1 with $H^{1}\left(\mathbb{T}_{C / k}\right)$ torsion, so $H^{i}(\mathbb{T})=0$ for $i>0$. So $\mathfrak{M}_{g, n, S} \rightarrow S$ is smooth at $p$ with relative tangent complex $\mathbb{T}$. The relative dimension is given by

$$
\begin{aligned}
\chi(\mathbb{T}) & =-\chi\left(C, \mathbb{T}_{C / k}\left(-x_{1}-\cdots-x_{n}\right)\right) \\
& =-\operatorname{deg}\left(\operatorname{det}\left(\mathbb{T}_{C / k}\left(-x_{1}-\cdots-x_{n}\right)\right)\right)+g-1 \\
& =-\operatorname{deg}\left(\omega_{C / k}^{-1}\left(-x_{1}-\cdots-x_{n}\right)\right)+g-1 \\
& =3 g-3+n
\end{aligned}
$$

by Riemann-Roch.
Definition 3.1.5. Let $\pi: X \rightarrow S$ be proper and representable. We say that an $n$-pointed prestable map $\left(f: C \rightarrow X, x_{1}, \ldots, x_{n}\right)$ over $S$ is stable if for every geometric point $s$ : Spec $k \rightarrow$ $S$, the prestable map $\left(f_{s}: C_{s} \rightarrow X_{s}, x_{1}(s), \ldots, x_{n}(s)\right)$ over $k$ has finite automorphism group. We write

$$
M_{g, n, S}(X) \subseteq \mathfrak{M}_{g, n, S}(X)
$$

for the open substack of stable maps.
Proposition 3.1.6. Let $X \rightarrow S$ be proper and representable and let ( $f: C \rightarrow X, x_{1}, \ldots, x_{n}$ ) be a prestable map to $X$ over $S$. The following are equivalent.
(1) The prestable map $\left(f, x_{1}, \ldots, x_{n}\right)$ is a stable map.
(2) The line bundle $\omega_{C / S}\left(x_{1}+\cdots x_{n}\right)$ on $C$ is ample relative to $X$, where $\omega_{C / S}$ is the relative dualising sheaf of $C \rightarrow S$.
(3) For every geometric point $s$ : Spec $k \rightarrow S$ and every irreducible component $C^{\prime} \subseteq C$ such that $C^{\prime}$ is contracted under $f$, the normalisation $\tilde{C}^{\prime}$ either has genus $\geq 2$, has genus 1 and at least one point mapping to a node or marked point in $C$, or has genus 0 and at least 3 points mapping to nodes or marked points in $C$.

Given a projective morphism $\pi: X \rightarrow S$ and an $S$-ample line bundle $\mathcal{O}_{X}(1)$ on $X$, we write

$$
M_{g, n, S}(X, d) \subseteq M_{g, n, S}(X)
$$

for the open and closed substack of stable maps $\left(f, x_{1}, \ldots, x_{n}\right)$ such that $\operatorname{deg} f^{*} \mathcal{O}_{X}(1)=d$.
Theorem 3.1.7. Let $\pi: X \rightarrow S$ be a projective morphism of stacks and fix an $S$-ample line bundle $\mathcal{O}_{X}(1)$ on $X$. Then:
(1) For all $g, n \in \mathbb{Z}_{\geq 0}$, the morphism $M_{g, n, S}(X) \rightarrow S$ satisfies the valuative criterion for properness.
(2) For any $d \in \mathbb{Z}$, the morphism $M_{g, n, S}(X, d) \rightarrow S$ is of finite type, hence proper.
(3) Let $U \subseteq M_{g, n, S}(X, d)$ be the open substack of points with trivial automorphism group scheme relative to $S$. Then $U$ is quasi-projective over $S$. In particular, if every point of $M_{g, n, S}(X, d)$ has trivial automorphism group scheme relative to $S$, then $M_{g, n, S}(X, d) \rightarrow$ $S$ is projective.

It is often useful to consider finer restrictions on the degree of a stable map than offered directly by Theorem 3.1.7. For example, if $X \rightarrow$ Spec $k$ is projective and $\operatorname{NS}(X)$ is the group of line bundles on $X$ modulo numerical equivalence, then for any $\beta \in \operatorname{Hom}(\operatorname{NS}(X), \mathbb{Z})$, Theorem 3.1.7 implies that the open and closed substack $M_{g, n}(X, \beta) \subseteq M_{g, n}(X)=M_{g, n, k}(X)$ is proper, and in particular of finite type, over Spec $k$. However, for families $X \rightarrow S$, the group $\mathrm{NS}(X)$ may not be large enough to capture all degree information. For example, let $E \rightarrow$ Spec $k$ be an elliptic curve over $k$, and consider the morphism $X=\operatorname{Spec} k \rightarrow$ $\mathbb{B} E=S$. Then $X \rightarrow S$ is a well-behaved family of smooth curves of genus 1 (and we have that $M_{g, n, S}(X)=M_{g, n}(E) / E$ is the stack of stable maps to $E$ modulo translation), but $\mathrm{NS}(X)=\mathrm{NS}(\operatorname{Spec} k)=0$. To work around this issue, we use the following hack.

Suppose we are given a smooth surjection $U \rightarrow S$ with connected geometric fibres, an abelian group $H$, and a homomorphism $\phi: H \rightarrow \mathrm{NS}_{U}\left(U \times_{S} X\right)$ to the group of line bundles on $U \times_{S} X$ modulo $U$-numerical equivalence. (Here we will say that two line bundles $L_{1}, L_{2} \in \operatorname{Pic}\left(U \times_{S} X\right)$ are $U$-numerically equivalent if for every geometric point $u$ : Spec $k \rightarrow U$ and every closed curve $C \subseteq X_{u}$, the restrictions of $L_{1}$ and $L_{2}$ to $C$ have the same degree.) If $s:$ Spec $k \rightarrow S$ is a geometric point and $f: C \rightarrow X_{s}$ is a prestable map, then there is a well-defined homomorphism

$$
\begin{aligned}
\operatorname{deg}(f)=\operatorname{deg}_{(U, \phi)}(f): H & \longrightarrow \mathbb{Z} \\
h & \longmapsto \operatorname{deg} f_{u}^{*} L_{h},
\end{aligned}
$$

where $u: \operatorname{Spec} k \rightarrow U$ is any lift of $s, f_{u}: C \rightarrow X_{s}=X_{u} \hookrightarrow U \times_{S} X$ the induced morphism, and $L_{h} \in \operatorname{Pic}\left(U \times_{S} X\right)$ any representative for $\phi(h) \in \mathrm{NS}_{U}\left(U \times_{S} X\right)$.

Definition 3.1.8. In the setup above, we call $(U, \phi)$ a degree datum, and we call $\operatorname{deg}_{(U, \phi)}(f)$ the degree of $f$ (with respect to $(U, \phi)$ ). If $\beta \in \operatorname{Hom}(H, \mathbb{Z})$, we write $\mathfrak{M}_{g, n, S}(X, \beta) \subseteq$
$\mathfrak{M}_{g, n, S}(X)$ and $M_{g, n, S}(X, \beta) \subseteq M_{g, n, S}(X)$ for the open and closed substacks of prestable and stable maps of degree $\beta$.

Corollary 3.1.9. Suppose that we are given $X \rightarrow S$ and a degree datum $(U, \phi)$ as above, and assume that $\phi(H) \subseteq \mathrm{NS}_{U}\left(U \times_{S} X\right)$ contains the class of a $U$-ample line bundle on $U \times_{S} X$. Then for all $\beta \in \operatorname{Hom}(H, \mathbb{Z})$, the stack $M_{g, n, S}(X, \beta)$ is proper, and in particular of finite type, over $S$.

Proof. By definition, we have

$$
U \times_{S} M_{g, n, S}(X, \beta)=M_{g, n, U}\left(U \times_{S} X, \beta\right)
$$

Since $U \rightarrow S$ is a smooth surjection, by descent for proper morphisms it suffices to show that $M_{g, n, U}\left(U \times_{S} X, \beta\right) \rightarrow U$ is proper. But if $h \in H$ is such that $\phi(h) \in \mathrm{NS}_{U}\left(U \times_{S} X\right)$ is the class of a $U$-ample line bundle $\mathcal{O}_{U \times_{S} X}(1)$, then $M_{g, n, U}\left(U \times_{S} X, \beta\right)$ is a union of connected components of $M_{g, n, U}\left(U \times_{S} X, \beta(h)\right)$, hence proper over $U$ by Theorem 3.1.7.

By construction, if $f: X \rightarrow Y$ is a morphism of proper representable stacks over $S$, then there is an induced morphism $\mathfrak{M}_{g, n, S}(X) \rightarrow \mathfrak{M}_{g, n, S}(Y)$ given by composition with $f$. For each $i \in\{1, \ldots, n\}$, there is also a morphism $\mathfrak{M}_{g, n, S}(X) \rightarrow \mathfrak{M}_{g, n-1, S}(X)$ given by forgetting the $i$ th marked point. In general, these morphisms do not restrict to morphisms $M_{g, n, S}(X) \rightarrow M_{g, n, S}(Y)$ or $M_{g, n, S}(X) \rightarrow M_{g, n-1, S}(X)$. This can be rectified, however, using the following construction.

Definition 3.1.10. Given a prestable map $\left(f: C \rightarrow X, x_{1}, \ldots, x_{n}\right)$ over $S$ and a representable morphism $U \rightarrow S$, we say that a morphism $g: C \rightarrow U$ over $S$ stabilises $\left(f, x_{1}, \ldots, x_{n}\right)$ if for every geometric point $s$ : Spec $k \rightarrow S$ and every irreducible component $C^{\prime}$ of $C_{s}$ such that $\left.\omega_{C / S}\left(x_{1}, \ldots, x_{n}\right)\right|_{C^{\prime}}$ is not ample relative to $X_{s}, g$ contracts $C^{\prime}$ to a $k$-point in $U_{s}$. A stabilisation of $\left(f, x_{1}, \ldots, x_{n}\right)$ is an initial object in the category of morphisms stabilising $\left(f, x_{1}, \ldots, x_{n}\right)$.

From the definition, it is clear that the stabilisation $g: C \rightarrow \tilde{C}$ of $\left(f, x_{1}, \ldots, x_{n}\right)$ is unique up to unique isomorphism if it exists, and that the morphism $f: C \rightarrow X$ factors as $g \circ \tilde{f}$ for some unique morphism $\tilde{f}: \tilde{C} \rightarrow X$.

Proposition 3.1.11. Let $\left(f: C \rightarrow X, x_{1}, \ldots, x_{n}\right)$ be an n-pointed prestable map of genus $g$ over $S$ and assume that either $f$ is non-constant on every geometric fibre of $C \rightarrow S$ or that $2 g+n \geq 3$. Then there exists a stabilisation $g: C \rightarrow \tilde{C}$ of $\left(f, x_{1}, \ldots, x_{n}\right)$ such that the tuple $\left(\tilde{f}: \tilde{C} \rightarrow X, g \circ x_{1}, \ldots, g \circ x_{n}\right)$ is an n-pointed stable map to $X$. Moreover, the formation of stabilisations commutes with base change.

Proof. This is proved in [BM, Proposition 3.10].
Remark 3.1.12. When $S=\operatorname{Spec} k$, the stabilisation of a prestable map $\left(f: C \rightarrow X, x_{1}, \ldots, x_{n}\right)$ can be constructed explicitly by contracting all rational components of $C$ contracted under $f$ that have at most 2 nodes and marked points combined.

Using Proposition 3.1.11, there is a canonical morphism $\mathfrak{M}_{g, n, S}(X)^{\prime} \rightarrow M_{g, n, S}(X)$ sending a prestable map to its stabilisation, where $\mathfrak{M}_{g, n, S}(X)^{\prime} \subseteq \mathfrak{M}_{g, n, S}(X)$ is the open and closed substack of maps satisfying the condition of Proposition 3.1.11. So if $f: X \rightarrow Y$ is a morphism of proper representable stacks over $S$, then we get a morphism $M_{g, n, S}(X)^{\prime} \rightarrow$ $\mathfrak{M}_{g, n, S}(Y) \rightarrow M_{g, n, S}(Y)$ by composition followed by stabilisation, and for all $i \in\{1, \ldots, n\}$ a morphism $M_{g, n, S}(X)^{\prime \prime} \rightarrow \mathfrak{M}_{g, n-1, S}(X)^{\prime} \rightarrow M_{g, n-1, S}(X)$ given by forgetting the $i$ th marked
point followed by stabilisation, where $M_{g, n, S}(X)^{\prime}, M_{g, n, S}(X)^{\prime \prime} \subseteq M_{g, n, S}(X)$ are the substacks where the relevant stabilisation morphisms are defined.

By construction, for any $i \in\{1, \ldots, n\}$, the morphism $M_{g, n, S}(X)^{\prime \prime} \rightarrow M_{g, n-1, S}(X)$ forgetting the $i$ th marked point fits into a commutative diagram

where $\left(C_{g, n, S}(X) \rightarrow M_{g, n, S}(X) \times{ }_{S} X, x_{1}, \ldots, x_{n}\right)$ is the universal $n$-pointed stable map and $C_{g, n, S}(X)^{\prime \prime}$ the preimage of $M_{g, n, S}(X)^{\prime \prime}$, such that the induced morphism

$$
C_{g, n, S}(X)^{\prime \prime} \longrightarrow C_{g, n-1, S}(X) \times_{M_{g, n-1, S}(X)} M_{g, n, S}(X)^{\prime \prime}
$$

is the stabilisation of $\left(C_{g, n, S}(X)^{\prime \prime} \rightarrow M_{g, n, S}(X)^{\prime \prime} \times_{S} X, x_{1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}\right)$. Composing the top arrow of (3.1.1) with the section $x_{i}: M_{g, n, S}(X) \rightarrow C_{g, n, S}(X)$, we get a morphism

$$
\begin{equation*}
M_{g, n, S}(X)^{\prime \prime} \longrightarrow C_{g, n-1, S}(X) \tag{3.1.2}
\end{equation*}
$$

Proposition 3.1.13. The morphism (3.1.2) is an isomorphism.
Proof. This is [BM, Corollary 4.6].

### 3.2 Dual graphs and gluing

One very useful property of the theory of stable maps is that maps with singular domain curves can be constructed by gluing together maps with simpler domains. In this section, we review the combinatorics governing this construction and study its geometric behaviour (Propositions 3.2.17 and 3.2.18) at the level of moduli stacks.

Our definitions mostly follow [BM, §1], although there are some differences in conventions for markings of graphs.

Definition 3.2.1. A graph is a tuple $\tau=\left(F_{\tau}, V_{\tau}, j_{\tau}, \partial_{\tau}\right)$, where $F_{\tau}$ and $V_{\tau}$ are sets, $j_{\tau}: F_{\tau} \rightarrow$ $F_{\tau}$ is an involution and $\partial_{\tau}: F_{\tau} \rightarrow V_{\tau}$ is a function. We call $F_{\tau}$ the set of flags (or half-edges), $V_{\tau}$ the set of vertices, $S_{\tau}=\left\{f \in F_{\tau} \mid j_{\tau}(f)=f\right\}$ the set of tails and $E_{\tau}=\left\{\left\{f_{1}, f_{2}\right\} \in E_{\tau} \mid\right.$ $\left.j_{\tau}\left(f_{1}\right)=f_{2} \neq f_{1}\right\}$ the set of edges. For $v \in V_{\tau}$, we also write $F_{\tau}(v)=\left\{f \in F_{\tau} \mid \partial_{\tau}(f)=v\right\}$ for the set of flags adjacent to $v$. We say that $v_{1}, v_{2} \in V_{\tau}$ are adjacent if there exists $\left\{f_{1}, f_{2}\right\} \in E_{\tau}$ such that $\partial_{\tau}\left(f_{i}\right)=v_{i}$. We say that $\tau$ is connected if all vertices are equivalent under the equivalence relation generated by adjacency.

If $\tau$ is a graph, we will draw a diagram representing $\tau$ as follows. For every vertex $v \in V_{\tau}$, we draw a corresponding node o. If $e=\left\{f_{1}, f_{2}\right\} \in E_{\tau}$, we draw a line segment connecting $\partial_{\tau}\left(f_{1}\right)$ and $\partial_{\tau}\left(f_{2}\right)$, and if $f \in S_{\tau}$ is a tail, we draw a line segment attached to $\partial_{\tau}(f)$ at one end. For example, we draw

for the graph with $V_{\tau}=\left\{v_{1}, v_{2}\right\}, F_{\tau}=\left\{f_{1}, f_{2}, f_{3}\right\}, j_{\tau}\left(f_{1}\right)=f_{1}, j_{\tau}\left(f_{2}\right)=f_{3}, j_{\tau}\left(f_{3}\right)=f_{2}$, $\partial_{\tau}\left(f_{1}\right)=v_{1}, \partial_{\tau}\left(f_{2}\right)=v_{1}$ and $\partial_{\tau}\left(f_{3}\right)=v_{2}$.

Definition 3.2.2. If $H$ is an abelian group, an $H$-graph is a tuple $(\tau, g, \beta)$, where $\tau$ is a connected graph, and $g: V_{\tau} \rightarrow \mathbb{Z}_{\geq 0}$ and $\beta: V_{\tau} \rightarrow \operatorname{Hom}(H, \mathbb{Z})$ are functions. The degree of $(\tau, g, \beta)$ is

$$
\operatorname{deg}(\tau)=\sum_{v \in V_{\tau}} \beta(v) \in \operatorname{Hom}(H, \mathbb{Z})
$$

and the genus of $(\tau, g, \beta)$ is

$$
g(\tau)=\sum_{v \in V_{\tau}} g(v)+\left|V_{\tau}\right|-\left|E_{\tau}\right|-1
$$

The main point of $H$-graphs is that they keep track of information about the irreducible components of a prestable map. In what follows, if $A$ is any finite set of size $|A|$, then $\mathfrak{M}_{g, A, S}(X) \cong \mathfrak{M}_{g,|A|, A}(X)$ and $M_{g, A, S}(X) \cong M_{g,|A|, S}(X)$ denote the stacks of prestable and stable maps $\left(f: C \rightarrow X,\left(x_{a}\right)_{a \in A}\right)$ with points marked by $A$, defined in the obvious way.

Definition 3.2.3. Suppose that $\left(U, \phi: H \rightarrow \mathrm{NS}_{U}\left(U \times_{S} X\right)\right.$ ) is a degree datum for a proper representable morphism $X \rightarrow S, s:$ Spec $k \rightarrow S$ is a geometric point and $(f: C \rightarrow$ $\left.X_{s},\left(x_{a}\right)_{a \in A}\right)$ is a prestable map over Spec $k$ with points marked by $A$. Write $\tilde{C} \rightarrow C$ for the normalisation of $C$. The dual graph of $f$ is the $H$-graph $\tau$ with $V_{\tau}$ equal to the set of connected components of $\tilde{C}, F_{\tau}$ equal to the union of $\left\{x_{a} \mid a \in A\right\}$ with the set of preimages of nodes of $C$ in $\tilde{C}, j_{\tau}: F_{\tau} \rightarrow F_{\tau}$ the involution fixing the marked points and interchanging the two preimages of each node, $\partial_{\tau}: F_{\tau} \rightarrow V_{\tau}$ the map sending a point in $\tilde{C}$ to the connected component it lies on, $g: V_{\tau} \rightarrow \mathbb{Z}_{\geq 0}$ the map sending a vertex $v \in V_{\tau}$ to the genus of the corresponding connected component $C_{v}$ of $\tilde{C}$, and $\beta: V_{\tau} \rightarrow \operatorname{Hom}(H, \mathbb{Z})$ the map sending $v \in V_{\tau}$ to the degree $\left.\operatorname{deg}_{(U, \phi)} f\right|_{C_{v}}$.

Remark 3.2.4. If $f: C \rightarrow X_{s}$ is a prestable map with dual graph $\tau$ as above, then $C$ has genus $g(\tau)$ and degree $\operatorname{deg}(\tau)$.

Definition 3.2.5. Let $X \rightarrow S$ be a proper representable morphism of stacks and fix a degree datum $\left(U, \phi: H \rightarrow \operatorname{NS}_{U}\left(U \times_{S} X\right)\right.$ ). If $\tau$ is an $H$-graph, then we define stacks $\mathfrak{M}_{S}(X, \tau)$ and $M_{S}(X, \tau)$ of $\tau$-marked prestable and stable maps to $X$ as the fibre products

and

where the products denote iterated fibre products over $S$, the bottom arrows are the natural diagonals, and the vertical arrows on the right send a family of (pre)stable maps ( $p_{v}: C_{v} \rightarrow$ $\left.X,\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)$ to $\left(\left(p_{\partial_{\tau}\left(f_{1}\right)}\left(x_{f_{1}}\right), p_{\partial_{\tau}\left(f_{2}\right)}\left(x_{f_{2}}\right)\right)\right)_{\left\{f_{1}, f_{2}\right\} \in E_{\tau}}$.

Remark 3.2.6. Morally, one should think of the stack $\mathfrak{M}_{S}(X, \tau)$ as the stack of prestable maps $\left(f: C \rightarrow X, x_{1}, \ldots, x_{n}\right)$ together with a partition of the irreducible components of $C$ into subcurves with marked points, intersections, genera and degrees labelled by $\tau$. This picture is partially realised by the following construction.

Definition 3.2.7. Let $p=\left(p_{v}: C_{v} \rightarrow X,\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)_{v \in V_{\tau}}$ be a $\tau$-marked prestable map to $X$ over $S$. If $S^{\prime} \rightarrow S$ is representable then a morphism to $S^{\prime}$ gluing $p$ is a morphism

$$
q=\left(q_{v}\right)_{v \in V_{\tau}}: \coprod_{v \in V_{\tau}} C_{v} \longrightarrow S^{\prime}
$$

together with a commutative diagram

for all $e=\left\{f_{1}, f_{2}\right\} \in E_{\tau}$. A gluing of $p$ is an initial object in the category of stacks $S^{\prime}$ representable over $S$ equipped with a morphism to $S^{\prime}$ gluing $p$.

Note that for any $\tau$-marked stable map $p$ as above, the morphism $p=\left(p_{v}\right)_{v \in V_{\tau}}: \coprod_{v \in V_{\tau}} C_{v} \rightarrow$ $X$ is canonically a morphism gluing $p$. So if $q: \coprod_{v \in V_{\tau}} C_{v} \rightarrow C$ is a gluing of $p$, then $p$ factors canonically as $p=\tilde{p} \circ q$ for some morphism

$$
\tilde{p}: C \longrightarrow X
$$

Proposition 3.2.8. Assume that $p=\left(p_{v}: C_{v} \rightarrow X,\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)_{v \in V_{\tau}}$ is a $\tau$-marked prestable map to $X$ over $S$. Then there exists a gluing $q: \coprod_{v \in V_{\tau}} C_{v} \rightarrow C$ of $p$ such that the tuple

$$
\left(\tilde{p}: C \rightarrow X,\left(q \circ x_{f}\right)_{f \in S_{\tau}}\right)
$$

is a prestable map to $X$ over $S$ of genus $g(\tau)$ and degree $\operatorname{deg}(\tau)$. Moreover, the formation of gluings commutes with base change.

Proof. This follows easily from [BM, Proposition 2.4 and Proposition 2.5].
Definition 3.2.9. We say that an $H$-graph $\tau$ is stable if for all $v \in V_{\tau}$ with $\beta(v)=0$, we have $2 g(v)+\left|F_{\tau}(v)\right| \geq 3$.

Proposition 3.2.8 ensures that there is a morphism

$$
\begin{equation*}
\mathfrak{M}_{S}(X, \tau) \longrightarrow \mathfrak{M}_{g(\tau), S_{\tau}, S}(X, \operatorname{deg}(\tau)) \tag{3.2.1}
\end{equation*}
$$

sending a $\tau$-marked prestable map to its gluing, which one can easily see restricts to a morphism

$$
M_{S}(X, \tau) \longrightarrow M_{g(\tau), S_{\tau}, S}(X, \operatorname{deg}(\tau))
$$

if the $H$-graph $\tau$ is stable. It is clear from the definitions that (3.2.1) gives a surjection from the open substack $\mathfrak{M}_{S}^{\circ}(X, \tau)$ where all domain curves are smooth onto the subset of prestable maps with dual graph $\tau$. The following relation between $H$-graphs keeps track of how degenerations of the domain curves in $\mathfrak{M}_{S}(X, \tau)$ affect the dual graph of the glued curve.

Definition 3.2.10. Let $\tau$ and $\tau^{\prime}$ be graphs. A contraction $\psi: \tau \rightarrow \tau^{\prime}$ is a pair of functions $\psi_{V}: V_{\tau} \rightarrow V_{\tau^{\prime}}$ and $\psi^{F}: F_{\tau^{\prime}} \rightarrow F_{\tau}$ (note the opposite directions for vertices and flags) such that
(1) $\psi_{V}$ is surjective and $\psi^{F}$ is injective
(2) the diagram

commutes,
(3) $j_{\tau} \circ \psi^{F}=\psi^{F} \circ j_{\tau^{\prime}}$,
(4) the induced map $\psi^{S}: S_{\tau^{\prime}} \rightarrow S_{\tau}$ is a bijection, and
(5) $\psi_{V}$ factors through a bijection $\left(V_{\tau} / \sim\right) \rightarrow V_{\tau^{\prime}}$, where $\sim$ is the equivalence relation generated by $\partial_{\tau}(f) \sim \partial_{\tau} \circ j_{\tau}(f)$ for $f \in F_{\tau} \backslash \psi^{F}\left(F_{\tau^{\prime}}\right)$.

If $\psi: \tau \rightarrow \tau^{\prime}$ is a contraction and $v^{\prime} \in V_{\tau^{\prime}}$ we write $\psi^{-1}\left(v^{\prime}\right)$ for the (automatically connected) graph with $V_{\psi^{-1}\left(v^{\prime}\right)}=\psi_{V}^{-1}\left(v^{\prime}\right), F_{\psi^{-1}\left(v^{\prime}\right)}=\left\{f \in F_{\tau} \mid \partial_{\tau}(f)=v^{\prime}\right\}, \partial_{\psi^{-1}\left(v^{\prime}\right)}=$ $\left.\partial_{\tau}\right|_{F_{\psi^{-1}\left(v^{\prime}\right)}}$ and $j_{\psi^{-1}\left(v^{\prime}\right)}$ defined by

$$
j_{\psi^{-1}\left(v^{\prime}\right)}(f)= \begin{cases}j_{\tau}(f), & \text { if } j_{\tau}(f) \in F_{\psi^{-1}\left(v^{\prime}\right)} \\ f, & \text { otherwise }\end{cases}
$$

for $f \in F_{\psi^{-1}\left(v^{\prime}\right)}$. Note that $\psi^{F}$ defines a bijection $S_{\psi^{-1}\left(v^{\prime}\right)} \cong F_{\tau^{\prime}}\left(v^{\prime}\right)$.
Remark 3.2.11. Intuitively, one should view a contraction $\psi=\left(\psi_{V}, \psi^{F}\right): \tau \rightarrow \tau^{\prime}$ as a map contracting the edges in $F_{\tau} \backslash \psi^{F}\left(F_{\tau^{\prime}}\right)$ and identifying their endpoints.

Definition 3.2.12. Let $H$ be an abelian group, and let $\tau$ and $\tau^{\prime}$ be $H$-graphs. A contraction $\psi: \tau \rightarrow \tau^{\prime}$ is a contraction of the underlying graphs such that for all $v^{\prime} \in V_{\tau^{\prime}}$ we have $\beta\left(v^{\prime}\right)=\operatorname{deg} \psi^{-1}\left(v^{\prime}\right)$ and $g\left(v^{\prime}\right)=g\left(\psi^{-1}\left(v^{\prime}\right)\right)$, where we view $\psi^{-1}\left(v^{\prime}\right) \subseteq \tau$ as an $H$-graph by restriction of $g$ and $\beta$ from $V_{\tau}$ to $V_{\psi^{-1}\left(v^{\prime}\right)}$.

Given $X \rightarrow S$, a degree datum $\left(U, \phi: H \rightarrow \mathrm{NS}_{U}\left(U \times_{S} X\right)\right)$ and a contraction $\psi: \tau \rightarrow \tau^{\prime}$, the gluing morphisms

$$
\prod_{v \in V_{\psi-1}\left(v^{\prime}\right)} \mathfrak{M}_{g(v), F_{\tau}(v), S}(X, \beta(v)) \longrightarrow \mathfrak{M}_{g\left(v^{\prime}\right), S_{\psi-1}\left(v^{\prime}\right), S}\left(X, \beta\left(v^{\prime}\right)\right)=\mathfrak{M}_{g\left(v^{\prime}\right), F_{\tau^{\prime}}\left(v^{\prime}\right), S}\left(X, \beta\left(v^{\prime}\right)\right)
$$

fit together to define a morphism

$$
\psi_{*}: \mathfrak{M}_{S}(X, \tau) \longrightarrow \mathfrak{M}_{S}\left(X, \tau^{\prime}\right)
$$

which, just as for (3.2.1), restricts to a morphism $M_{S}(X, \tau) \rightarrow M_{S}\left(X, \tau^{\prime}\right)$ if $\tau$ is a stable $H$-graph. It is easy to see that if $\psi: \tau \rightarrow \tau^{\prime}$ and $\psi^{\prime}: \tau^{\prime} \rightarrow \tau^{\prime \prime}$ are contractions, then we have canonically $\left(\psi^{\prime} \circ \psi\right)_{*}=\left(\psi^{\prime}\right)_{*} \circ \psi_{*}$, where composition of contractions is defined in the obvious way.

Remark 3.2.13. If $\tau^{\prime}$ is an $H$-graph with one vertex and no edges, and $\psi: \tau \rightarrow \tau^{\prime}$ is a contraction, then we have $\mathfrak{M}_{S}\left(X, \tau^{\prime}\right)=\mathfrak{M}_{g(\tau), S_{\tau}, S}(X, \operatorname{deg}(\tau))$, and $\psi_{*}$ agrees with (3.2.1).

Definition 3.2.14. Let $\tau$ be an $H$-graph, let $p=\left(\left(p_{v}: C_{v} \rightarrow X_{s},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)\right)_{v \in V_{\tau}}$ be a $\tau$-marked prestable map, and let $\tau_{v}$ denote the dual graph of the prestable map ( $p_{v}: C_{v} \rightarrow$ $\left.X_{s},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)$. The dual graph of $\tau$ is the contraction $\psi_{p}: \tau_{p} \rightarrow \tau$, where $\tau_{p}$ is $H$-graph
with vertices $V_{\tau_{p}}=\coprod_{v \in V_{\tau}} V_{\tau_{v}}$, flags $F_{\tau_{p}}=\coprod_{v \in V_{\tau}} F_{\tau_{v}}, \partial_{\tau_{p}}=\coprod_{v \in V_{\tau}} \partial_{\tau_{v}}, j_{\tau_{p}}: F_{\tau_{p}} \rightarrow F_{\tau_{p}}$ defined by

$$
j_{\tau_{p}}(f)= \begin{cases}j_{\tau_{v}}(f), & \text { if } f \notin S_{\tau_{v}} \\ j_{\tau}(f), & \text { if } f \in S_{\tau_{v}}=F_{\tau}(v),\end{cases}
$$

for $f \in F_{\tau_{v}}, v \in V_{\tau}$, and the functions $\beta$ and $g$ are inherited from the functions on $V_{\tau_{v}}$ for $v \in V_{\tau}$ in the obvious way. The contraction $\psi_{p}$ is defined by $\left(\psi_{p}\right)_{V}\left(v^{\prime}\right)=v$ for $v^{\prime} \in V_{\tau_{v}} \subseteq V_{\tau_{p}}$, and $\psi_{p}^{F}(f)=x_{f} \in F_{\tau_{v}}$ for $f \in F_{\tau}(v)$ and $v \in V_{\tau}$.

Remark 3.2.15. The dual graph $\tau_{p}$ of a $\tau$-marked prestable map $p$ is nothing but the dual graph of the gluing of $p$.

We make the following observation.
Lemma 3.2.16. Suppose that $\psi: \tau \rightarrow \tau^{\prime}$ is a contraction of H-graphs. Then for any $\tau^{\prime}$-marked prestable map $p=\left(p_{v^{\prime}}: C_{v^{\prime}} \rightarrow X_{s},\left(x_{f}\right)_{f \in F_{\tau^{\prime}}\left(v^{\prime}\right)}\right)_{v^{\prime} \in V_{\tau^{\prime}}}$ over a geometric point $s$ : $\operatorname{Spec} k \rightarrow S$, the groupoid of $k$-points of the preimage $\psi_{*}^{-1}(p)$ is isomorphic to the set of contractions $\tau_{p} \rightarrow \tau$ over $\tau^{\prime}$, where $\tau_{p} \rightarrow \tau^{\prime}$ is the dual graph of $p$. Moreover, the subset of $\tau$-marked prestable maps in $\psi_{*}^{-1}(p)$ with smooth domain curves corresponds to the subset of contractions $\tau_{p} \rightarrow \tau$ that are isomorphisms (i.e., bijections on both vertices and flags).

Proposition 3.2.17. Let $\psi: \tau \rightarrow \tau^{\prime}$ be a contraction of H-graphs. Then the morphism

$$
\psi_{*}: \mathfrak{M}_{S}(X, \tau) \longrightarrow \mathfrak{M}_{S}\left(X, \tau^{\prime}\right)
$$

is finite and unramified.
Proof. First note that we can assume without loss of generality that $F_{\tau} \backslash \psi^{F}\left(F_{\tau^{\prime}}\right)$ consists of a single edge $e=\left\{f_{1}, f_{2}\right\}$, since all contractions are compositions of contractions with this property. In this case, there is a pullback square

where $v^{\prime}=\psi_{V}\left(\partial_{\tau}\left(f_{1}\right)\right)=\psi_{V}\left(\partial_{\tau}\left(f_{2}\right)\right) \in V_{\tau^{\prime}}$. It therefore suffices to prove the claim when $\tau$ has exactly one edge and $\tau^{\prime}$ has a single vertex and no edges.

We first show that $\psi_{*}$ is unramified. Since $\psi_{*}$ is locally of finite type, it suffices to show that it is formally unramified at every geometric point of $\mathfrak{M}_{S}(X, \tau)$.

Suppose that $s: \operatorname{Spec} k \rightarrow S$ is a geometric point and that $p=\left(p_{v}: C_{v} \rightarrow X_{s},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)_{v \in V_{\tau}}$ is a $\tau$-marked stable map to $X_{s}$ over Spec $k$ with gluing $\left(\tilde{p}: C \rightarrow X,\left(x_{f}\right)_{f \in S_{\tau}}\right)$, and consider the morphism

$$
\begin{equation*}
\mathfrak{M}_{\tau, l o c} \longrightarrow \mathfrak{M}_{\tau^{\prime}, l o c}, \tag{3.2.2}
\end{equation*}
$$

of functors from the category $\mathrm{Art}_{k}$ of Artinian local rings with residue field $k$ to the 2category of groupoids, where

$$
\mathfrak{M}_{\tau, l o c}(A)=\operatorname{Hom}\left(\operatorname{Spec} A, \mathfrak{M}_{S}(X, \tau)\right) \times_{\operatorname{Hom}\left(\operatorname{Spec} k, \mathfrak{M}_{S}(X, \tau)\right)}\{p\}
$$

and

$$
\mathfrak{M}_{\tau^{\prime}, l o c}(A)=\operatorname{Hom}\left(\operatorname{Spec} A, \mathfrak{M}_{S}\left(X, \tau^{\prime}\right)\right) \times_{\operatorname{Hom}\left(\operatorname{Spec} k, \mathfrak{M}_{S}\left(X, \tau^{\prime}\right)\right)}\{\tilde{p}\} .
$$

There is a Cartesian diagram

where for $A \in \operatorname{Art}_{k}, \mathfrak{D}_{\tau, e}(A)$ is the groupoid of flat deformations of the augmented complete local rings $R_{f_{1}}=\widehat{\mathcal{O}}_{C_{\partial_{\tau}\left(f_{1}\right)}, x_{f_{1}}} \cong k \llbracket x_{f_{1}} \rrbracket$ and $R_{f_{2}}=\widehat{\mathcal{O}}_{C_{\partial_{\tau}\left(f_{2}\right)}, x_{f_{2}}} \cong k \llbracket x_{f_{2}} \rrbracket$ and $\mathfrak{D}_{e}(A)$ is the groupoid of flat deformations over $A$ of the local ring $R_{e}=\widehat{\mathcal{O}}_{C, x_{e}} \cong k \llbracket x_{f_{1}}, x_{f_{2}} \rrbracket /\left(x_{f_{1}} x_{f_{2}}\right)$. But a straightforward deformation theory computation shows that the morphism $\mathfrak{D}_{\tau, e} \rightarrow \mathfrak{D}_{e}$ has a tangent-obstruction complex $\mathbb{T}=k[-1]$, and is hence formally unramified. So (3.2.2) is formally unramified, and hence so is $\psi_{*}$.

We next show that $\psi_{*}$ satisfies the valuative criterion for properness. Let $R$ be a complete discrete valuation ring with fraction field $K$, and suppose we are given $\operatorname{Spec} R \rightarrow S$, a prestable map $p=\left(f: C \rightarrow X_{R},\left(x_{f}\right)_{f \in S_{\tau^{\prime}}}\right)$ over $\operatorname{Spec} R$ and a $\tau$-marked prestable map $\tilde{p}_{K}=\left(\left(f_{v, K}: C_{v, K} \rightarrow X_{K},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)\right)_{v \in V_{\tau}}$ over Spec $K$ with gluing $q_{K}: \coprod_{v \in V_{\tau}} C_{v, K} \rightarrow$ $C_{K}$. First note that $q_{K}$ defines a collection of stable maps to $C_{K}$, and that any extension of $\tilde{p}_{K}$ to a $\tau$-marked prestable map over $\operatorname{Spec} R$ with gluing $p$ defines an extension of this collection of stable maps to $C$ over $\operatorname{Spec} R$. So the uniqueness part of the formal criterion for properness follows from the corresponding property for stable maps to $C$. For the existence part, again by the formal criterion of properness of stable maps to $C$, after replacing $R$ with a finite extension if necessary, we can extend $q_{K}$ to a $\tau$-marked stable map $q=\left(q_{v}: C_{v} \rightarrow\right.$ $\left.C,\left(\bar{x}_{f}\right)_{f \in F_{\tau}(v)}\right)_{v \in V_{\tau}}$ over Spec $R$, and the composition with $f: C \rightarrow X_{R}$ produces a $\tau$-marked prestable map $\tilde{p}=\left(\tilde{p}_{v}: C_{v} \rightarrow X,\left(\bar{x}_{f}\right)_{f \in F_{\tau}(v)}\right)_{v \in V_{\tau}}$ to $X_{R}$. The gluing of $q$ is a stable map to $C$ that is an isomorphism over $\operatorname{Spec} K$, and hence an isomorphism on all of $\operatorname{Spec} R$, so we deduce that $p$ is the gluing of $\tilde{p}$.

Finally, by Lemma 3.2.16, the groupoids of points of all geometric fibres of $\psi_{*}$ are finite sets with no automorphisms. Since $\psi_{*}$ is unramified and in particular relatively DeligneMumford, $\psi_{*}$ is therefore representable. Moreover, since $\psi_{*}$ satisfies the valuative criterion for properness, it is universally closed. So if $U \rightarrow \mathfrak{M}_{S}\left(X, \tau^{\prime}\right)$ is a morphism from a scheme and $x \in U$, since $\psi_{*}^{-1}(x)$ is finite we can find a finite type open $V \subseteq \mathfrak{M}_{S}(X, \tau) \times_{\mathfrak{M}_{S}\left(X, \tau^{\prime}\right)} U$ containing $\psi_{*}^{-1}(x)$. Setting

$$
U^{\prime}=U \backslash \pi\left(\mathfrak{M}_{S}(X, \tau) \times_{\mathfrak{M}_{S}\left(X, \tau^{\prime}\right)} U \backslash V\right),
$$

for $\pi: \mathfrak{M}_{S}(X, \tau) \times_{\mathfrak{M}_{S}\left(X, \tau^{\prime}\right)} U \rightarrow U$ the natural projection, we have that $U^{\prime} \subseteq U$ is open and contains $x$ and that $\psi_{*}^{-1}\left(U^{\prime}\right) \subseteq V$ is of finite type. So the morphism $\psi_{*}$ is of finite type (hence of finite presentation since everything is locally Noetherian). So $\psi_{*}$ is proper, representable and quasi-finite, hence finite.

In the following proposition, we write $\mathfrak{M}_{g, n, S}^{\circ}(X, \beta) \subseteq \mathfrak{M}_{g, n, S}(X, \beta)$ and $M_{g, n, S}^{\circ}(X, \beta) \subseteq$ $M_{g, n, S}(X, \beta)$ for the open substacks of (pre)stable maps where the domain curve is smooth, and $\mathfrak{M}_{S}^{\circ}(X, \tau) \subseteq \mathfrak{M}_{S}(X, \tau)$ and $M_{S}^{\circ}(X, \tau) \subseteq M_{S}(X, \tau)$ for the preimages of

$$
\prod_{v \in V_{\tau}} \mathfrak{M}_{g(v), F_{\tau}(v), S}^{\circ}(X, \beta(v)) .
$$

Proposition 3.2.18. Let $\tau$ be an $H$-graph. For any contraction of $H$-graphs $\psi: \tau^{\prime} \rightarrow \tau$, the morphism

$$
\begin{equation*}
\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}_{\tau}\left(\tau^{\prime}\right) \longrightarrow \mathfrak{M}_{S}(X, \tau) \tag{3.2.3}
\end{equation*}
$$

induced by $\psi_{*}$ is a locally closed immersion, and the morphism

$$
\begin{equation*}
\coprod_{\tau^{\prime} \rightarrow \tau} \mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}_{\tau}\left(\tau^{\prime}\right) \longrightarrow \mathfrak{M}_{S}(X, \tau) \tag{3.2.4}
\end{equation*}
$$

is bijective on geometric points, where the coproduct is taken over all isomorphism classes of $H$-graphs with a contraction onto $\tau$. Moreover, if the image of $\phi: H \rightarrow \mathrm{NS}_{U}\left(U \times_{S} X\right)$ contains the class of a $U$-ample line bundle and the $H$-graph $\tau$ is stable, then the morphism

$$
\begin{equation*}
\coprod_{\tau^{\prime} \rightarrow \tau} M_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}_{\tau}\left(\tau^{\prime}\right) \longrightarrow M_{S}(X, \tau) \tag{3.2.5}
\end{equation*}
$$

is also bijective on geometric points, where the coproduct is now taken over all isomorphism classes of stable $H$-graphs with a contraction onto $\tau$.

Proof. Fix a geometric point $p=\left(\left(p_{v}: C_{v} \rightarrow X_{s},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)\right)_{v \in V_{\tau}}$ of $\mathfrak{M}_{S}^{\circ}(X, \tau)$ over $s$ : Spec $k \rightarrow S$. Then Lemma 3.2.16 implies that, up to isomorphism, there exists a unique contraction $\psi: \tau^{\prime} \rightarrow \tau$ of $H$-graphs such that $p$ is in the image of $\left.\psi_{*}\right|_{\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right)}$, where we take $\tau^{\prime}$ to be the dual graph of $p$, and that $\operatorname{Aut}_{\tau}\left(\tau^{\prime}\right)$ acts freely and transitively on the fibre $\psi_{*}^{-1}(p)$. This implies that (3.2.4) is bijective on geometric points, and that (3.2.3) is a locally closed immersion since $\psi_{*}$ is finite and unramified. Finally, to show that (3.2.5) is bijective on geometric points, we note that under the hypothesis on the degree datum, the dual graph of a $\tau$-marked stable map is stable, and that for $\tau^{\prime}$ stable, a prestable map in $M_{S}^{\circ}\left(X, \tau^{\prime}\right)$ is stable if and only if its image under $\psi_{*}$ is stable.

Remark 3.2.19. We stress that the decomposition (3.2.4) is in general weaker than a stratification: it is not necessarily the case that the closure of one term is a union of others. However, we will see that in good cases (e.g., Propositions 3.3.7 and 3.4.13) we do have such a stratification.

Corollary 3.2.20. Let $R$ be an integral local ring, $\operatorname{Spec} R \rightarrow S$ a morphism, and $p a$ prestable map over $\operatorname{Spec} R$. Then there exists a contraction $\tau \rightarrow \tau^{\prime}$, where $\tau$ is the dual graph of the geometric special fibre of $p$ and $\tau^{\prime}$ the dual graph of the geometric generic fibre.

Proof. Since the geometric generic fibre of $p$ has dual graph $\tau^{\prime}$, the generic point of Spec $R$ is in the image of the morphism

$$
\begin{equation*}
\mathfrak{M}_{S}\left(X, \tau^{\prime}\right) \times_{\mathfrak{M}_{g\left(\tau^{\prime}\right), S} \tau^{\prime}, S}\left(X, \operatorname{deg}\left(\tau^{\prime}\right)\right), \operatorname{Spec} R \longrightarrow \operatorname{Spec} R . \tag{3.2.6}
\end{equation*}
$$

Since (3.2.6) is finite by Proposition 3.2.17, it is therefore surjective. So by Proposition 3.2.18, there exists a contraction of $H$-graphs $\psi: \tau^{\prime \prime} \rightarrow \tau^{\prime}$, a $\tau^{\prime \prime}$-marked prestable map $q$ in $\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime \prime}\right)$, and an identification of the geometric special fibre of $p$ with the gluing of $\psi_{*}(q)$. In particular, $\tau^{\prime \prime}$ is identified with the dual graph $\tau$, so $\psi$ defines a contraction $\tau \rightarrow \tau^{\prime}$ as claimed.

Corollary 3.2.21. Assume that $\tau$ is a stable $H$-graph with $\operatorname{Aut}(\tau)=\{1\}$. Then the gluing map

$$
\begin{equation*}
M_{S}(X, \tau) \longrightarrow M_{g(\tau), S_{\tau}, S}(X, \operatorname{deg}(\tau)) \tag{3.2.7}
\end{equation*}
$$

is a closed immersion if and only if for all contractions stable $H$-graphs $\tau^{\prime}$ with $M_{S}^{\circ}\left(X, \tau^{\prime}\right) \neq$ $\emptyset$, there is at most one contraction $\psi: \tau^{\prime} \rightarrow \tau$.

Proof. By Proposition 3.2.17, (3.2.7) is a closed immersion if and only if it is injective on points. By Proposition 3.2.18, this is equivalent to requiring that

$$
\left(\coprod_{\psi: \tau^{\prime} \rightarrow \tau} M_{S}^{\circ}\left(X, \tau^{\prime}\right)\right) / \operatorname{Aut}\left(\tau^{\prime}\right) \longrightarrow M_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right)
$$

is a bijection for all stable $H$-graphs $\tau^{\prime}$ with $M^{\circ}\left(X, \tau^{\prime}\right) \neq \emptyset$ such that there is some contraction $\tau^{\prime} \rightarrow \tau$. But this is clearly equivalent to the condition in the statement, so the corollary follows.

### 3.3 Prestable degenerations

In this section, we study degenerations of the identity in the stack of prestable maps to a curve. These will form the domain curves for the Kontsevich-Mori compactifications of sections of flag variety bundles.

Definition 3.3.1. Let $X \rightarrow S$ be a prestable curve over $S$ of genus $g(X)$. We say that a prestable map $f: C \rightarrow X$ is a prestable degeneration if the stabilisation of $f$ is the identity $\operatorname{id}_{X}: X \rightarrow X$. We write $\mathfrak{D e g}_{S}(X) \subseteq \mathfrak{M}_{g(X), S}(X)$ for the substack of prestable degenerations.

Proposition 3.3.2. The stack $\mathfrak{D e g}_{S}(X)$ is an open and closed substack of $\mathfrak{M}_{g(X), S}(X)$.
Proof. Note that there is a Cartesian square

where the vertical morphism on the right is the stabilisation morphism. The section $S \rightarrow$ $M_{g(X), S}(X)$ classifying the stable map $\operatorname{id}_{X}: X \rightarrow X$ is an open immersion, since $\mathrm{id}_{X}$ is an open point in $\mathfrak{M}_{g(X), S}(X)$ with trivial automorphism group scheme, and closed since $M_{g(X), S}(X) \rightarrow S$ is separated by Theorem 3.1.7. So the pullback square (3.3.1) implies that $\mathfrak{D e g}_{S}(X) \rightarrow \mathfrak{M}_{g(X), S}(X)$ is an open and closed immersion, so we are done.

Lemma 3.3.3. Let $f: C \rightarrow X$ be a prestable degeneration over $S$. Then the morphism $\mathcal{O}_{X} \rightarrow \mathbb{R} f_{*} \mathcal{O}_{C}$ is a quasi-isomorphism.

Proof. By base change, it suffices to prove the lemma in the case when $S=\operatorname{Spec} k$ for $k$ an algebraically closed field. By induction on the number of irreducible components of $X$, we can reduce to the case where $f$ is the morphism contracting a single rational component $C_{0}$ of $C$ onto a point $x \in X$. Writing $u: X_{u}=\operatorname{Spec} \widehat{\mathcal{O}}_{X, x} \rightarrow X$ for the canonical morphism and $f_{u}: C_{u}=C \times_{X} X_{u} \rightarrow \operatorname{Spec} X_{u}$, it suffices by faithfully flat descent to show that the morphism $\mathcal{O} \rightarrow u^{*} \mathbb{R} f_{*} \mathcal{O}_{C}=\mathbb{R} f_{u^{*}} \mathcal{O}_{C_{u}}$ is a quasi-isomorphism.

If $X$ is smooth at $x$, then there is a decomposition $C_{u}=X_{u} \cup \mathbb{P}_{k}^{1}$ and an exact sequence

$$
0 \longrightarrow \mathcal{O}_{C_{u}} \longrightarrow i_{*} \mathcal{O}_{X_{u}} \oplus j_{*} \mathcal{O}_{\mathbb{P}_{k}^{1}} \longrightarrow \mathcal{O}_{x} \longrightarrow 0
$$

where $i$ and $j$ are the inclusions. So we have an exact triangle

$$
\mathbb{R} f_{u_{*}} \mathcal{O}_{C_{u}} \longrightarrow \mathcal{O}_{X_{u}} \oplus \mathcal{O}_{x} \longrightarrow \mathcal{O}_{x} \longrightarrow \mathbb{R} f_{u_{*}} \mathcal{O}_{C_{u}}[1]
$$

from which the claim follows.
Conversely, if $x \in X$ is a node, then we have decompositions $X_{u}=X_{1} \cup_{x} X_{2}$ and $C_{u}=X_{1} \cup_{x, p} \mathbb{P}_{k}^{1} \cup_{q, x} X_{2}$, where $\mathbb{P}_{k}^{1}$ is glued to $X_{1}$ at $p \in \mathbb{P}_{k}^{1}$ and to $X_{2}$ at $q \in \mathbb{P}_{k}^{1}$. So we have an exact sequence

$$
0 \longrightarrow \mathcal{O}_{C_{u}} \longrightarrow i_{1_{*}} \mathcal{O}_{X_{1}} \oplus i_{2_{*}} \mathcal{O}_{X_{2}} \oplus j_{*} \mathcal{O}_{\mathbb{P}_{k}^{1}} \longrightarrow \mathcal{O}_{p} \oplus \mathcal{O}_{q} \longrightarrow 0
$$

where $i_{1}, i_{2}$ and $j$ are the inclusions of the irreducible components of $C_{u}$, and hence an exact triangle

$$
\mathbb{R} f_{u_{*}} \mathcal{O}_{C_{u}} \longrightarrow i_{1 *}^{\prime} \mathcal{O}_{X_{1}} \oplus i_{2 *}^{\prime} \mathcal{O}_{X_{2}} \oplus \mathcal{O}_{x} \longrightarrow \mathcal{O}_{x} \oplus \mathcal{O}_{x} \longrightarrow \mathbb{R} f_{u_{*}} \mathcal{O}_{C_{u}}[1]
$$

where $i_{1}^{\prime}$ and $i_{2}^{\prime}$ are the inclusions of the irreducible components of $X_{u}$, and the claim also follows in this case.

If $X \rightarrow S$ is a smooth curve, then there is a degree datum $(U, \phi)$ for $X$ over $S$, where $U=X$ and $\phi: \mathbb{Z} \rightarrow \mathrm{NS}_{U}\left(U \times_{S} X\right)=\mathrm{NS}_{X}\left(X \times_{S} X\right)$ given by $\phi(d)=d\left[\mathcal{O}\left(\Delta_{X / S}\right)\right]$, where $\Delta_{X / S} \hookrightarrow X \times_{S} X$ is the diagonal. Note that $\mathcal{O}\left(\Delta_{X / S}\right)$ is an ample line bundle on $X \times_{S} X$.

Lemma 3.3.4. Assume $X \rightarrow S$ is a smooth curve of genus $g$ and let $(U, \phi)$ be the degree datum defined above. Then we have

$$
\mathfrak{D e g}_{S}(X)=\mathfrak{M}_{g, S}(X, 1) \subseteq \mathfrak{M}_{g, S}(X)
$$

as open substacks.
Proof. It is clear from the definitions that $\operatorname{Deg}_{S}(X) \subseteq \mathfrak{M}_{g, S}(X, 1)$. For the reverse inclusion, assume that $s$ : Spec $k \rightarrow S$ is a geometric point and $f: C \rightarrow X_{s}$ is a prestable map of genus $g$ and degree 1 with respect to $(U, \phi)$. Then the stabilisation $\tilde{f}: \tilde{C} \rightarrow X_{s}$ is a stable map of degree 1 and genus $g$, so, since $X$ is smooth, the normalisation of $\tilde{C}$ has a unique component mapping isomorphically to $X_{s}$, all other components are rational and contracted to points in $X_{s}$, and the dual graph of $C$ is a tree. Stability of $\tilde{f}$ then implies that there are no rational contracted components, so $\tilde{f}$ is an isomorphism, which proves that $f$ defines a point in $\mathfrak{D e g}_{S}(X)$.

Lemma 3.3.5. Let $X \rightarrow S$ be a smooth curve over $S$ of genus $g$, endowed with the degree datum $(U, \phi)$ above. If $\tau$ is any $\mathbb{Z}$-graph of degree 1 and genus $g$, then the following are equivalent.
(1) The stack $\mathfrak{M}_{S}^{\circ}(X, \tau)$ is nonempty.
(2) The stack $\mathfrak{M}_{S}(X, \tau)$ is nonempty.
(3) There exists a unique vertex $v \in V_{\tau}$ with $\beta(v) \neq 0$, and this unique vertex satisfies $g(v)=g$.

Proof. The implication (1) $\Rightarrow(2)$ is clear. We prove $(2) \Rightarrow(3) \Rightarrow(1)$.
Assume (2) is satisfied. Then there exists a $\tau$-marked prestable map $p=\left(\left(p_{v}: C_{v} \rightarrow\right.\right.$ $\left.\left.X_{s},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)\right)_{v \in V_{\tau}}$ over a geometric point $s: \operatorname{Spec} k \rightarrow S$. So $\beta(v)=\operatorname{deg}\left(p_{v}\right) \geq 0$ for all $v \in V_{\tau}$, and hence there exists $v \in V_{\tau}$ with $\beta(v)=1$ and $\beta\left(v^{\prime}\right)=0$ for $v^{\prime} \neq v$ (since $\operatorname{deg}(\tau)=1)$. So $p_{v}: C_{v} \rightarrow X_{s}$ has degree 1 , so there must exist an irreducible component of $C_{v}$ mapping isomorphically to $X_{s}$. So $g(v) \geq g$, from which we deduce that $g(v)=g$, since $g(v) \leq g(\tau)=g$. So (3) is satisfied.

Now assume that (3) is satisfied. Then $\beta(v)=\operatorname{deg}(\tau)=1$, and the dual graph of $\tau$ is a tree with $g\left(v^{\prime}\right)=0$ for $v^{\prime} \in V_{\tau} \backslash\{v\}$. If $\tau$ has a single vertex, then for any geometric point $s:$ Spec $k \rightarrow S$ and any distinct $k$-points $\left(x_{f}\right)_{f \in S_{\tau}}$ of $X$, the tuple ( $\mathrm{id}_{X_{s}}: X_{s} \rightarrow$ $\left.X_{s},\left(x_{f}\right)_{f \in S_{\tau}}\right)$ defines a point of $\mathfrak{M}_{S}^{\circ}(X, \tau)$ over $s$, so $\mathfrak{M}_{S}(X, \tau) \neq \emptyset$. If $\tau$ has more than one vertex, then choosing a leaf $v^{\prime} \neq v$, we have an isomorphism

$$
\mathfrak{M}_{S}^{\circ}(X, \tau) \cong \mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) \times_{X} \mathfrak{M}_{0,1, S}^{\circ}(X, 0)=\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) \times_{S} \mathfrak{M}_{0,1, S}^{\circ}
$$

where $\tau^{\prime}$ is the $\mathbb{Z}$-graph with $V_{\tau^{\prime}}=V_{\tau} \backslash\{v\}, F_{\tau^{\prime}}=F_{\tau} \backslash F_{\tau}(v), \partial_{\tau^{\prime}}=\partial_{\tau} \mid F_{\tau^{\prime}}$ and

$$
j_{\tau^{\prime}}(f)=\left\{\begin{array}{ll}
j_{\tau}(f), & \text { if } j_{\tau}(f) \in F_{\tau^{\prime}} \\
f, & \text { otherwise }
\end{array} .\right.
$$

So $\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right)$ is nonempty by induction on the number of vertices, and hence so is $\mathfrak{M}_{S}^{\circ}(X, \tau)$. So (1) is satisfied.

Convention 3.3.6. If $\tau$ is a $\mathbb{Z}$-graph (of genus $g$ ) satisfying the equivalent conditions of Lemma 3.3.5, we will represent the functions $g: V_{\tau} \rightarrow \mathbb{Z}_{\geq 0}$ and $\beta: V_{\tau} \rightarrow \mathbb{Z}$ using a filled circle • for the unique vertex $v$ with $\beta(v)=1$ and $g(v)=g$ and empty circles $\circ$ for the remaining vertices $v^{\prime}$ with $g\left(v^{\prime}\right)=\beta\left(v^{\prime}\right)=0$. So, for example, the graph

is the dual graph of a degeneration with one component mapping isomorphically to $X_{s}$, and two rational components mapping to different points of $X_{s}$.

In the following two propositions we assume $X \rightarrow S$ is a smooth curve over $S$ and write $f: \mathcal{C} \rightarrow \operatorname{Deg}_{S}(X) \times{ }_{S} X$ for the universal prestable degeneration, $\pi: \mathfrak{D e g}_{S}(X) \times{ }_{S} X \rightarrow$ $\mathfrak{D} \operatorname{eg}_{S}(X)$ for the natural projection, and $D \subseteq \mathfrak{D e g}_{S}(X)$ for the closed substack of prestable degenerations with singular domain curve. We also write $\mathfrak{D} \operatorname{eg}_{S}(X)^{\leq 1} \subseteq \mathfrak{D e g}_{S}(X)$ for the open substack of prestable degenerations such that the domain curve has at most one node, $\mathcal{C}^{\leq 1}=\pi^{-1} f^{-1}\left(\operatorname{Deg}_{S}(X)^{\leq 1}\right)$ and $D^{\leq 1}=D \cap \mathfrak{D} \operatorname{eg}_{S}(X)^{\leq 1}$.

Proposition 3.3.7. Assume that $X \rightarrow S$ is a smooth curve over $S$. Then we have the following.
(1) The stack $\mathfrak{D} \operatorname{eg}_{S}(X)$ is smooth over $S$.
(2) The closed substack $D \subseteq \mathfrak{D e g}_{S}(X)$ is a reduced divisor with normal crossings relative to $S$.
(3) For every $n \geq 0$, the open stratum $\operatorname{Deg}_{S}(X)^{(n)}$ of points where $D$ is locally isomorphic to an intersection of $n$ coordinate hyperplanes in an affine space is given by

$$
\mathfrak{D e g}_{S}(X)^{(n)}=\coprod_{\tau} \mathfrak{M}_{S}^{\circ}(X, \tau) / \operatorname{Aut}(\tau)
$$

where the coproduct is over $\mathbb{Z}$-graphs $\tau$ satisfying the equivalent conditions of Lemma 3.3.5 with $S_{\tau}=\emptyset$ and $\left|E_{\tau}\right|=n$.
(4) For every $\mathbb{Z}$-graph $\tau$ satisfying the equivalent conditions of Lemma 3.3.5 with $S_{\tau}=\emptyset$, we have

$$
\overline{\mathfrak{M}_{S}^{\circ}(X, \tau) / \operatorname{Aut}(\tau)}=\bigcup_{\tau^{\prime} \rightarrow \tau} \mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right)
$$

where the closure is taken in $\operatorname{Deg}_{S}(X)$ and the union is over all isomorphism classes of $\mathbb{Z}$-graphs satisfying the equivalent conditions of Lemma 3.3.5 such that there exists a contraction $\tau^{\prime} \rightarrow \tau$.

Proof. The proof follows similar lines to the proofs of [DM, Corollary 1.9 and Theorem 5.2].
Fix a geometric point $s: \operatorname{Spec} k \rightarrow S$ corresponding to a prestable map $p: C \rightarrow X_{s}$ and consider the deformation functors

$$
\mathfrak{D}_{l o c}, S_{l o c}: \mathrm{Art}_{k} \longrightarrow \mathrm{Grpd}
$$

given by

$$
\mathfrak{D}_{l o c}(A)=\operatorname{Hom}\left(\operatorname{Spec} A, \mathfrak{D e g}_{S}(X)\right) \times_{\operatorname{Hom}\left(\operatorname{Spec} k, \mathfrak{D e g}_{S}(X)\right)}\left\{p: C \rightarrow X_{s}\right\},
$$

and

$$
S_{l o c}(A)=\operatorname{Hom}(\operatorname{Spec} A, S) \times_{\operatorname{Hom}(\operatorname{Spec} k, S)}\{s\},
$$

where $\mathrm{Art}_{k}$ is the category of Artinian local rings with residue field $k$, and Grpd is the 2 category of groupoids. Let $\left\{x_{1}, \ldots, x_{n}\right\} \subseteq C(k)$ denote the set of nodes, and for each $i$, let $\mathfrak{D}_{i}$ denote the functor sending $A \in \operatorname{Art}_{k}$ to the groupoid of flat deformations of the completed local ring $R_{i}=\widehat{\mathcal{O}}_{C, x_{i}} \cong k \llbracket y_{i}, z_{i} \rrbracket /\left(y_{i} z_{i}\right)$ over $A$. We claim that the natural morphism

$$
\mathfrak{D}_{l o c} \longrightarrow S_{l o c} \times \prod_{i=1}^{n} \mathfrak{D}_{i}
$$

is formally smooth.
To see this, first note that by deformation theory of schemes and of local rings, it is enough to show that the kernel in the derived category of $k$-vector spaces of the natural morphism

$$
\mathbb{T}_{\mathfrak{D}_{\text {loc }} / S_{\text {loc }}} \longrightarrow \bigoplus_{i} \mathbb{T}_{\mathfrak{D}_{i}}
$$

has vanishing $H^{j}$ for $j>0$, where

$$
\mathbb{T}_{\mathfrak{D}_{\text {loc }} / S_{\text {loc }}}=\mathbb{R} \Gamma\left(C, \mathbb{T}_{C / X_{s}}[1]\right) \quad \text { and } \quad \mathbb{T}_{\mathfrak{D}_{i}}=\mathbb{T}_{R_{i} / k}[1] \cong\left[R_{i} \oplus R_{i} \xrightarrow{\left(y_{i} z_{i}\right)} R_{i}\right],
$$

where we have used the identification of $R_{i}$ with $k \llbracket y_{i}, z_{i} \rrbracket /\left(y_{i} z_{i}\right)$ to write out the tangent complex of $R_{i}$ explicitly as a complex in degrees -1 and 0 . The cohomology vanishing reduces easily to the claim that

$$
H^{0}\left(\mathbb{T}_{\mathfrak{D}_{\text {loc }} / S_{\text {loc }}}\right)=H^{1}\left(C, \mathbb{T}_{C / X_{s}}\right) \longrightarrow H^{0}\left(C, H^{1}\left(\mathbb{T}_{C}\right)\right)=\bigoplus_{i} H^{0}\left(\mathbb{T}_{\mathfrak{D}_{i}}\right)
$$

is surjective, which is equivalent to the vanishing of $\mathbb{H}^{2}(C, K)$, where

$$
K=\operatorname{ker}\left(\mathbb{T}_{C / X_{s}} \rightarrow H^{1}\left(\mathbb{T}_{C}\right)[-1]\right)=\left[q_{*} T_{\tilde{C}}(-N) \rightarrow p^{*} T_{X_{s}}\right]
$$

is a complex on $C$ in degrees 0 and 1 , where $q: \tilde{C} \rightarrow C$ is the normalisation of $C$ and $N \subseteq \tilde{C}$ is the divisor of preimages of nodes of $C$. But, using Lemma 3.3.3 to conclude that $\mathbb{R} p_{*} \mathcal{O}_{C}=\mathcal{O}_{X_{s}}$, we have

$$
\mathbb{R} \Gamma(C, K)=\mathbb{R} \Gamma\left(\tilde{C}^{\prime}, T_{\tilde{C}^{\prime}}\left(-N^{\prime}\right)\right) \oplus H^{0}\left(N_{X}, T_{X_{s}} \mid N_{X}\right)[-1],
$$

which manifestly has vanishing $H^{2}$, where $\tilde{C}^{\prime} \subseteq \tilde{C}$ is the union of rational components contracted under $p, N^{\prime}=N \cap \tilde{C}^{\prime}$ and $N_{X}=N \cap X_{s}$ is the intersection of $N$ with the unique connected component of $\tilde{C}$ mapped isomorphically to $X_{s}$ under $p$.

Now the product $\prod_{i=1}^{n} \mathfrak{D}_{i}$ is formally smooth, with a miniversal formal deformation given by

$$
\left(\mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n} \rrbracket \longrightarrow \frac{\mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n}, y_{i}, z_{i} \rrbracket}{\left(y_{i} z_{i}-t_{i}\right)}\right)_{i=1, \ldots, n}
$$

where $\mathfrak{o}_{k}$ is the complete regular local ring with residue field $k$ defined in [DM, p. 79]. It follows that $\mathfrak{D} \mathrm{eg}_{S}(X) \rightarrow S$ is formally smooth at $p$, so (1) follows.

To prove (2), choose any formally smooth morphism $\operatorname{Spf} A \rightarrow \operatorname{Spf} \mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n} \rrbracket \times{ }_{\prod_{i=1}^{n}} \mathfrak{D}_{i}$ $\mathfrak{D}_{l o c}$ sending the closed point to the natural base point, with $A$ a complete Noetherian local
ring with residue field $k$. Then we have a formally smooth morphisms $\operatorname{Spec} A \rightarrow \operatorname{Deg}_{S}(X)$ and Spec $A \rightarrow$ Spec $\mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n} \rrbracket \times S$ corresponding to a prestable degeneration $p_{A}: C_{A} \rightarrow$ $X_{A}$ with closed fibre $p$ and isomorphisms

$$
\widehat{\mathcal{O}}_{C_{A}, x_{i}} \cong \frac{A \llbracket y_{i}, z_{i} \rrbracket}{\left(y_{i} z_{i}-t_{i}\right)}
$$

as $A$-algebras. In particular, we have $D \times_{\mathfrak{D e g}_{S}(X)} \operatorname{Spec} A$ is the locus $t_{1} \cdots t_{n}=0$, which is a divisor with normal crossings relative to $S$. So this proves (2). This also shows that the stratum $\mathfrak{D e g}_{S}(X)^{(n)}$ is the locus of prestable degenerations with exactly $n$ nodes, from which (3) follows as well.

To prove (4), we first note that Corollary 3.2.20 implies that

$$
\overline{\mathfrak{M}_{S}^{\circ}(X, \tau) / \operatorname{Aut}(\tau)} \subseteq \bigcup_{\tau^{\prime} \rightarrow \tau} \mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right)
$$

To prove the reverse inclusion, suppose that $\psi: \tau^{\prime} \rightarrow \tau$ is a contraction satisfying the conditions of (4) and that the prestable degeneration $p$ chosen above is in the image of $\mathfrak{M}_{S}^{\circ}\left(X, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right) \rightarrow \mathfrak{D} \operatorname{eg}_{S}(X)$, i.e., that $p$ has dual graph $\tau^{\prime}$. If we choose the labelling of the nodes of $C$ so that $x_{i+1}, x_{i+2}, \ldots, x_{n}$ are the nodes corresponding to the edges contracted by $\psi$, then the morphism

$$
\operatorname{Spec} A /\left(t_{1}, \ldots, t_{i}\right) \longleftrightarrow \operatorname{Spec} A \longrightarrow \operatorname{Deg}_{S}(X)
$$

factors through a morphism to $\mathfrak{M}_{S}(X, \tau)$, such that the restriction to

$$
\operatorname{Spec} A /\left(t_{1}, \ldots, t_{i}\right)\left[t_{i+1}^{-1}, \ldots, t_{n}^{-1}\right]
$$

(which is nonempty since $\operatorname{Spec} A \rightarrow \operatorname{Spec} \mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n} \rrbracket \times S$ is formally smooth) factors through $\mathfrak{M}_{S}^{\circ}(X, \tau) \subseteq \mathfrak{M}_{S}(X, \tau)$. So $p$ is in the closure of the image of $\mathfrak{M}_{S}^{\circ}(X, \tau) \rightarrow \operatorname{Deg}_{S}(X)$, which proves (4).

Proposition 3.3.8. Assume that $X \rightarrow S$ is a smooth curve over $S$. Then we have the following.
(1) The stack $\mathcal{C}$ is smooth over $S$.
(2) The preimage of $D$ in $\mathcal{C}$ is a reduced divisor with normal crossings relative to $S$.
(3) The preimage of $D$ in $\mathcal{C}$ decomposes as

$$
f^{-1} \pi^{-1}(D)=D^{\prime} \cup \operatorname{Exc}
$$

where Exc $\subseteq \mathcal{C}$ is the locus of points around which $f: \mathcal{C} \rightarrow \operatorname{Deg}_{S}(X) \times_{S} X$ is not an isomorphism, and $D^{\prime}$ is the proper transform of $\pi^{-1}(D)$ under $f$.
(4) The morphism $\left.f\right|_{\mathcal{C} \leq 1}: \mathcal{C} \leq 1 \rightarrow \operatorname{Deg}_{S}(X)^{\leq 1} \times_{S} X$ is the blowup at the image $f\left(\operatorname{Exc}^{\leq 1}\right)$ of the divisor $\operatorname{Exc}^{\leq 1}=\operatorname{Exc} \cap \mathcal{C} \leq 1$, and the projection $\mathfrak{D} \operatorname{eg}_{S}(X) \times_{S} X \rightarrow \mathfrak{D e g}_{S}(X)$ maps $f\left(\mathrm{Exc}^{\leq 1}\right)$ isomorphically onto $D \leq 1$.

Proof. To prove (1) and (2), first note that it suffices to check each property after pulling back to a formally smooth neighbourhood of every geometric point. So fix a prestable degeneration $p: C \rightarrow X_{s}$ over a geometric point $s: \operatorname{Spec} k \rightarrow S$ and a $k$-point $x: \operatorname{Spec} k \rightarrow C \hookrightarrow \mathcal{C}$.

If $x$ is a smooth point of $C$, then the morphism $\pi \circ f: \mathcal{C} \rightarrow \mathfrak{D e g}_{S}(X)$ is smooth at $x$, so (1) and (2) hold in a neighbourhood of $x$ by Proposition 3.3.7. So assume that $x=x_{j}$ is a node.

Keeping the notation of the proof of Proposition 3.3.7, let $\mathcal{C}_{\text {loc }}: \operatorname{Art}_{k} \rightarrow \operatorname{Grpd}$ be the functor given by

$$
\mathcal{C}_{l o c}(A)=\operatorname{Hom}(\operatorname{Spec} A, \mathcal{C}) \times_{\operatorname{Hom}(\operatorname{Spec} k, \mathcal{C})}\{x\} .
$$

We show that

$$
\mathcal{C}_{l o c} \longrightarrow S_{l o c} \times \mathfrak{D}_{j}^{\prime} \times \prod_{i \neq j} \mathfrak{D}_{i}
$$

is formally smooth as follows, where $\mathfrak{D}_{j}^{\prime}$ is the functor sending $A \in \operatorname{Art}_{k}$ to the groupoid of flat deformations $\bar{R}_{j}$ of $R_{j}=\widehat{\mathcal{O}}_{C, x_{j}}$ over $A$ equipped with an augmentation homomorphism $\bar{R}_{j} \rightarrow A$ restricting to $x$ at the special fibre. By a similar argument to the proof of formal smoothness of $\mathfrak{D}_{\text {loc }} \rightarrow \prod_{i=1}^{n} \mathfrak{D}_{i}$ in Proposition 3.3.7, we reduce to showing that $\mathbb{H}^{2}\left(C, K^{\prime}\right)=$ 0 , where

$$
K^{\prime}=\operatorname{ker}\left(\mathbb{T}_{C / X_{s}}^{\prime} \rightarrow H^{1}\left(\mathbb{T}_{C}^{\prime}\right)[-1]\right)
$$

where $\mathbb{T}_{C}^{\prime}=\operatorname{ker}\left(\left.\mathbb{T}_{C} \rightarrow \mathbb{T}_{C}\right|_{x}\right)$ and $\mathbb{T}_{C / X s}^{\prime}=\operatorname{ker}\left(\mathbb{T}_{C}^{\prime} \rightarrow p^{*} \mathbb{T}_{X_{s}}\right)$. A direct local computation shows that $H^{0}\left(\mathbb{T}_{C}^{\prime}\right)=H^{0}\left(\mathbb{T}_{C}\right)$, from which it follows easily that $K^{\prime}=K=\operatorname{ker}\left(\mathbb{T}_{C / X} \rightarrow\right.$ $H^{1}\left(\mathbb{T}_{C}\right)[-1]$ ), which we already showed had vanishing $\mathbb{H}^{2}$ in the proof of Proposition 3.3.7.

There is a commutative diagram

where, in the leftmost square, the horizontal arrows are formally smooth, the left vertical arrow is given by $t_{j} \mapsto u_{j} v_{j}$, the bottom arrow is the miniversal deformation in the proof of Proposition 3.3.7, and the top arrow is given on the $\mathfrak{D}_{j}^{\prime}$-factor by the algebra

$$
\mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{j-1}, u_{j}, v_{j}, t_{j+1}, \ldots t_{n} \rrbracket \longrightarrow \frac{\mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{j-1}, u_{j}, v_{j}, t_{j+1}, \ldots t_{n}, y_{j}, z_{j} \rrbracket}{\left(y_{j} z_{j}-u_{j} v_{j}\right)},
$$

with augmentation sending $y_{j}$ to $u_{j}$ and $z_{j}$ to $v_{j}$. In particular, $\mathfrak{D}_{j}^{\prime} \times \prod_{i \neq j} \mathfrak{D}_{i}$ is formally smooth, so this proves that (1) holds near $x$. Moreover, if we write

$$
Y=\operatorname{Spf} \mathfrak{o}_{k} \llbracket t_{1}, \ldots, u_{j}, v_{j}, \ldots, t_{n} \rrbracket \times_{\mathfrak{D}_{j}^{\prime} \times \prod_{i \neq j} \mathfrak{D}_{i}} \mathcal{C}_{l o c}
$$

and

$$
Z=\operatorname{Spf} \mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{j}, \ldots, t_{n} \rrbracket \times_{\Pi_{i} \mathfrak{D}_{i}} \mathfrak{D}_{l o c}
$$

then choosing versal deformations $\operatorname{Spf} A \rightarrow Z$ and $\operatorname{Spf} B \rightarrow Y \times{ }_{Z} \operatorname{Spf} A$, we have a commutative diagram

where the horizontal arrows are all formally smooth. Since $D \times_{\operatorname{Deg}_{S}(X)} \operatorname{Spec} A$ is the divisor $t_{1} \cdots t_{n}=0$, it follows that $f^{-1} \pi^{-1}(D) \times_{\mathcal{C}} \operatorname{Spec} B$ is the divisor $t_{1} \cdots t_{j-1} u_{j} v_{j} t_{j+1} \cdots t_{n}=0$, which proves that (2) holds near $x$. So (1) and (2) hold everywhere on $\mathcal{C}$.

The claim (3) follows immediately from the fact that $f: \mathcal{C} \rightarrow \operatorname{Deg}_{S}(X) \times_{S} X$ is a representable morphism between smooth stacks over $S$ that is an isomorphism outside the divisor $f^{-1} \pi^{-1}(D)$.

Finally, to prove (4), observe that it suffices to check the claims after pulling back to any atlas of $\mathfrak{D e g}_{S}(X) \leq 1$. It is straightforward to check that the morphism $\mathbb{A}_{S}^{1} \times_{S} X \rightarrow$ $\mathfrak{D} \operatorname{eg}_{S}(X)^{\leq 1}$ given by the prestable map

$$
\operatorname{Bl}_{\{0\} \times_{S} \Delta}\left(\mathbb{A}_{S}^{1} \times_{S} X \times_{S} X\right) \longrightarrow \mathbb{A}_{S}^{1} \times_{S} X \times_{S} X
$$

gives such an atlas, where Bl denotes the blowup at the given substack, and $\Delta=\Delta_{X / S} \subseteq$ $X \times_{S} X$ denotes the diagonal. The claims of (4) are now obvious by construction after pulling back to this atlas.

### 3.4 Kontsevich-Mori compactifications

In this section, we introduce the long awaited Kontsevich-Mori compactifications of the stack of principal bundles under a parabolic subgroup of a reductive group.

Definition 3.4.1. Let $G$ be a reductive group, $P \subseteq G$ a parabolic subgroup, $X \rightarrow S$ a smooth proper curve of genus $g$ over a stack $S$, and $\xi_{G}^{u n i} \rightarrow \operatorname{Bun}_{G / S}(X) \times_{S} X$ the universal $G$-bundle. The Kontsevich-Mori compactification of $\operatorname{Bun}_{P / S}(X)$ is the fibre product

$$
\operatorname{KM}_{P, G / S}(X)=M_{g, \operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P\right) \times_{\mathfrak{M}_{g, S}(X)} \mathfrak{D e g}_{S}(X)
$$

If $\mu \in \mathbb{X}_{*}\left(T_{P}\right)=\mathbb{X}_{*}(P /[P, P])$, we write

$$
\mathrm{KM}_{P, G / S}^{\mu}(X) \subseteq \mathrm{KM}_{P, G / S}(X)
$$

for the open and closed substack of stable maps $\sigma: C \rightarrow \xi_{G}^{u n i} / P$ such that $\operatorname{deg} \sigma^{*} \mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right)=$ $\langle\lambda, \mu\rangle$ for all $\lambda \in \mathbb{X}^{*}\left(T_{P}\right)$, where $\mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right)$ is the line bundle $\mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right)=\xi_{G}^{u n i} \times{ }^{G} \mathbb{Z}_{\lambda}$ on $\xi_{G}^{u n i} / P$.

Remark 3.4.2. In more down to earth terms, the Kontsevich-Mori compactification is the stack of tuples $\left(s, \xi_{G}, C, \sigma\right)$, where $s \in S$ is a point of $S, \xi_{G} \rightarrow X_{s}$ is a $G$-bundle, $C$ is a prestable curve, and $\sigma: C \rightarrow \xi_{G} / P$ is a stable map such that the prestable map $C \rightarrow \xi_{G} / P \rightarrow X_{s}$ is a prestable degeneration of $X_{s}$. Note that if we take $s$ to be defined over an algebraically closed field $k$, then $C=X_{s} \cup \bigcup_{i} C_{i}$ has a unique irreducible component mapping to a section of $\xi_{G} / P \rightarrow X_{s}$, and a number of rational components $C_{i} \cong \mathbb{P}_{k}^{1}$ mapping into fibres of the $G / P$-bundle $\xi_{G} / P \rightarrow X_{s}$.

Remark 3.4.3. Note that the partial flag variety bundle $\xi_{G}^{u n i} / P=\xi_{G}^{u n i} \times{ }^{G} F_{t(P)} \rightarrow$ $\operatorname{Bun}_{G / S}(X) \times_{S} X$ and the line bundles $\mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right)=\xi_{G}^{u n i} \times{ }^{G} \mathcal{L}_{\lambda}$ depend only on the type $t(P)$ of $P$ up to canonical isomorphism, and hence so do the Kontsevich-Mori compactifications $\mathrm{KM}_{P, G / S}^{\mu}(X)$.

The degree datum of Lemma 3.3.4 for $X \rightarrow S$ has the following analogue for $\xi_{G}^{u n i} / P \rightarrow S$. Let $U=\operatorname{Bun}_{G / S}(X) \times{ }_{S} X$ and define

$$
\phi: \mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z} \longrightarrow \mathrm{NS}_{U}\left(U \times_{\operatorname{Bun}_{G / S}(X)} \xi_{G}^{u n i} / P\right)
$$

by

$$
\phi(\lambda, d)=\left[p^{*} \mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right)\right]+d\left[q^{*} \mathcal{O}\left(\Delta_{X / S}\right)\right]
$$

where $p: U \times_{\operatorname{Bun}_{G / S}(X)} \xi_{G}^{u n i} / P \rightarrow \xi_{G}^{u n i} / P$ is the natural projection, $q$ is the morphism

$$
U \times_{\operatorname{Bun}_{G / S}(X)} \xi_{G}^{u n i} / P \longrightarrow U \times_{S} X=\operatorname{Bun}_{G / S}(X) \times_{S} X \times_{S} X \longrightarrow X \times_{S} X
$$

and $\Delta_{X / S} \subseteq X \times_{S} X$ is the diagonal divisor.

Lemma 3.4.4. With respect to the degree datum $(U, \phi)$ above, we have

$$
\mathrm{KM}_{P, G / S}^{\mu}(X)=M_{g, S}(X,(\mu, 1))
$$

where we identify $\operatorname{Hom}\left(\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}, \mathbb{Z}\right)$ with $\mathbb{X}_{*}\left(T_{P}\right) \oplus \mathbb{Z}$ in the usual way.
Proof. This is an immediate consequence of Lemma 3.3.4 and the definitions.
Proposition 3.4.5. In the situation of Definition 3.4.1, the morphism

$$
\operatorname{KM}_{P, G / S}^{\mu}(X) \longrightarrow \operatorname{Bun}_{G / S}(X)
$$

is proper with finite relative stabilisers.
Proof. Let $(U, \phi)$ be the degree datum of Lemma 3.4.4. If we choose $\lambda \in \mathbb{X}^{*}\left(T_{P}\right)$ so that $\mathcal{L}_{\lambda} \in \operatorname{Pic}(G / P)$ is ample, then

$$
\phi(\lambda, 1)=\left[p^{*} \mathcal{L}_{\lambda}\left(\xi_{G}^{u n i}\right) \otimes q^{*} \mathcal{O}\left(\Delta_{X / S}\right)\right] \in \mathrm{NS}_{U}\left(U \times_{\operatorname{Bun}_{G / S}(X)} \xi_{G}^{u n i} / P\right)
$$

is the class of a $U$-ample line bundle. So the claim follows by Corollary 3.1.9 and Lemma 3.4.4.

Definition 3.4.6. Let $s: \operatorname{Spec} k \rightarrow E$ be a geometric point, $\xi_{G} \rightarrow E_{s}$ a $G$-bundle and $\sigma: C \rightarrow \xi_{G} / P$ a stable map. We write $[\sigma] \in \mathbb{X}_{*}\left(T_{P}\right)$ for the projection of the degree $\operatorname{deg}_{(U, \phi)} \sigma \in \mathbb{X}_{*}\left(T_{P}\right) \oplus \mathbb{Z}$ to the first factor, where $(U, \phi)$ is the degree datum of Lemma 3.4.4. We will often abuse terminology slightly and refer to $[\sigma]$ as the degree of $\sigma$.

Proposition 3.4.7. The morphism

$$
\operatorname{KM}_{P, G / S}(X) \longrightarrow \mathfrak{D e g}_{S}(X)
$$

is smooth.
Proof. The stack $\mathrm{KM}_{P, G / S}(X)$ is an open substack of

$$
\operatorname{Bun}_{G / \mathfrak{D} \operatorname{eg}_{S}(X)}\left(\operatorname{Deg}_{S}(X) \times_{S} X\right) \times \times_{\operatorname{Bun}_{G / \mathcal{D} \operatorname{eg}_{S}(X)}(\mathcal{C})} \operatorname{Bun}_{P / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C})
$$

where $\mathcal{C} \rightarrow \mathfrak{D e g}_{S}(X) \times{ }_{S} X$ is the universal prestable degeneration of $X$. Since

$$
\operatorname{Bun}_{P / \mathfrak{D e g}_{S}(X)}(\mathcal{C}) \longrightarrow \mathfrak{D e g}_{S}(X)
$$

is smooth, it therefore suffices to show that the morphism

$$
\begin{equation*}
\mathfrak{D e g}_{S}(X) \times_{S} \operatorname{Bun}_{G / S}(X)=\operatorname{Bun}_{G / \operatorname{Deg}_{S}(X)}\left(\mathfrak{D e g}_{S}(X) \times_{S} X\right) \longrightarrow \operatorname{Bun}_{G / \mathfrak{D e g}_{S}(X)}(\mathcal{C}) \tag{3.4.1}
\end{equation*}
$$

defined by pullback of $G$-bundles is smooth. We in fact show that (3.4.1) is étale. By deformation theory for $G$-bundles, it is enough to show that if $s$ : $\operatorname{Spec} k \rightarrow S$ is a geometric point, $f: C \rightarrow X_{s}$ is a prestable degeneration, and $\xi_{G} \rightarrow X_{s}$ is a principal $G$-bundle, then the canonical morphism

$$
\mathbb{R} \Gamma\left(X_{s}, \xi_{G} \times{ }^{G} \mathfrak{g}\right) \longrightarrow \mathbb{R} \Gamma\left(X_{s}, \mathbb{R} f_{*} \mathcal{O}_{C} \otimes\left(\xi_{G} \times{ }^{G} \mathfrak{g}\right)\right)=\mathbb{R} \Gamma\left(C, f^{*}\left(\xi_{G} \times{ }^{G} \mathfrak{g}\right)\right)
$$

is a quasi-isomorphism. But this holds by Lemma 3.3.3, so we are done.
Corollary 3.4.8. The stack $\mathrm{KM}_{P, G / S}(X)$ is smooth over $S$, and contains $\operatorname{Bun}_{P / S}(X)$ as a dense open substack.

Proof. Proposition 3.3.7 shows that $\operatorname{Deg}_{S}(X)$ is smooth over $S$ and that the open immersion $S \rightarrow \mathfrak{D e g}_{S}(X)$ classifying the identity $\operatorname{id}_{X}: X \rightarrow X$ is dense. So the claim now follows from Proposition 3.4.7 and the natural identification of $\operatorname{Bun}_{P / S}(X)$ with the stack of tuples $\left(s, \xi_{G}, \sigma\right)$, where $s \in S, \xi_{G} \rightarrow X_{s}$ is a $G$-bundle and $\sigma: X_{s} \rightarrow \xi_{G} / P$ is a section.

Propositions 3.3.7 and 3.4.7 imply that the complement of $\operatorname{Bun}_{P / S}(X)$ in $\mathrm{KM}_{P, G / S}(X)$ is a divisor with normal crossings relative to $S$. In order to study this divisor in more detail, we recall the following facts about stable maps to partial flag varieties.

Recall that there is a homomorphism

$$
\phi^{\prime}: \mathbb{X}^{*}\left(T_{P}\right)=\operatorname{Pic}^{G}(G / P) \longrightarrow \operatorname{Pic}(G / P)=\mathrm{NS}_{\mathrm{Spec} \mathbb{Z}}(G / P)=\mathbb{X}^{*}\left(T_{P}^{s c}\right)
$$

which is surjective after tensoring with $\mathbb{Q}$. Here $T^{s c} \subseteq T$ is the subtorus of the abstract Cartan $T$ with cocharacter group $\mathbb{Z} \Phi^{\vee} \subseteq \mathbb{X}_{*}(T)$, and $T_{P}^{s c}$ is the quotient of $T^{s c}$ and subtorus of $T_{P}$ with character group

$$
\mathbb{X}^{*}\left(T_{P}^{s c}\right)=\left\{\lambda \in \mathbb{X}^{*}\left(T^{s c}\right) \mid\left\langle\lambda, \alpha_{i}^{\vee}\right\rangle=0 \text { for } \alpha_{i} \in t(P)\right\}
$$

With $U^{\prime}=\operatorname{Spec} \mathbb{Z}$, the homomorphism $\phi^{\prime}$ defines a degree datum $\left(U^{\prime}, \phi^{\prime}\right)$ for $G / P$ over $\operatorname{Spec} \mathbb{Z}$.

Proposition 3.4.9. Let $P \subseteq G$ be a parabolic subgroup and let $\mu \in \mathbb{X}_{*}\left(T_{P}\right)$. The morphism

$$
\begin{equation*}
M_{0,1}(G / P, \mu) \longrightarrow G / P \times \mathfrak{M}_{0,1} \tag{3.4.2}
\end{equation*}
$$

given on the two factors by evaluation and forgetting the map to $G / P$ is smooth of relative dimension $\left\langle 2 \rho_{P}, \mu\right\rangle$, where $2 \rho_{P} \in \mathbb{X}^{*}\left(T_{P}\right) \subseteq \mathbb{X}^{*}(T)$ is the sum of all positive roots that are not roots of $P$. In particular, the morphism $M_{0,1}(G / P, \mu) \rightarrow G / P$ is smooth of relative dimension $\left\langle 2 \rho_{P}, \mu\right\rangle-2$.

Proof. To prove that (3.4.2) is smooth of the required relative dimension, note that there is a tangent-obstruction complex for (3.4.2) given at a 1-pointed stable map $\left(f: C \rightarrow(G / P)_{k}, x\right)$ defined over an algebraically closed field $k$ by

$$
\mathbb{T}=\mathbb{R} \Gamma\left(C, f^{*} T_{G / P}(-x)\right)
$$

Since $T_{G / P}$ is generated by global sections and $C$ is rational, we have $H^{i}\left(C, f^{*} T_{G / P}(-x)\right)=0$ for $i>0$, so (3.4.2) is smooth, and $\mathbb{T}$ is its relative tangent bundle at $(f, x)$. By RiemannRoch, the relative dimension is

$$
\operatorname{deg}\left(f^{*} T_{G / P}(-x)\right)+\operatorname{rank} f^{*} T_{G / P}(-x)=\operatorname{deg} f^{*} T_{G / P}=\left\langle 2 \rho_{P}, \mu\right\rangle
$$

as claimed.
Proposition 3.4.10. Let $\alpha_{i} \in \Delta$ be a simple root of $G$ and $B \subseteq G$ a Borel subgroup. Then the morphism

$$
M_{0,1}\left(G / B, \alpha_{i}^{\vee}\right) \longrightarrow G / B
$$

given by evaluation at the marked point is an isomorphism, and

$$
M_{0,1}\left(G / B, \alpha_{i}^{\vee}\right)=M_{0,1}^{\circ}\left(G / B, \alpha_{i}^{\vee}\right)
$$

Proof. Let $P$ be the minimal parabolic containing $B$ with $t(P)=\Delta \backslash\left\{\alpha_{i}\right\}$. Note that the morphism $\pi: G / B \rightarrow G / P$ is a family of smooth curves of genus 0 and that the identity $G / B \rightarrow G / B$ defines a stable map to $G / B$ over $G / P$ with degree $\alpha_{i}^{\vee}$. We claim that this
is the universal stable map of genus 0 and degree $\alpha_{i}^{\vee}$, so that $G / P=M_{0}\left(G / B, \alpha_{i}^{\vee}\right)$. The proposition then follows by Proposition 3.1.13 and the fact that all the domain curves are smooth.

To prove the claim, suppose that $U$ is any scheme and $f: C \rightarrow U \times G / B$ is a stable map over $U$ of genus 0 and degree $\alpha_{i}^{\vee}$. For every line bundle $\mathcal{L}_{\lambda}$ on $G / P$, we have $\operatorname{deg} f^{*} \pi^{*} \mathcal{L}_{\lambda}=0$ on the fibres of $C \rightarrow U$. So, since $G / P$ is projective, the morphism

$$
C \longrightarrow G / B \longrightarrow G / P
$$

factors through a unique morphism $U \rightarrow G / P$, and the morphism $C \rightarrow U \times_{G / P} G / B$ is a genus 0 stable map of degree 1 to a smooth curve of genus 0 over $S$, hence an isomorphism. So $(\pi: G / B \rightarrow G / P$, id: $G / B \rightarrow G / B)$ is the universal stable map to $G / B$ as claimed.

Lemma 3.4.11. Let $k$ be an algebraically closed field and let $\tau$ be a stable $\mathbb{X}^{*}\left(T_{P}\right)$-graph of genus 0 . Then $M_{k}\left((G / P)_{k}, \tau\right) \neq \emptyset$ if and only if for all $v \in V_{\tau}$, we have $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0} \subseteq$ $\mathbb{X}_{*}\left(T_{P}\right)$, where

$$
\mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}=\left\{\mu \in \mathbb{X}_{*}\left(T_{P}^{s c}\right) \mid\left\langle\varpi_{i}, \mu\right\rangle \geq 0 \text { for all } \alpha_{i} \in t(P)\right\}
$$

Proof. First suppose that $M_{k}\left((G / P)_{k}, \tau\right) \neq \emptyset$. Then for all $v \in V_{\tau}$, there exists a stable $\operatorname{map}\left(p_{v}: C_{v} \rightarrow(G / P)_{k},\left(x_{f}\right)_{f \in F_{\tau}(v)}\right)$ over Spec $k$ of degree $\beta(v)$. So by definition of degree, $\beta(v)$ is the image of $\mu \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)$ under the inclusion

$$
\mathbb{X}_{*}\left(T_{P}^{s c}\right)=\operatorname{Hom}(\mathrm{NS}(G / P), \mathbb{Z}) \hookrightarrow \operatorname{Hom}\left(\mathbb{X}^{*}\left(T_{P}\right), \mathbb{Z}\right)=\mathbb{X}_{*}\left(T_{P}\right),
$$

where $\langle\lambda, \mu\rangle=\operatorname{deg} p_{v}^{*} \mathcal{L}_{\lambda}$ for $\lambda \in \mathbb{X}^{*}\left(T_{P}^{s c}\right)$. Moreover, for all $\alpha_{i} \in t(P)$, we have $\left\langle\varpi_{i}, \mu\right\rangle \geq 0$ since the line bundle $\mathcal{L}_{\varpi_{i}}$ on $G / P$ is nef. So $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}$ as claimed.

Conversely, suppose that $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}$ for all $v \in V_{\tau}$. Then we can find a contraction $\tau^{\prime} \rightarrow \tau$ where $\tau^{\prime}$ is a stable $\mathbb{X}^{*}\left(T_{P}\right)$-graph of genus 0 such that $\beta\left(v^{\prime}\right) \in\{0\} \cup\left\{\alpha_{i}^{\vee} \mid \alpha_{i} \in t(P)\right\}$ for all $v^{\prime} \in V_{\tau^{\prime}}$. Since Proposition 3.4.10 implies that there exists an $n$-pointed stable map of genus 0 and degree $\alpha_{i}^{\vee}$ through any point in $(G / P)_{k}$ for any $\alpha_{i} \in t(P)$ and any $n \geq 0$, it follows by induction on the number of vertices of $\tau^{\prime}$ that $M_{k}\left((G / P)_{k}, \tau^{\prime}\right) \neq \emptyset$, and hence that $M_{k}\left((G / P)_{k}, \tau\right) \neq \emptyset$ as claimed.

Lemma 3.4.12. Let $\tau$ be a stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graph of genus $g$ such that the underlying $\mathbb{Z}$-graph $\tau_{0}$ has degree 1. Then $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right) \neq \emptyset$ if and only if $\tau_{0}$ satisfies the equivalent conditions of Lemma 3.3.5 and for every $v \in V_{\tau}$ with $\beta(v) \in \mathbb{X}_{*}\left(T_{P}\right) \subseteq \mathbb{X}_{*}\left(T_{P}\right) \oplus \mathbb{Z}$, we have $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0} \subseteq \mathbb{X}_{*}\left(T_{P}\right)$.

Proof. It is clear from Lemmas 3.3.5 and 3.4.11 that if $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right) \neq \emptyset$ then the claimed conditions must be satisfied. For the converse, assume that $\tau_{0}$ satisfies the conditions of Lemma 3.3.5 and that for every $v \in V_{\tau}$ with $\beta(v) \in \mathbb{X}_{*}\left(T_{P}\right)$ we have $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}$. We prove that $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right)$ is nonempty by induction on the number of vertices of $\tau$.

If $\tau$ has a single vertex $v$, then we have $\beta(v)=(1, \mu)$ for some $\mu \in \mathbb{X}_{*}\left(T_{P}\right)$. Choose any geometric point $s: \operatorname{Spec} k \rightarrow S$, any $T_{P}$-bundle $\xi_{T_{P}}$ on $X_{s}$ of degree $\mu$, and any lift of $\xi_{T_{P}}$ to a $P$-bundle $\xi_{P} \rightarrow X_{s}$. Then setting $\xi_{G}=\xi_{P} \times{ }^{P} G$ and choosing distinct points $x_{f} \in X_{s}$ for $f \in S_{\tau}$, we have a $\tau$-marked stable map $\left(\sigma: X_{s} \rightarrow \xi_{G} / P,\left(x_{f}\right)_{f \in F_{\tau}}\right)$, where $\sigma$ is the canonical section defined by the reduction $\xi_{P}$.

If $\tau$ has more than one vertex, then since the underlying graph of $\tau$ is a tree, we can choose a leaf $v \in V_{\tau}$ with $\beta(v) \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}$. Then writing $\tau_{0}^{\prime}$ and $\tau_{1}^{\prime}$ for the graphs with $V_{\tau_{0}^{\prime}}=\{v\}, F_{\tau_{0}^{\prime}}=F_{\tau}(v), V_{\tau_{1}^{\prime}}=V_{\tau} \backslash\{v\}$ and $F_{\tau_{1}^{\prime}}=F_{\tau} \backslash F_{\tau}(v)$, we have a Cartesian diagram

where $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{1}^{\prime}\right) \neq \emptyset$ by induction and the right vertical arrow is surjective by Lemma 3.4.11. So $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right) \neq \emptyset$ and we are done.

Proposition 3.4.13. Let $X \rightarrow S$ be a smooth curve over $S$. We have the following.
(1) The gluing morphism

$$
\coprod_{\tau} M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right) / \operatorname{Aut}(\tau) \longrightarrow \operatorname{KM}_{P, G / S}(X)
$$

is a bijection on points, where the coproduct is taken over all stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graphs $\tau$ with $S_{\tau}=\emptyset$ satisfying the conditions of Lemma 3.4.12.
(2) For every stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graph $\tau$ as above, we have

$$
\overline{M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right) / \operatorname{Aut}(\tau)}=\bigcup_{\tau^{\prime}} M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right),
$$

where the union is over all stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graphs $\tau^{\prime}$ satisfying the conditions of Lemma 3.4.12 such that there exists a contraction $\tau^{\prime} \rightarrow \tau$.

Proof. The claim (1) follows immediately from Proposition 3.2.18 and Lemma 3.4.12. To prove (2), note that by Corollary 3.2.20, it suffices to show that

$$
M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau^{\prime}\right) / \operatorname{Aut}\left(\tau^{\prime}\right) \subseteq \overline{M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right) / \operatorname{Aut}(\tau)}
$$

for all appropriate $\tau^{\prime}$ with a contraction $\tau^{\prime} \rightarrow \tau$. Fix such a $\tau^{\prime}$, a contraction $\psi: \tau^{\prime} \rightarrow \tau$ and a geometric point $p^{\prime}$ in $M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau^{\prime}\right)$. Writing $\psi_{0}: \tau_{0}^{\prime} \rightarrow \tau_{0}$ for the contraction of the underlying $\mathbb{Z}$-graphs, and $p_{0}^{\prime} \in \mathfrak{D e g}_{S}(X)$ for the image of $p^{\prime}$, from the proof of Proposition 3.3.7, (4) it is clear that there exists a complete discrete valuation ring $R$ and a morphism $p_{0}: \operatorname{Spec} R \rightarrow \mathfrak{M}_{S}\left(X, \tau_{0}\right)$ such that the generic fibre factors through $\mathfrak{M}_{S}^{\circ}\left(X, \tau_{0}\right)$ and the closed fibre is $\left(\psi_{0}\right)_{*}\left(p_{0}^{\prime}\right)$. Since there is a Cartesian diagram

where the coproduct is over all stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graphs with underlying $\mathbb{Z}$-graph $\tau_{0}$, the morphism $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right) \rightarrow \mathfrak{M}_{S}\left(X, \tau_{0}\right)$ is smooth by Proposition 3.4.7. So we can lift $p_{0}$ to a morphism $p: \operatorname{Spec} R \rightarrow M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau\right)$ with closed fibre $p^{\prime}$ and generic fibre factoring through $M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right)$. So the gluing of $p^{\prime}$ is in the closure of $M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right) / \operatorname{Aut}(\tau)$ and we are done.

We have the following corollary of Proposition 3.4.13, where for $\lambda, \lambda^{\prime} \in \mathbb{X}_{*}\left(T_{P}\right)$ we write $\lambda^{\prime} \leq \lambda$ if $\lambda-\lambda^{\prime} \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0}$.

Corollary 3.4.14. A G-bundle $\xi_{G} \rightarrow X_{s}$ is in the image of $\mathrm{KM}_{P, G / S}^{\lambda}(X) \rightarrow \operatorname{Bun}_{G / S}(X)$ if and only if there exists a reduction of $\xi_{G}$ to a $P$-bundle of degree $\lambda^{\prime} \leq \lambda$.

Proof. It is immediate from Proposition 3.4.13 that the restriction of a stable map $\sigma: C \rightarrow$ $\xi_{G} / P$ in $\mathrm{KM}_{P, G / S}^{\lambda}(X)$ to the irreducible component mapping isomorphically to $X_{s}$ defines a reduction of $\xi_{G}$ to a $P$-bundle of degree $\lambda^{\prime} \leq \lambda$. Conversely, if such a reduction exists, then the proof of Lemma 3.4.12 shows that we can complete the corresponding section of $\xi_{G} / P \rightarrow X_{s}$ to a stable map in $\mathrm{KM}_{P, G / S}^{\lambda}(X)$.

Convention 3.4.15. If a $\mathbb{X}^{*}(T) \oplus \mathbb{Z}$-graph $\tau$ is the dual graph of a stable map in $\mathrm{KM}_{P, G / S}(X)$ with respect to the degree datum of Lemma 3.4.4, we will draw $\tau$ by labelling each vertex $v$ of the underlying $\mathbb{Z}$-graph (drawn according to Convention 3.3.6) with the projection of $\beta(v) \in \mathbb{X}_{*}\left(T_{P}\right) \oplus \mathbb{Z}$ to $\mathbb{X}_{*}\left(T_{P}\right)$.

For $\lambda \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}=\mathbb{X}_{*}\left(T_{P}^{s c}\right)_{\geq 0} \backslash\{0\}$, write $D_{\lambda, P}^{\mu, \circ} \subseteq \mathrm{KM}_{P, G / S}^{\mu}(X)$ for the locally closed substack of stable maps with dual graph $\tau_{\lambda}^{\mu}$ given by

$$
\tau_{\lambda}^{\mu}=\underset{\mu-\lambda}{\bullet}
$$

In other words, $D_{\lambda, P}^{\mu, \circ}$ is the image of $M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \rightarrow \mathrm{KM}_{P, G / S}^{\mu}(X)$. We write $D_{\lambda, P}^{\mu}$ for the closure of $D_{\lambda, P}^{\mu, o}$ in $\mathrm{KM}_{P, G / S}^{\mu}(X)$, and

$$
D_{P}^{\mu}=\bigcup_{\lambda \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}} D_{\lambda, P}^{\mu}
$$

Note that Proposition 3.4.13 implies that $D_{P}^{\mu}$ is equal to the complement of $\operatorname{Bun}_{P / S}^{\mu}(X) \subseteq$ $\mathrm{KM}_{P, G / S}^{\mu}(X)$ since every stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graph with more than one vertex appearing in $\mathrm{KM}_{P, G / S}^{\mu}(X)$ admits a contraction onto $\tau_{\lambda}^{\mu}$ for some $\lambda \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}$.

Proposition 3.4.16. We have the following.
(1) The closed substack $D_{P}^{\mu} \subseteq \mathrm{KM}_{P, G / S}^{\mu}(X)$ is a divisor with normal crossings.
(2) If $\lambda_{1}, \ldots, \lambda_{m} \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}$are distinct elements and $j_{1}, \ldots, j_{m} \in \mathbb{Z}_{>0}$, then the open stratum where $D_{P}^{\mu}$ is locally the intersection of $j_{i}$ branches of $D_{\lambda_{i}, P}$ for $i=1, \ldots, m$ is given by

$$
\left(D_{P}^{\mu}\right)^{\left(\lambda_{1}^{\left.j_{1} \ldots \lambda_{m}^{j_{m}}\right)}=\coprod_{\tau} M_{\operatorname{Bun}_{G / S}(X)}^{\circ}\left(\xi_{G}^{u n i} / P, \tau\right) / \operatorname{Aut}(\tau),{ }^{\circ}\right)}
$$

where the coproduct is over all stable $\mathbb{X}^{*}\left(T_{P}\right) \oplus \mathbb{Z}$-graphs $\tau$ satisfying the conditions of Lemma 3.4.12 such that there are exactly $j_{i}$ contractions $\tau \rightarrow \tau_{\lambda_{i}}^{\mu}$ for each $i=1, \ldots, m$, and no contraction $\tau \rightarrow \tau_{\lambda}^{\mu}$ for $\lambda \notin\left\{\lambda_{1}, \ldots, \lambda_{m}\right\}$.

Proof. The claim (1) is immediate from Propositions 3.3.7 and 3.4.7.
To prove (2), fix a geometric point $p$ in the open stratum $D_{P}^{(n)} \subseteq D$ where $D_{P}^{\mu}$ is locally isomorphic to an intersection of $n$ coordinate hyperplanes in an affine space, let $\tau$ be the dual graph of $p$, and write $p^{\prime}$ for the $\tau$-marked stable map with gluing $p$. Then by Proposition 3.3.7, (3), the graph $\tau$ has exactly $n$ edges. Writing $E_{\tau}=\left\{e_{1}, \ldots, e_{n}\right\}$, for every $i=1, \ldots, n$ there exists a unique $\lambda_{(i)} \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}$and a contraction $\psi_{i}: \tau \rightarrow \tau_{\lambda_{(i)}}^{\mu}$ such that $e_{i}$ is the unique edge not contracted under $\psi_{i}$. So there are $\lambda_{1}, \ldots, \lambda_{m} \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}$and $j_{1}, \ldots, j_{m} \in \mathbb{Z}_{>0}$, unique up to reordering, such that $\tau$ has exactly $j_{i}$ contractions onto $\tau_{\lambda_{i}}^{\mu}$ for each $i$ and no contractions onto $\tau_{\lambda}^{\mu}$ for $\lambda \notin\left\{\lambda_{1}, \ldots, \lambda_{m}\right\}$. So it suffices to show that $p$ lies in the stratum $\left(D_{P}^{\mu}\right)^{\left(\lambda_{1}^{j_{1}} \ldots \lambda_{m}^{j_{m}}\right)}$ where $j_{i}$ branches of $D_{\lambda_{i}, P}^{\mu}$ intersect for $i=1, \ldots, m$.

From the proof of Proposition 3.3.7 (3), there exists a complete Noetherian local ring $A$ and formally smooth morphisms $\bar{p}_{0}: \operatorname{Spec} A \rightarrow \operatorname{Deg}_{S}(X)$ and $\operatorname{Spec} A \rightarrow \operatorname{Spec} \mathfrak{o}_{k} \llbracket t_{1}, \ldots, t_{n} \rrbracket \times$ $S$ over $S$ such that $\bar{p}^{-1}(D)$ is the locus $t_{1} \cdots t_{n}=0$, and for each $i=1, \ldots, n$, we have a commutative diagram

where $A_{i}=A /\left(t_{i}\right),\left(\tau_{\lambda_{(i)}}^{\mu}\right)_{0}$ is the $\mathbb{Z}$-graph underlying $\tau_{\lambda_{(i)}}^{\mu}$ and $\left(\bar{p}_{i}\right)_{0}$ maps the locus $\prod_{j \neq i} t_{i} \neq$ 0 into $\mathfrak{M}_{S}^{\circ}\left(X,\left(\tau_{\lambda_{(i)}}^{\mu}\right)_{0}\right)$ and the closed point to the the gluing $\left(\psi_{i}\right)_{0_{*}}\left(p_{0}^{\prime}\right)$ of the image $p_{0}^{\prime}$ of $p^{\prime}$ in $\mathfrak{M}_{S}\left(X, \tau_{0}\right)$ with respect to the contraction $\left(\psi_{i}\right)_{0}: \tau_{0} \rightarrow\left(\tau_{\lambda_{(i)}}^{\mu}\right)_{0}$ underlying $\psi_{i}$.

Since the diagram

realises $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda_{(i)}}^{\mu}\right)$ as a connected component of the fibre product, Proposition 3.4.7 ensures that, after replacing $\operatorname{Spec} A$ with some formally smooth cover if necessary, the diagram (3.4.3) lifts to a diagram

such that $\bar{p}$ sends the closed point to $p$ and $\bar{p}_{i}$ sends the closed point to the gluing of $p^{\prime}$ under $\psi_{i}$. So in particular $\bar{p}$ sends the locus $\prod_{j \neq i} t_{i} \neq 0$ into $D_{\lambda_{(i)}}^{\circ}$, and we conclude that $p$ lies in $\left(D_{P}^{\mu}\right)^{\left(\lambda_{1}^{j_{1}} \ldots \lambda_{m}^{j_{m}}\right)}$ as claimed.

### 3.5 Blow down morphisms

An important feature of the Kontsevich-Mori compactification is the existence of a morphism

$$
\operatorname{KM}_{P, G / S}(X) \longrightarrow \operatorname{Bun}_{T_{P} / S}(X)
$$

extending the natural morphism $\operatorname{Bun}_{P / S}(X) \rightarrow \operatorname{Bun}_{T_{P} / S}(X)$. In this section, we define this morphism and study some of its basic properties.

For simplicity, we will assume that the base stack $S$ is regular. Since any family of smooth curves of genus $g$ is pulled back from one over the smooth $\mathbb{Z}$-stack $\mathfrak{M}_{g}^{\circ}$, one can carry through these constructions for a general base $S$, if desired, by pulling back from this universal base.

Let $T$ be a split torus over $\operatorname{Spec} \mathbb{Z}$, and let $\xi_{T}^{u n i} \rightarrow \operatorname{Bun}_{T / \mathcal{D e g}_{S}(X)}(\mathcal{C})$ be the universal $T$-bundle, where $f: \mathcal{C} \rightarrow \operatorname{Deg}_{S}(X) \times{ }_{S} X$ is the universal prestable degeneration of $X$. By Proposition 3.3.8, there is an open substack $U=\left(\mathfrak{D e g}_{S}(X) \times{ }_{S} X\right) \backslash f(\operatorname{Exc}) \subseteq \operatorname{Deg}_{S}(X) \times_{S} X$ whose complement has codimension 2, such that morphism $f^{-1}(U) \rightarrow U$ is an isomorphism. Since $S$ is regular, so is $\operatorname{Bun}_{T / \mathcal{D e g}_{S}(X)}(\mathcal{C}) \times_{S} X$, so the restriction of $\xi_{T}^{u n i}$ to
$\operatorname{Bun}_{T / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times_{\mathfrak{D e g}_{S}(X)} f^{-1}(U) \cong \operatorname{Bun}_{T / \mathcal{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times_{\mathfrak{D e g}_{S}(X)} U \subseteq \operatorname{Bun}_{T / \mathcal{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times_{S} X$
extends uniquely to a $T$-bundle $\operatorname{Bl}\left(\xi_{T}^{u n i}\right)$ on $\operatorname{Bun}_{T / \mathfrak{D e g}}^{S}(X)(\mathcal{C}) \times{ }_{S} X$ since $T$ is a torus. The $T$-bundle $\mathrm{Bl}\left(\xi_{T}^{u n i}\right)$ determines a morphism

$$
\mathrm{Bl}_{T}: \operatorname{Bun}_{T / \mathfrak{D e g}}^{S}(X)(\mathcal{C}) \longrightarrow \operatorname{Bun}_{T / S}(X)
$$

Definition 3.5.1. In the setup above, we call the morphism $\mathrm{Bl}_{T}$ the blow down morphism for $T$. If $P \subseteq G$ is a parabolic subgroup of a reductive group $G$ with associated torus $T_{P}=P /[P, P]$, we define the blow down morphism for $P$ to be the composition

$$
\mathrm{Bl}_{P}: \mathrm{KM}_{P, G / S}(X) \longrightarrow \operatorname{Bun}_{T_{P} / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \xrightarrow{\mathrm{Bl}_{T_{P}}} \operatorname{Bun}_{T_{P} / S}(X) .
$$

The blow down of a $T$-bundle can also be described in terms of its associated line bundles.
Lemma 3.5.2. Let $\lambda \in \mathbb{X}^{*}(T)$ be a character. Then

$$
\begin{equation*}
\lambda\left(\operatorname{Bl}\left(\xi_{T}^{u n i}\right)\right)=\operatorname{det} \mathbb{R} f_{*} \lambda\left(\xi_{T}\right), \tag{3.5.1}
\end{equation*}
$$

where det denotes the determinant of a perfect complex, and by abuse of notation we write

$$
f: \operatorname{Bun}_{T / \operatorname{Deg}_{S}(X)}(\mathcal{C}) \times_{\mathfrak{D e g}_{S}(X)} \mathcal{C} \rightarrow \operatorname{Bun}_{T / \operatorname{Deg}_{S}(X)}(\mathcal{C}) \times_{S} X
$$

for the pullback of the morphism $f: \mathcal{C} \rightarrow \operatorname{Deg}_{S}(X) \times{ }_{S} X$.
Proof. Since $\operatorname{Bun}_{T / \operatorname{Deg}_{S}(X)}(\mathcal{C}) \times_{S} X$ is regular, this follows from the fact that both sides of (3.5.1) agree when restricted to $\operatorname{Bun}_{T / \mathfrak{D e g}}^{S}(X)(\mathcal{C}) \times \mathfrak{D e g}_{S}(X) U$.

Proposition 3.5.3. The morphism

$$
\mathrm{Bl}_{T}^{\prime}: \operatorname{Bun}_{T / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \longrightarrow \mathfrak{D e g}_{S}(X) \times_{S} \operatorname{Bun}_{T / S}(X)
$$

is étale.
Proof. The claim for general $T$ reduces easily to the case where $T=\mathbb{G}_{m}$. In this case, we need to show that the morphism of tangent complexes

$$
\begin{equation*}
\mathbb{R} p_{*} \mathbb{R} f_{*} \mathcal{O}[1]=\mathbb{T}_{\operatorname{Bun}_{\mathbb{G}_{m} / \mathcal{D e g} \mathbb{g}_{S}(X)}(\mathcal{C}) / \operatorname{Deg}_{S}(X)} \longrightarrow \mathrm{Bl}_{T}^{*} \mathbb{T}_{\operatorname{Bun}_{G_{m} / S}(X) / S}=\mathbb{R} p_{*} \mathcal{O}[1] \tag{3.5.2}
\end{equation*}
$$

is a quasi-isomorphism, where $p: \operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D e g}}^{S(X)},(\mathcal{C}) \times{ }_{S} X \rightarrow \operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D} \mathrm{eg}_{S}(X)}(\mathcal{C})$ is the projection onto the first factor and $f: \operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times \times_{\operatorname{Deg}_{S}(X)} \mathcal{C} \rightarrow \operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times_{S} X$ is the pullback of the universal morphism as above. From the description of $\mathrm{Bl}_{T}$ from Lemma 3.5.2, it follows that (3.5.2) is obtained by pushing forward the morphism

$$
\begin{equation*}
\mathbb{R} f_{*} \mathcal{O}[1]=\mathbb{R} f_{*} \underline{\operatorname{End}}(L)[1] \longrightarrow \mathbb{R} \underline{\operatorname{Hom}}\left(\mathbb{R} f_{*} L, \mathbb{R} f_{*} L\right)[1] \xrightarrow{\operatorname{Tr}} \mathcal{O}[1] \tag{3.5.3}
\end{equation*}
$$

on $\operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D e g} \operatorname{li}_{S}(X)}(\mathcal{C}) \times{ }_{S} X$, where $L$ is the universal line bundle on $\operatorname{Bun}_{\mathbb{G}_{m} / \mathfrak{D} \operatorname{eg}_{S}(X)}(\mathcal{C}) \times_{\mathfrak{D e g}_{S}(X)}$ $\mathcal{C}$. But since $\mathbb{R} f_{*} \mathcal{O}=\mathcal{O}$, and (3.5.3) is an isomorphism on an open substack whose complement has codimension 2 , it is necessarily an isomorphism everywhere. So (3.5.2) is a quasi-isomorphism as claimed.

Corollary 3.5.4. Let $P$ be a parabolic subgroup of a reductive group $G$. Then for any $\lambda \in \mathbb{X}_{*}\left(T_{P}\right)$, the blow down morphism $\mathrm{Bl}_{P}$ restricts to a smooth morphism

$$
\mathrm{KM}_{P, G / S}^{\lambda}(X) \longrightarrow \operatorname{Bun}_{T / S}^{\lambda}(X)
$$

extending the natural morphism $\operatorname{Bun}_{P / S}^{\lambda}(X) \rightarrow \operatorname{Bun}_{T / S}^{\lambda}(X)$.

Proof. Note that $\mathrm{Bl}_{P}$ restricts to the canonical morphism $\operatorname{Bun}_{P / S}(X) \rightarrow \operatorname{Bun}_{T_{P} / S}(X)$ by definition, and hence sends $P$-bundles of degree $\lambda$ on $\mathcal{C}$ to $T_{P}$-bundles of degree $\lambda$ on $X$ since $\operatorname{Bun}_{P / S}(X) \subseteq \mathrm{KM}_{P, G / S}(X)$ is dense. It remains to show that $\mathrm{Bl}_{P}$ is smooth. By Proposition 3.5.3 and Proposition 3.3.7 (1), the only thing left to check is that the composition

$$
\operatorname{KM}_{P, G / S}(X) \rightarrow \operatorname{Bun}_{P / \operatorname{Deg}_{S}(X)}(\mathcal{C}) \rightarrow \operatorname{Bun}_{T_{P} / \mathcal{D e g}_{S}(X)}(\mathcal{C})
$$

is smooth. The first morphism is étale since it is the pullback of the étale morphism $\operatorname{Bun}_{G / S}(X) \times_{S} \operatorname{Deg}_{S}(X) \rightarrow \operatorname{Bun}_{G / \mathfrak{D e g}_{S}(X)}(\mathcal{C})$. Smoothness of the second morphism reduces to the fact that for every prestable degeneration $g: C \rightarrow X_{s}$ and every $P$-bundle $\xi_{P}$ on $C$, we have

$$
H^{i}\left(C, \xi_{P} \times^{P}[\mathfrak{p}, \mathfrak{p}]\right)=0 \quad \text { for } i>1,
$$

where $\mathfrak{p}=\operatorname{Lie}(P)$.
We conclude this section with the following observation about the connection between the blow down morphism and gluing.

Proposition 3.5.5. Let $\mu \in \mathbb{X}_{*}\left(T_{P}\right)$ and $\lambda \in \mathbb{X}_{*}\left(T_{P}^{s c}\right)_{+}$. Then there is a commutative diagram

where the vertical morphism on the left is given by the natural forgetful map

$$
M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \longrightarrow M_{g, 1, \operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P,(\mu-\lambda, 1)\right)
$$

composed with the map forgetting the marked point and stabilising on the first factor and evaluating at the marked point and composing with the natural map to $X$ on the second, and the horizontal morphism on the bottom right is given by $\left(\xi_{T}, x\right) \mapsto \xi_{T} \otimes \lambda(\mathcal{O}(x))$.

Proof. The two morphisms

$$
M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \longrightarrow \operatorname{Bun}_{T_{P} / S}^{\mu}(X)
$$

classify $T_{P}$-bundles on $M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \times_{S} X$ with an isomorphism $\phi$ outside the section

$$
M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \rightarrow M_{\operatorname{Bun}_{G / S}(X)}\left(\xi_{G}^{u n i} / P, \tau_{\lambda}^{\mu}\right) \times_{S} X
$$

given by evaluation at the marked point on the genus $g$ domain curve and projection from $\xi_{G}^{u n i} / P$ to $X$. Since both $T$-bundles have degree $\mu$ on every fibre, it follows that $\phi$ extends to a global isomorphism, which gives the 2-isomorphism making the diagram commute.

### 3.6 Applications to $\mathrm{Bun}_{G}$

In this section, we give some basic applications of the theory of Kontsevich-Mori compactifications to the structure theory of $\operatorname{Bun}_{G / S}(X)$ for a reductive group $G$. We begin with the proof that the locus of semistable bundles is open (Proposition 2.5.13).

Proof of Proposition 2.5.13. Let $P_{1}, \ldots, P_{l}$ denote the maximal parabolics of $G$ with types $t\left(P_{i}\right)=\left\{\alpha_{i}\right\}$. Call a character $\lambda \in \mathbb{X}_{*}\left(T_{P_{i}}\right)$ primitive if it is not a positive multiple of another character, and let $\lambda_{i} \in \mathbb{X}^{*}\left(P_{i}\right)$ be the unique primitive nonzero dominant characters vanishing on $Z(G)^{\circ}$. The natural homomorphism

$$
\frac{\mathbb{X}_{*}\left(T_{P_{i}}\right)}{\mathbb{Z} \alpha_{i}^{\vee}} \longrightarrow \mathbb{X}_{*}(G /[G, G])
$$

is surjective with finite kernel, and for every $\mu_{0} \in \mathbb{X}_{*}\left(T_{P_{i}}\right) / \mathbb{Z} \alpha_{i}^{\vee}$, there is a unique lift $\mu \in \mathbb{X}_{*}\left(T_{P_{i}}\right)$ such that $\left\langle\lambda_{i}, \mu\right\rangle<0$ and $\mu$ is maximal among lifts with this property with respect to the partial order of Corollary 3.4.14. Corollary 3.4.14 then implies that for every $\mu \in \mathbb{X}_{*}(G /[G, G])$, there exist finitely many $\mu_{1}, \ldots, \mu_{n_{i}} \in \mathbb{X}_{*}\left(T_{P_{i}}\right)$ such that the image of

$$
\coprod_{j=1}^{n_{i}} \mathrm{KM}_{P_{i} / S}^{\mu_{j}}(X) \longrightarrow \operatorname{Bun}_{G / S}^{\mu}(X)
$$

is equal to the locus of $G$-bundles $\xi_{G} \rightarrow X_{s}$ admitting a section $\sigma: X_{s} \rightarrow \xi_{G} / P_{i}$ of degree $\mu^{\prime}$ satisfying $\left\langle\lambda_{i}, \mu^{\prime}\right\rangle<0$. So the locus of semistable bundles in $\operatorname{Bun}_{G / S}^{\mu}(X) \subseteq \operatorname{Bun}_{G / S}(X)$ is equal to the complement of the image of

$$
\coprod_{i=1}^{l} \coprod_{j=1}^{n_{i}} \mathrm{KM}_{P_{i} / S}^{\mu_{j}}(X) \longrightarrow \operatorname{Bun}_{G / S}(X)
$$

Since this morphism is proper and hence has closed image, openness of semistable bundles follows.

Proposition 3.6.1. Assume that $G$ is a reductive group such that the semisimple group $[G, G]$ is simply connected. Then for any $\mu \in \mathbb{X}_{*}(G /[G, G])$, the morphism $\operatorname{Bun}_{G / S}^{\mu}(X) \rightarrow S$ has connected fibres.

Proof. Since the statement concerns only geometric fibres of $\operatorname{Bun}_{G / S}(X) \rightarrow S$, we may assume without loss of generality that $S=\operatorname{Spec} k$ for some algebraically closed field $k$. We will also fix a Borel subgroup $B$ and maximal torus $T \subseteq B$.

First, observe that since $G$ is connected and reductive and $k$ is algebraically closed, [BS, §8.6] shows that any $G$-bundle $\xi_{G} \rightarrow X$ is trivial at the generic fibre, so there exists a section $X \rightarrow \xi_{G} / B$ since $X$ is a curve and $\xi_{G} / B$ is proper. So the morphism $\operatorname{Bun}_{B}(X) \rightarrow$ $\operatorname{Bun}_{G}(X)$ is surjective.

Now suppose that $\xi_{G}$ and $\eta_{G}$ are $G$-bundles on $X$ of the same degree $\mu \in \mathbb{X}_{*}(G /[G, G])$. We need to show that $\xi_{G}$ and $\eta_{G}$ belong to the same connected component of $\operatorname{Bun}_{G}(X)$. To see this, choose $B$-reductions $\xi_{B}$ and $\eta_{B}$ of $\xi_{G}$ and $\eta_{G}$ of degrees $\lambda_{1}, \lambda_{2} \in \mathbb{X}_{*}(T)$. Since $[G, G]$ is simply connected, we have a short exact sequence

$$
0 \longrightarrow \mathbb{Z} \Phi^{\vee} \longrightarrow \mathbb{X}_{*}(T) \longrightarrow \mathbb{X}_{*}(G /[G, G]) \longrightarrow 0
$$

and hence $\lambda_{2}-\lambda_{1} \in \mathbb{Z} \Phi^{\vee}$. So there exists $\lambda \in \mathbb{X}_{*}(T)$ such that $\lambda_{1} \leq \lambda$ and $\lambda_{2} \leq \lambda$. Therefore, $\xi_{G}$ and $\eta_{G}$ are both in the image of the morphism

$$
\mathrm{KM}_{B, G}^{\lambda}(X) \longrightarrow \operatorname{Bun}_{G}(X)
$$

by Corollary 3.4.14. But $\mathrm{KM}_{B, G}^{\lambda}(X)$ is connected, since its dense open substack $\operatorname{Bun}_{B}^{\lambda}(X)$ is by Proposition 2.4.2 and Corollary 2.4.5, so $\xi_{G}$ and $\eta_{G}$ belong to the same connected component as claimed.

Remark 3.6.2. The same proof shows that the morphism $\operatorname{Bun}_{G / S}^{\lambda}(X) \rightarrow \operatorname{Bun}_{(G /[G, G]) / S}^{\lambda}(X)$ has connected fibres when $[G, G]$ is simply connected.

Remark 3.6.3. When the derived group $[G, G]$ is not simply connected, one can show that the degree of a Borel reduction of $\xi_{G}$ modulo $\mathbb{Z} \Phi^{\vee}$ depends only on $\xi_{G}$. The proof of Proposition 3.6.1 then shows that this invariant in $\mathbb{X}_{*}(T) / \mathbb{Z} \Phi^{\vee}$ singles out the connected components of $\operatorname{Bun}_{G}(X)$ in the general case.

Proposition 3.6.4. Assume that $[G, G]$ is simply connected and $X \rightarrow S$ is a family of smooth curves of genus $\leq 1$, let $P \subseteq G$ be a parabolic subgroup containing a Borel subgroup $B$, and let $\lambda \in \mathbb{X}_{*}(T)$ be such that $\langle\alpha, \lambda\rangle \geq 0$ for all $\alpha \in \Phi_{+}$such that $\alpha$ is a root of $P$. Then the morphism

$$
\mathrm{KM}_{B, G / S}^{\lambda}(X) \longrightarrow \mathrm{KM}_{P, G / S}^{\lambda}(X)
$$

is surjective, where $\lambda^{\prime}$ is the image of $\lambda$ in $\mathbb{X}_{*}\left(T_{P}\right)$. In particular, any degree $\lambda P$-bundle on a geometric fibre of $X \rightarrow S$ has a reduction to $B$ of degree $\leq \lambda$.

Proof. For simplicity, we can assume without loss of generality that $S=\operatorname{Spec} k$ for $k$ an algebraically closed field. We also fix a maximal torus $T \subseteq B$.

We first remark that by the assumption on $\lambda$, for a generic $T$-bundle of degree $\lambda$, we have that $\xi_{T} \times{ }^{T} \mathfrak{p} / \mathfrak{b}$ is a direct sum of nontrivial line bundles of nonnegative degree, and hence $H^{1}\left(X, \xi_{T} \times^{T} \mathfrak{p} / \mathfrak{b}\right)=0$, where $\mathfrak{p}=\operatorname{Lie}(P)$ and $\mathfrak{b}=\operatorname{Lie}(B)$. So the morphism $\operatorname{Bun}_{B}^{\lambda}(X) \rightarrow$ $\operatorname{Bun}_{P}^{\lambda^{\prime}}(X)$ is smooth at the point $\xi_{B}=\xi_{T} \times^{T} B$ for such a $T$-bundle.

Hence, there is a nonempty open subset $U \subseteq \operatorname{Bun}_{B}^{\lambda}(X)$ such that the morphism $U \rightarrow$ $\operatorname{Bun}_{P}^{\lambda^{\prime}}(X)$ is smooth. Since smooth morphisms are open, we conclude that the image of $\mathrm{KM}_{B, G}^{\lambda}(X) \rightarrow \mathrm{KM}_{P, G}^{\lambda^{\prime}}(X)$ contains an open substack. Since Corollary 2.4.5 and Proposition 3.6.1 imply that $\mathrm{KM}_{P, G}^{\lambda^{\prime}}(X)$ is smooth and connected and $\mathrm{KM}_{B, G}^{\lambda}(X) \rightarrow \mathrm{KM}_{P, G}^{\lambda^{\prime}}(X)$ is proper, we deduce that $\mathrm{KM}_{B, G}^{\lambda}(X) \rightarrow \mathrm{KM}_{P, G}^{\lambda^{\prime}}(X)$ is surjective as claimed.

### 3.7 Bruhat cells for parabolic bundles

Let $G$ be a reductive group and $P, P^{\prime} \subseteq G$ parabolic subgroups containing a Borel $B$ and maximal torus $T \subseteq B$. The Bruhat decomposition

$$
\begin{equation*}
G / P^{\prime}=\coprod_{w \in W_{P} \backslash W / W_{P^{\prime}}} P w P^{\prime} / P^{\prime} \tag{3.7.1}
\end{equation*}
$$

into $P$-orbits is an important tool in the study of the partial flag variety $G / P^{\prime}$. Here $w$ ranges over any fixed set of double coset representatives for the Weyl groups $W_{P}=W_{L}$ and $W_{P^{\prime}}=W_{L^{\prime}}$ of the Levi factors $L \subseteq P$ and $L^{\prime} \subseteq P^{\prime}$ inside the Weyl group $W=N_{G}(T) / T$ of $G$. In this section, we study the natural cells in the stack

$$
\operatorname{Bun}_{P / S}(X) \times_{\operatorname{Bun}_{G / S}(X)} \operatorname{Bun}_{P^{\prime} / S}(X)
$$

coming from the decomposition (3.7.1) for $X \rightarrow S$ a smooth curve.
It will be convenient for us to have a standard set of double coset representatives. Define

$$
\begin{equation*}
W_{P, P^{\prime}}^{0}=\left\{w \in W \mid w^{-1} \alpha_{i} \in \Phi_{+} \text {and } w \alpha_{j} \in \Phi_{+} \text {for } \alpha_{i} \in \Delta \backslash t(P) \text { and } \alpha_{j} \in \Delta \backslash t\left(P^{\prime}\right)\right\} \tag{3.7.2}
\end{equation*}
$$

Proposition 3.7.1. The set $W_{P, P^{\prime}}^{0}$ is a complete set of coset representatives for $W_{P}$ and $W_{P}^{\prime}$ in $W$.

Proof. First notice that
$W_{P, P^{\prime}}^{0}=\left\{w \in W \mid \ell\left(s_{i} w\right)>\ell(w)\right.$ and $\ell\left(w s_{j}\right)>\ell(w)$ for $\alpha_{i} \in \Delta \backslash t(P)$ and $\left.\alpha_{j} \in \Delta \backslash t\left(P^{\prime}\right)\right\}$, so in particular, if $w \in W$ has minimal length among elements of $W_{P} w W_{P^{\prime}}$, then $w \in W_{P, P^{\prime}}^{0}$. So every double coset has a representative in $W_{P, P^{\prime}}^{0}$.

To prove uniqueness of this representative, assume that $w \in W_{P, P^{\prime}}^{0}$ has minimal length in $W_{P} w W_{P^{\prime}}$ and that $w^{\prime} \in W_{P} w W_{P^{\prime}} \cap W_{P, P^{\prime}}^{0}$. Then we can write

$$
w^{\prime}=s_{i_{1}} \cdots s_{i_{m}} w s_{j_{1}} \cdots s_{j_{n}}
$$

where $\alpha_{i_{k}} \in \Delta \backslash t(P)$ and $\alpha_{j_{k}} \in \Delta \backslash t\left(P^{\prime}\right)$ for each $k$, and $s_{i_{1}} \cdots s_{i_{m}}$ and $s_{j_{1}} \cdots s_{j_{n}}$ are reduced words. We prove by induction on $m+n$ that $w=w^{\prime}$. If $m+n=0$, then this is clear, and if $m+n>0$ we can assume without loss of generality that $m>0$. Choose any reduced word $w=s_{k_{1}} \cdots s_{k_{p}}$ for $w$. Since $w^{\prime} \in W_{P, P^{\prime}}^{0}$,

$$
\ell\left(w^{\prime}\right)=\ell\left(s_{i_{1}} s_{i_{2}} \cdots s_{i_{m}} s_{k_{1}} \cdots s_{k_{p}} s_{j_{1}} \cdots s_{j_{n}}\right)<\ell\left(s_{i_{2}} \cdots s_{i_{m}} s_{k_{1}} \cdots s_{k_{p}} s_{j_{1}} \cdots s_{j_{n}}\right)=\ell\left(s_{i_{1}} w^{\prime}\right)
$$

So by the deletion property for Coxeter groups, reducedness of $s_{i_{1}} \cdots s_{i_{m}}, s_{k_{1}} \cdots s_{k_{p}}$ and $s_{j_{1}} \cdots s_{j_{n}}$, and the minimality of the length of $w$ in its double coset, there must exist an index $j_{q}$ such that

$$
w^{\prime}=s_{i_{2}} \cdots s_{i_{m}} s_{k_{1}} \cdots s_{k_{p}} s_{j_{1}} \cdots \widehat{s}_{j_{q}} \cdots s_{j_{n}}=s_{i_{2}} \cdots s_{i_{m}} w s_{j_{1}} \cdots \widehat{s}_{j_{q}} \cdots s_{j_{n}}
$$

So $w^{\prime}=w$ by induction.
We remark that the coset representatives $W_{P}^{0}=W_{P, B}^{0}$ have the following nice property.
Proposition 3.7.2. If $w \in W_{P}^{0}$, then $L \cap w B w^{-1}=L \cap B$, and hence $L \cap B \subseteq L \cap w P^{\prime} w^{-1}$.
Proof. Note that since $w^{-1} \alpha_{i} \in \Phi_{+}$for all $\alpha_{i} \in \Delta \backslash t(P)$, we have that $\alpha_{i}$ is a root of $w B w^{-1}$ for all $\alpha_{i} \in \Delta \backslash t(P)$, and hence $L \cap B \subseteq L \cap w B w^{-1}$. Since $L \cap w B w^{-1}$ is a Borel subgroup of $L$, it follows that $L \cap B=L \cap w B w^{-1}$ as claimed.

Let $S$ be any stack and let $X \rightarrow S$ be a smooth curve. The partial Bruhat decomposition on $G / P^{\prime}$ gives a decomposition

$$
\mathbb{B} P \times_{\mathbb{B} G} \mathbb{B} P^{\prime}=P \backslash G / P^{\prime}=\coprod_{w \in W_{P} \backslash W / W_{P^{\prime}}} P \backslash P w P^{\prime} / P^{\prime} \cong \coprod_{w \in W_{P} \backslash W / W_{P^{\prime}}} \mathbb{B}\left(P \cap w P^{\prime} w^{-1}\right)
$$

into disjoint locally closed substacks, and hence a family of disjoint locally closed substacks

$$
\operatorname{Bun}_{P \cap w P^{\prime} w^{-1} / S}(X) \longleftrightarrow \operatorname{Bun}_{P / S}(X) \times_{\operatorname{Bun}_{G / S}(X)} \operatorname{Bun}_{P^{\prime} / S}(X)
$$

for $w \in W_{P} \backslash W / W_{P^{\prime}}$.
Definition 3.7.3. If $w \in W_{P} \backslash W / W_{P^{\prime}}$ and $\lambda \in \mathbb{X}_{*}\left(T_{P^{\prime}}\right)$, the associated Bruhat cell is

$$
\begin{aligned}
C_{P, P^{\prime} / S}^{w, \lambda}(X) & =\operatorname{Bun}_{P \cap w P^{\prime} w^{-1} / S}(X) \times \times_{\operatorname{Bun}_{P^{\prime} / S}(X)} \operatorname{Bun}_{P^{\prime} / S}^{\lambda}(X) \\
& \subseteq \operatorname{Bun}_{P / S}(X) \times_{\operatorname{Bun}_{G / S}(X)} \operatorname{Bun}_{P^{\prime} / S}^{\lambda}(X) .
\end{aligned}
$$

There is a natural decomposition of $C_{P, P^{\prime} / S}^{w, \lambda}(X)$ in terms of the degree of the associated $P \cap w P^{\prime} w^{-1}$-bundle. In the following proposition, we write $T_{P \cap w P^{\prime} w^{-1}}=\left(P \cap w P^{\prime} w^{-1}\right) /[P \cap$ $\left.w P^{\prime} w^{-1}, P \cap w P^{\prime} w^{-1}\right]$, and let

$$
j_{w}: T_{P \cap w P^{\prime} w^{-1}} \longrightarrow T_{P^{\prime}}
$$

to be the natural homomorphism induced by the homomorphism

$$
\bar{w}^{-1}(-) \bar{w}: P \cap w P^{\prime} w^{-1} \longrightarrow P^{\prime}
$$

for any choice of lift $\bar{w} \in N_{G}(T)$ of $w \in W=N_{G}(T) / T$.
Proposition 3.7.4. The Bruhat cell $C_{P, P^{\prime} / S}^{w, \lambda}(X)$ decomposes as a disjoint union

$$
C_{P, P^{\prime} / S}^{w, \lambda}(X)=\coprod_{\mu \in j_{w}^{-1}(\lambda)} \operatorname{Bun}_{P \cap w P^{\prime} w^{-1} / S}^{\mu}(X)
$$

Proof. There are identifications $P \backslash P w P^{\prime} / P^{\prime} \cong \mathbb{B}\left(P \cap w P^{\prime} w^{-1}\right.$ ) (resp., $P \backslash P w P^{\prime} / P^{\prime} \cong$ $\left.\mathbb{B}\left(w^{-1} P w \cap P^{\prime}\right)\right)$ coming from the fact that $P$ (resp., $P^{\prime}$ ) acts transitively on $P w P^{\prime} / P^{\prime}$ (resp., $P \backslash P w P^{\prime}$ ) so that the stabiliser of $w P^{\prime} / P^{\prime}$ (resp., $P \backslash P w$ ) is $P \cap w P^{\prime} w^{-1}$ (resp., $\left.w^{-1} P w \cap P^{\prime}\right)$. If we choose any lift $\bar{w} \in N_{G}(T)$ for $w$, then these both lift to a transitive action of $P \times P^{\prime}$ on $P w P^{\prime}$ such that the stabiliser of $\bar{w}$ is

$$
\left\{\left(g_{1}, g_{2}\right) \in P \times P^{\prime} \mid g_{1}^{-1} \bar{w} g_{2}=\bar{w}\right\}
$$

So we have an identification

$$
P \backslash P w P^{\prime} / P^{\prime} \cong \mathbb{B}\left\{\left(g_{1}, g_{2}\right) \in P \times P^{\prime} \mid g_{1}^{-1} \bar{w} g_{2}=\bar{w}\right\}
$$

lifting the two identifications above. It follows that the morphism

$$
\mathbb{B}\left(P \cap w P^{\prime} w^{-1}\right) \xrightarrow{\sim} \mathbb{B}\left(w^{-1} P w \cap P^{\prime}\right) \longrightarrow \mathbb{B} P^{\prime}
$$

is induced by the homomorphism $\bar{w}^{-1}(-) \bar{w}: P \cap w P^{\prime} w^{-1} \rightarrow P$, from which the result follows immediately.

The decomposition $P=L \ltimes R_{u}(P)$ gives a description of the Bruhat cell $C_{P, P^{\prime} / S}^{w, \lambda}(X)$ in terms of $L$ and $R_{u}(P)$. In the following proposition, if $\mu \in \mathbb{X}_{*}\left(T_{P \cap w P^{\prime} w^{-1}}\right)$, then we write $\xi_{L}$ and $\xi_{L \cap w P^{\prime} w^{-1}}$ respectively for the universal $L$-bundle and $L \cap w P^{\prime} w^{-1}$ bundle on

$$
\operatorname{Bun}_{L \cap w P^{\prime} w^{-1} / S}^{\mu}(X) \times_{\operatorname{Bun}_{L / S}(X)} \operatorname{Bun}_{P / S}(X) \times_{S} X
$$

$\mathcal{U}=\xi_{L} \times{ }^{L} R_{u}(P)$ for the associated unipotent group scheme, $\mathcal{U}_{w}=\xi_{L \cap w P^{\prime} w^{-1}} \times{ }^{L \cap w P^{\prime} w^{-1}}$ $\left(R_{u}(P) \cap w P^{\prime} w^{-1}\right) \subseteq \mathcal{U}$, and $\xi_{\mathcal{U}}=\xi_{P} / L$ for the associated $\mathcal{U}$-bundle.

Proposition 3.7.5. In the setup above, there is an isomorphism

$$
\operatorname{Bun}_{P \cap w P^{\prime} w^{-1} / S}^{\mu}(X) \cong \Gamma_{M}\left(M \times_{S} X, \xi_{\mathcal{U}} / \mathcal{U}_{w}\right)
$$

where

$$
M=\operatorname{Bun}_{L \cap w P^{\prime} w^{-1} / S}^{\mu}(X) \times_{\operatorname{Bun}_{L / S}(X)} \operatorname{Bun}_{P / S}(X)
$$

Proof. We can have a natural identification

$$
M=\operatorname{Bun}_{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P) / S}^{\mu}(X) .
$$

Since

$$
P \cap w P^{\prime} w^{-1}=L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P) \cap w P^{\prime} w^{-1} \subseteq L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)
$$

is a subgroup, by Proposition 2.3.6 we have

$$
\operatorname{Bun}_{P \cap w P^{\prime} w^{-1} / S}^{\mu}(X)=\Gamma_{M}\left(M \times_{S} X, \xi_{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)} \times^{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)} N\right),
$$

where

$$
N=\frac{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)}{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P) \cap w P^{\prime} w^{-1}} .
$$

But the isomorphism $R_{u}(P) /\left(R_{u}(P) \cap w P^{\prime} w^{-1}\right) \cong N$ induces an isomorphism

$$
\xi_{\mathcal{U}} / \mathcal{U}_{w} \xrightarrow{\sim} \xi_{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)} \times{ }^{L \cap w P^{\prime} w^{-1} \ltimes R_{u}(P)} N,
$$

so this proves the proposition.
Unlike the Bruhat cells for the flag variety, the cells $C_{P, P^{\prime} / S}^{w, \lambda}(X)$ do not cover the stack

$$
\operatorname{Bun}_{P / S}(X) \times \times_{\operatorname{Bun}_{G / S}(X)} \operatorname{Bun}_{P^{\prime} / S}^{\lambda}(X) .
$$

However, by giving bounds on the degrees of sections of flag variety bundles, the following proposition can often be used to show that they do cover the preimages of certain substacks of interest in $\operatorname{Bun}_{P / S}(X)$.

In the proposition below, we write

$$
C_{P, P^{\prime} / S}^{w, \lambda}(X)_{\xi_{P}}=\left\{\xi_{P}\right\} \times_{\operatorname{Bun}_{P / S}(X)} C_{P, P^{\prime} / S}^{w, \lambda}(X)
$$

for $\xi_{P} \in \operatorname{Bun}_{P / S}(X)$.
Proposition 3.7.6. Let $\xi_{P} \rightarrow X_{s}$ be a P-bundle on a geometric fibre of $X \rightarrow S$, and suppose there exists a point in $\operatorname{Bun}_{P / S}(X) \times_{\operatorname{Bun}_{G / S}(X)} \operatorname{Bun}_{P^{\prime} / S}^{\lambda}(X)$ over $\xi_{P}$ that does not lie in any Bruhat cell. Then there exists $w \in W_{P, P^{\prime}}^{0} \backslash\{1\}$ and $\lambda^{\prime}<\lambda$ such that $C_{P, P^{\prime} / S}^{w, \lambda^{\prime}}(X)_{\xi_{L} \times{ }^{L} P} \neq \emptyset$, where $\xi_{L}=\xi_{P} \times{ }^{P} L$ is the associated $L$-bundle.

Proof. We can assume without loss of generality that $S=$ Spec $k$ for some algebraically closed field $k$.

The preimage of $\xi_{P}$ in $\operatorname{Bun}_{P}(X) \times$ Bun $_{G}(X) \operatorname{Bun}_{P^{\prime}}(X)$ is the space of sections of the partial flag variety bundle $\xi_{P} \times^{P} G / P^{\prime} \rightarrow X$, and for all $w \in W_{P, P^{\prime}}^{0}$, the preimage $C_{P, P^{\prime}}^{w, \lambda}(X)_{\xi_{P}}$ of $\xi_{P}$ in the Bruhat cell $C_{P, P^{\prime}}^{w, \lambda}(X)$ is the space of sections of $\xi_{P} \times^{P} P w P^{\prime} / P^{\prime} \rightarrow X$ of degree $\lambda$. So the assumption of the proposition is equivalent to the assumption that we have a section $\sigma: X \rightarrow \xi_{P} \times{ }^{P} G / P^{\prime}$ of degree $\lambda$ that does not factor through any Bruhat cell $\xi_{P} \times{ }^{P} P w P^{\prime} / P^{\prime}$.

The strategy of the proof is to construct a degeneration of $\xi_{P}$ to the bundle $\xi_{L} \times{ }^{L} P$, together with a degeneration of $\sigma$ to a stable map $\sigma^{\prime}: C \rightarrow \xi_{L} \times^{L} G / P^{\prime}$ such that the restriction of $\sigma^{\prime}$ to the irreducible component of $C$ mapping isomorphically onto $X$ factors through some Bruhat cell. We then deduce the degree bounds by decomposing the degree $\lambda$ of $\sigma^{\prime}$ into contributions from each irreducible component of $C$.

To construct the degeneration, first choose a cocharacter $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right) \subseteq \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ of the centre of $L$ such that $\mu$ is a Harder-Narasimhan vector for $P$, and consider the induced action

$$
\begin{align*}
\mathbb{G}_{m} \times P & \longrightarrow P  \tag{3.7.3}\\
(t, p) & \longmapsto \mu(t) p \mu(t)^{-1} .
\end{align*}
$$

Since multiplication defines a $\mathbb{G}_{m}$-equivariant isomorphism of schemes $L \times R_{u}(P) \cong P$, and since $\mathbb{G}_{m}$ acts trivially on $L$ and with strictly positive weights on the affine space $R_{u}(P)$, the morphism (3.7.3) extends uniquely to a morphism of schemes

$$
\begin{equation*}
\mathbb{A}^{1} \times P \longrightarrow P, \tag{3.7.4}
\end{equation*}
$$

which restricts to the morphism $P \rightarrow L \rightarrow P$ over $0 \in \mathbb{A}^{1}$. Since (3.7.3) defines a homomorphism $\mathbb{G}_{m} \times P \rightarrow \mathbb{G}_{m} \times P$ of group schemes over $\mathbb{G}_{m}$, by continuity (3.7.4) defines a homomorphism $\mathbb{A}^{1} \times P \rightarrow \mathbb{A}^{1} \times P$ of group schemes over $\mathbb{A}^{1}$. So we get a morphism $\mathbb{A}^{1} \times \operatorname{Bun}_{P}(X) \rightarrow \operatorname{Bun}_{P}(X)$ extending the action of $\mathbb{G}_{m}$ on $\operatorname{Bun}_{P}(X)$, and hence a $\mathbb{G}_{m^{-}}$ equivariant morphism

$$
\mathbb{A}_{k}^{1} \longrightarrow \operatorname{Bun}_{P}(X)
$$

by restricting to $\mathbb{A}^{1} \times\left\{\xi_{P}\right\}$, which sends $1 \in \mathbb{A}_{k}^{1}$ to $\xi_{P}$ and the origin $0 \in \mathbb{A}_{k}^{1}$ to the $P$-bundle $\xi_{L} \times{ }^{L} P$. We write $\bar{\xi}_{P}$ for the corresponding $P$-bundle on $\mathbb{A}_{k}^{1} \times{ }_{k} X$.

The action of $\mathbb{G}_{m}$ on $\left(\xi_{P}, \sigma\right)$, viewed as a point in the stack $\operatorname{Bun}_{P}(X) \times \operatorname{Bun}_{G}(X) \operatorname{Bun}_{P^{\prime}}^{\lambda}(X)$, defines a morphism

$$
\mathbb{G}_{m, k} \longrightarrow \mathbb{A}_{k}^{1} \times \times_{\operatorname{Bun}_{G}(X)} \operatorname{Bun}_{P^{\prime}}^{\lambda}(X) \subseteq \mathbb{A}_{k}^{1} \times \times_{\operatorname{Bun}_{G}(X)} \operatorname{KM}_{P^{\prime}, G}^{\lambda}(X)
$$

Since $\mathrm{KM}_{P^{\prime}, G}^{\lambda}(X) \rightarrow \operatorname{Bun}_{G}(X)$ is proper, by the valuative criterion for properness, there exists morphism $\operatorname{Spec} R \rightarrow \mathbb{A}_{k}^{1}$, with $R$ a complete discrete valuation ring finite over $\widehat{\mathcal{O}}_{\mathbb{A}_{k}^{1}, 0}$, and a commutative diagram

where $K$ is the field of fractions of $R$. Write $x \in \operatorname{Spec} R$ for the closed point; since $k$ is algebraically closed by assumption, the point $x$ is defined over $k$. We also write

$$
\bar{\sigma}: C \longrightarrow \operatorname{Spec} R \times_{\mathbb{A}_{k}^{1}}\left(\bar{\xi}_{P} \times{ }^{P} G / P^{\prime}\right)
$$

for the family of stable maps classified by the morphism $\operatorname{Spec} R \rightarrow \mathbb{A}_{k}^{1} \times{ }_{\operatorname{Bun}_{G}(X)} \mathrm{KM}_{P^{\prime}, G}^{\lambda}(X)$.
We claim that the restriction $\left.\bar{\sigma}_{x}\right|_{X}$ of $\bar{\sigma}_{x}$ to the unique component of the fibre $C_{x}$ of $C \rightarrow$ Spec $R$ over $x$ mapping isomorphically to $X$ factors through some Bruhat cell $\xi_{L} \times{ }^{L} P w P^{\prime} / P^{\prime}$ with $w \in W_{P, P^{\prime}}^{0} \backslash\{1\}$. To see this, observe that since each Bruhat cell is open in its closure, we must have that $U=\sigma^{-1}\left(\xi_{P} \times{ }^{P} P w P^{\prime} / P^{\prime}\right) \subseteq X$ is open and dense for some $w \in W_{P, P^{\prime}}^{0}$. Since $\sigma$ does not factor through $\xi_{P} \times{ }^{P} P w P^{\prime} / P^{\prime}$, we must also have $\sigma^{-1}\left(\xi_{P} \times{ }^{P} D_{w}\right) \neq \emptyset$, where $D_{w}$ is the complement of $P w P^{\prime} / P^{\prime}$ in its closure. Since $D_{1}=\emptyset$, this in particular implies that $w \neq 1$.

Let $p: C \rightarrow X$ be the natural morphism. Since $\mathbb{G}_{m}$ acts on the fibres of $P w P^{\prime} / P^{\prime} \rightarrow$ $L /\left(L \cap w P^{\prime} w^{-1}\right)$ with strictly positive weights, it is clear that $\left.\bar{\sigma}\right|_{p^{-1}(U)}$ is a section of $\xi_{L} \times^{L}$ $L /\left(L \cap w P^{\prime} w^{-1}\right)$ over $U$ and that $p^{-1}(U)_{x}$ is a dense open subset of $X \subseteq C_{x}$. Since $L \cap w P^{\prime} w^{-1} \subseteq L$ is a parabolic subgroup by Proposition 3.7.2, $L w P^{\prime} / P^{\prime} \cong L /\left(L \cap w P^{\prime} w^{-1}\right)$ is proper over $\operatorname{Spec} \mathbb{Z}$, hence closed in $G / P^{\prime}$. So $\left.\bar{\sigma}_{x}\right|_{X}$ factors through $\xi_{L} \times{ }^{L} L w P^{\prime} / P^{\prime} \subseteq$ $\xi_{L} \times{ }^{L} P w P^{\prime} / P^{\prime}$ as claimed.

It remains to prove that $\left.\bar{\sigma}_{x}\right|_{X}$ has degree $\lambda^{\prime}<\lambda$. To see this, note that since $D_{w}$ is proper over $\operatorname{Spec} \mathbb{Z}$ and $P$-invariant, $C \rightarrow \operatorname{Spec} R$ is proper, and $\sigma^{-1}\left(\xi_{P} \times{ }^{P} D_{w}\right) \neq \emptyset$, it follows that $\bar{\sigma}_{x}^{-1}\left(\xi_{L} \times{ }^{L} D_{w}\right) \neq \emptyset$ also. So $C_{x}$ must have an irreducible component different from $X$. So Proposition 3.4.13 now implies that the degree $\lambda^{\prime}$ of $\left.\bar{\sigma}_{x}\right|_{X}$ is less than the degree $\lambda$ of $\bar{\sigma}_{x}$, so we are done.

## Chapter 4

## The elliptic Grothendieck-Springer resolution

We are now ready to begin our study of elliptic Springer theory in earnest. From now on, we fix a split simply connected simple group $G$ over $\mathbb{Z}$ of rank $l$ with maximal torus and Borel subgroup $T \subseteq B \subseteq G$ and a family $E \rightarrow S$ of smooth curves of genus 1 over a connected regular stack $S$. For the sake of brevity, for any group scheme $H$, we will write $\operatorname{Bun}_{H}=$ $\operatorname{Bun}_{H / S}(E)$, and for $P \subseteq G$ a parabolic subgroups, we will write $\mathrm{KM}_{P, G}=\mathrm{KM}_{P, G / S}(E)$.

The main aim of this chapter is to construct the elliptic Grothendieck-Springer resolution

advertised in the introduction. In §4.1, we define some of the basic objects and morphisms appearing in (4.0.1). In $\S 4.2$ we write down some technical results on extending and descending line bundles and their sections in families, which we apply in $\S 4.3$ to prove the elliptic Chevalley isomorphism relating line bundles on $\operatorname{Bun}_{G}$ to certain line bundles on an abelian variety $Y$. In $\S 4.4$, we classify the relevant line bundles on $Y$ and thus obtain a useful description of the generator of $\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)$ via the elliptic Chevalley isomorphism. We then pull everything together in $\S 4.5$ to construct the diagram (4.0.1). Finally, in $\S 4.6$, we show how the methods of this chapter can be used to give explicit descriptions of the canonical bundles of the stacks $\operatorname{Bun}_{G}$ and $\widetilde{\operatorname{Bun}}_{G}$.

### 4.1 The basic objects

In this section, we define some of the objects appearing in (4.0.1) and introduce some other useful bits and pieces of notation.

The most important definition is the following: we let $\widetilde{\operatorname{Bun}}_{G}$ be the Kontsevich-Mori compactification $\widetilde{\operatorname{Bun}}_{G}=\mathrm{KM}_{B, G}^{0}$ of the stack of degree $0 B$-bundles, and we let $\psi: \widetilde{\operatorname{Bun}}_{G} \rightarrow$ $\operatorname{Bun}_{G}$ be the canonical morphism. By Propositions 3.4.5 and 3.6.4, we have the following.

Proposition 4.1.1. The morphism $\psi: \widetilde{\operatorname{Bun}}_{G} \rightarrow \operatorname{Bun}_{G}$ is proper and surjective, with finite relative stabilisers.

Set $\widetilde{\operatorname{Bun}_{G}^{s s}}=\psi^{-1}\left(\operatorname{Bun}_{G}^{s s}\right)$, where $\operatorname{Bun}_{G}^{s s} \subseteq \operatorname{Bun}_{G}$ is the open substack of semistable bundles. One pleasing property of $\widetilde{\operatorname{Bun}}_{G}$ is that this coincides with the locus of stable maps with smooth domain curve.

Proposition 4.1.2. We have $\widetilde{\operatorname{Bun}}_{G}^{s s}=\operatorname{Bun}_{B}^{0}$ as open substacks of $\widetilde{\operatorname{Bun}}{ }_{G}=\mathrm{KM}_{B, G}^{0}$.
Proof. The claim reduces easily to the following: given a geometric point $s$ : Spec $k \rightarrow S$, a $G$-bundle $\xi_{G} \rightarrow E_{s}$ and a stable map $\sigma: C \rightarrow \xi_{G} / B$ classified by a point in $\mathrm{KM}_{B, G}^{0}$, the $G$-bundle $\xi_{G}$ is semistable if and only if $C \rightarrow E_{s}$ is an isomorphism.

Assume first that $C \rightarrow E_{s}$ is an isomorphism. Then $\sigma$ is a section $\sigma: E_{s} \rightarrow \xi_{G} / B$ of degree 0 , so we can write $\xi_{G}=\xi_{B} \times{ }^{B} G$ for some degree $0 B$-bundle $\xi_{B} \rightarrow E_{s}$ with associated $T$-bundle $\xi_{T}=\xi_{B} \times{ }^{B} T$. Proposition 3.7.4 implies that for any standard parabolic $P \subseteq G, C_{B, P / S}^{w, \lambda}\left(E_{s}\right)_{\xi_{T} \times^{T} B}=\emptyset$ unless $\lambda=0$, so Proposition 3.7.6 shows that for any section $\sigma: E_{s} \rightarrow \xi_{G} / P$, the degree of the corresponding $P$-bundle is $\geq 0$. So $\xi_{G}$ is semistable.

Conversely, assume that $C \rightarrow E_{s}$ is not an isomorphism. Then there exists a unique irreducible component of $C$ mapping isomorphically to $E_{s}$, and the restriction of $\sigma$ to this irreducible component defines a section of degree

$$
\left[\left.\sigma\right|_{E_{s}}\right]<[\sigma]=0 .
$$

If $\xi_{B}$ denotes the $B$-bundle corresponding to $\left.\sigma\right|_{E_{s}}: E_{s} \rightarrow \xi_{G} / B$, then it follows that there exists a dominant character $\lambda$ of $B$ such that

$$
\operatorname{deg} \xi_{B} \times^{B} \mathbb{Z}_{\lambda}=\operatorname{deg}\left(\left.\sigma\right|_{E_{s}}\right)^{*} \mathcal{L}_{\lambda}^{\xi_{G}}=\left\langle\lambda,\left[\left.\sigma\right|_{E_{s}}\right]\right\rangle<0
$$

so $\xi_{G}$ is unstable.
Writing $D_{\lambda}=D_{B, \lambda}^{0} \subseteq \mathrm{KM}_{B, G}^{0}=\widetilde{\operatorname{Bun}}_{G}$ for $\lambda \in \mathbb{X}_{*}(T)_{+}$, we have the following corollary of Propositions 3.4.16 and 4.1.2.

Corollary 4.1.3. The closed substack $\bigcup_{\lambda \in \mathbb{X}_{*}(T)_{+}} D_{\lambda} \subseteq \widetilde{\operatorname{Bun}}_{G}$ is a divisor with normal crossings, equal to the complement of $\widetilde{\operatorname{Bun}_{G}}$.

At various points, we will need to work with rigidified stacks of principal bundles. (See Definition 2.2.6.) Note that since $Z(G)$ acts trivially on any partial flag variety $G / P$, it follows that we have an action $\mathrm{KM}_{P, G} \times \mathbb{B} Z(G) \rightarrow \mathrm{KM}_{P, G}$ satisfying the conditions of Proposition 2.2.5, and hence a rigidification with respect to $Z(G)$. For this and subsequent chapters, we fix the following convention.

Convention 4.1.4. If $H \subseteq G$ is any subgroup containing the centre $Z(G)$, then by $\operatorname{Bun}_{H, r i g}$ we will always mean the rigidified stack underlying $\operatorname{Bun}_{H}$ with respect to $Z(G) \subseteq Z(H)$. Similarly, if $P \subseteq G$ is a parabolic subgroup, then $\mathrm{KM}_{P, G, \text { rig }}^{\lambda}$ will denote the rigidification of $\mathrm{KM}_{P, G}^{\lambda}$ with respect to $Z(G)$. We also write $\widetilde{\operatorname{Bun}}_{G, r i g}=\mathrm{KM}_{B, G, r i g}^{0}$.

If $T^{\prime}$ is any torus, we will also need notation for the rigidified stack of $T^{\prime}$-bundles on $E$ with respect to the whole group $T^{\prime}$.

Definition 4.1.5. Let $T^{\prime}$ be a split torus over $\operatorname{Spec} \mathbb{Z}$. If $\lambda \in \mathbb{X}^{*}\left(T^{\prime}\right)$, we write $Y_{T^{\prime}}^{\lambda}$ for the rigidification of $\operatorname{Bun}_{T^{\prime} / S}^{\lambda}$ with respect to $T^{\prime}$. If $P \subseteq H$ is a parabolic subgroup of a reductive group $H$ with $T^{\prime}=P /[P, P]$, we will also write $Y_{P}^{\lambda}=Y_{T^{\prime}}^{\lambda}$ and $Y_{P}=Y_{P}^{0}$. Finally, we will write $Y^{\lambda}=Y_{T}^{\lambda}=Y_{B}^{\lambda}$ for the rigidification of $\operatorname{Bun}_{T}^{\lambda}$ with respect to $T$, and $Y=Y^{0}$.

Note that the blow down morphism $\mathrm{Bl}_{B}: \mathrm{KM}_{B, G / S}^{0}(E) \rightarrow \mathrm{Bun}_{T}^{0}$ gives a morphism

$$
\bar{\chi}: \widetilde{\operatorname{Bun}}_{G} \longrightarrow Y,
$$

which is smooth by Corollary 3.5.4.
Remark 4.1.6. Proposition 2.5 . 1 shows that $Y \rightarrow S$ is the family of $l$-dimensional abelian varieties

$$
Y=\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{X}^{*}(T), \operatorname{Pic}_{S}^{0}(E)\right) \cong \operatorname{Pic}_{S}^{0}(E)^{l}
$$

### 4.2 Some results on extension and descent

In this section, we collect a few general results on extending line bundles and their sections from big open substacks, and on descending line bundles along ramified Galois coverings, which we will need in the proof of the elliptic Chevalley isomorphism. All the results here are surely well known, but we include them in full here for completeness.

It will be convenient to package the data of line bundles and their sections as follows. For any stack $X$ over $S$, we will write $\mathcal{P i c}(X)$ for the category of line bundles on $X$. This is a symmetric monoidal category under tensor product, which has an enrichment over the category $\mathcal{O}_{\mathrm{Sch}_{/ S}}-\bmod$ of sheaves of $\mathcal{O}$-modules on the category of (locally Noetherian) schemes over $S$ defined by the formula

$$
\underline{\operatorname{Hom}}\left(L, L^{\prime}\right)=\pi_{X *}\left(L^{\vee} \otimes L^{\prime}\right)
$$

for $L, L^{\prime} \in \mathcal{P i c}(X)$, where $\pi_{X}: X \rightarrow S$ is the structure morphism. If $\Gamma$ is a finite group acting on $X$ over $S$, then we write $\mathcal{P} i c^{\Gamma}(X)$ for the category of $\Gamma$-linearised line bundles on $X$. This is again a symmetric monoidal category with enrichment over $\mathcal{O}_{\text {Sch } / S}-\bmod$ defined by the formula

$$
\underline{\operatorname{Hom}}\left(L, L^{\prime}\right)=\pi_{X *}\left(L^{\vee} \otimes L^{\prime}\right)^{\Gamma} .
$$

If $X$ is proper and representable over $S$, then $\mathcal{P i c}(X)$ and $\mathcal{P i c} c^{\Gamma}(X)$ are actually enriched over the full subcategory $\operatorname{Coh}(S) \subseteq \mathcal{O}_{\operatorname{Sch}_{/ S}}-\bmod$ of coherent sheaves on $S$.

Definition 4.2.1. Let $\pi_{X}: X \rightarrow S$ be a smooth morphism of stacks, and let $U \subseteq X$ be an open substack. We say that $U$ is big relative to $S$ if for every geometric point $s$ : Spec $k \rightarrow S$, the complement of the open substack $\pi_{X}^{-1}(s) \cap U \subseteq \pi_{X}^{-1}(s)$ has codimension at least 2 in the fibre $\pi_{X}^{-1}(s)$.

Lemma 4.2.2. Let $\pi_{X}: X \rightarrow S$ be a smooth morphism of stacks, and let $U \subseteq X$ be a big open substack relative to $S$. Then the restriction functor

$$
\begin{equation*}
\mathcal{P i c}(X) \longrightarrow \mathcal{P} i c(U) \tag{4.2.1}
\end{equation*}
$$

is an equivalence of symmetric monoidal categories enriched over $\mathcal{O}_{\mathrm{Sch}_{/ S}}-\bmod$.
Remark 4.2.3. If we restrict to the lisse-étale site of $S$ instead of the big site Sch $/ S$, then the corresponding statement is easy since every smooth chart of $X$ is regular.

Proof of Lemma 4.2.2. We first note that since we have assumed that $S$ is regular and $\pi_{X}$ is smooth, the stack $X$ is also regular, so (4.2.1) is essentially surjective since the codimension of $U$ in $X$ is at least 2. It therefore remains to prove that (4.2.1) is fully faithful as an enriched functor, i.e., that for all line bundles $L, L^{\prime} \in \mathcal{P} i c(X)$, the morphism

$$
\underline{\operatorname{Hom}}_{\mathcal{P} i c(X)}\left(L, L^{\prime}\right)=\pi_{X *}\left(L^{\vee} \otimes L^{\prime}\right) \longrightarrow \pi_{U *}\left(\left.\left(L^{\vee} \otimes L^{\prime}\right)\right|_{U}\right)=\underline{\operatorname{Hom}}_{\mathcal{P i c}(U)}\left(\left.L\right|_{U},\left.L^{\prime}\right|_{U}\right)
$$

is an isomorphism of sheaves of $\mathcal{O}$-modules on $\mathrm{Sch}_{/ S}$. This reduces immediately to the claim that for every line bundle $L$ on $X$ and every morphism $S^{\prime} \rightarrow S$ with $S^{\prime}$ a locally Noetherian scheme, the morphism

$$
p_{*} f^{*} L \longrightarrow p_{U_{*}}\left(\left.f_{U}^{*} L\right|_{U}\right)
$$

is an isomorphism of sheaves on $S^{\prime}$, where $p, f, p_{U}$ and $f_{U}$ are as in the Cartesian diagrams


By smooth descent, we reduce to the case where $S^{\prime}$ is Noetherian, $S^{\prime} \times_{S} X$ is a scheme, and $f^{*} L$ is the trivial line bundle. The result in this case now follows from Lemma 4.2.4.

Lemma 4.2.4. Let $S^{\prime}$ be a Noetherian scheme, let $p: X \rightarrow S^{\prime}$ be a smooth morphism of schemes, and let $U \subseteq X$ be a big open subset over $S^{\prime}$. Then the restriction map

$$
\begin{equation*}
H^{0}\left(X, \mathcal{O}_{X}\right) \longrightarrow H^{0}\left(U, \mathcal{O}_{U}\right) \tag{4.2.2}
\end{equation*}
$$

is an isomorphism.
Proof. Given a quasi-coherent sheaf $F$ on $S^{\prime}$, say that $F$ has the unique extension property if the morphism

$$
H^{0}\left(X, p^{*} F\right) \longrightarrow H^{0}\left(U,\left.p^{*} F\right|_{U}\right)
$$

is an isomorphism. If $s \in S^{\prime}$ is any scheme-theoretic point with residue field $\kappa(s)$, then, since the fibre $p^{-1}(s)$ is smooth over $\kappa(s)$, hence normal, and the complement of $p^{-1}(s) \cap U$ in $p^{-1}(s)$ has codimension at least 2 , any vector space over $\kappa(s)=\mathcal{O}_{S^{\prime}, s} / \mathfrak{m}_{s}$ has the unique extension property. It is also easy to show that, given a short exact sequence

$$
0 \longrightarrow F \longrightarrow F^{\prime} \longrightarrow F^{\prime \prime} \longrightarrow 0
$$

of quasi-coherent sheaves on $S^{\prime}$, if $F$ and $F^{\prime \prime}$ have the unique extension property, then so does $F^{\prime}$. So inductively, we deduce that $\mathcal{O}_{S^{\prime}, s} / \mathfrak{m}_{s}^{n}$ has the unique extension property for all $n \in \mathbb{Z}_{>0}$. It follows that the morphism

$$
H^{0}\left(\widehat{X}_{s}, \mathcal{O}_{\widehat{X}_{s}}\right) \longrightarrow H^{0}\left(\widehat{U}_{s}, \mathcal{O}_{\widehat{U}_{s}}\right)
$$

is an isomorphism, where $\widehat{X}_{s}$ and $\widehat{U}_{s}$ are the formal completions of $X$ and $U$ along $p^{-1}(s)$.
To prove injectivity of (4.2.2), simply observe that there is a commutative diagram

such that vertical arrows are injective and the bottom arrow is an isomorphism as argued above.

To prove surjectivity of (4.2.2), by Noetherian induction, it suffices to show that given $f \in H^{0}\left(U, \mathcal{O}_{U}\right)$, if $U \neq X$ then there exists an open set $U^{\prime} \subseteq X$ properly containing $U$ and an extension $f^{\prime} \in H^{0}\left(U^{\prime}, \mathcal{O}_{U^{\prime}}\right)$ of $f$ to $U^{\prime}$.

Assume we are in the situation above, and let $x \in X \backslash U$ be the generic point of any irreducible component of $X \backslash U$. We show below that there exists $\widehat{f} \in \mathcal{O}_{X, x}$ such that the restriction to $U \cap \operatorname{Spec} \mathcal{O}_{X, x}=\operatorname{Spec} \mathcal{O}_{X, x} \backslash\{x\}$ agrees with the restriction of $f$. It then follows that $\widehat{f}$ is the germ at $x$ of some $f^{\prime} \in H^{0}\left(U^{\prime}, \mathcal{O}_{U^{\prime}}\right)$ extending $f$ to some open set $U^{\prime} \subseteq X$ containing $x$ and $U$.

We first remark that that $\operatorname{Spec} \mathcal{O}_{X, x} \backslash\{x\}$ is covered by affine open sets $D(g)=\operatorname{Spec} \mathcal{O}_{X, x}\left[g^{-1}\right]$ for $g \in \mathfrak{m}_{x}$ but $g \notin \mathfrak{m}_{s} \mathcal{O}_{X, x}$, where $s=p(x) \in S^{\prime}$. To see this, let $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{X, x}$ be a prime ideal different from $\mathfrak{m}_{x}$. If $\mathfrak{m}_{s} \mathcal{O}_{X, x} \subseteq \mathfrak{p}$, then taking any $g \in \mathfrak{m}_{x} \backslash \mathfrak{p}$, we have $g \notin \mathfrak{m}_{s} \mathcal{O}_{X, x}$ and $\mathfrak{p} \in D(g)$. If $\mathfrak{m}_{s} \mathcal{O}_{X, x} \nsubseteq \mathfrak{p}$, choose $h \in \mathfrak{m}_{s} \mathcal{O}_{X, x}$ with $h \notin \mathfrak{p}$ and any $g \in \mathfrak{m}_{x} \backslash \mathfrak{m}_{s} \mathcal{O}_{X, x}$. (Note that $\mathfrak{m}_{x} \neq \mathfrak{m}_{s} \mathcal{O}_{X, x}$ since $U$ is dense in every fibre.) If $g \notin \mathfrak{p}$, then $\mathfrak{p} \in D(g)$, and if $g \in \mathfrak{p}$, then $g+h \notin \mathfrak{m}_{s} \mathcal{O}_{X, x}$ and $g+h \notin \mathfrak{p}$, so $\mathfrak{p} \in D(g+h)$.

Now let $\widehat{f} \in\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}$ be the germ at $x$ of the unique lift in $H^{0}\left(\widehat{X}_{s}, \mathcal{O}_{\widehat{X}_{s}}\right)$ of the image of $f$ in $H^{0}\left(\widehat{U}_{s}, \mathcal{O}_{\widehat{U}_{s}}\right)$. Here $\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}^{\wedge}$ denotes the completion of $\mathcal{O}_{X, x}$ with respect to the $\mathfrak{m}_{s}$-adic topology. We show that in fact $\widehat{f} \in \mathcal{O}_{X, x} \subseteq\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}^{\wedge}$, and that the restriction to any $D(g)$ with $g \in \mathfrak{m}_{x} \backslash \mathfrak{m}_{s} \mathcal{O}_{X, x}$ agrees with $f$. Since $U \cap \operatorname{Spec} \mathcal{O}_{X, x}$ is covered by such $D(g)$, this will complete the proof.

Choose any $g \in \mathfrak{m}_{x} \backslash \mathfrak{m}_{s} \mathcal{O}_{X, x}$. From Lemma 4.2.5, we have a commutative diagram of topological rings

in which every morphism is the inclusion of a subring with the subspace topology. It follows from [GD, Chapitre 0, Corollaire 7.3.5] that $\mathcal{O}_{X, x}$ is closed in $g^{-n} \mathcal{O}_{X, x}$ for all $n \geq 0$, and hence in $\mathcal{O}_{X, x}\left[g^{-1}\right]$, so we have

$$
\widehat{f} \in\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}^{\wedge} \cap \mathcal{O}_{X, x}\left[g^{-1}\right]=\mathcal{O}_{X, x}
$$

as claimed. Since $\widehat{f}$ agrees with $f$ on $D(g)$ by construction, this completes the proof.
Lemma 4.2.5. In the setup of Lemma 4.2.4, fix a point $x \in X$, set $s=p(x) \in S^{\prime}$, and assume that $M \subseteq \mathcal{O}_{X, x}$ is a multiplicative set with $M \cap \mathfrak{m}_{s} \mathcal{O}_{X, x}=\emptyset$. Then the localisation morphism

$$
i_{M}: M^{-1} \mathcal{O}_{X, x} \longrightarrow\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}
$$

is injective and satisfies $i_{M}^{-1}\left(\mathfrak{m}_{s}^{n}\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}\right)=\mathfrak{m}_{s}^{n} M^{-1} \mathcal{O}_{X, x}$ for all $n \geq 0$. Moreover, $M^{-1} \mathcal{O}_{X, x}$ is separated for the $\mathfrak{m}_{s}$-adic topology.

Proof. By induction on $n$, we can assume that

$$
i_{M}^{-1}\left(\mathfrak{m}_{s}^{n}\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}\right) \subseteq \mathfrak{m}_{s}^{n-1} M^{-1} \mathcal{O}_{X, x}
$$

So by flatness of $X \rightarrow S^{\prime}$, the claim reduces to the assertion that

$$
\begin{aligned}
\frac{\mathfrak{m}_{s}^{n-1} M^{-1} \mathcal{O}_{X, x}}{\mathfrak{m}_{s}^{n} M^{-1} \mathcal{O}_{X, x}}=\frac{\mathfrak{m}_{s}^{n-1}}{\mathfrak{m}_{s}^{n}} \otimes_{\kappa(s)} M^{-1} & \left(\frac{\mathcal{O}_{X, x}}{\mathfrak{m}_{s} \mathcal{O}_{X, x}}\right) \\
& \longrightarrow \frac{\mathfrak{m}_{s}^{n-1}}{\mathfrak{m}_{s}^{n}} \otimes_{\kappa(s)} \operatorname{Frac}\left(\frac{\mathcal{O}_{X, x}}{\mathfrak{m}_{s} \mathcal{O}_{X, x}}\right)=\frac{\mathfrak{m}_{s}^{n-1}\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}}{\mathfrak{m}_{s}^{n}\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}}
\end{aligned}
$$

is injective, which is clear from the fact that $\mathfrak{m}_{s} \mathcal{O}_{X, x}$ is prime (as $X \rightarrow S^{\prime}$ is smooth).
For the remaining statements, note that the above with $M=\{1\}$ shows that $\mathcal{O}_{X, x} \rightarrow$ $\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}$ is injective, since the local ring $\mathcal{O}_{X, x}$ is separated for the $\mathfrak{m}_{s}$-adic topology. So $i_{M}$ is also injective for any $M$. Since $\left(\mathcal{O}_{X, x}\right)_{\mathfrak{m}_{s}}$ is separated, this implies that $M^{-1} \mathcal{O}_{X, x}$ is also separated for the $\mathfrak{m}_{s}$-adic topology.

Remark 4.2.6. The same argument as the proof of Lemma 4.2 .2 shows that $\mathcal{P i c}(X) \rightarrow$ $\operatorname{Pic}(U)$ is faithful whenever $U \subseteq X$ is open and dense in every fibre of $X \rightarrow S$.

Definition 4.2.7. Let $f: X \rightarrow Z$ be a morphism of smooth stacks over $S$, with $Z$ connected. We say that $f$ is a ramified Galois covering relative to $S$ with Galois group $\Gamma$ if
(1) the morphism $f$ is representable and finite, and
(2) there exists an open substack $U \subseteq Z$ such that $f^{-1}(U) \rightarrow U$ is an étale Galois covering with Galois group $\Gamma$, and $U$ is dense in every fibre of $Z \rightarrow S$.

Remark 4.2.8. Note that if $f: X \rightarrow Z$ is a ramified Galois covering relative to $S$, then $f$ is automatically flat, since it is a finite morphism between regular stacks of the same dimension.

Lemma 4.2.9. Let $f: X \rightarrow Z$ be a ramified Galois covering relative to $S$ with Galois group $\Gamma$, and let $U \subseteq Z$ be as in Definition 4.2.7. Then the action of $\Gamma$ on $f^{-1}(U)$ extends uniquely to an action on $X$ over $Z$.

Proof. Since for any smooth (connected) chart $V \rightarrow Z$, the pullback $V \times_{Z} X \rightarrow V$ is a ramified Galois covering relative to $S$ with Galois group $\Gamma$, by descent for morphisms of stacks, it suffices to prove the claim in the case where $Z$ (and hence $X$ ) is a regular affine scheme. So we can assume $Z=\operatorname{Spec} A$ and $X=\operatorname{Spec} B$, with $A \rightarrow B$ a finite flat extension of regular rings, with Spec $A$ connected. By assumption, we have $\operatorname{Spec} K \otimes_{A} B \rightarrow \operatorname{Spec} K$ a Galois covering with Galois group $\Gamma$, where $K=\operatorname{Frac}(A)$ is the fraction field of $A$. Since $B \subseteq K \otimes_{A} B$ is the subring of elements integral over $A$, it follows that $B$ is preserved by the action of $\Gamma$, which completes the proof.

Definition 4.2.10. Let $f: X \rightarrow Z$ be a ramified Galois covering with Galois group $\Gamma$, and let $L$ be a $\Gamma$-linearised line bundle on $X$. We say that $L$ is good if for every $\gamma \in \Gamma$, the morphism

$$
\gamma:\left.\left.L\right|_{X_{(1)}^{\gamma}} ^{\gamma} \longrightarrow L\right|_{X_{(1)}^{\gamma}} ^{\gamma}
$$

is the identity, where $X_{(1)}^{\gamma} \subseteq X^{\gamma}$ denotes the open substack of points in the fixed locus $X^{\gamma}$ (relative to $Z$ ) at which $X^{\gamma} \subseteq X$ has codimension $\leq 1$. We write $\operatorname{Pic}^{\Gamma}(X)_{\text {good }} \subseteq \operatorname{Pic}^{\Gamma}(X)$ for the subgroup of good $\Gamma$-linearised line bundles, and $\mathcal{P} i c^{\Gamma}(X)_{\text {good }} \subseteq \mathcal{P} i c^{\Gamma}(X)$ for the corresponding full subcategory.

Remark 4.2.11. It is important in Definition 4.2 .10 that we take fixed loci relative to $Z$ and not $S$. The fixed locus $X^{\gamma}$ relative to $Z$ is by definition the fibre product

which is a closed substack of $X$ since $X$ is representable and separated over $Z$. Taking fixed loci relative to $S$ would amount to replacing $\Delta_{X / Z}$ with $\Delta_{X / S}$, which will not be a closed immersion if $X \rightarrow S$ is not representable.

We now state our main descent result for ramified Galois coverings. For simplicity, we have restricted to the case of line bundles, and to ramified Galois coverings in which fixed loci intersect in high codimension.

Proposition 4.2.12. Let $f: X \rightarrow Z$ be a ramified Galois covering of smooth stacks over $S$, with Galois group $\Gamma$. Assume that for any $\gamma, \gamma^{\prime} \in \Gamma \backslash\{1\}$ with $\gamma \neq \gamma^{\prime}$, the intersection of the fixed loci (relative to $Z$ ) $X^{\gamma} \cap X^{\gamma^{\prime}} \subseteq X$ has codimension at least 2. Then the pullback functor $\mathcal{P i c}(Z) \rightarrow \mathcal{P i c}{ }^{\Gamma}(X)$ factors through an equivalence

$$
\mathcal{P} i c(Z) \xrightarrow{\sim} \mathcal{P i c}^{\Gamma}(X)_{\mathrm{good}}
$$

of categories enriched over $\mathcal{O}_{\text {Sch }_{/ S}}-\bmod$.

Proof. We first prove that $\mathcal{P i c}(Z) \rightarrow \mathcal{P} i c^{\Gamma}(X)$ is fully faithful as a functor between categories enriched over $\mathcal{O}_{\mathrm{Sch} / S}$-mod. This is equivalent to the claim that, for every line bundle $L$ on $Z$, the natural morphism

$$
\pi_{Z *} L \longrightarrow\left(\pi_{X *} f^{*} L\right)^{\Gamma}
$$

is an isomorphism of sheaves on $\mathrm{Sch}_{/ S}$, i.e., that

$$
H^{0}\left(S^{\prime} \times_{S} Z, \operatorname{pr}_{Z}^{*} L\right) \longrightarrow H^{0}\left(S^{\prime} \times_{S} X, \operatorname{pr}_{X}^{*} f^{*} L\right)^{\Gamma}
$$

is an isomorphism for every morphism $S^{\prime} \rightarrow S$ with $S^{\prime}$ a locally Noetherian scheme. By smooth descent, we may reduce to the case where $S^{\prime} \times{ }_{S} X=\operatorname{Spec} C, S^{\prime} \times{ }_{S} Z=\operatorname{Spec} B$ and $S^{\prime}=\operatorname{Spec} A$ are all Noetherien affine schemes, and $S^{\prime} \times_{S} Z$ is connected. Since $X \rightarrow Z$ is faithfully flat and $\operatorname{pr}_{Z}^{*} L$ is a flat $\mathcal{O}_{\text {Spec } B \text {-module, by flat descent we have an exact sequence }}$

$$
0 \longrightarrow H^{0}\left(\operatorname{Spec} B, \operatorname{pr}_{Z}^{*} L\right) \longrightarrow H^{0}\left(\operatorname{Spec} C, \operatorname{pr}_{X}^{*} f^{*} L\right) \longrightarrow H^{0}\left(\operatorname{Spec} C, \operatorname{pr}_{X}^{*} f^{*} L\right) \otimes_{B} C,
$$

so we can reduce to showing that the morphism

$$
\begin{align*}
C \otimes_{B} C & \longrightarrow \bigoplus_{\gamma \in \Gamma} C  \tag{4.2.3}\\
c_{1} \otimes c_{2} & \longmapsto\left(c_{1} \gamma\left(c_{2}\right)\right)_{\gamma \in \Gamma}
\end{align*}
$$

is injective.
If $A$ is regular, then so is $B$, so writing $K$ for the fraction field of $B$, we have a commutative diagram

where the vertical morphisms are injective by flatness of $C$ over $B$, and the bottom morphism is an isomorphism since $\operatorname{Spec} C \rightarrow \operatorname{Spec} B$ is generically an étale Galois cover. So injectivity of (4.2.3) in this case follows.

In general, using injectivity of (4.2.3) when $A$ is a field and the argument at the start of the proof of Lemma 4.2.4, we deduce that for every prime ideal $\mathfrak{p} \subseteq A$, the morphism

$$
\widehat{C}_{\mathfrak{p}} \otimes_{\widehat{B}_{\mathfrak{p}}} \widehat{C}_{\mathfrak{p}} \longrightarrow \bigoplus_{\gamma \in \Gamma} \widehat{C}_{\mathfrak{p}}
$$

is injective, where $\widehat{B}_{\mathfrak{p}}$ and $\widehat{C}_{\mathfrak{p}}$ are the $\mathfrak{p}$-adic completions of $B$ and $C$. Injectivity of (4.2.3) now follows from injectivity of the vertical arrows in the commutative diagram


It remains to prove that the essential image of the functor $\mathcal{P} i c(Z) \rightarrow \mathcal{P} i c^{\Gamma}(X)$ is $\mathcal{P} i c^{\Gamma}(X)_{\text {good }}$, i.e., that every good $\Gamma$-linearised line bundle $L$ on $X$ descends to a line bundle $L_{Z}$ on $Z$. By smooth descent, it suffices to prove this in the case where $S=\operatorname{Spec} A, Z=\operatorname{Spec} B$ and $X=\operatorname{Spec} C$ are regular affine schemes, and $Z$ is connected.

By definition, there is a dense open subset $U \subseteq Z$ such that $f^{-1}(U) \rightarrow U$ is a Galois covering with Galois group $\Gamma$. So $\left.L\right|_{f^{-1}(U)}$ descends to a line bundle $L_{U}$ on $U$. By Noetherian induction, to construct $L_{Z}$ it suffices to prove that for any open subset $V \subsetneq Z$ containing $U$ and line bundle $L_{V}$ on $V$ with $\left.f^{*} L_{V} \cong L\right|_{f^{-1}(V)}$ (as $\Gamma$-linearised line bundles), there exists an open $V^{\prime} \supsetneq V$ and a line bundle $L_{V^{\prime}}$ on $V^{\prime}$ with $\left.f^{*} L_{V^{\prime}} \cong L\right|_{f^{-1}\left(V^{\prime}\right)}$.

Assume we are given $V$ and $L_{V}$ as above. If the codimension of $Z \backslash V$ in $Z$ is at least 2, then since $X$ and $Z$ are regular there exists a unique extension $L_{Z}$ of $L_{V}$ to $V^{\prime}=Z$, which necessarily satisfies $f^{*} L_{Z} \cong L$. If not, then there is a point $z \in Z \backslash V$ of codimension 1 in $Z$. We claim that $\left.L\right|_{f^{-1}\left(\operatorname{Spec} \mathcal{O}_{z, z}\right)}$ descends to a line bundle $L_{z}$ on $\operatorname{Spec} \mathcal{O}_{Z, z}$. Assuming the claim, there is a canonical isomorphism between the restrictions of $L_{z}$ and $L_{V}$ to the generic point of $Z$. Since $L_{z}$ and this isomorphism must be defined over some open sets in $Z$, we can glue to a line bundle $L_{V^{\prime \prime}}$ on some open set $V^{\prime \prime}$ containing $V$ and $z$. The isomorphisms $\left.f^{*} L_{V} \cong L\right|_{f^{-1}(V)}$ and $\left.f^{*} L_{z} \cong L\right|_{f^{-1}\left(\operatorname{Spec} \mathcal{O}_{Z, z}\right)}$ agree on the generic point of $Z$, so define an isomorphism $\left.f^{*} L_{V^{\prime}} \cong L\right|_{V^{\prime}}$ where $L_{V^{\prime}}$ is the restriction of $L_{V^{\prime \prime}}$ to some open subset containing $V$ and $z$.

To complete the proof, it therefore remains to prove the claim that $\left.L\right|_{f^{-1}\left(\operatorname{Spec} \mathcal{O}_{z, z}\right)}$ descends to $\operatorname{Spec} \mathcal{O}_{Z, z}$. For brevity, write $B=\mathcal{O}_{Z, z}, f^{-1}\left(\operatorname{Spec} \mathcal{O}_{Z, z}\right)=\operatorname{Spec} C$ and $M=$ $\left.L\right|_{f^{-1}\left(\operatorname{Spec} \mathcal{O}_{z, z}\right)}$ (viewed as $\Gamma$-linearised $C$-module). By the general machinery of faithfully flat descent, it suffices to show that the isomorphism

$$
\begin{aligned}
\phi: & \bigoplus_{\gamma \in \Gamma} C \otimes_{\gamma, C} M \xrightarrow{\sim} \bigoplus_{\gamma \in \Gamma} M \otimes_{C, \text { id }} C \\
& \left(c_{\gamma} \otimes m_{\gamma}\right)_{\gamma \in \Gamma} \longmapsto\left(\gamma\left(m_{\gamma}\right) \otimes c_{\gamma}\right)_{\gamma \in \Gamma}
\end{aligned}
$$

restricts to an isomorphism

$$
\phi^{\prime}: C \otimes_{B} M \xrightarrow{\sim} M \otimes_{B} C,
$$

satisfying a cocycle condition, under the inclusions

$$
\begin{aligned}
C \otimes_{B} M & \bigoplus_{\gamma \in \Gamma} C \otimes_{\gamma, C} M \\
c \otimes m & \longmapsto(c \otimes m)_{\gamma \in \Gamma}=\left(1 \otimes \gamma^{-1}(c) m\right)_{\gamma \in \Gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
& M \otimes_{B} C \longleftrightarrow \bigoplus_{\gamma \in \Gamma} M \otimes_{C, \text { id }} C \\
& m \otimes c \longmapsto(m \otimes c)_{\gamma \in \Gamma}=(c m \otimes 1)_{\gamma \in \Gamma}
\end{aligned}
$$

Since the morphism

$$
\begin{aligned}
& C \otimes_{B} C \otimes_{B} C \bigoplus_{\gamma, \gamma^{\prime} \in \Gamma} C \\
& c_{1} \otimes c_{2} \otimes c_{3} \longmapsto c_{1} \gamma\left(c_{2}\right) \gamma^{\prime}\left(c_{3}\right)
\end{aligned}
$$

is injective (this follows by essentially the same argument as injectivity of (4.2.3)), the cocycle condition for $\phi^{\prime}$, if it exists, follows from the cocycle condition for $\phi$.

To show that $\phi$ restricts as desired, we first show that the sequence

$$
\begin{gather*}
0 \longrightarrow C \otimes_{B} C \xrightarrow{\alpha} \bigoplus_{\gamma \in \Gamma} C \stackrel{\beta}{\longrightarrow} \bigoplus_{\gamma, \gamma^{\prime} \in \Gamma} C / C\left(\gamma-\gamma^{\prime}\right)(C)  \tag{4.2.4}\\
\left(c_{\gamma}\right)_{\gamma \in \Gamma} \longmapsto\left(c_{\gamma}-c_{\gamma^{\prime}}\right)_{\gamma, \gamma^{\prime} \in \Gamma}
\end{gather*}
$$

is exact, where $\alpha$ is (4.2.3). We have already shown that $\alpha$ is injective, so suppose that $\left(c_{\gamma}\right)_{\gamma \in \Gamma}$ satisfies $c_{\gamma}-c_{\gamma^{\prime}} \in C\left(\gamma-\gamma^{\prime}\right)(C)$ for all $\gamma, \gamma^{\prime} \in \Gamma$. We will show that $\left(c_{\gamma}\right)$ is in the image of $\alpha$ by showing that for every subset $\Gamma^{\prime} \subseteq \Gamma$ containing the identity $1 \in \Gamma,\left(c_{\gamma}\right)$ is in the image of $\alpha$ modulo the ideal

$$
I_{\Gamma^{\prime}}=\left\{\left(c_{\gamma}^{\prime}\right)_{\gamma \in \Gamma} \mid c_{\gamma}^{\prime}=0 \text { for } \gamma \in \Gamma^{\prime}\right\} .
$$

We then deduce that $\left(c_{\gamma}\right)$ is in the image of $\alpha$ by setting $\Gamma^{\prime}=\Gamma$.
We work by induction on the size of $\Gamma^{\prime}$. For the base case, suppose that $\Gamma^{\prime}=\{1\}$. Then we have $\left(c_{\gamma}\right)-\alpha\left(c_{1} \otimes 1\right) \in I_{\Gamma^{\prime}}$, so the claim holds. So suppose $\Gamma^{\prime}=\Gamma^{\prime \prime} \cup\left\{\gamma_{0}\right\}$ and that the claim holds for $\Gamma^{\prime \prime} \subseteq \Gamma$. By the induction hypothesis, we may assume without loss of generality that $\left(c_{\gamma}\right) \in I_{\Gamma^{\prime \prime}}$. So by assumption we have

$$
c_{\gamma_{0}} \in \bigcap_{\gamma \in \Gamma^{\prime \prime}} C\left(\gamma-\gamma_{0}\right)(C) .
$$

Note that $C\left(\gamma-\gamma_{0}\right)(C)$ is the ideal defining the fixed locus of $\gamma \gamma_{0}^{-1}$ in Spec $C$. Since $C$ has dimension 1 and the intersections of these fixed loci have codimension at least 2 in $X$, it follows that

$$
\bigcap_{\gamma \in \Gamma^{\prime \prime}} C\left(\gamma-\gamma_{0}\right)(C)=\prod_{\gamma \in \Gamma^{\prime \prime}} C\left(\gamma-\gamma_{0}\right)(C) .
$$

So we can write

$$
c_{\gamma_{0}}=\sum_{i} \prod_{\gamma \in \Gamma^{\prime \prime}} c_{\gamma, i}\left(\gamma-\gamma_{0}\right)\left(c_{\gamma, i}^{\prime}\right) .
$$

Writing

$$
d=\sum_{i} \prod_{\gamma^{\prime} \in \Gamma^{\prime \prime}}\left(c_{\gamma, i} \gamma\left(c_{\gamma, i}^{\prime}\right) \otimes 1-c_{\gamma, i} \otimes c_{\gamma, i}^{\prime}\right) \in C \otimes_{B} C
$$

we have $\left(c_{\gamma}\right)_{\gamma \in \Gamma}-\alpha(d) \in I_{\Gamma^{\prime}}$, so the claim is proved by induction.
Since $M$ is flat, tensoring (4.2.4) with $M$ over each factor of $C$ in $C \otimes_{B} C$ gives a pair of exact sequences


To prove that $\phi$ restricts to a morphism $\phi^{\prime}$ as shown, it suffices to construct a morphism $\phi^{\prime \prime}$ as shown such that the square on the right commutes. We define $\phi^{\prime \prime}$ by the formula

$$
\phi^{\prime \prime}\left(\left(c_{\gamma, \gamma^{\prime}} \otimes m_{\gamma, \gamma^{\prime}}\right)_{\gamma, \gamma^{\prime} \in \Gamma}\right)=\left(\gamma\left(m_{\gamma, \gamma^{\prime}}\right) \otimes c_{\gamma, \gamma^{\prime}}\right)_{\gamma \in \Gamma} .
$$

This is well-defined, and the condition that the necessary diagram commutes is precisely the condition that the $\Gamma$-linearisation on $M$ is good. So $\phi$ restricts to a descent datum $\phi^{\prime}, M$ descends to $B$, and we are done.

### 4.3 The Chevalley isomorphism

The classical Chevalley isomorphisms $\mathfrak{g} / / G \cong \mathfrak{t} / / W$ and $G / / G \cong T / / W$ are essential ingredients in the construction of the additive and multiplicative Grothendieck-Springer resolutions as simultaneous resolutions, as they provide the base change maps $\mathfrak{t} \rightarrow \mathfrak{g} / / G$ and $T \rightarrow G / / G$.

In this section, we prove an elliptic analogue of these statements, which is one of the main results of this thesis.

One can think of the classical (say, additive) Chevalley isomorphism as an isomorphism between the ring of regular functions on the stack $\mathfrak{g} / G$ and the ring of $W$-invariant functions on the affine variety $\mathfrak{t}$. So at first glance, the elliptic Chevalley isomorphism should identify the ring of regular functions on the stack $\operatorname{Bun}_{G}$ with $W$-invariant regular functions on some variety. However, since the coarse moduli space of semistable $G$-bundles is projective rather than affine, there are not enough global regular functions on $\mathrm{Bun}_{G}$ to make such a statement particularly useful. Instead, the elliptic Chevalley isomorphism that we will prove gives an identification of $\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)$ with a subgroup of the group of $W$-linearised line bundles on the abelian variety $Y$, so that the space of global sections of a line bundle on $\operatorname{Bun}_{G}$ is naturally isomorphic to the space of $W$-invariant sections of the corresponding line bundle on $Y$.

Remark 4.3.1. The Weyl group $W$ acts naturally on the torus $T$, and hence on the abelian variety $Y$ over $S$. Explicitly, this action is given by

$$
s_{\alpha}(y)=y-\alpha^{\vee}(\alpha(y))
$$

for $\alpha \in \Phi$, where we use the natural group structure on $Y$, and for $\lambda \in \mathbb{X}^{*}(T)$ (resp., $\mu \in \mathbb{X}_{*}(T)$ ), we write $\lambda: Y \rightarrow \operatorname{Pic}_{S}^{0}(E)$ (resp., $\mu: \operatorname{Pic}_{S}^{0}(E) \rightarrow Y$ ) for the morphism induced by $\lambda: T \rightarrow \mathbb{G}_{m}$ (resp., $\mu: \mathbb{G}_{m} \rightarrow T$ ).

Definition 4.3.2. Let $L$ be a $W$-linearised line bundle on $Y$. We say that $L$ is good if for every root $\alpha \in \Phi_{+}$, the morphism

$$
s_{\alpha}:\left.\left.L\right|_{Y^{s_{\alpha}}} \longrightarrow L\right|_{Y^{s_{\alpha}}}
$$

is the identity, where $Y^{s_{\alpha}} \subseteq Y$ is the fixed locus of $s_{\alpha}: Y \rightarrow Y$. We write $\operatorname{Pic}^{W}(Y)_{\text {good }} \subseteq$ $\mathrm{Pic}^{W}(Y)$ for the subgroup of good $W$-linearised line bundles, and $\mathcal{P} i c^{W}(Y)_{\text {good }} \subseteq \mathcal{P} i c^{W}(Y)$ for the corresponding full subcategory.

Remark 4.3.3. Over the smooth locus of $Y / / W$, the morphism $Y \rightarrow Y / / W$ is a ramified Galois covering. Definition 4.3.2 is consistent with Definition 4.2.10 over this locus, since for $w \in W$ we have $Y_{(1)}^{w} \neq \emptyset$ if and only if $w=s_{\alpha}$ is the reflection in some root $\alpha \in \Phi_{+}$.

Theorem 4.3.4 (Elliptic Chevalley isomorphism). There are equivalences

$$
\mathcal{P} i c\left(\operatorname{Bun}_{G}\right) \simeq \mathcal{P} i c\left(\operatorname{Bun}_{G, r i g}\right) \simeq \mathcal{P} i c^{W}(Y)_{\operatorname{good}}
$$

of symmetric monoidal categories enriched over $\mathcal{O}_{\mathrm{Sch}_{/ S}}-\bmod$.
Remark 4.3.5. In more down to earth terms, Theorem 4.3.4 states that there are isomorphisms

$$
\operatorname{Pic}\left(\operatorname{Bun}_{G}\right) \cong \operatorname{Pic}\left(\operatorname{Bun}_{G, r i g}\right) \cong \operatorname{Pic}^{W}(Y)_{\operatorname{good}}
$$

of abelian groups, and isomorphisms

$$
\pi_{\mathrm{Bun}_{G} *} L_{\mathrm{Bun}_{G}} \cong \pi_{\mathrm{Bun}_{G, r i g *}} L_{\mathrm{Bun}_{G, r i g}} \cong\left(\pi_{Y *} L\right)^{W}
$$

of sheaves of $\mathcal{O}$-modules on $\mathrm{Sch}_{/ S}$, compatible with tensor products, for $L \in \operatorname{Pic}^{W}(Y)_{\text {good }}$ corresponding to $L_{\mathrm{Bun}_{G}} \in \operatorname{Pic}\left(\operatorname{Bun}_{G}\right)$ and $L_{\mathrm{Bun}_{G, r i g}} \in \operatorname{Pic}\left(\operatorname{Bun}_{G, \text { rig }}\right)$.

Proof of Theorem 4.3.4. We give the outline of the proof here, and fill in the details in the rest of the section.

First, by Lemma 4.2.2 and Proposition 2.6.8, the restriction functors

$$
\mathcal{P} i c\left(\operatorname{Bun}_{G}\right) \longrightarrow \mathcal{P} i c\left(\operatorname{Bun}_{G}^{s s}\right) \quad \text { and } \quad \mathcal{P} i c\left(\operatorname{Bun}_{G, r i g}\right) \longrightarrow \mathcal{P} i c\left(\operatorname{Bun}_{G, r i g}^{s s}\right)
$$

are equivalences of symmetric monoidal categories enriched over $\mathcal{O}_{\mathrm{Sch}_{/ S}}-$ mod. So it suffices to prove the theorem with $\operatorname{Bun}_{G}^{s s}$ and $\operatorname{Bun}_{G, r i g}^{s s}$ in place of $\mathrm{Bun}_{G}$ and $\operatorname{Bun}_{G, r i g}$.

Consider the commutative diagram

and its rigidification

where $\operatorname{Bun}_{G}^{s s, r e g} \subseteq \operatorname{Bun}_{G}^{s s}$ and $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \subseteq{\widetilde{\operatorname{Bun}_{G}}}^{s s}$ are the big open substacks of regular semistable bundles (see Definition 4.3.7 and Proposition 4.3.15). By Lemma 4.2.2 again, restriction of line bundles gives equivalences

$$
\mathcal{P} i c\left(\operatorname{Bun}_{G}^{s s}\right) \xrightarrow{\sim} \mathcal{P} i c\left(\operatorname{Bun}_{G}^{s s, r e g}\right) \quad \text { and } \quad \mathcal{P} i c\left(\operatorname{Bun}_{G, r i g}^{s s}\right) \xrightarrow{\sim} \mathcal{P i c}\left(\operatorname{Bun}_{G, r i g}^{s s, r e g}\right) .
$$

By Proposition 4.3.14, the morphisms $\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s, \text { reg }} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s s, \text { reg }}$ and $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \rightarrow \operatorname{Bun}_{G}^{s s, \text { reg }}$ are ramified Galois coverings with Galois group $W$ satisfying the hypotheses of Proposition 4.2.12 such that the Galois action covers the natural $W$-action on $Y$, so there are equivalences

$$
\mathcal{P} i c\left(\operatorname{Bun}_{G}^{s s, r e g}\right) \xrightarrow{\sim} \mathcal{P} i c^{W}\left(\widetilde{\operatorname{Bun}}_{G}^{\text {ss,reg }}\right)_{\text {good }} \quad \text { and } \quad \mathcal{P} i c\left(\operatorname{Bun}_{G, r i g}^{s s, r e g}\right) \xrightarrow{\sim} \mathcal{P} i c^{W}\left(\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s s, r e g}\right)_{\text {good }}
$$

of symmetric monoidal categories enriched over $\mathcal{O}_{\mathrm{Sch}_{/ S}}-\bmod$. But by Proposition 4.3 .17 the natural pullback functors give equivalences

$$
\mathcal{P} i c^{W}(Y)_{\text {good }} \xrightarrow{\sim} \mathcal{P} i c^{W}\left({\widetilde{\operatorname{Bun}_{G}}}^{s, \text { reg }}\right)_{\text {good }}
$$

and

$$
\mathcal{P} i c^{W}(Y)_{\mathrm{good}} \xrightarrow{\sim} \mathcal{P} i c^{W}\left({\widetilde{\operatorname{Bun}_{G, r i g}}}_{\text {ss,reg }}\right)_{\text {good }}
$$

which completes the proof.
Remark 4.3.6. From the proof, it is clear that the equivalence

$$
\mathcal{P} i c\left(\operatorname{Bun}_{G, \text { rig }}\right) \xrightarrow{\sim} \mathcal{P} i c\left(\operatorname{Bun}_{G}\right)
$$

of Theorem 4.3.4 is just the obvious pullback functor.
The rest of this section is concerned with proving the various propositions and lemmas quoted in the proof of Theorem 4.3.4. We begin by introducing and studying the substack of regular semistable bundles.

Definition 4.3.7. We say that a semistable $G$-bundle $\xi_{G} \in \operatorname{Bun}_{G}^{s s}$ is regular if $\operatorname{dim} \psi^{-1}\left(\xi_{G}\right)=$ 0 . We write $\operatorname{Bun}_{G}^{s s, r e g} \subseteq \operatorname{Bun}_{G}^{s s}$ for the open substack of regular semistable bundles. We also write

$$
\widetilde{\operatorname{Bun}}_{G}^{s s, r e g}=\psi^{-1}\left(\operatorname{Bun}_{G}^{s s, r e g}\right)
$$

Remark 4.3.8. There is another notion of regular semistable bundle in use in the literature, namely that of a semistable principal bundle whose automorphism group has minimal dimension $l=\operatorname{rank} G$. We will see later on (Proposition 5.5.5) that this notion agrees with ours.

In classical Springer theory, the simplest regular elements to describe are the regular semisimple ones. The same is true in our context.

Definition 4.3.9. We say that a point $y: \operatorname{Spec} k \rightarrow Y$ over $s: \operatorname{Spec} k \rightarrow S$ is strictly regular if for every root $\alpha \in \Phi_{+}$, we have $\alpha(y) \neq 0 \in \operatorname{Pic}^{0}\left(Y_{s}\right)$. We write $Y^{\text {sreg }}$ for the open subset of regular points and $\operatorname{Bun}_{T}^{0, \text { sreg }}$ and $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }}$ for the preimages in $\operatorname{Bun}_{T}^{0}$ and $\widetilde{\operatorname{Bun}}_{G}^{s s}$ respectively. We call a $G$-bundle $\xi_{G} \in \operatorname{Bun}_{G}^{s s}$ strictly regular, or regular semisimple, if it lies in the image $\operatorname{Bun}_{G}^{s s, \text { sreg }}=\psi\left({\widetilde{\operatorname{Bun}_{G}}}^{s s, \text { sreg }}\right)$.

Lemma 4.3.10. The morphism $\widetilde{\operatorname{Bun}}_{G}^{\text {ss,sreg }} \rightarrow \mathrm{Bun}_{T}^{0, \text { sreg }}$ is an isomorphism.
Proof. Since $\widetilde{\operatorname{Bun}}_{G}^{\text {ss,sreg }} \rightarrow \operatorname{Bun}_{T}^{0, \text { sreg }}$ is smooth, it suffices to show that each geometric fibre is trivial. But this is clear from Lemma 4.3 .11 below, so we are done.

Lemma 4.3.11. Fix a geometric point $s: \operatorname{Spec} k \rightarrow S$ and a degree $0 T$-bundle $\xi_{T}$ on $E_{s}$ corresponding to $y \in Y_{s}$, and let $U \subseteq R_{u}(B)$ be a unipotent closed subgroup scheme that is invariant under conjugation by $T$. Assume that for all $\alpha \in \Phi_{-}$such that $\alpha(y)=0$, we have $U_{\alpha} \subseteq U$, where $U_{\alpha}=\mathbb{G}_{a}$ is the root subgroup corresponding to $\alpha$. Then the induced bundle morphism

$$
\operatorname{Bun}_{T U}\left(E_{s}\right)_{\xi_{T}} \longrightarrow \operatorname{Bun}_{B}\left(E_{s}\right)_{\xi_{T}}
$$

is an isomorphism, where the subscript denotes the fibre over $\xi_{T}$ of the natural morphism to $\operatorname{Bun}_{T}\left(E_{s}\right)$.

Proof. Since the statement only concerns individual geometric fibres of $E \rightarrow S$, we can assume that $S=\operatorname{Spec} k$.

Writing $\mathcal{R}$ and $\mathcal{U}$ for the group schemes $\xi_{T} \times{ }^{T} R_{u}(B)$ and $\xi_{T} \times{ }^{T} U$, we have canonical isomorphisms

$$
\operatorname{Bun}_{T U}(E)_{\xi_{T}} \cong \operatorname{Bun}_{\mathcal{U}} \quad \text { and } \quad \operatorname{Bun}_{B}(E)_{\xi_{T}} \cong \operatorname{Bun}_{\mathcal{R}}
$$

so it suffices to show that the natural morphism $\operatorname{Bun}_{\mathcal{U}} \rightarrow \operatorname{Bun}_{\mathcal{R}}$ is an isomorphism.
Let $R_{u}(B)=R_{u}(B)^{\geq 1} \supseteq R_{u}(B)^{\geq 2} \supseteq \cdots$ be the filtration on $R_{u}(B)$ according to root height, and $U^{\geq i}=U \cap R_{u}(B)^{\geq i}$ for all $i$. Then writing $\mathcal{R}^{\geq i}=\xi_{T} \times^{T} R_{u}(B)^{\geq i}$ and $\mathcal{U}^{\geq i}=\xi_{T} \times^{T} U^{\geq i}$, we show by induction on $i$ that

$$
\begin{equation*}
\operatorname{Bun}_{\mathcal{U} / \mathcal{U} \geq i} \longrightarrow \operatorname{Bun}_{\mathcal{R} / \mathcal{R} \geq i} \tag{4.3.1}
\end{equation*}
$$

is an isomorphism for all $i$, and the statement then follows. Clearly this is true for $i=1$, so suppose $i>1$. Then we have a commutative diagram of central extensions of group schemes on $E$


Since $\mathcal{U}^{\geq i-1} / \mathcal{U}^{\geq i}$ and $\mathcal{R}^{\geq i-1} / \mathcal{R}^{\geq i}$ are direct sums of degree 0 line bundles such that $\mathcal{U}^{\geq i-1} / \mathcal{U}^{\geq i}$ contains all trivial summands of $\mathcal{R}^{\geq i-1} / \mathcal{R}^{\geq i}$, the induced morphism

$$
\operatorname{Bun}_{\mathcal{U} \geq i-1} / \mathcal{U} \geq i \xrightarrow{\sim} \operatorname{Bun}_{\mathcal{R} \geq i-1 / \mathcal{R} \geq i}
$$

is an isomorphism, and by induction, the induced morphism

$$
\operatorname{Bun}_{\mathcal{U} / \mathcal{U} \geq i-1} \xrightarrow{\sim} \operatorname{Bun}_{\mathcal{R} / \mathcal{R} \geq i-1}
$$

is also an isomorphism. So Proposition 2.4.2 implies that (4.3.1) is an isomorphism as claimed.

In the following lemma, for $\xi_{B} \rightarrow E_{s}$ a $B$-bundle and $w \in W$, we write

$$
C^{w}=\left\{\xi_{B}\right\} \times_{\operatorname{Bun}_{B}} C_{B, B / S}^{w, 0}(E) .
$$

Lemma 4.3.12. Let $\xi_{B} \rightarrow E_{s}$ be a $B$-bundle of degree 0 on a geometric fibre of $E \rightarrow S$, and let $\xi_{G}=\xi_{B} \times{ }^{B} G$ be the induced $G$-bundle. Then the morphism

$$
\coprod_{w \in W} C^{w} \longrightarrow\left\{\xi_{G}\right\} \times_{\operatorname{Bun}_{G}} \operatorname{Bun}_{B}^{0}=\psi^{-1}\left(\xi_{G}\right)
$$

is surjective.
Proof. Let $\xi_{T}=\xi_{B} \times{ }^{B} T$. Since $C_{B, B / S}^{w, \lambda}(E)_{\xi_{T} \times{ }^{T} B}=\emptyset$ unless $\lambda=0$, the lemma follows from Proposition 3.7.6.

Proposition 4.3.13. We have $\psi^{-1}\left(\operatorname{Bun}_{G}^{s s, \text { sreg }}\right)=\widetilde{\operatorname{Bun}_{G}}{ }^{\text {ss,sreg }}$, and the morphism $\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s, \text { sreg }} \rightarrow$ $\operatorname{Bun}_{G, \text { rig }}^{s s, \text { sreg }}$ (and hence also $\widetilde{\mathrm{Bun}}_{G}^{s s, \text { sreg }} \rightarrow \operatorname{Bun}_{G}^{s s, \text { sreg }}$ ) is an étale Galois covering with Galois group $W$. In particular, every strictly regular $G$-bundle is regular.

Proof. We first show that $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }} \rightarrow \operatorname{Bun}_{G}^{s s}$ is étale. To see this, let $\xi_{B} \rightarrow E_{s}$ be the $B$-bundle classified by a geometric point of $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }}$ over $s$ : Spec $k \rightarrow S$, and observe that the relative tangent complex at $\xi_{B}$ is given by

$$
\mathbb{T}_{\widetilde{\operatorname{Bun}}_{G} / \operatorname{Bun}_{G}, \xi_{B}}=\mathbb{R} \Gamma\left(E_{S}, \xi_{B} \times{ }^{B} \mathfrak{g} / \mathfrak{b}\right)
$$

The $B$-module $\mathfrak{g} / \mathfrak{b}$ has a filtration with subquotients isomorphic to $\mathfrak{g}_{\alpha}=\mathbb{Z}_{\alpha}$ for $\alpha \in \Phi_{+}$. Since the associated $T$-bundle is strictly regular, the line bundles $\xi_{B} \times{ }^{B} \mathbb{Z}_{\alpha}$ are nontrivial of degree 0 , so have vanishing cohomology. So $\mathbb{T}_{\operatorname{Bun}_{G} / \operatorname{Bun}_{G}, \xi_{B}}=0$, which implies that $\psi$ is étale at $\xi_{B}$ as claimed.

We next compute the fibre of $\psi$ over a geometric point $\xi_{G} \in \operatorname{Bun}_{G}^{s s, \text { sreg }}$ over $s: \operatorname{Spec} k \rightarrow$ $S$. Let $\xi_{B}$ be a lift of $\xi_{G}$ to $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }}$ with associated $T$-bundle $\xi_{T}$. By Lemma 4.3.12, we have a decomposition

$$
\psi^{-1}\left(\xi_{G}\right)=\coprod_{w \in W} C^{w}
$$

into locally closed subsets indexed by the Weyl group $W$. By Propositions 3.7.4 and 3.7.5, we have

$$
C^{w}=\Gamma\left(E_{s}, \xi_{\mathcal{U}} / \mathcal{U}_{w}\right)=\operatorname{Bun}_{\mathcal{U}_{w}} \times_{\operatorname{Bun}_{\mathcal{U}}}\left\{\xi_{\mathcal{U}}\right\}
$$

where $\mathcal{U}$ and $\mathcal{U}_{w}$ are the unipotent group schemes $\mathcal{U}=\xi_{T} \times{ }^{T} R_{u}(B)$ and $\mathcal{U}_{w}=\xi_{T} \times{ }^{T}\left(R_{u}(B) \cap\right.$ $w B w^{-1}$ ) on $E_{s}$, and $\xi_{\mathcal{U}}=\xi_{B} / T$ is the $\mathcal{U}$-bundle corresponding to $\xi_{B}$. But since $\xi_{T}$ is strictly regular, Lemma 4.3 .11 shows that $\operatorname{Bun}_{\mathcal{U}}\left(E_{s}\right)=\operatorname{Bun}_{\mathcal{U}_{w}}\left(E_{s}\right)=\operatorname{Spec} k$, so $C^{w} \cong \operatorname{Spec} k$ as well. Since $\psi^{-1}\left(\xi_{G}\right)$ is reduced by étaleness of $\psi$, we therefore have an isomorphism

$$
W \times \operatorname{Spec} k \xrightarrow{\sim} \psi^{-1}\left(\xi_{G}\right)
$$

sending $w \in W$ to $C^{w}$, such that the composition with $\psi^{-1}\left(\xi_{G}\right) \rightarrow Y$ sends $w \in W$ to $j_{w}(y)=w^{-1} y$, where $y=\bar{\chi}\left(\xi_{B}\right)$. In particular, since $w^{-1} y \in Y^{\text {sreg }}$, we have $\psi^{-1}\left(\xi_{G}\right) \subseteq$ $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }}$, so this proves $\psi^{-1}\left(\operatorname{Bun}_{G}^{\text {ss,sreg }}\right)=\widetilde{\operatorname{Bun}}_{G}^{\text {ss,sreg }}$.

Since $\psi$ is proper, this implies that $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }} \rightarrow \operatorname{Bun}_{G}^{s s, \text { sreg }}$ is finite étale, and hence so is $\widetilde{\mathrm{Bun}}_{G, \text { rig }}^{s s, \text { sreg }} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s, \text { sreg }}$. To prove that it is a Galois cover with Galois group $W$, we need to show that $W$ acts on $\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s, \text { sreg }}$ over $\operatorname{Bun}_{G, \text { rig }}^{s s, \text { sreg }}$, freely and transitively on some (hence every) fibre. By Lemma 4.3.10, we can identify $\widetilde{\text { Bun }}_{G, \text { rig }}^{s, s r e g} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s s, \text { sreg }}$ with the morphism Bun ${ }_{T, \text { rig }}^{0, \text { sreg }} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s s, \text { sreg }}$ given by inducing along our chosen embedding $T \hookrightarrow G$. Since $N_{G}(T)$ acts on $\mathbb{B}(T / Z(G))$ over $\mathbb{B}(G / Z(G))$, it acts on $\mathbb{B} T$ over $\mathbb{B} G$ preserving the $\mathbb{B} Z(G)$-action. So we get an action of $N_{G}(T)$ on $\operatorname{Bun}_{T}^{0, \text { sreg }}$ over $\mathrm{Bun}_{G}^{s s, \text { sreg }}$ also preserving the $\mathbb{B} Z(G)$-action, and hence on $\operatorname{Bun}_{T, \text { rig }}^{0, \text { sreg }}$ over $\operatorname{Bun}_{G, \text { rig }}^{s s, \text { sreg }}$. Since $\mathrm{Bun}_{T, \text { rig }}^{0, \text { sreg }} \rightarrow \operatorname{Bun}_{G, r i g}^{s s, \text { sreg }}$ is étale, the connected subgroup $T \subseteq N_{G}(T)$ must act trivially, so this factors through an action of $W$.

By construction, the morphism $\mathrm{Bun}_{T, \text { rig }}^{0, \text { sreg }} \rightarrow Y^{\text {sreg }}$ is $W$-equivariant. Choose a geometric point $y$ : Spec $k \rightarrow Y^{\text {sreg }}$ over $s: \operatorname{Spec} k \rightarrow S$ such that the stabiliser of $y$ under $W$ is trivial, and let $\xi_{T}$ be the corresponding $T$-bundle. Then $\psi^{-1}\left(\xi_{T} \times{ }^{T} G\right)$ maps isomorphically onto the $W$-orbit of $y$ in $Y_{s}$, so in particular, the $W$-action on $\psi^{-1}\left(\xi_{T} \times{ }^{T} G\right)$ is free and transitive, so we are done.

Proposition 4.3.14. The morphisms $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \rightarrow \operatorname{Bun}_{G}^{s s, \text { reg }}$ and $\widetilde{\operatorname{Bun}_{G, \text { rig }}^{s s, r e g}} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s s, \text { reg }}$ are ramified Galois coverings with Galois group $W$ satisfying the conditions of Proposition 4.2.12.

Proof. Since $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { sreg }} \subseteq \widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}$ is dense in every fibre over $S$, and both morphisms are proper and quasi-finite, hence finite, both morphisms are ramified Galois coverings with Galois group $W$ by Proposition 4.3.13. It suffices to check the conditions of Proposition 4.2.12 for $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \rightarrow \operatorname{Bun}_{G}^{s s, \text { reg }}$, as the claims for $\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s s, \text { reg }} \rightarrow \operatorname{Bun}_{G, \text { rig }}^{s s, \text { reg }}$ then follow by descent.

We need to show that for every $w, w^{\prime} \in W \backslash\{1\}$ with $w \neq w^{\prime}$, the intersection of the $\operatorname{Bun}_{G}^{s s, \text { reg }}$-relative fixed loci $\left({\widetilde{\operatorname{Bun}_{G}}}^{s s, \text { reg }}\right)^{w} \cap\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)^{w^{\prime}}$ has codimension at least 2 in every fibre over $S$. Observe that since $\widetilde{\mathrm{Bun}}_{G}^{\text {ss,sreg }} \rightarrow Y^{\text {sreg }}$ is $W$-equivariant and $Y \rightarrow S$ is representable and separated, it follows by continuity that $\bar{\chi}^{\text {reg }}: \widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \rightarrow Y$ is also $W$-equivariant. So $\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)^{w} \subseteq \bar{\chi}^{-1}\left(Y^{w}\right)$ for all $w \in W$, where $Y^{w}$ denotes the fixed locus relative to $S$. Since $\bar{\chi}$ is smooth and $Y^{w} \cap Y^{w}$ has codimension at least 2 in every fibre, the result now follows.

Proposition 4.3.15. The open substacks

$$
\operatorname{Bun}_{G}^{s s, \text { reg }} \subseteq \operatorname{Bun}_{G}^{s s} \quad \text { and } \quad \widetilde{\operatorname{Bun}}_{G}^{s, \text { reg }} \subseteq{\widetilde{\operatorname{Bun}_{G}}}^{s s}
$$

are big relative to $S$.
The proof of Proposition 4.3.15 relies on the following construction of $G$-bundles that are regular semistable but not strictly regular.

Let $s$ : Spec $k \rightarrow S$ be a geometric point, $\alpha \in \Phi_{+}$a positive root, and let $y \in Y_{s}$ satisfy $\alpha(y)=0$ and $\beta(y) \neq 0$ for all $\beta \in \Phi_{+} \backslash\{\alpha\}$. Then by Lemma 4.3.11 and Proposition 2.4.2, the fibre of $\widetilde{\operatorname{Bun}}_{G}^{s s} \rightarrow Y$ over $y$ is

$$
\widetilde{\operatorname{Bun}}_{G, y}^{s s} \cong \operatorname{Bun}_{B, y} \cong \operatorname{Bun}_{T U_{-\alpha}}\left(E_{s}\right)_{y} \cong \operatorname{Bun}_{U_{-\alpha}}\left(E_{s}\right) / T
$$

Since $H^{1}\left(E_{s}, \mathcal{O}\right)=k$, Proposition 2.4.1 implies that there is a unique $k$-point of $\operatorname{Bun}_{U_{-\alpha}}\left(E_{s}\right) / T$ corresponding to a nontrivial $U_{-\alpha}$-bundle. Let $\xi_{T U_{-\alpha}}$ be the corresponding point of $\operatorname{Bun}_{T U_{-\alpha}}\left(E_{s}\right)_{y}$ and let $\xi_{G}=\xi_{T U_{-\alpha}} \times{ }^{T U_{-\alpha}} G$.

Lemma 4.3.16. Let $\xi_{G}$ be the $G$-bundle defined above. Then the fibre $\psi^{-1}\left(\xi_{G}\right)$ has exactly $|W| / 2 k$-points. In particular, $\xi_{G}$ is regular semistable.

Proof. Observe that since the subgroup $T U_{-\alpha} \subseteq G$ is conjugate under $N_{G}(T) \subseteq G$ to $T U_{-\alpha_{i}}$ for some $\alpha_{i} \in \Delta$, we can assume without loss of generality that $\alpha=\alpha_{i}$ is a simple root.

Writing $\xi_{B}=\xi_{T U_{-\alpha_{i}}} \times{ }^{T U_{-\alpha_{i}}} B$, we have $\xi_{G}=\xi_{B} \times{ }^{B} G$, so by Lemma 4.3.12 we get a decomposition

$$
\psi^{-1}\left(\xi_{G}\right)=\coprod_{w \in W} C^{w}
$$

into locally closed subschemes, where by Proposition 3.7 .5 we can identify $C^{w}$ with the space of sections of

$$
\xi_{B} \times{ }^{B} B w B / B=\xi_{T U_{-\alpha_{i}}} \times{ }^{T U_{-\alpha_{i}}} R_{u}(B) /\left(R_{u}(B) \cap w B w^{-1}\right)
$$

If $U_{-\alpha_{i}} \nsubseteq R_{u}(B) \cap w B w^{-1}$, i.e., if $w^{-1} \alpha_{i} \in \Phi_{-}$, then there is a $T U_{-\alpha_{i}}$-equivariant morphism

$$
R_{u}(B) /\left(R_{u}(B) \cap w B w^{-1}\right) \longrightarrow U_{-\alpha_{i}}
$$

so $\xi_{T U_{-\alpha_{i}}} \times T U_{-\alpha_{i}} R_{u}(B) /\left(R_{u}(B) \cap w B w^{-1}\right)$ has no sections since $\xi_{T U_{-\alpha_{i}}} \times{ }^{T U_{-\alpha_{i}}} U_{-\alpha_{i}}=\xi_{U_{-\alpha_{i}}}$ has none, and hence $C^{w}=\emptyset$. If $U_{-\alpha_{i}} \subseteq R_{u}(B) \cap w B w^{-1}$, i.e., if $w^{-1} \alpha_{i} \in \Phi_{+}$, then the natural morphisms

$$
\operatorname{Bun}_{T U_{-\alpha_{i}}}\left(E_{s}\right)_{\xi_{T}} \longrightarrow \operatorname{Bun}_{T R_{u}(B) \cap w B w^{-1}}\left(E_{s}\right)_{\xi_{T}} \longrightarrow \operatorname{Bun}_{B}\left(E_{s}\right)_{\xi_{T}}
$$

are isomorphisms by Lemma 4.3.11, where $\xi_{T} \rightarrow E_{s}$ is the $T$-bundle corresponding to $y$, which implies that $C^{w}=$ Spec $k$. Since there are exactly $|W| / 2$ elements of $W$ satisfying $w^{-1} \alpha_{i} \in \Phi_{+}$, this proves the lemma.
Proof of Proposition 4.3.15. It suffices to prove the statement for $\widetilde{\operatorname{Bun}}_{G}^{\text {ss,reg }} \subseteq \widetilde{\operatorname{Bun}}_{G}^{s s}$; the statement for $\operatorname{Bun}_{G}^{s s, \text { reg }} \subseteq \operatorname{Bun}_{G}^{s s}$ then follows immediately. Since the property of being big is defined fibrewise, it suffices to prove the claim when $S=\operatorname{Spec} k$ for some algebraically closed field $k$.

We need to show that the complement of $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}$ in $\widetilde{\operatorname{Bun}}_{G}^{s s}$ has codimension at least
 $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \cap\left(\bar{\chi}^{s s}\right)^{-1}(X)$ is dense in $\left(\bar{\chi}^{s s}\right)^{-1}(X)$ for all irreducible components $X$ of $Y \backslash$ $Y^{\text {sreg }}=\bigcup_{\alpha \in \Phi_{+}} Y^{s_{\alpha}}$. But $\left(\bar{\chi}^{s s}\right)^{-1}\left(X^{\text {red }}\right)$ is smooth and connected, hence irreducible, and $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \cap\left(\bar{\chi}^{s s}\right)^{-1}(X)$ is open. So it suffices to show that $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \cap\left(\bar{\chi}^{s s}\right)^{-1}(X)$ is nonempty. But there exists $\alpha \in \Phi_{+}$such that the generic point $y$ of $X \subseteq Y$ satisfies $\alpha(y)=0$ and $\beta(y) \neq 0$ for all $\beta \in \Phi_{+} \backslash\{\alpha\}$, so this follows from Lemma 4.3.16.

Proposition 4.3.17. We have the following.
(1) The pullback functors

$$
\mathcal{P i c}(Y) \longrightarrow \mathcal{P} i c\left(\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s s, r e g}\right) \quad \text { and } \quad \mathcal{P} i c(Y) \longrightarrow \mathcal{P i c}\left(\widetilde{\operatorname{Bun}}_{G}^{s s, r e g}\right)
$$

are fully faithful as functors enriched over $\mathcal{O}_{\mathrm{Sch}_{/ S}}-\bmod$.
(2) The pullback functors

$$
\begin{equation*}
\mathcal{P} i c^{W}(Y) \longrightarrow \mathcal{P} i c^{W}\left({\widetilde{\mathrm{Bun}_{G, r i g}}}^{\text {ss,reg }}\right) \longrightarrow \mathcal{P} i c^{W}\left(\widetilde{\mathrm{Bun}}_{G}^{\text {ss,reg }}\right) \tag{4.3.2}
\end{equation*}
$$

are equivalences of categories enriched over $\mathcal{O}_{\text {Sch }_{/ S}}-\bmod$.
(3) The equivalences (4.3.2) restrict to equivalences

Proof. Note that since all enriched categories in the statement satisfy flat descent on $S$, we can assume without loss of generality that $S$ is a connected regular scheme and that $E \rightarrow S$ has a section $O_{E}: S \rightarrow E$.

To prove (1), note that by Proposition 4.3.15 and Lemma 4.2.2, it suffices to prove that the enriched functors

$$
\mathcal{P i c}(Y) \longrightarrow \mathcal{P} i c\left(\widetilde{\operatorname{Bun}}_{G, r i g}^{s s}\right) \quad \text { and } \quad \mathcal{P} i c(Y) \longrightarrow \mathcal{P} i c\left(\widetilde{\operatorname{Bun}}_{G}^{s s}\right)
$$

are fully faithful. We will in fact show that

$$
\bar{\chi}_{*} \mathcal{O}_{\widetilde{\operatorname{Bun}}_{G, r i g}^{s s}}^{s,}=\bar{\chi}_{*} \mathcal{O}_{\widetilde{\operatorname{Bun}}_{G}^{s s}}^{s s}=\mathcal{O}_{Y}
$$

as sheaves of $\mathcal{O}$-modules on $\mathrm{Sch}_{/ Y}$. Fully faithfulness then follows since

$$
\underline{\operatorname{Hom}}\left(\bar{\chi}^{*} L_{1}, \bar{\chi}^{*} L_{2}\right)=\pi_{Y *} \bar{\chi}_{*} \bar{\chi}^{*}\left(L_{1}^{-1} \otimes L_{2}\right)=\pi_{Y_{*}}\left(L_{1}^{-1} \otimes L_{2} \otimes \bar{\chi}_{*} \mathcal{O}\right)
$$

Recall that there is a universal degree $0 T$-bundle $\xi_{T, Y}$ on $Y \times_{S} E$ such that the pullback to $Y$ along $O_{E}$ is trivial. This induces canonical isomorphisms $\operatorname{Bun}_{T}^{0}=Y \times$ $\mathbb{B} T$ (resp., $\operatorname{Bun}_{T, \text { rig }}^{0}=Y \times \mathbb{B}(T / Z(G))$ ) and $\widetilde{\operatorname{Bun}}_{G}^{s s}=\operatorname{Bun}_{\mathcal{R} / Y}(E) / T$ (resp., ${\underset{\operatorname{Bun}}{G, \text { rig }}}^{s s}=$ $\left.\operatorname{Bun}_{\mathcal{R} / Y}(E) /(T / Z(G))\right)$, where $\mathcal{R} \rightarrow Y \times_{S} E$ is the unipotent group scheme $\xi_{T, Y} \times^{T} R_{u}(B)$.

Fix a cocharacter $\lambda: \mathbb{G}_{m} \rightarrow T$ such that $\left\langle\alpha_{i}, \lambda\right\rangle>0$ for all simple roots $\alpha_{i} \in \Delta$, and write $\lambda\left(\mathcal{O}\left(O_{E}\right)\right)$ for the $T$-bundle on $E$ induced from the $\mathbb{G}_{m}$-bundle corresponding to the line bundle $\mathcal{O}\left(O_{E}\right)$ on $E$. Since the weights of $T$ acting on $R_{u}(B)$ are strictly negative linear combinations of the $\alpha_{i}$, the morphism $\mathcal{O} \rightarrow \mathcal{O}\left(O_{E}\right)$ induces a morphism of group schemes $\mathcal{R}^{\prime}=\lambda\left(\mathcal{O}\left(O_{E}\right)\right) \times{ }^{T} \mathcal{R} \rightarrow \mathcal{R}$, which satisfies the conditions of Proposition 2.4.7 over $Y$. Corollary 2.4.3 shows that $\operatorname{Bun}_{\mathcal{R}^{\prime} / Y}(E) \rightarrow Y$ is an affine space bundle and Proposition 2.4.7 shows that $\operatorname{Bun}_{\mathcal{R} / Y}(E)=\operatorname{Bun}_{\mathcal{R}^{\prime} / Y}(E) / U$ for some unipotent group scheme $U=$ $\Gamma_{Y}\left(Y \times_{S} E, \mathcal{R} / \mathcal{R}^{\prime}\right)$ on $Y$. Moreover, $T$ acts on the fibres of $\operatorname{Bun}_{\mathcal{R}^{\prime} / Y}(E)$ and $U$ over $Y$ with nonzero weights in $\mathbb{Z}_{\leq 0} \Delta$, so the claim follows by direct computation using the Čech complex for the covering $\operatorname{Bun}_{\mathcal{R}^{\prime} / Y}(E) \rightarrow \operatorname{Bun}_{\mathcal{R} / Y}(E)$ to compute the pushforward of $\mathcal{O}$ and then taking $T$-invariants.

To prove (2), note that (1) implies that the functors

$$
\mathcal{P i c} c^{W}(Y) \longrightarrow \mathcal{P i c} c^{W}\left(\widetilde{\operatorname{Bun}}_{G, \text { rig }}^{s, r e g}\right) \quad \text { and } \quad \mathcal{P i c}{ }^{W}(Y) \longrightarrow \mathcal{P i c}^{W}\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)
$$

are fully faithful as enriched functors, and so it is enough to prove that they are essentially surjective.

Fix a $W$-linearised line bundle $L$ on $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}$. Restricting $L$ to $\widetilde{\operatorname{Bun}_{G}^{s s, \text { sreg }}}=\operatorname{Bun}_{T}^{0, \text { sreg }}=$ $Y^{\text {sreg }} \times \mathbb{B} T$ and using the isomorphism

$$
\operatorname{Pic}^{W}\left(Y^{\text {sreg }} \times \mathbb{B} T\right) \cong \operatorname{Pic}^{W}\left(Y^{\text {sreg }}\right) \oplus \mathbb{X}^{*}(T)^{W}=\operatorname{Pic}^{W}\left(Y^{\text {sreg }}\right)
$$

gives a $W$-linearised line bundle on $Y^{\text {sreg }}$, which can be extended (non-uniquely) to a line bundle $L_{0}$ on $Y$. By construction,

$$
L=\left(\left(\bar{\chi}^{\text {reg }}\right)^{*} L_{0}\right)\left(\sum_{i} n_{i} \bar{D}_{i}\right)
$$

where $n_{i} \in \mathbb{Z}$ and $\bar{D}_{i} \subseteq{\widetilde{\operatorname{Bun}_{G}}}^{\text {ss,reg }}$ are irreducible divisors in the complement of $\widetilde{\operatorname{Bun}}_{G}^{\text {ss,sreg }}$. But it is clear from the discussion preceding Lemma 4.3.16 that for any irreducible divisor $D$ in $Y$ in the complement of $Y^{\text {sreg }},\left(\bar{\chi}^{\text {reg }}\right)^{-1}(D)$ is nonempty and irreducible. So we must have $\bar{D}_{i}=\left(\bar{\chi}^{\text {reg }}\right)^{-1}\left(D_{i}\right)$ for some divisors $D_{i}$ on $Y$, and hence $L=\left(\bar{\chi}^{\text {reg }}\right)^{*}\left(L^{\prime}\right)$, where

$$
L^{\prime}=L_{0}\left(\sum_{i} n_{i} D_{i}\right)
$$

Now (1) implies that the $W$-linearisation on $L$ necessarily descends to a $W$-linearisation on $L^{\prime}$, so we have shown that $\mathcal{P i c}^{W}(Y) \rightarrow \mathcal{P} i c^{W}\left({\widetilde{\operatorname{Bun}_{G}}}^{s s, \text { reg }}\right)$ is essentially surjective. An identical argument shows that $\mathcal{P} i c^{W}(Y) \rightarrow \mathcal{P} i c^{W}\left(\widetilde{\operatorname{Bun}}_{\text {G, rig }}^{s, r e g}\right)$ is essentially surjective, so this proves (2).

To prove (3), first note that it is clear from the definitions that a $W$-linearised line bundle on $\widetilde{\mathrm{Bun}}_{G, \text { rig }}^{\text {sser }}$ is good if and only if its pullback to $\widetilde{\mathrm{Bun}}_{G}^{\text {ss,reg }}$ is good. So it suffices to prove that a $W$-linearised line bundle $L$ on $Y$ is good in the sense of Definition 4.3.2 if and only if the $W$-linearised line bundle $\left(\bar{\chi}^{\text {reg }}\right)^{*} L$ on $\widetilde{\mathrm{Bun}}_{G}^{s s, \text { reg }}$ is good in the sense of Definition 4.2.10.

If $w \in W$, then $\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)^{w} \subseteq\left(\bar{\chi}^{\text {reg }}\right)^{-1}\left(Y^{w}\right)$, so either $w=s_{\alpha}$ for some $\alpha \in \Phi_{+}$or $\left(\widetilde{\operatorname{Bun}}_{G}^{s s, r e g}\right)^{w} \subseteq \widetilde{\operatorname{Bun}}_{G}^{s s, r e g}$ has codimension at least 2. It is clear from this and the definitions that if $L$ is good then $\left(\bar{\chi}^{\text {reg }}\right)^{*} L$ is good also. Conversely, suppose that $\left(\bar{\chi}^{\text {reg }}\right)^{*} L$ is good. To show that $L$ is good, it suffices to show that for every $\alpha \in \Phi_{+}$and every generic point $y \in Y^{s_{\alpha}}$ of an irreducible component with codimension 1 in $Y$, the morphism

$$
s_{\alpha}:\left.\left.L\right|_{\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}} \longrightarrow L\right|_{\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}}
$$

is the identity, where $\mathcal{O}_{Y, y}^{h}$ is a strict Henselisation of the local ring $\mathcal{O}_{Y, y}$. We show below that the morphism

$$
\begin{equation*}
\left(\widetilde{\operatorname{Bun}}_{G}^{s s, r e g} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}=\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)^{s_{\alpha}} \times_{Y^{s_{\alpha}}}\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}} \longrightarrow\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}} \tag{4.3.3}
\end{equation*}
$$

admits a section, where the fixed locus on the left is relative to $\mathrm{Bun}_{G}$, from which the result follows since $\left(\bar{\chi}^{\text {reg }}\right)^{*} L$ is good.

We first claim that there exists some section $\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \rightarrow \widetilde{\operatorname{Bun}}_{G}^{\text {ss,reg }} \times{ }_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$. To see this, note that by Lemma 4.3.11, we have

$$
\widetilde{\operatorname{Bun}}_{G}^{s s} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \cong \operatorname{Bun}_{T U_{-\alpha} / \mathcal{O}_{Y, y}^{h}}(E)_{\xi_{T}} / T \cong \operatorname{Bun}_{\mathcal{U}_{-\alpha} / \mathcal{O}_{Y, y}^{h}}(E) / T
$$

where $\xi_{T} \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} E$ is the restriction of the universal $T$-bundle (trivialised along $O_{E}$ ) on $Y \times_{S} E$, and $\mathcal{U}_{-\alpha}=\xi_{T} \times{ }^{T} U_{-\alpha}$. Applying Proposition 2.4.7 to the exact sequence

$$
0 \longrightarrow \mathcal{U}_{-\alpha} \longrightarrow \mathcal{U}_{-\alpha}\left(O_{E}\right) \longrightarrow O_{E *} O_{E}^{*} \mathcal{U}_{-\alpha}\left(O_{E}\right) \longrightarrow 0
$$

of sheaves on $\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times{ }_{S} E$, we have a morphism

$$
\begin{equation*}
O_{E}^{*} \mathcal{U}_{-\alpha}\left(O_{E}\right) \longrightarrow \operatorname{Bun}_{\mathcal{U}_{-\alpha} / \mathcal{O}_{Y, y}^{h}}(E) \longrightarrow{\widetilde{\operatorname{Bun}_{G}}}^{s s} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \tag{4.3.4}
\end{equation*}
$$

given by the action of $O_{E}^{*} \mathcal{U}_{-\alpha}\left(O_{E}\right)=\Gamma_{\text {Spec } \mathcal{O}_{Y, y}^{h}}\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} E, \mathcal{U}_{-\alpha}\left(O_{E}\right) / \mathcal{U}_{-\alpha}\right)$ on the trivial bundle. Since every line bundle on Spec $\mathcal{O}_{Y, y}^{h}$ is trivial, there exists a nonvanishing section of $O_{E}^{*} \mathcal{U}_{-\alpha}\left(O_{E}\right)$, which gives the desired section Spec $\mathcal{O}_{Y, y}^{h} \rightarrow \widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \subseteq$ $\widetilde{\operatorname{Bun}}_{G}^{s s} \times{ }_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$ after composition with (4.3.4). Note that this does indeed factor through $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}$ by Lemma 4.3 .16 since the $U_{-\alpha}$-bundle at the closed point in $\operatorname{Spec} \mathcal{O}_{Y, y}^{h}$ is nontrivial.

Writing $\xi_{B} \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} E$ for the $B$-bundle induced by the section constructed above, we remark that the group scheme $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$ is flat over $\operatorname{Spec} \mathcal{O}_{Y, y}^{h}$. To see this, note that there is a short exact sequence

$$
1 \longrightarrow H \longrightarrow \underline{\operatorname{Aut}}\left(\xi_{B}\right) \longrightarrow \operatorname{ker}\left(\alpha: T \rightarrow \mathbb{G}_{m}\right) \times \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \longrightarrow 1
$$

of group schemes over $\operatorname{Spec} \mathcal{O}_{Y, y}^{h}$, such that the special fibre of $H$ is $\mathbb{G}_{a}$ and the generic fibre is $\mathbb{G}_{m}$. Since both are irreducible of dimension 1, flatness of $H$ and hence Aut $\left(\xi_{B}\right)$ follows since $\mathcal{O}_{Y, y}^{h}$ is a discrete valuation ring. Since every fibre of $\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$ contains exactly one isomorphism class of regular bundles, it follows that the canonical morphism

$$
\begin{equation*}
\mathbb{B}_{\mathrm{Spec} \mathcal{O}_{Y, y}^{h}} \operatorname{Aut}\left(\xi_{B}\right) \longrightarrow{\widetilde{\operatorname{Bun}_{G}}}^{s s, \text { reg }} \times_{Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \tag{4.3.5}
\end{equation*}
$$

is an isomorphism.
In terms of the isomorphism (4.3.5), the automorphism

$$
s_{\alpha}: \widetilde{\operatorname{Bun}}_{G}^{\text {ss,reg }} \times{ }_{Y}\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}} \longrightarrow \widetilde{\operatorname{Bun}}_{G}^{\text {ss,reg }} \times_{Y}\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}
$$

sends an $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$-torsor $\theta$ to $\theta \times \underline{\operatorname{Aut}}\left(\xi_{B}\right) \eta$ for some $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$-torsor $\eta \rightarrow\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}$ under the commutative group scheme $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$, equipped with an additional commuting action of $\underline{\text { Aut }}\left(\xi_{B}\right)$ on the left. Since $s_{\alpha}$ is an automorphism relative to $\operatorname{Bun}_{G}$, we are also given an Aut $\left(\xi_{B}\right)$-equivariant isomorphism of $G$-bundles

$$
g: \eta \times\left.\left.\underline{\operatorname{Aut}\left(\xi_{B}\right)} \xi_{G}\right|_{\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}} \longrightarrow \xi_{G}\right|_{\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}}
$$

where $\xi_{G}=\xi_{B} \times{ }^{B} G$. Since the group scheme $\operatorname{Aut}\left(\xi_{B}\right) \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$ is affine and $\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}=$ Spec $\mathcal{O}_{Y^{s_{\alpha}, y}}^{h}$ is the spectrum of an Artinian local ring with separably closed residue field, the torsor $\eta$ is necessarily trivial. Fixing a trivialisation, $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$-action acts on $\eta$ on the left through a group automorphism $s_{\alpha}: \underline{\operatorname{Aut}}\left(\xi_{B}\right) \rightarrow \underline{\operatorname{Aut}}\left(\xi_{B}\right)$, and the isomorphism $g$ is equivalent to a section

$$
g^{\prime}:\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}} \longrightarrow \underline{\operatorname{Aut}}\left(\xi_{G}\right)
$$

such that $\left(g^{\prime}\right)^{-1}(-) g^{\prime}: \underline{\operatorname{Aut}}\left(\xi_{G}\right) \rightarrow_{s s, r e g}^{\operatorname{Aut}}\left(\xi_{G}\right)$ restricts to $s_{\alpha}: \underline{\operatorname{Aut}}\left(\xi_{B}\right) \rightarrow \underset{s s, \text { reg }}{\operatorname{Aut}}\left(\xi_{B}\right)$. The given section $\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}} \rightarrow \widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }} \times_{Y}\left(\operatorname{Spec} \mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}$ factors through $\left(\widetilde{\operatorname{Bun}}_{G}^{s s, \text { reg }}\right)^{s_{\alpha}} \times_{Y}$ (Spec $\left.\mathcal{O}_{Y, y}^{h}\right)^{s_{\alpha}}$ if and only if $g^{\prime}$ factors through $\underline{\operatorname{Aut}}\left(\xi_{B}\right) \subseteq \underline{\operatorname{Aut}}\left(\xi_{G}\right)$. To complete the proof, it is therefore enough to prove that $\underline{\operatorname{Aut}}\left(\xi_{G}\right)=\underline{\operatorname{Aut}}\left(\xi_{B}\right)$.

Write $\psi^{-1}\left(\xi_{G}\right)=\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times$ Bun $_{G} \widetilde{\operatorname{Bun}}_{G}$. Then by construction there is a section Spec $\mathcal{O}_{Y, y}^{h} \rightarrow \psi^{-1}\left(\xi_{G}\right)$ corresponding to the reduction $\xi_{B}$ of $G$ to $B$ whose stabiliser under the natural action of $\underline{\operatorname{Aut}}\left(\xi_{G}\right)$ is $\underline{\operatorname{Aut}}\left(\xi_{B}\right)$. The morphism $\psi^{-1}\left(\xi_{G}\right) \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times{ }_{S}$ $\widetilde{\operatorname{Bun}}_{G} \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} Y$ is $\underline{\operatorname{Aut}}\left(\xi_{G}\right)$-equivariant for the trivial action on $Y$, so the action on $\psi^{-1}\left(\xi_{G}\right)$ restricts to an action on $\psi^{-1}\left(\xi_{G}\right) \times_{\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times{ }_{S} Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$. We claim that $\psi^{-1}\left(\xi_{G}\right) \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} Y$ is a closed immersion. Given the claim, it follows that $\psi^{-1}\left(\xi_{G}\right) \times_{\operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times{ }_{S} Y} \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$ is also a closed immersion, and hence an isomorphism since it has a section. So $\underline{\text { Aut }}\left(\xi_{G}\right)$ must stabilise the natural section of $\psi^{-1}\left(\xi_{G}\right)$, which proves that $\underline{\operatorname{Aut}}\left(\xi_{G}\right)=\underline{\operatorname{Aut}}\left(\xi_{B}\right)$.

It remains to prove the claim that $\psi^{-1}\left(\xi_{G}\right) \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h} \times_{S} Y$ is a closed immersion. Since it separates points over the special fibre by Lemma 4.3.16 and $W$-equivariance, it suffices to show that the restriction to special fibres is unramified. This is equivalent to the claim that for every $\xi_{B}^{\prime}$ in the special fibre of $\psi^{-1}\left(\xi_{G}\right) \rightarrow \operatorname{Spec} \mathcal{O}_{Y, y}^{h}$, the morphism

$$
\begin{equation*}
H^{0}\left(E_{s}, \xi_{B}^{\prime} \times^{B} \mathfrak{g} / \mathfrak{b}\right) \longrightarrow H^{1}\left(E_{s}, \mathfrak{t} \otimes \mathcal{O}_{E_{s}}\right) \tag{4.3.6}
\end{equation*}
$$

induced by the extension of $B$-modules

$$
0 \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{g} / R_{u}(\mathfrak{b}) \longrightarrow \mathfrak{g} / \mathfrak{b} \longrightarrow 0
$$

is injective. The proof of Lemma 4.3.16 shows that there exists $w \in W$ such that $w^{-1} \alpha \in \Phi_{+}$ and $\xi_{B}^{\prime}$ lies over $w^{-1} y \in Y$. It follows that $\xi_{B}^{\prime}$ is induced from a $T U_{-w^{-1} \alpha}$-bundle $\xi_{T U_{-w^{-1} \alpha}}$ such that $\xi_{T U_{-w^{-1} \alpha}}$ has nontrivial associated $U_{-w^{-1} \alpha}$-bundle. So writing $\beta=w^{-1} \alpha$, we can identify (4.3.6) with the morphism

$$
\begin{equation*}
H^{0}\left(E_{s}, \xi_{T U_{-\beta}} \times^{T U_{-\beta}} \mathfrak{u}_{\beta}\right) \longrightarrow H^{1}\left(E_{s}, \mathfrak{t} \otimes \mathcal{O}_{E_{s}}\right) \tag{4.3.7}
\end{equation*}
$$

induced by the extension

$$
0 \longrightarrow \mathfrak{t} \longrightarrow\left(\mathfrak{t}+\rho_{\beta}\left(\mathfrak{s l}_{2}\right)\right) / \mathfrak{u}_{-\beta} \longrightarrow \mathfrak{u}_{\beta} \longrightarrow 0
$$

of $T U_{-\beta}$-modules. But we can write $\mathfrak{t}=\mathfrak{t}^{\prime} \oplus \beta^{\vee}(\mathbb{Z})$ so that the induced extension

$$
0 \longrightarrow \beta^{\vee}(\mathbb{Z}) \longrightarrow V \longrightarrow \mathfrak{u}_{\beta} \longrightarrow 0
$$

is the canonical non-split one, so injectivity of (4.3.7) follows. This proves the claim, and hence the proposition.

### 4.4 The theta bundle

In this section, we use Theorem 4.3.4 to compute $\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)$ by computing the group $\operatorname{Pic}^{W}(Y)_{\text {good }}$. The computations show that $\mathrm{Pic}^{W}(Y)_{\text {good }}$ is generated by $\operatorname{Pic}(S)$ and a single ample line bundle $\Theta_{Y}$. This corresponds to a canonical line bundle $\Theta_{\operatorname{Bun}_{G}}$ on $\mathrm{Bun}_{G}$, which we call the theta bundle.

Definition 4.4.1. Let $L$ be any line bundle on $Y$. The quadratic class of $L$ is the function

$$
\begin{aligned}
q(L): \mathbb{X}_{*}(T) & \longrightarrow \mathbb{Z} \\
\lambda & \longmapsto \operatorname{deg}_{\operatorname{Pic}^{0}\left(E_{s}\right)}\left(\lambda^{*} L\right)
\end{aligned}
$$

where the degree is taken over any geometric fibre of the relative Picard scheme $\operatorname{Pic}_{S}^{0}(E) \rightarrow$ $S$.

The following lemma motivates the terminology "quadratic class".
Lemma 4.4.2. Let $L$ be a line bundle on $Y$. Then $q(L)$ is a quadratic form, i.e., $q(L)(-\lambda)=$ $q(L)(\lambda)$ for all $\lambda \in \mathbb{X}_{*}(T)$ and the map

$$
\begin{aligned}
Q(L): \mathbb{X}_{*}(T) \times \mathbb{X}_{*}(T) & \longrightarrow \mathbb{Q} \\
(\lambda, \mu) & \longmapsto \frac{1}{2}(q(L)(\lambda+\mu)-q(L)(\lambda)-q(L)(\mu))
\end{aligned}
$$

is symmetric and bilinear.
Proof. Since $q(L)$ is computed on a geometric fibre, we can assume for simplicity that $S=\operatorname{Spec} k$ for some algebraically closed field $k$. We have

$$
q(L)(-\lambda)=\operatorname{deg}_{\operatorname{Pic}^{0}(E)}(-\lambda)^{*} L=\operatorname{deg}_{\operatorname{Pic}^{0}(E)}[-1]^{*} \lambda^{*} L=\operatorname{deg}_{\operatorname{Pic}^{0}(E)} \lambda^{*} L
$$

since $[-1]: \operatorname{Pic}^{0}(E) \rightarrow \operatorname{Pic}^{0}(E)$ has degree 1 .

For the map $Q(L)$, symmetry is obvious. Bilinearity is equivalent to $Q(L)(0,0)=0$, which is true by inspection, and the statement that for all $\lambda, \mu, \nu \in \mathbb{X}_{*}(T)$ the line bundle on $\operatorname{Pic}^{0}(E)$

$$
(\lambda+\mu+\nu)^{*} L \otimes(\lambda+\mu)^{*} L^{-1} \otimes(\lambda+\nu)^{*} L^{-1} \otimes(\mu+\nu)^{*} L^{-1} \otimes \lambda^{*} L \otimes \mu^{*} L \otimes \nu^{*} L
$$

has degree 0 . But this line bundle is trivial by the theorem of the cube [M, $\S I I .6]$, so we are done.

Remark 4.4.3. Let $L$ be a line bundle on $Y$. Note that Lemma 4.4.2 implies that $q(L)(0)=$ $-Q(L)(0,0)=0$, and hence

$$
q(L)(\lambda)=-\frac{1}{2}(q(L)(\lambda-\lambda)-q(L)(\lambda)-q(L)(-\lambda))=-Q(L)(\lambda,-\lambda)=Q(L)(\lambda, \lambda)
$$

So the datum of the function $q(L): \mathbb{X}_{*}(T) \rightarrow \mathbb{Z}$ is equivalent to the datum of the bilinear form $Q(L) \in \operatorname{Hom}\left(\operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right), \mathbb{Q}\right)$. For this reason, we will often refer to $Q(L)$ as the quadratic class of $L$.

Remark 4.4.4. One might hope that the quadratic class $q(L)$ determines the first Chern class $c_{1}(L)$. This is not true in general: for example, it fails for the elliptic curve $E=$ $\mathbb{C} /(\mathbb{Z}+i \mathbb{Z})$ over $S=\operatorname{Spec} \mathbb{C}$, the group $G=S L_{3}$ and the line bundle $L$ constructed as follows. Identify $\operatorname{Pic}^{0}(E)$ with $E$ and hence $Y$ with $E \times E$ via the canonical principal polarisation. Let $P$ be the line bundle on $E \times E$ defining the polarisation on $E$, and let $L=(\mathrm{id}, i)^{*} P$ be the pullback of $P$ under the automorphism (id, $\left.i\right): E \times E \rightarrow E \times E$. Then $q(L)=0$ but $c_{1}(L) \neq 0$.

Lemma 4.4.5. Let $L$ be a good $W$-linearised line bundle on $Y$. Then $Q(L)$ lies in the image of

$$
\operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W} \subseteq \operatorname{Hom}\left(\operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right), \mathbb{Z}\right) \subseteq \operatorname{Hom}\left(\operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right), \mathbb{Q}\right)
$$

under the inclusion sending $\lambda \mu \in \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)$ to the bilinear map

$$
\begin{aligned}
\mathbb{X}_{*}(T) \times \mathbb{X}_{*}(T) & \longrightarrow \mathbb{Z} \\
\left(\lambda^{\prime}, \mu^{\prime}\right) & \longmapsto\left\langle\lambda, \lambda^{\prime}\right\rangle\left\langle\mu, \mu^{\prime}\right\rangle+\left\langle\lambda, \mu^{\prime}\right\rangle\left\langle\mu, \lambda^{\prime}\right\rangle
\end{aligned}
$$

Proof. As in Lemma 4.4.2, we may assume $S=\operatorname{Spec} k$. Since $Q(L)$ is manifestly $W$ invariant, by elementary linear algebra, it is enough to show that if $\alpha_{i}^{\vee}, \alpha_{j}^{\vee} \in \Delta^{\vee}$ are simple coroots, then $Q(L)\left(\alpha_{i}^{\vee}, \alpha_{i}^{\vee}\right) \in 2 \mathbb{Z}$ and $Q(L)\left(\alpha_{i}^{\vee}, \alpha_{j}^{\vee}\right) \in \mathbb{Z}$.

If $\alpha_{i}^{\vee} \in \Delta^{\vee}$, then Lemma 4.4.6 below implies that

$$
Q(L)\left(\alpha_{i}^{\vee}, \alpha_{i}^{\vee}\right)=\operatorname{deg}_{\operatorname{Pic}^{0}(E)}\left(\alpha_{i}^{\vee}\right)^{*} L \in 2 \mathbb{Z}
$$

If $\alpha_{j}^{\vee} \in \Delta^{\vee}$ is another simple coroot, then invariance of $q(L)$ and hence $Q(L)$ under $s_{i} \in W$ implies that

$$
Q(L)\left(\alpha_{i}^{\vee}, \alpha_{j}^{\vee}\right)=\frac{Q(L)\left(\alpha_{i}^{\vee}, \alpha_{i}^{\vee}\right)}{2}\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle \in \mathbb{Z}
$$

so we are done.

In the following lemma, we write $O_{\operatorname{Pic}_{S}^{0}(E)}$ for the origin in $\operatorname{Pic}_{S}^{0}(E)$, in order to distinguish it from the zero divisor 0 . For the sake of clarity, we will also write $O_{Y}: S \rightarrow Y$ for the section corresponding to the trivial $T$-bundle.

Lemma 4.4.6. Let $L$ be a $W$-linearised line bundle on $Y$ such that the action of $W$ on $O_{Y}^{*} L$ is trivial. Then the $W$-linearisation on $L$ is good if and only if for all simple coroots $\alpha_{i}^{\vee},\left(\alpha_{i}^{\vee}\right)^{*} L \cong \pi_{\mathrm{Pic}_{S}^{0}(E)}^{*} L^{\prime}\left(2 d \cdot O_{\mathrm{Pic}_{S}^{0}(E)}\right)$ for some $d \in \mathbb{Z}$ and some line bundle $L^{\prime}$ on $S$.
Proof. Since every reflection in $W$ is conjugate to a simple reflection, the line bundle $L$ is good if and only if

$$
s_{i}:\left.\left.L\right|_{Y^{s_{i}}} \longrightarrow L\right|_{Y^{s_{i}}}
$$

is the identity for all simple reflections $s_{i}$.
Fix a simple reflection $s_{i}$. Observe that since $\mathbb{X}_{*}(T)_{\mathbb{Q}}^{s_{i}}+\mathbb{Q} \alpha_{i}^{\vee}=\mathbb{X}_{*}(T)_{\mathbb{Q}}$, the morphism $f: \operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}} \times_{S} \operatorname{Pic}_{S}^{0}(E)=\operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}}\left(\mathbb{X}_{*}(T)^{s_{i}}+\mathbb{Z} \alpha_{i}^{\vee}\right) \longrightarrow \operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)=Y$ is an isogeny of abelian varieties over $S$, and $f \circ s_{i}=r \circ f$, where

$$
r=(\operatorname{id},[-1]): \operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}} \times_{S} \operatorname{Pic}_{S}^{0}(E) \longrightarrow \operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}} \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)
$$

Since $r$ acts trivially on $\operatorname{ker}(f)$, it follows that

$$
f^{-1}\left(Y^{s_{i}}\right)=\operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}} \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)[2]
$$

and so the action of $s_{i}$ on $\left.L\right|_{Y^{s_{i}}}$ is trivial if and only if the action of $r$ on

$$
\left.f^{*} L\right|_{\operatorname{Pic}_{S}^{0}(E) \otimes \mathbb{X}_{*}(T)^{s_{i}} \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)[2]}
$$

is trivial. Since the action of $r$ is given by a global regular function on $\operatorname{Pic}_{S}^{0}(E) \otimes \mathbb{X}_{*}(T)^{s_{i}} \times{ }_{S}$ $\operatorname{Pic}_{S}^{0}(E)[2]$, which is necessarily pulled back from a regular function on $\operatorname{Pic}_{S}^{0}(E)[2]$, it suffices to check that $r$ acts as the identity on the fibre of

$$
\operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}} \times_{S} \operatorname{Pic}_{S}^{0}(E)[2] \longrightarrow \operatorname{Pic}_{S}^{0}(E) \otimes_{\mathbb{Z}} \mathbb{X}_{*}(T)^{s_{i}}
$$

over any section of the structure map to $S$. Taking the fibre over the natural origin, the restriction of $f$ here is $\alpha_{i}^{\vee}: \operatorname{Pic}_{S}^{0}(E) \rightarrow Y$. So by Lemma 4.4.7 below, $r$ acts as the identity if and only if $\left(\alpha_{i}^{\vee}\right)^{*} L=\pi_{\mathrm{Pic}_{S}^{0}(E)}^{*} L^{\prime}\left(2 d \cdot O_{\operatorname{Pic}_{S}^{0}(E)}\right)$, which proves the lemma.

Lemma 4.4.7. Let $L$ be a line bundle on $\operatorname{Pic}_{S}^{0}(E)$ with $[-1]^{*} L \cong L$, and let $\sigma:[-1]^{*} L \rightarrow L$ be the unique isomorphism acting as the identity on the pullback of $L$ along the section $O_{\operatorname{Pic}_{S}^{0}(E)}: S \rightarrow \operatorname{Pic}_{S}^{0}(E)$ corresponding to the trivial line bundle on $E$. Then $\sigma$ acts as the identity on $\left.L\right|_{\operatorname{Pic}_{S}^{0}(E)[2]}$ if and only if $L=\pi^{*} L^{\prime} \otimes \mathcal{O}\left(2 d \cdot O_{\operatorname{Pic}_{S}^{0}(E)}\right)$ for some $d \in \mathbb{Z}$ and some line bundle $L^{\prime}$ on $S$, where $\pi: \operatorname{Pic}_{S}^{0}(E) \rightarrow S$ is the structure morphism.

Proof. The morphism $\sigma$ acts as the identity on $\left.L\right|_{\operatorname{Pic}_{S}^{0}(E)[2]}$ if and only if $L=f^{*} L^{\prime \prime}$ for some line bundle $L^{\prime \prime}$ on the $\mathbb{P}^{1}$-bundle $\mathbb{P}=\mathbb{P}_{S}\left(\pi_{*} \mathcal{O}\left(2 \cdot O_{\operatorname{Pic}_{S}^{0}(E)}\right)\right)^{\vee}$ over $S$, where $f$ is the canonical morphism. But every line bundle $L^{\prime \prime}$ on $\mathbb{P}$ is of the form $L^{\prime \prime}=\pi_{\mathbb{P}}^{*} L^{\prime}\left(d \cdot f\left(O_{\operatorname{Pic}_{S}^{0}(E)}\right)\right)$ for some $d \in \mathbb{Z}$ and some line bundle $L^{\prime}$ on $S$, so $f^{*} L^{\prime \prime}=\pi^{*} L^{\prime}\left(2 d \cdot O_{\operatorname{Pic}_{S}^{0}(E)}\right)$ and we are done.

Lemma 4.4.8. Let $M$ be a finitely generated free abelian group. Then the abelian group $\operatorname{Sym}^{2}(M)^{\vee}=\operatorname{Hom}_{\mathbb{Z}}\left(\operatorname{Sym}^{2}(M), \mathbb{Z}\right)$ is generated by

$$
\begin{aligned}
\lambda^{2}: \operatorname{Sym}^{2}(M) & \longrightarrow \mathbb{Z} \\
m n & \longmapsto \lambda(m) \lambda(n)
\end{aligned}
$$

for $\lambda \in M^{\vee}$, with relations

$$
\lambda^{2}=(-\lambda)^{2} \quad \text { and } \quad(\lambda+\mu+\nu)^{2}-(\lambda+\mu)^{2}-(\lambda+\nu)^{2}-(\mu+\nu)^{2}+\lambda^{2}+\mu^{2}+\nu^{2}=0
$$

for $\lambda, \mu, \nu \in M^{\vee}$.

Proof. Choose a basis $e_{1}, \ldots, e_{n}$ for $M$. Then $\operatorname{Sym}^{2}(M)$ has basis $\left\{e_{i j}=e_{i} e_{j} \mid 1 \leq i \leq\right.$ $j \leq n\}$, so $\operatorname{Sym}^{2}(M)^{\vee}$ has basis $\left\{e_{i j}^{*} \mid 1 \leq i \leq j \leq n\right\}$, where $e_{i j}^{*}\left(e_{i^{\prime} j^{\prime}}\right)=\delta_{i, i^{\prime}} \delta_{j, j^{\prime}}$. Since $e_{i i}^{*}=\left(e_{i}^{*}\right)^{2}$ and $e_{i j}^{*}=\left(e_{i}^{*}+e_{j}^{*}\right)^{2}-\left(e_{i}^{*}\right)^{2}-\left(e_{j}^{*}\right)^{2}$, this shows that $\operatorname{Sym}^{2}(M)^{\vee}$ is generated by $\lambda^{2}$ for $\lambda \in M^{\vee}$.

Now let $N$ be the abelian group with generators $[\lambda]^{2}$ for $\lambda \in M^{\vee}$, and relations

$$
\begin{equation*}
[\lambda]^{2}=[-\lambda]^{2} \quad \text { and } \quad[\lambda+\mu+\nu]^{2}-[\lambda+\mu]^{2}-[\lambda+\nu]^{2}-[\mu+\nu]^{2}+[\lambda]^{2}+[\mu]^{2}+[\nu]^{2}=0 \tag{4.4.1}
\end{equation*}
$$

for all $\lambda, \mu, \nu \in M^{\vee}$. We have shown that the map $N \rightarrow \operatorname{Sym}^{2}(M)^{\vee}$ sending $[\lambda]^{2}$ to $\lambda^{2}$ is surjective. To show that it is injective, let $P$ be the span of $\left[e_{i j}^{*}\right] \in N$ for $1 \leq i \leq j \leq n$, where

$$
\left[e_{i i}^{*}\right]=\left[e_{i}^{*}\right]^{2} \quad \text { and } \quad\left[e_{i j}^{*}\right]=\left[e_{i}^{*}+e_{j}^{*}\right]^{2}-\left[e_{i}^{*}\right]^{2}-\left[e_{j}^{*}\right]^{2}
$$

for $1 \leq i<j \leq n$. The morphism $P \rightarrow \operatorname{Sym}^{2}(M)^{\vee}$ is injective, since it has a retraction

$$
\begin{aligned}
\operatorname{Sym}^{2}(M)^{\vee} & \longrightarrow P \\
Q & \longmapsto \sum_{1 \leq i \leq j \leq n} Q\left(e_{i}, e_{j}\right)\left[e_{i j}^{*}\right] .
\end{aligned}
$$

So it suffices to prove that $P=N$, i.e., that for all $\lambda \in M^{\vee},[\lambda]^{2} \in P$.
Writing $\lambda=\sum_{i=1}^{n} a_{i} e_{i}^{*}, a_{i} \in \mathbb{Z}$, we have tautologically that

$$
[\lambda]^{2}=\sum_{i=1}^{n}\left(\left[a_{i} e_{i}^{*}\right]^{2}+R\left(a_{i} e_{i}^{*}, \sum_{i<j} a_{j} e_{j}^{*}\right)\right)
$$

where

$$
R(\lambda, \mu)=[\lambda+\mu]^{2}-[\lambda]^{2}-[\mu]^{2},
$$

for $\lambda, \mu \in M^{\vee}$. The relations for $N$ imply that $R: M^{\vee} \times M^{\vee} \rightarrow N$ is symmetric and bilinear, so

$$
[\lambda]^{2}=\sum_{i=1}^{n}\left[a_{i} e_{i}^{*}\right]^{2}+\sum_{i<j} a_{i} a_{j} R\left(e_{i}^{*}, e_{j}^{*}\right)=\sum_{i=1}^{n}\left[a_{i} e_{i}^{*}\right]^{2}+\sum_{i<j} a_{i} a_{j}\left[e_{i j}^{*}\right] .
$$

So it remains to show that $\left[a e_{i}^{*}\right]^{2}$ is in $P$ for all $i$ and all $a \in \mathbb{Z}$. We note that

$$
R\left(e_{i}^{*}, e_{i}^{*}\right)=-R\left(e_{i}^{*},-e_{i}^{*}\right)=-[0]^{2}+\left[e_{i}^{*}\right]^{2}+\left[-e_{i}^{*}\right]^{2}=2\left[e_{i}^{*}\right]^{2},
$$

and hence that

$$
\left[(a+1) e_{i}^{*}\right]^{2}=\left[a e_{i}^{*}\right]^{2}+\left[e_{i}^{*}\right]^{2}+R\left(a e_{i}^{*}, e_{i}^{*}\right)=\left[a e_{i}^{*}\right]^{2}+(2 a+1)\left[e_{i}^{*}\right]^{2}
$$

for all $a \in \mathbb{Z}$. Since $[0]^{2}=0$ (set $\lambda=\mu=\nu=0$ in the second relation of (4.4.1)), we see by induction that $\left[a e_{i}^{*}\right]^{2}=a^{2}\left[e_{i}^{*}\right]^{2} \in P$ for all $a \in \mathbb{Z}$, so we are done.

Proposition 4.4.9. The homomorphism

$$
\begin{equation*}
\left(Q, O_{Y}^{*}\right): \operatorname{Pic}^{W}(Y)_{\mathrm{good}} \longrightarrow \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W} \oplus \operatorname{Pic}(S) \tag{4.4.2}
\end{equation*}
$$

is an isomorphism of abelian groups.
Proof. We show that (4.4.2) is both injective and surjective.
For injectivity, suppose that $L$ is a good $W$-linearised line bundle on $Y$ such that $Q(L)=$ 0 and $O_{Y}^{*} L \cong \mathcal{O}_{S}$. To show that $L$ is trivial, it is enough to show that $L$ is trivial on every geometric fibre of $Y \rightarrow S$, since this implies by Grauert's Theorem that $L$ is pulled back
from a line bundle on $S$. So we can assume for this part of the proof that $S=\operatorname{Spec} k$ for some algebraically closed field $k$.

We first claim that for any two simple coroots $\alpha_{i}^{\vee}$ and $\alpha_{j}^{\vee}$, the pullback $L^{\prime}=\left(\alpha_{i}^{\vee}, \alpha_{j}^{\vee}\right)^{*} L$ of $L$ under

$$
\left(\alpha_{i}^{\vee}, \alpha_{j}^{\vee}\right): \operatorname{Pic}^{0}(E) \times \operatorname{Pic}^{0}(E) \longrightarrow Y
$$

is trivial. To see this, it suffices to show that $\left(\alpha_{j}^{\vee}\right)^{*} L=\mathcal{O}$, and that $L^{\prime}$ is trivial restricted to every $k$-fibre of the second projection $\operatorname{Pic}^{0}(E) \times \operatorname{Pic}^{0}(E) \rightarrow \operatorname{Pic}^{0}(E)$. To see the first condition, apply Lemma 4.4.7 to the morphism

$$
[-1]^{*}\left(\alpha_{j}^{\vee}\right)^{*} L=\left(\alpha_{j}^{\vee}\right)^{*} s_{j}^{*} L \xrightarrow{\sim}\left(\alpha_{j}^{\vee}\right)^{*} L,
$$

and use the fact that $\left(\alpha_{j}^{\vee}\right)^{*} L$ has degree $q(L)\left(\alpha_{j}^{\vee}\right)=0$. For the second, let $x_{2} \in \operatorname{Pic}^{0}(E)$ be a $k$-point, and consider the restriction $L_{x_{2}}^{\prime}$ of $L^{\prime}$ to $\operatorname{Pic}^{0}(E) \times\left\{x_{2}\right\}$. Define $\sigma: \operatorname{Pic}^{0}(E) \rightarrow$ $\operatorname{Pic}^{0}(E)$ by

$$
\sigma\left(x_{1}\right)=-x_{1}-\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle x_{2}
$$

Then the diagram

commutes, where $i_{x_{2}}$ is given by $i_{x_{2}}\left(x_{1}\right)=\alpha_{i}^{\vee}\left(x_{1}\right)+\alpha_{j}^{\vee}\left(x_{2}\right)$. So the isomorphism $s_{i}^{*} L \rightarrow L$ gives an isomorphism $\sigma^{*} L_{x_{2}}^{\prime} \rightarrow L_{x_{2}}^{\prime}$ acting as the identity on $\operatorname{Pic}^{0}(E)^{\sigma}$. But since $k$ is algebraically closed, $\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle x_{2}$ has a square root in $E(k)$, so $\sigma$ is a conjugate of [ -1 ] by a translation. So we can apply Lemma 4.4.7 to conclude that $L_{x_{2}}^{\prime}=\mathcal{O}$. So $L^{\prime}$ is trivial as claimed.

To complete the proof of injectivity, we need to show that in fact $L=\mathcal{O}_{Y}$. We prove by induction on $n \in \mathbb{Z}_{>0}$ that for all $i_{1}, \ldots, i_{n} \in\{1, \ldots, l\}$, the line bundle

$$
L^{\prime}=\left(\alpha_{i_{1}}^{\vee}, \ldots, \alpha_{i_{n}}^{\vee}\right)^{*} L
$$

on $\operatorname{Pic}^{0}(E)^{n}$ is trivial. We have shown this for $n=1$ or 2 . For $n>2$, we write $\operatorname{Pic}^{0}(E)^{n}=$ $\operatorname{Pic}^{0}(E)^{n-2} \times \operatorname{Pic}^{0}(E) \times \operatorname{Pic}^{0}(E)$, and observe that by induction, the restrictions of $L^{\prime}$ to $\operatorname{Pic}^{0}(E)^{n-2} \times \operatorname{Pic}^{0}(E) \times\left\{O_{\operatorname{Pic}^{0}(E)}\right\}, \operatorname{Pic}^{0}(E)^{n-2} \times\left\{O_{\operatorname{Pic}^{0}(E)}\right\} \times \operatorname{Pic}^{0}(E)$ and $\left\{O_{\operatorname{Pic}^{0}(E)}\right\}^{n} \times$ $\operatorname{Pic}^{0}(E) \times \operatorname{Pic}^{0}(E)$ are all trivial. So $L^{\prime}$ is trivial as claimed by the theorem of the cube. Setting $n=l$ and $\left\{i_{1}, \ldots, i_{n}\right\}=\{1, \ldots, l\}$, we conclude that $L$ is trivial, and hence that (4.4.2) is injective.

To prove surjectivity, we first claim that there is a homomorphism

$$
\phi: \operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right)^{\vee} \longrightarrow \operatorname{Pic}(Y)
$$

sending $\lambda^{2} \in \operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right)^{\vee}$ to the line bundle

$$
\phi\left(\lambda^{2}\right)=\lambda^{*} \mathcal{O}\left(O_{\operatorname{Pic}_{S}^{0}(E)}\right) \otimes \pi_{Y}^{*} O_{\operatorname{Pic}_{S}^{0}(E)}^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)
$$

for $\lambda \in \mathbb{X}^{*}(T)$. By Lemma 4.4.8, it suffices to check that for all $\lambda, \mu, \nu \in \mathbb{X}^{*}(T)$, we have

$$
\phi\left(\lambda^{2}\right)=\phi\left((-\lambda)^{2}\right)
$$

and
$\phi\left((\lambda+\mu+\nu)^{2}\right) \otimes \phi\left((\lambda+\mu)^{2}\right)^{-1} \otimes \phi\left((\lambda+\nu)^{2}\right)^{-1} \otimes \phi\left((\mu+\nu)^{2}\right)^{-1} \otimes \phi\left(\lambda^{2}\right) \otimes \phi\left(\mu^{2}\right) \otimes \phi\left(\nu^{2}\right)=\mathcal{O}_{Y}$.

Since it is clear that $O_{Y}^{*} \phi\left(\lambda^{2}\right)=\mathcal{O}_{S}$ for every $\lambda \in \mathbb{X}^{*}(T)$, it suffices to check these relations on every geometric fibre over $S$. The first holds since $[-1]^{*} \mathcal{O}\left(O_{\operatorname{Pic}_{S}^{0}(E)}\right)=\mathcal{O}\left(O_{\operatorname{Pic}_{S}^{0}(E)}\right)$ and the second holds by the theorem of the cube, so the homomorphism $\phi$ is indeed well-defined. By construction, it is also clear that $Q(\phi(P))=P$ for all $P \in \operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right)^{\vee}$.

Assume now that $P \in \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W} \subseteq \operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right)^{\vee}$ and $L \in \operatorname{Pic}(S)$. We need to find a good $W$-linearised line bundle $L^{\prime}$ on $Y$ such that $Q\left(L^{\prime}\right)=P$ and $O_{Y}^{*} L^{\prime}=L$. Note that since the homomorphism $\phi$ is $W$-equivariant by construction, the line bundle $\phi(P)$ is $W$-invariant, so $w^{*} \phi(P) \cong \phi(P)$ for all $w \in W$. We can turn this into a $W$-linearisation by taking

$$
w^{*}: w^{*} \phi(P) \xrightarrow{\sim} \phi(P)
$$

to be the unique isomorphism acting as the identity on $O_{Y}^{*} \phi(P)$. We let $L^{\prime}=\phi(P) \otimes \pi_{Y}^{*} L$. It is clear that $Q\left(L^{\prime}\right)=P$ and $O_{Y}^{*} L^{\prime}=L$, so it remains to show that $\phi(P)$, and hence $L^{\prime}$, is good. By Lemma 4.4.6, it suffices to show that for every simple coroot $\alpha_{i}^{\vee}$ and every geometric point $s: \operatorname{Spec} k \rightarrow S$, we have

$$
\left.\left(\alpha_{i}^{\vee}\right)^{*} \phi(P)\right|_{\mathrm{Pic}^{0}\left(E_{s}\right)}=\mathcal{O}\left(2 d \cdot O_{\operatorname{Pic}^{0}\left(E_{s}\right)}\right)
$$

for some $d \in \mathbb{Z}$. But by construction, it is clear that

$$
\left.\left(\alpha_{i}^{\vee}\right)^{*} \phi(P)\right|_{\mathrm{Pic}^{0}\left(E_{s}\right)}=\mathcal{O}\left(P\left(\alpha_{i}^{\vee}, \alpha_{i}^{\vee}\right) \cdot O_{\mathrm{Pic}^{0}\left(E_{s}\right)}\right)
$$

and $P\left(\alpha_{i}^{\vee}, \alpha_{i}^{\vee}\right) \in 2 \mathbb{Z}$ since $P \in \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right) \subseteq \operatorname{Sym}^{2}\left(\mathbb{X}_{*}(T)\right)^{\vee}$. So $\phi(P)$ is good, and hence (4.4.2) is surjective as claimed.

Proposition 4.4.9 allows us to compute the $\operatorname{Picard}$ group $\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)$. In the statement below, we write $(\mid) \in \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W}$ for the normalised Killing form. This is the unique $W$-invariant symmetric bilinear form on $\mathbb{X}_{*}(T)$ satisfying $\left(\alpha^{\vee} \mid \alpha^{\vee}\right)=2$ for $\alpha^{\vee}$ a short coroot.

Corollary 4.4.10. The Picard group of $\operatorname{Bun}_{G}$ is

$$
\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)=\mathbb{Z}\left[\Theta_{\operatorname{Bun}_{G}}\right] \oplus \operatorname{Pic}(S)
$$

where $\Theta_{\operatorname{Bun}_{G}}$ is the unique line bundle satisfying

$$
\begin{equation*}
\left.\psi^{*}\left(\Theta_{\operatorname{Bun}_{G}}\right)\right|_{\operatorname{Bun}_{G}^{s s}} ^{s,}=\left.\bar{\chi}^{*} \Theta_{Y}\right|_{\operatorname{Bun}_{G}^{s}} ^{s} \tag{4.4.3}
\end{equation*}
$$

where $\Theta_{Y}$ is the unique good $W$-linearised line bundle on $Y$ with $Q\left(\Theta_{Y}\right)=(\mid)$ and $O_{Y}^{*} \Theta_{Y}=$ $\mathcal{O}_{S}$. Moreover, there is an isomorphism of graded $\mathcal{O}_{S}$-algebras

$$
\bigoplus_{d \geq 0} \pi_{\operatorname{Bun}_{G *}} \Theta_{\operatorname{Bun}_{G}}^{\otimes d}=\bigoplus_{d \geq 0}\left(\pi_{Y *} \Theta_{Y}^{\otimes d}\right)^{W} .
$$

Proof. It is an elementary and well-known observation that since $G$ is simply connected and simple, we have $\operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W}=\mathbb{Z}(\mid)$. So Proposition 4.4.9 gives

$$
\operatorname{Pic}^{W}(Y)_{\text {good }}=\mathbb{Z}\left[\Theta_{Y}\right] \oplus \operatorname{Pic}(S)
$$

and hence Theorem 4.3.4 gives

$$
\operatorname{Pic}\left(\operatorname{Bun}_{G}\right)=\mathbb{Z}\left[\Theta_{\operatorname{Bun}_{G}}\right] \oplus \operatorname{Pic}(S),
$$

where $\Theta_{\operatorname{Bun}_{G}}$ is the image of $\Theta_{Y}$ under the Chevalley isomorphism. The construction of the Chevalley isomorphism shows that (4.4.3) is satisfied. Moreover, if $L$ is any other line bundle on $\operatorname{Bun}_{G}$ satisfying $\left.\psi^{*} L\right|_{\operatorname{Bun}_{G}^{s s}} ^{s}=\left.\bar{\chi}^{*} \Theta_{Y}\right|_{\operatorname{Bun}_{G}^{s,}} ^{s,}$, then writing $L$ as the image of a good
$W$-linearised line bundle $L_{Y}$ under the Chevalley isomorphism, we must have $L_{Y} \cong \Theta_{Y}$ as line bundles on $Y$ by Proposition 4.3.17, and hence as $W$-linearised ones. So (4.4.3) indeed characterises $\Theta_{\text {Bun }_{G}}$. Finally, the isomorphism of graded algebras follows immediately from the fact that the Chevalley isomorphism is an equivalence of enriched symmetric monoidal categories.

Remark 4.4.11. Of course, Corollary 4.4.10 also holds with $\mathrm{Bun}_{G, \text { rig }}$ in place of $\mathrm{Bun}_{G}$, with the same proof.

Remark 4.4.12. When $S=\operatorname{Spec} k$, Corollary 4.4 .10 shows that $\operatorname{Pic}\left(\operatorname{Bun}_{G}\right) \cong \mathbb{Z}$, recovering a special case of a theorem of Y. Laszlo and C. Sorger [LS]. We remark that Laszlo and Sorger's proof is very different to ours: it uses the uniformisation of $\mathrm{Bun}_{G}$ by an affine Grassmannian, rather than our method using the relation to the abelian variety $Y$.

We conclude by remarking on the following very basic property of the line bundle $\Theta_{Y}$.
Proposition 4.4.13. The line bundle $\Theta_{Y}$ is ample relative to $S$.
Proof. Since $Y \rightarrow S$ is proper and flat, it suffices to prove that $\Theta_{Y}$ is ample on every geometric fibre. So we can assume for the proof that $S=\operatorname{Spec} k$ for $k$ an algebraically closed field.

Since $Y$ is an abelian variety, $Y$ is projective over $k$, so there exists some ample line bundle $L$ on $Y$. The ample line bundle $L^{\prime}=\bigotimes_{w \in W} w^{*} L$ is canonically $W$-linearised, and it is easy to see that the $W$-linearisation on $L^{\prime \prime}=\left(L^{\prime}\right)^{\otimes 2}$ is therefore good. So $L^{\prime \prime}$ is a good $W$-linearised ample line bundle on $Y$, and therefore a positive multiple of $\Theta_{Y}$ by Proposition 4.4.9. Hence $\Theta_{Y}$ is ample as claimed.

### 4.5 The coarse quotient map and the fundamental diagram

In this section, we apply Corollary 4.4.10 to construct the elliptic Grothendieck-Springer resolution as a commutative diagram

and give an explicit formula for the divisor $\tilde{\chi}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$.
We first remark that by Corollary 4.4.10, there is a tautological $\mathbb{G}_{m}$-equivariant morphism

$$
\begin{equation*}
\Theta_{\text {Bun }_{G}}^{-1} \longrightarrow \operatorname{Spec}_{S} \bigoplus_{d \geq 0} \pi_{\operatorname{Bun}_{G *}} \Theta_{\operatorname{Bun}_{G}}^{\otimes d} \cong \operatorname{Spec}_{S} \bigoplus_{d \geq 0}\left(\pi_{Y *} \Theta_{Y}^{\otimes d}\right)^{W}=\widehat{Y} / / W, \tag{4.5.2}
\end{equation*}
$$

where $\widehat{Y}$ is the cone over $Y$ given by contracting the zero section of $\Theta_{Y}^{-1}$ to $S$. Deleting the zero section of $\Theta_{\text {Bun }_{G}}^{-1}$ and taking the quotient of both sides of (4.5.2) by $\mathbb{G}_{m}$ therefore gives a morphism

$$
\chi: \operatorname{Bun}_{G} \longrightarrow(\widehat{Y} / / W) / \mathbb{G}_{m} .
$$

Definition 4.5.1. The morphism $\chi$ above is called the coarse quotient map for Bun $_{G}$.
We next construct the morphism $\tilde{\chi}$. By construction of the elliptic Chevalley isomorphism, if $L_{Y}$ is a good $W$-linearised line bundle on $Y$ and $L_{\text {Bun }_{G}}$ is its image under the Chevalley isomorphism, then there is a functorial isomorphism

$$
\gamma^{s s}:\left.\left.\psi^{*} L_{\mathrm{Bun}_{G}}\right|_{{\underset{\operatorname{Bun}}{G}}_{s s}^{s}} \xrightarrow{\sim} \bar{\chi}^{*} L_{Y}\right|_{\mathrm{Bun}_{G}} ^{s s},
$$

compatible with the isomorphisms $\pi_{\mathrm{Bun}_{G} *} L_{\mathrm{Bun}_{G}} \cong\left(\pi_{Y *} L_{Y}\right)^{W}$. (Recall that $\bar{\chi}: \widetilde{\operatorname{Bun}_{G}} \rightarrow Y$ is the blow down morphism $\mathrm{Bl}_{B}$ composed with the rigidification map $\mathrm{Bun}_{T}^{0} \rightarrow Y$.) We therefore have a rational map of line bundles

$$
\gamma: \psi^{*} L_{\mathrm{Bun}_{G}}--\rightarrow \bar{\chi}^{*} L_{Y} .
$$

We prove at the end of this subsection (Corollary 4.5.9) that when $L_{Y}=\Theta_{Y}^{-1}$, the rational map $\gamma$ has divisor of zeroes and poles

$$
\sum_{\lambda \in \mathbb{X}_{*}(T)_{+}} \frac{1}{2}(\lambda \mid \lambda) D_{\lambda} .
$$

In particular, $\gamma$ is in fact a morphism of line bundles, and so defines a morphism

$$
\tilde{\chi}: \widetilde{\operatorname{Bun}}_{G} \longrightarrow \Theta_{Y}^{-1} / \mathbb{G}_{m} .
$$

Corollary 4.5.2. The morphisms $\chi$ and $\tilde{\chi}$ constructed above fit into a commutative diagram (4.5.1).

Proof. This follows immediately from the definitions and the functoriality of the isomorphism $\gamma^{s s}$.

Remark 4.5.3. Since the elliptic Chevalley isomorphism holds for the rigidified stack $\operatorname{Bun}_{G, r i g}$ as well, we also have a rigidified version

of the diagram (4.5.1).
We remark that the coarse quotient map gives a GIT-style characterisation of semistable and unstable $G$-bundles.

Proposition 4.5.4. Let $\xi_{G} \rightarrow E_{s}$ be a $G$-bundle on a geometric fibre of $E \rightarrow S$. Then $\xi_{G}$ is unstable if and only if $\chi\left(\xi_{G}\right) \in\left(\widehat{Y}_{s} / / W\right) / \mathbb{G}_{m}$ is equal to the image of the cone point of $\widehat{Y}_{s}$.

Proof. Write $0_{\widehat{Y}} \subseteq \widehat{Y}$ for the family of cone points over $S$ and $q: \Theta_{Y}^{-1} \rightarrow \widehat{Y}$ for the tautological morphism. Since $\Theta_{Y}$ is ample, we have $q^{-1}\left(\widehat{Y} \backslash 0_{\widehat{Y}}\right)=\Theta_{Y}^{-1} \backslash 0_{\Theta_{Y}^{-1}}$, and by Corollary 4.5.9, we have $\tilde{\chi}^{-1}\left(\Theta_{Y}^{-1} \backslash 0_{\Theta_{Y}^{-1}}\right)=\widetilde{\operatorname{Bun}}_{G}^{s s}$. So

$$
\chi^{-1}\left(\left(\left(\widehat{Y} \backslash 0_{\widehat{Y}}\right) / / W\right) / \mathbb{G}_{m}\right)=\psi\left(\tilde{\chi}^{-1}\left(\left(\Theta_{Y}^{-1} \backslash 0_{\Theta_{Y}^{-1}}\right) / \mathbb{G}_{m}\right)\right)=\psi\left(\widetilde{\operatorname{Bun}}_{G}^{s s}\right)=\operatorname{Bun}_{G}^{s s},
$$

which proves the proposition.
Proposition 4.5.5. The morphism $\tilde{\chi}: \widetilde{\operatorname{Bun}}_{G} \rightarrow \Theta_{Y}^{-1} / \mathbb{G}_{m}$ is flat, and all fibres of the morphism $\chi: \operatorname{Bun}_{G} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ have dimension $-l=\operatorname{dim}\left(\operatorname{Bun}_{G}\right)-\operatorname{dim}\left((\widehat{Y} / / W) / \mathbb{G}_{m}\right)$.

Proof. To prove that $\tilde{\chi}$ is flat, note that since $\gamma^{s s}:\left.\left.\psi^{*} \Theta_{\text {Bun }_{G}}^{-1}\right|_{\operatorname{Bun}_{G}^{s s}} \rightarrow \bar{\chi}^{*} \Theta_{Y}^{-1}\right|_{\text {Bun }_{G}^{s s}}$ is an isomorphism, and since $\bar{\chi}$ is smooth by Corollary 3.5.4, hence flat, we can apply Lemma 4.5.7 to conclude that the morphism on total spaces $\psi^{*} \Theta_{\operatorname{Bun}_{G}}^{-1} \rightarrow \Theta_{Y}^{-1}$ is flat, and hence so is $\tilde{\chi}$.

It remains to prove that the fibres of $\chi$ have dimension $-l$. Fix a geometric point $x:$ Spec $k \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ over $s:$ Spec $k \rightarrow S$ and consider the fibre $\chi^{-1}(x)$. We know by straightforward comparison of dimensions that $\operatorname{dim} \chi^{-1}(x) \geq-l$, so it suffices to show that $\operatorname{dim} \chi^{-1}(x) \leq-l$. Since the morphism $q: \Theta_{Y}^{-1} / \mathbb{G}_{m} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ is finite away from the zero section and $\tilde{\chi}$ is flat, if $x$ is not the image of the cone point $0_{\widehat{Y}_{s}}$, then

$$
\operatorname{dim} \chi^{-1}(x) \leq \operatorname{dim} \psi^{-1} \chi^{-1}(x)=\operatorname{dim} \tilde{\chi}^{-1} q^{-1}(x)=-l
$$

so we are done. On the other hand, if $x$ is the image of the cone point, then $\chi^{-1}(x)$ is a $\mathbb{G}_{m}$-torsor over the locus of unstable $G$-bundles on $E_{s}$ by Proposition 4.5.4. So

$$
\operatorname{dim} \chi^{-1}(x)=-\operatorname{codim}_{\operatorname{Bun}_{G}\left(E_{s}\right)} \operatorname{Bun}_{G}^{\text {unstable }}\left(E_{s}\right)+1=-l
$$

by Proposition 2.6.8.
Remark 4.5.6. We will show later on (Corollary 5.5.1) that $\widehat{Y} / / W$ is in fact an affine space bundle over $S$, and in particular regular. Together with Proposition 4.5.5 and [E, Theorem 18.16], this imples that the coarse quotient map $\chi$ is flat.

Lemma 4.5.7. Let $f: Z_{1} \rightarrow Z_{2}$ be a flat morphism of regular stacks, let $L_{1}$ and $L_{2}$ be line bundles on $Z_{1}$ and $Z_{2}$ respectively, and let $g: L_{1} \rightarrow f^{*} L_{2}$ be a morphism of line bundles. If $g$ does not vanish identically along any irreducible component of any fibre of $f$, then the induced morphism of total spaces $L_{1} \rightarrow L_{2}$ is flat.

Proof. Since flatness can be checked locally on the source in the smooth topology, it suffices to prove the lemma when $Z_{1}=\operatorname{Spec} R_{1}$ and $Z_{2}=\operatorname{Spec} R_{2}$, with $R_{1}, R_{2}$ regular local rings, $L_{1}=\mathcal{O}_{R_{1}}$ and $L_{2}=\mathcal{O}_{R_{2}}$ are trivial, and $f$ is induced by a flat local homomorphism $\phi: R_{2} \rightarrow R_{1}$. The morphism $g$ is then given by multiplication by some element $r \in R_{1}$, and the induced morphism on total spaces is the spectrum of the $R_{2}$-linear homomorphism

$$
\begin{aligned}
\phi^{\prime}: R_{2}[y] & \longmapsto R_{1}[x] \\
y & \longmapsto r x .
\end{aligned}
$$

Since $R_{1}[x]$ and $R_{2}[y]$ are regular rings, by [E, Theorem 18.16], it suffices to show that every closed fibre has dimension $d=\operatorname{dim} R_{1}[x]-\operatorname{dim} R_{2}[y]=\operatorname{dim} R_{1}-\operatorname{dim} R_{2}$. If $\mathfrak{m}$ is a maximal ideal of $R_{2}[y]$, then writing $k$ for the residue field of $R_{2}$ and $K=R_{2}[y] / \mathfrak{m}$, we have that $K$ is a finite field extension of $k$, and

$$
\frac{R_{1}[x]}{\phi^{\prime}(\mathfrak{m}) R_{1}[x]}=\frac{\left(K \otimes_{R_{2}} R_{1}\right)[x]}{(r x-y)} .
$$

Since $g$ is flat, $\operatorname{dim}\left(K \otimes_{R_{2}} R_{1}[x]\right)=\operatorname{dim}\left(k \otimes_{R_{2}} R_{1}\right)+1=d+1$, so it suffices to show that $r x-y \in K \otimes_{R_{2}} R_{1}[x]$ is neither 0 nor a zero-divisor. But this follows from the fact that $g$ does not vanish along any irreducible component of $\operatorname{Spec} k \otimes_{R_{2}} R_{1}$, so we are done.

The rest of this section is devoted to proving the following theorem, and hence Corollary 4.5.9, which was used above to construct the morphism $\tilde{\chi}$.

Theorem 4.5.8. Let $L_{\operatorname{Bun}_{G}}$ be a line bundle on $\operatorname{Bun}_{G}$ and let $L_{Y}$ be the corresponding good $W$-linearised line bundle on $Y$. Then the rational map of line bundles $\gamma: \psi^{*} L_{\mathrm{Bun}_{G}} \rightarrow \bar{\chi}^{*} L_{Y}$ induced by the isomorphism

$$
\gamma^{s s}:\left.\left.\psi^{*}\left(L_{\mathrm{Bun}_{G}}\right)\right|_{\widetilde{\operatorname{Bun}}_{G}^{s s}} ^{s} \xrightarrow{\sim} \bar{\chi}^{*} L_{Y}\right|_{\widetilde{\operatorname{Bun}}_{G}^{s s}} ^{s s}
$$

has divisor of zeroes and poles

$$
-\sum_{\lambda \in \mathbb{X}_{*}(T)_{+}} \frac{1}{2} Q\left(L_{Y}\right)(\lambda, \lambda) D_{\lambda} .
$$

Corollary 4.5.9. When $L_{\operatorname{Bun}_{G}}=\Theta_{\operatorname{Bun}_{G}}^{-1}$, the map $\gamma$ of Theorem 4.5 .8 is regular, and has divisor of zeroes

$$
\operatorname{div}(\gamma)=\tilde{\chi}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)=\sum_{\lambda \in \mathbb{X}_{*}(T)_{+}} \frac{1}{2}(\lambda \mid \lambda) D_{\lambda} .
$$

The key idea behind the proof of Theorem 4.5.8 is to reduce to the case of a specific choice of line bundle $L_{\mathrm{Bun}_{G}}$ that we can describe explicitly as the determinant of a perfect complex. The following lemma plays an important role.

Lemma 4.5.10. Let $X$ be an algebraic stack, let $D \subseteq X$ be an effective Cartier divisor on $X$, let $U=X \backslash D$, and let $i: D \hookrightarrow X$ denote the inclusion. If $\mathcal{E}$ is a perfect complex on $D$, then the rational map of line bundles

$$
g: \mathcal{O}_{X}=\operatorname{det} 0 \rightarrow \operatorname{det} \mathbb{R} i_{*} \mathcal{E}
$$

induced by the quasi-isomorphism $\left.\left.0\right|_{U} \cong \mathbb{R} i_{*} \mathcal{E}\right|_{U}$ induces an isomorphism

$$
\mathcal{O}_{X}(\chi(\mathcal{E}) D) \xrightarrow{\sim} \operatorname{det} \mathbb{R} i_{*} \mathcal{E}
$$

where $\chi$ denotes the Euler characteristic of a perfect complex.
Proof. We need to show that the divisor of zeros and poles of $g$ is $\chi(\mathcal{E}) D$. Since both are local on $X$ and additive in $\mathcal{E}$ under exact triangles, it suffices to check this when $\mathcal{E}=\mathcal{O}_{D}$. In this case, an explicit free resolution for $\mathbb{R} i_{*} \mathcal{E}$ is

$$
\mathbb{R} i_{*} \mathcal{E}=\left[\mathcal{O}_{X}(-D) \longrightarrow \mathcal{O}_{X}\right]
$$

which gives an isomorphism

$$
\mathcal{O}_{X} \otimes \mathcal{O}_{X}(-D)^{\vee} \cong \operatorname{det} \mathbb{R} i_{*} \mathcal{E}
$$

The map $g$ is given by

$$
\mathcal{O}_{X} \xrightarrow{\sim} \mathcal{O}_{X}(-D) \otimes \mathcal{O}_{X}(-D)^{\vee} \longrightarrow \mathcal{O}_{X} \otimes \mathcal{O}_{X}(-D)^{\vee}=\operatorname{det} \mathbb{R} i_{*} \mathcal{E}
$$

which does indeed have divisor $D=\chi(\mathcal{E}) D$ as claimed.
Lemma 4.5.11. Let $X$ be a smooth stack over $S$, let $Z \subseteq X \times{ }_{S} E$ be a smooth substack of codimension 2 mapping isomorphically to a divisor $D \subseteq X$ under the first projection $\operatorname{pr}_{X}: X \times_{S} E \rightarrow X$, and let $f: C \rightarrow X \times_{S} E$ be the blowup along $Z$. If $L$ is a line bundle on $C$ such that $L$ restricted to any exceptional fibre of $f$ has degree $d$, then the canonical rational map

$$
\begin{equation*}
\operatorname{det} \mathbb{R} \mathrm{pr}_{X *} \mathbb{R} f_{*} L-\rightarrow \operatorname{det} \mathbb{R} \operatorname{pr}_{X *} \operatorname{det} \mathbb{R} f_{*} L \tag{4.5.3}
\end{equation*}
$$

induced by the quasi-isomorphism $\left.\left.\mathbb{R} f_{*} L\right|_{X \times_{S} E \backslash Z} \cong \operatorname{det} \mathbb{R} f_{*} L\right|_{X \times_{S} E \backslash Z}$ has divisor

$$
\frac{d(d+1)}{2} D
$$

Proof. We first observe that if $d=0$, then the claim is true since $\mathbb{R} f_{*} L$ is a line bundle and hence (4.5.3) is an isomorphism.

For a general line bundle $L$, write $\operatorname{div}(L)$ for the divisor of (4.5.3). Consider the exact sequence

$$
\left.0 \longrightarrow L(-\mathrm{Exc}) \longrightarrow L \longrightarrow L\right|_{\mathrm{Exc}} \longrightarrow 0
$$

where Exc $=f^{-1}(Z)$ is the exceptional divisor. The morphism $\mathbb{R} f_{*} L(-$ Exc $) \rightarrow \mathbb{R} f_{*} L$ induces a commutative diagram

of rational maps of line bundles, where the bottom arrow is an isomorphism since $Z$ has codimension 2. We deduce that

$$
\operatorname{div}(L(-\operatorname{Exc}))=\operatorname{div}(L)+\operatorname{div}(g)
$$

But $\operatorname{div}(g)=\operatorname{div}\left(g^{\prime}\right)$, where

$$
g^{\prime}: \mathcal{O}_{X}-->\operatorname{det}\left(\mathbb{R p r}_{X *} \mathbb{R} f_{*} L(-\mathrm{Exc})\right)^{-1} \otimes \mathbb{R p r}_{X *} \mathbb{R} f_{*} L=\operatorname{det} \mathbb{R p r}_{X *} \mathbb{R} f_{*}\left(\left.L\right|_{\mathrm{Exc}}\right)
$$

is the rational morphism induced by the quasi-isomorphism $\left.\mathbb{R p r}_{X *} \mathbb{R} f_{*}\left(\left.L\right|_{\text {Exc }}\right)\right|_{X \backslash D} \simeq 0$. But $\mathbb{R} \operatorname{pr}_{X *} \mathbb{R} f_{*}\left(\left.L\right|_{\text {Exc }}\right)$ is the pushforward from $D$ of a perfect complex with Euler characteristic $d+1$, so Lemma 4.5.10 gives $\operatorname{div}\left(g^{\prime}\right)=(d+1) D$ and hence

$$
\operatorname{div}(L(-\operatorname{Exc}))=\operatorname{div}(L)+(d+1) D
$$

Since $L$ (-Exc) has degree $d+1$ restricted to any exceptional fibre of $f$, the lemma now follows easily by induction on the absolute value of $d$.

The next lemma identifies the $W$-linearised line bundle on $Y$ and quadratic form corresponding to a determinant line bundle on $\mathrm{Bun}_{G}$.

Lemma 4.5.12. Let $V$ be a representation of $G$, and let $L_{\operatorname{Bun}_{G}}=\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{G} *}\left(\xi_{G}^{u n i} \times^{G} V\right)$, where $\xi_{G}^{u n i} \rightarrow \operatorname{Bun}_{G} \times_{S} E$ is the universal $G$-bundle, and $\operatorname{pr}_{\operatorname{Bun}_{G}}: \operatorname{Bun}_{G} \times_{S} E \rightarrow \operatorname{Bun}_{G}$ is the canonical projection. Then the corresponding good $W$-linearised line bundle on $Y$ is given by

$$
L_{Y}=\bigotimes_{\lambda \in \mathbb{X}^{*}(T)} \lambda^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)^{\otimes \operatorname{dim} V_{\lambda}},
$$

and hence

$$
q\left(L_{Y}\right)(\mu)=-\sum_{\lambda \in \mathbb{X}^{*}(T)} \operatorname{dim} V_{\lambda}\langle\lambda, \mu\rangle^{2}
$$

for $\mu \in \mathbb{X}_{*}(T)$, where

$$
V=\bigoplus_{\lambda \in \mathbb{X}^{*}(T)} V_{\lambda}
$$

is the weight space decomposition of $V$ under the action of $T$.
Proof. We have

$$
\psi^{*} L_{\operatorname{Bun}_{G}}=\operatorname{det} \mathbb{R p r}_{\widetilde{\operatorname{Bun}_{G} *}}\left(\left(\psi^{*} \xi_{G}^{u n i}\right) \times^{G} V\right)=\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{G *}} \mathbb{R} f_{*} f^{*}\left(\left(\psi^{*} \xi_{G}^{u n i}\right) \times{ }^{G} V\right)
$$

where $\operatorname{pr}_{\operatorname{Bun}_{G}}: \widetilde{\operatorname{Bun}_{G}} \times_{S} E \rightarrow \widetilde{\operatorname{Bun}_{G}}$ is the first projection and $f: \widetilde{\operatorname{Bun}_{G}} \times{ }_{\operatorname{Deg}_{S}(E)} \mathcal{C} \rightarrow$ $\widetilde{\operatorname{Bun}}_{G} \times{ }_{S} E$ is the pullback of the universal prestable degeneration of $E$ over $\widetilde{\operatorname{Bun}}_{G}$. Since the $G$-linearised vector bundle $V \otimes \mathcal{O}_{F}$ on the flag variety $F=G / B$ has a $G$-equivariant filtration with associated quotients isomorphic to $V_{\lambda} \otimes \mathcal{L}_{\lambda}$ for $\lambda \in \mathbb{X}^{*}(T)$, we get an isomorphism

$$
\begin{aligned}
\psi^{*} L_{\mathrm{Bun}_{G}} & \cong \bigotimes_{\lambda \in \mathbb{X}^{*}(T)} \operatorname{det} \mathbb{R p r}_{\widetilde{\operatorname{Bun}}_{G} *} \mathbb{R} f_{*}\left(V_{\lambda} \otimes\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{\lambda}\right)\right) \\
& =\bigotimes_{\lambda \in \mathbb{X}^{*}(T)}\left(\operatorname{det} \mathbb{R p r}_{\mathrm{Bun}_{G}} * \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{\lambda}\right)\right)^{\otimes \operatorname{dim} V_{\lambda}},
\end{aligned}
$$

where $\xi_{T, \mathcal{C}}$ is the induced $T$-bundle on $\widetilde{\operatorname{Bun}_{G}} \times_{\mathfrak{D e g}_{S}(E)} \mathcal{C}$. So restricting to $\widetilde{\operatorname{Bun}_{G}^{s s}}$ gives

$$
\begin{equation*}
\left.\left.\psi^{*} L_{\operatorname{Bun}_{G}}\right|_{\widetilde{\operatorname{Bun}}_{G}^{s s}} ^{s}=\bigotimes_{\lambda \in \mathbb{X}^{*}(T)} \operatorname{Bl}_{B}^{*}\left(\operatorname{det} \mathbb{R}_{\operatorname{Bun}}^{\operatorname{Bun}_{T}^{0}}{ }^{( } \xi_{T}^{u n i} \times \mathbb{Z}_{\lambda}\right)\right)^{\otimes \operatorname{dim} V_{\lambda}}{\widetilde{\operatorname{Bun}_{G}}}_{s,}^{s s} \tag{4.5.4}
\end{equation*}
$$

where $\xi_{T}^{u n i}$ is the universal $T$-bundle on $\operatorname{Bun}_{T}^{0} \times_{S} E$ and $\mathrm{Bl}_{B}: \widetilde{\operatorname{Bun}_{G}} \rightarrow \operatorname{Bun}_{T}^{0}$ is the blow down morphism. Now, for all $\lambda \in \mathbb{X}^{*}(T)$, we have

$$
\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{T}^{0} *}\left(\xi_{T}^{u n i} \times{ }^{T} \mathbb{Z}_{\lambda}\right)=\lambda^{*} \operatorname{det} \mathbb{R}^{\operatorname{pr}_{\operatorname{Bun}_{G_{m} / S}^{0}}(E) *} 1 M,
$$

where $M$ is the universal line bundle on $\operatorname{Bun}_{\mathbb{G}_{m} / S}^{0}(E) \times_{S} E$. But

$$
\mathbb{R p r}_{\operatorname{Bun}_{\mathbb{G}_{m} / S}^{0}(E) *} M=\left(\mathbb{R}^{1} \operatorname{pr}_{\operatorname{Bun}_{G_{m} / S}^{0}(E) *} M\right)[-1]
$$

is the pushforward of a perfect complex of Euler characteristic -1 on the Cartier divisor $O_{\text {Bun }_{G_{m} / S}^{0}(E)}$ corresponding to the trivial bundle, so by Lemma 4.5.10, we have

$$
\operatorname{det} \mathbb{R} \operatorname{pr}_{\operatorname{Bun}_{G_{m} / S}^{0}}(E) * M=\mathcal{O}\left(-O_{\operatorname{Bun}_{G_{m} / S}^{0}}(E)\right)=q^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)
$$

where $q: \operatorname{Bun}_{\mathbb{G}_{m} / S}^{0}(E) \rightarrow \operatorname{Pic}_{S}^{0}(E)$ is the canonical quotient by $\mathbb{B} \mathbb{G}_{m}$. So (4.5.4) gives

$$
\left.\psi^{*} L_{\operatorname{Bun}_{G}}\right|_{\operatorname{Bun}_{G}^{s s}} ^{s s}=\left.\bar{\chi}^{*} \bigotimes_{\lambda \in \mathbb{X}^{*}(T)} \lambda^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)^{\otimes \operatorname{dim} V_{\lambda}}\right|_{\widetilde{\operatorname{Bun}}_{G}^{s s}} ^{s}
$$

from which the result follows immediately.
Proof of Theorem 4.5.8. We first remark that since the truth or falsehood of the statement is unchanged if we raise $L_{\mathrm{Bun}_{G}}$ to a nonzero power or tensor with a line bundle on $S$, by Corollary 4.4.10, it suffices to prove the claim for any single $L_{\operatorname{Bun}_{G}}$ with $q\left(L_{Y}\right) \neq 0$. So choose any nontrivial representation $V$ of $G$ and set

$$
L_{\operatorname{Bun}_{G}}=\operatorname{det} \mathbb{R} \operatorname{pr}_{\operatorname{Bun}_{G} *}\left(\xi_{G}^{u n i} \times{ }^{G} V\right),
$$

as in Lemma 4.5.12. Keeping the notation from that proof, since $\pi_{\text {Bun }_{G}^{s s} *} \mathcal{O}=\pi_{Y *} \mathcal{O}_{Y}=\mathcal{O}_{S}$ by Proposition 4.3.17 (1), the rational map $\gamma$ must coincide up to rescaling by a nonvanishing function on $S$ with the rational map

$$
\begin{equation*}
\bigotimes_{\lambda \in \mathbb{X}^{*}(T)}\left(\operatorname{det} \mathbb{R p r}_{\widetilde{\operatorname{Bun}}_{G^{*}}}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{\lambda}\right)\right)^{\otimes \operatorname{dim} V_{\lambda}} \rightarrow \bigotimes_{\lambda \in \mathbb{X}^{*}(T)}\left(\operatorname{det} \mathbb{R}_{\operatorname{pr}}^{\widetilde{\operatorname{Bun}}_{G} *} \operatorname{det}^{R} f_{*}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{\lambda}\right)\right)^{\otimes \operatorname{dim} V_{\lambda}} \tag{4.5.5}
\end{equation*}
$$

given by the quasi-isomorphisms

$$
\left.\left.\operatorname{det} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times{ }^{T} \mathbb{Z}_{\lambda}\right)\right|_{\widehat{\operatorname{Bun}}_{G}^{s s}} \cong \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times{ }^{T} \mathbb{Z}_{\lambda}\right)\right|_{\widetilde{\operatorname{Bun}}_{G}^{s s}} ^{s,}
$$

for $\lambda \in \mathbb{X}^{*}(T)$, where we recall that $\operatorname{Bl}_{B}^{*}\left(\xi_{T}^{u n i} \times^{T} \mathbb{Z}_{\lambda}\right)=\operatorname{det} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times{ }^{T} \mathbb{Z}_{\lambda}\right)$ by Lemma 3.5.2 and the definition of $\mathrm{Bl}_{B}:{\mathrm{Bun}_{G} \rightarrow \mathrm{Bun}_{T}^{0} \text {. To complete the proof, observe that Proposition }}^{2}$ 3.4.13 implies that

$$
\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\leq 1}=\bigcup_{\mu \in \mathbb{X}_{*}(T)_{+}}\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\leq 1}
$$

where

$$
\left.\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\leq 1}=\widetilde{\operatorname{Bun}}_{G} \times \mathfrak{D e g}_{S}(E) \mathfrak{D e g}_{S}(E)\right)^{\leq 1} \subseteq \widetilde{\operatorname{Bun}}_{G}
$$

is the open substack of points where the nodal domain curve $C$ has at most 1 node, and for $\mu \in \mathbb{X}_{*}(T)_{+},\left(\widetilde{\operatorname{Bun}_{G}}\right)_{\bar{\mu}}^{\leq 1} \subseteq\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\leq 1}$ is the open substack of stable maps with either smooth
domain or dual graph $\tau_{\mu}^{0}$. Since the complement of $\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\leq 1}$ in $\widetilde{\operatorname{Bun}}_{G}$ has codimension 2, it suffices to show that for any $\mu \in \mathbb{X}_{*}(T)_{+}$, the restriction of (4.5.5) to $\left(\widetilde{\operatorname{Bun}}_{G}\right)_{\mu}^{\leq 1}$ has divisor

$$
\frac{1}{2} \sum_{\lambda \in \mathbb{X}^{*}(T)} \operatorname{dim} V_{\lambda}\langle\lambda, \mu\rangle^{2} D_{\mu}
$$

By Lemma 4.5.11, the map

$$
\operatorname{det} \mathbb{R} \operatorname{pr}_{\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\frac{\leq 1}{\mu}} 1 *} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{\lambda}\right)-\rightarrow \operatorname{det} \mathbb{R} \operatorname{pr}_{\left(\widetilde{\operatorname{Bun}}_{G}\right)^{\frac{\leq 1}{\mu}}{ }^{1}} \operatorname{det} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times{ }^{T} \mathbb{Z}_{\lambda}\right)
$$

has divisor

$$
\frac{\langle\lambda, \mu\rangle(\langle\lambda, \mu\rangle+1)}{2} D_{\mu},
$$

so taking the tensor product over all $\lambda \in \mathbb{X}^{*}(T)$, we find that the restriction of (4.5.5) has divisor

$$
\left(\sum_{\lambda \in \mathbb{X}^{*}(T)} \frac{\operatorname{dim} V_{\lambda}}{2}\langle\lambda, \mu\rangle^{2}+\sum_{\lambda \in \mathbb{X}^{*}(T)} \frac{\operatorname{dim} V_{\lambda}}{2}\langle\lambda, \mu\rangle\right) D_{\mu}=\frac{1}{2} \sum_{\lambda \in \mathbb{X}^{*}(T)} \operatorname{dim} V_{\lambda}\langle\lambda, \mu\rangle^{2} D_{\mu}
$$

as required, since

$$
\sum_{\lambda \in \mathbb{X}^{*}(T)}\left(\operatorname{dim} V_{\lambda}\right) \lambda \in \mathbb{X}^{*}(T)^{W}=\{0\} .
$$

### 4.6 The canonical bundles of $\operatorname{Bun}_{G}$ and $\widetilde{\operatorname{Bun}}_{G}$

As an application of the methods used in the proof of Theorem 4.5.8, we compute the canonical bundles $K_{\operatorname{Bun}_{G} / S}=\operatorname{det} \mathbb{L}_{\operatorname{Bun}_{G} / S}$ and $K_{\operatorname{Bun}_{G} / S}=\operatorname{det} \mathbb{L}_{\mathbb{B u n}_{G} / S}$ of $\operatorname{Bun}_{G}$ and $\widetilde{\operatorname{Bun}_{G}}$ relative to $S$.

The aim is to prove the following theorem. In the statement below, we write $\omega \in \operatorname{Pic}(S)$ for the line bundle

$$
\omega=\pi_{E *} K_{E / S}=\pi_{\operatorname{Pic}_{S}^{0}(E) *} K_{\operatorname{Pic}_{S}^{0}(E) / S}=O_{\operatorname{Pic}_{S}^{0}(E)}^{*} K_{\mathrm{Pic}_{S}^{0}(E) / S}
$$

Given $(R, L) \in \operatorname{Sym}^{2}\left(\mathbb{X}^{*}(T)\right)^{W} \oplus \operatorname{Pic}(S)$, we write $\mathcal{L}(R, L)$ for the line bundle on $\operatorname{Bun}_{G}$ associated to $(R, L)$ by Proposition 4.4.9 and Theorem 4.3.4.

Theorem 4.6.1. The relative canonical bundles of $\mathrm{Bun}_{G}$ and $\widetilde{\mathrm{Bun}}_{G}$ are given by

$$
K_{\mathrm{Bun}_{G} / S}=\mathcal{L}\left(-\sum_{\alpha \in \Phi} \alpha^{2}, \omega^{\otimes \operatorname{dim} G}\right)=\mathcal{L}\left(-2 \sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} G}\right)
$$

and

$$
K_{\widehat{\operatorname{Bun}}_{G} / S}=\psi^{*} \mathcal{L}\left(-\sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} B}\right) \otimes \mathcal{O}\left(\sum_{\mu \in \mathbb{X}_{*}(T)_{+}}(-2+\langle\rho, \mu\rangle) D_{\mu}\right) .
$$

Lemma 4.6.2. The line bundle

$$
L_{Y}=\bigotimes_{\alpha \in \Phi_{+}} \alpha^{*} \mathcal{O}\left(-O_{\mathrm{Pic}_{S}^{0}(E)}\right)
$$

admits a (necessarily unique) good $W$-linearisation.

Proof. We first note that $s_{i}^{*} L_{Y} \cong L_{Y}$ for all simple reflections $s_{i}$, since

$$
s_{i}^{*} L_{Y}=\bigotimes_{\alpha \in \Phi_{+} \backslash\left\{\alpha_{i}\right\}} \alpha^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right) \otimes\left(-\alpha_{i}\right)^{*} \mathcal{O}\left(-O_{\mathrm{Pic}_{S}^{0}(E)}\right)
$$

and

$$
\left(-\alpha_{i}\right)^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right) \cong \alpha_{i}^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)
$$

So $L_{Y}$ admits a unique $W$-linearisation such that the action of $W$ on $O_{Y}^{*} L$ is trivial. It remains to show that this linearisation is good. For all simple coroots $\alpha_{i}^{\vee}$, we have

$$
\left(\alpha_{i}^{\vee}\right)^{*} L_{Y}=\bigotimes_{\alpha \in \Phi_{+}}\left(\alpha_{i}^{\vee}\right)^{*} \alpha^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)=\mathcal{O}\left(-\sum_{\alpha \in \Phi_{+}}\left\langle\alpha, \alpha_{i}^{\vee}\right\rangle^{2} O_{\operatorname{Pic}_{S}^{0}(E)}\right)
$$

But

$$
\sum_{\alpha \in \Phi_{+}}\left\langle\alpha, \alpha_{i}^{\vee}\right\rangle^{2} \equiv \sum_{\alpha \in \Phi_{+}}\left\langle\alpha, \alpha_{i}^{\vee}\right\rangle \equiv\left\langle 2 \rho, \alpha_{i}^{\vee}\right\rangle \equiv 0 \quad \bmod 2,
$$

so the $W$-linearisation on $L_{Y}$ is indeed good by Lemma 4.4.6.
Lemma 4.6.3. There is an isomorphism

$$
K_{\mathfrak{D} \mathrm{eg}_{S}(E) / S} \cong \mathcal{O}(-2 D)
$$

where $D \subseteq \mathfrak{D e g}_{S}(E)$ is the locus of singular curves.
For the proof of Lemma 4.6.3, we will use the following description of the relative tangent complex of a blow up.

Lemma 4.6.4. Let $X$ be a stack, let $Z \subseteq X$ be a regularly embedded closed substack of codimension $d$, and let $\pi: \tilde{X} \rightarrow X$ be the blowup of $X$ along $Z$. Then there is an exact triangle

$$
\begin{equation*}
\mathbb{T}_{\tilde{X} / X} \longrightarrow \mathbb{R} i_{*} N_{\mathrm{Exc} / \tilde{X}} \longrightarrow \mathbb{R} i_{*} \pi^{*} N_{Z / X} \longrightarrow \mathbb{T}_{\tilde{X} / X}[1] \tag{4.6.1}
\end{equation*}
$$

in $D(\tilde{X})$, where $\operatorname{Exc}=\pi^{-1}(Z)$ is the exceptional divisor, $i: \operatorname{Exc} \hookrightarrow \tilde{X}$ is the inclusion, and $N_{U / V}$ denotes the normal bundle of $U$ in $V$.

Proof. Consider the exact triangle

$$
\mathbb{T}_{\mathrm{Exc} / \tilde{X}} \longrightarrow \mathbb{T}_{\mathrm{Exc} / X} \longrightarrow \mathbb{L} i^{*} \mathbb{T}_{\tilde{X} / X} \longrightarrow \mathbb{T}_{\mathrm{Exc} / \tilde{X}}[1]
$$

in $D$ (Exc). Pushing forward to $\tilde{X}$ and letting $M$ be the derived kernel (cocone) of $\mathbb{R} i_{*} \mathbb{T}_{\text {Exc } / X} \oplus$ $\mathbb{T}_{\tilde{X} / X} \rightarrow \mathbb{R} i_{*} \mathbb{L} i^{*} \mathbb{T}_{\tilde{X} / X}$ gives a morphism

of exact triangles in $D(\tilde{X})$. We claim that the composition

$$
M \longrightarrow \mathbb{R} i_{*} \mathbb{T}_{\mathrm{Exc} / X} \longrightarrow \mathbb{R} i_{*} \mathbb{L} \pi^{*} \mathbb{T}_{Z / X}
$$

is an isomorphism in $D(\tilde{X})$. Given the claim, the top row of (4.6.2) can be rewritten as an exact triangle

$$
\mathbb{R} i_{*} N_{\mathrm{Exc} / \tilde{X}}[-1] \longrightarrow \mathbb{R} i_{*} \pi^{*} N_{Z / X}[-1] \longrightarrow \mathbb{T}_{\tilde{X} / X} \longrightarrow \mathbb{R} i_{*} N_{\mathrm{Exc} / \tilde{X}}
$$

from which we get (4.6.1) by rotation.
Since the claim is local on $X$ for the smooth topology, we may assume without loss of generality that $X=\operatorname{Spec} R$ is an affine scheme and that $Z$ is defined by the ideal $\left(x_{1}, \ldots, x_{d}\right)$ where $x_{1}, \ldots, x_{d}$ is a regular sequence in $R$. Then $\tilde{X} \subseteq X \times \mathbb{P}^{d-1}$ is the closed subscheme defined by the ideal

$$
\left(x_{i} X_{j}-x_{j} X_{i} \mid 1 \leq i<j \leq d\right)
$$

where $X_{1}, \ldots, X_{d}$ are the homogeneous coordinates of $\mathbb{P}^{d-1}$. The tangent complex $\mathbb{T}_{\tilde{X} / X}$ is given by

$$
\mathbb{T}_{\tilde{X} / X}=\left[\left.T_{X \times \mathbb{P}^{d-1} / X}\right|_{\tilde{X}} \rightarrow N_{\tilde{X} / X \times \mathbb{P}^{d-1}}\right]
$$

and the bottom row of (4.6.2) is given by the exact sequence of complexes


So the morpism $M \rightarrow \mathbb{R} i_{*} \mathbb{L} \pi^{*} \mathbb{T}_{Z / S}$ is given by

where $M^{\prime}$ is the fibre product


It remains to show that (4.6.3) is a quasi-isomorphism (and hence an isomorphism in $D(\tilde{X})$ ). Observing that the canonical map $N_{\mathrm{Exc} / X \times \mathbb{P}^{d-1}} \rightarrow \pi^{*} N_{Z / X}$ is an isomorphism, we have a commutative diagram

where the rightmost square is Cartesian. To show that (4.6.3) is a quasi-isomorphism, it suffices to show that the bottom row of (4.6.4), and hence the top, is exact. But this follows from a direct computation in local affine coordinates on $\mathbb{P}^{d-1}$, so we are done.

Proof of Lemma 4.6.3. Let $\mathfrak{D e g}_{S}(E)^{\leq 1} \subseteq \mathfrak{D e g}_{S}(E)$ denote the locus of curves with at most one node. Since $\mathfrak{D} \mathrm{eg}_{S}(E)$ is smooth and the complement of $\mathfrak{D} \mathrm{eg}_{S}(E) \leq 1$ is a closed substack with codimension 2 by Proposition 3.3.7, it suffices to prove the claim for $\mathfrak{D} \operatorname{eg}_{S}(E) \leq 1$.

The tangent complex of $\mathfrak{D e g}{ }_{S}(E) \leq 1$ is

$$
\mathbb{T}_{\operatorname{Deg}_{S}(E) \leq 1}=\mathbb{R} \operatorname{pr}_{\mathfrak{D e g}_{S}(E) \leq 1 *} \mathbb{R} f_{*} \mathbb{T}_{\mathcal{C} \leq 1 / \mathfrak{D e g}}^{S}(E) \leq 1 \times_{S} E[1]
$$

where $f: \mathcal{C} \rightarrow \mathfrak{D e g}_{S}(E) \times{ }_{S} E$ is the universal degeneration and $\mathcal{C} \leq 1=f^{-1}\left(\mathfrak{D e g}_{S}(E) \leq 1\right)$. So we have

$$
K_{\mathfrak{D e g}_{S}(E) \leq 1 / S}=\operatorname{det} \mathbb{R p r}_{\mathfrak{D e g}_{S}(E) \leq 1 *} \mathbb{R} f_{*} \mathbb{T}_{\mathcal{C} \leq 1 / E \times_{S} \mathfrak{D e g}_{S}(E) \leq 1}
$$

Since $f: \mathcal{C} \leq 1 \rightarrow \mathcal{D e g}_{S}(E)^{\leq 1} \times_{S} E$ is the blow up along a closed substack $Z$ of codimension 2 (mapping isomorphically to $D^{\leq 1}$ under the projection $\operatorname{pr}_{\mathfrak{D e g}_{S}(E)}$ ), by Lemma 4.6.4, there is an exact triangle

$$
\begin{equation*}
\mathbb{T}_{\mathcal{C} \leq 1 / \mathfrak{D e g} g_{S}(E) \leq 1 \times_{S} E} \longrightarrow i_{*} N_{\mathrm{Exc} / \mathcal{C} \leq 1} \longrightarrow i_{*} g^{*} N_{Z / \mathfrak{D e g}}^{S}(E) \leq 1 \times_{S} E \longrightarrow \mathbb{T}_{\mathcal{C} \leq 1 / \mathcal{D} \operatorname{eg}_{S}(E) \leq 1 \times_{S} E}[1] \tag{4.6.5}
\end{equation*}
$$

where $\operatorname{Exc}=f^{-1}(Z)$ is the exceptional divisor, $i: \operatorname{Exc} \rightarrow \mathcal{C} \leq 1$ is the natural inclusion, and $g$ : Exc $\rightarrow Z$ is the restriction of $f$. By Lemma 4.5.10, we have

$$
\begin{aligned}
\operatorname{det} \mathbb{R p r}_{\mathfrak{\operatorname { D e g } _ { S } ( E ) \leq 1 *}} \mathbb{R} f_{*} i_{*} N_{\mathrm{Exc} / \mathcal{C} \leq 1} & =\operatorname{det}\left(\left.\mathbb{R} j_{*} \mathbb{R} p_{*} \mathcal{O}(\mathrm{Exc})\right|_{\mathrm{Exc}}\right) \\
& =\mathcal{O}\left(\chi\left(\left.\mathbb{R} p_{*} \mathcal{O}(\mathrm{Exc})\right|_{\mathrm{Exc}}\right) D^{\leq 1}\right) \\
& =\mathcal{O}\left(\chi\left(\mathbb{P}^{1}, \mathcal{O}(-1)\right) D^{\leq 1}\right)=\mathcal{O}
\end{aligned}
$$

where $D^{\leq 1}=D \cap \mathfrak{D e g}_{S}(E)^{\leq 1}, j: D^{\leq 1} \rightarrow \mathfrak{D e g}_{S}(E)^{\leq 1}$ is the inclusion, and $p=\operatorname{pr}_{\mathfrak{D e g}_{S}(E) \leq 1} \circ$ $g: \operatorname{Exc} \rightarrow D^{\leq 1}$. (Note that $p$ is a $\mathbb{P}^{1}$-bundle.) Similarly,

$$
\begin{aligned}
\operatorname{det} \mathbb{R p r}_{\mathfrak{D e g}_{S}(E) \leq 1 *} \mathbb{R} f_{*} i_{*} g^{*} N_{Z / \mathfrak{D} \operatorname{eg}_{S}(E) \leq 1 \times_{S} E} & =\operatorname{det}\left(\mathbb{R} j_{*} \mathbb{R} p_{*} g^{*} N_{Z / \mathfrak{D e g}}^{S}(E) \leq 1 \times_{S} E\right. \\
& =\mathcal{O}\left(\chi\left(\mathbb{R} p_{*} g^{*} N_{Z / \mathfrak{D e g}}^{S}(E) \leq 1 \times_{S} E\right) D^{\leq 1}\right) \\
& =\mathcal{O}\left(\chi\left(\mathbb{P}^{1}, \mathcal{O}^{\oplus 2}\right) D^{\leq 1}\right)=\mathcal{O}\left(2 D^{\leq 1}\right)
\end{aligned}
$$

From the exact triangle (4.6.5), we deduce that

$$
K_{\mathfrak{D e g}_{S}(E) \leq 1 / S}=\operatorname{det} \mathbb{R p r}_{\operatorname{Deg}_{S}(E) \leq 1 *} \mathbb{R} f_{*} \mathbb{T}_{\mathcal{C} \leq 1 / \mathfrak{D e g}}^{S}(E) \leq 1 \times_{S} E=\mathcal{O}\left(-2 D^{\leq 1}\right)
$$

which proves the lemma.
Proof of Theorem 4.6.1. We have

$$
K_{\operatorname{Bun}_{G} / S}=\operatorname{det}\left(\mathbb{R p r}_{\operatorname{Bun}_{G} *}\left(\xi_{G}^{u n i} \times{ }^{G} \mathfrak{g}[1]\right)\right)^{\vee}=\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{G} *}\left(\xi_{G}^{u n i} \times{ }^{G} \mathfrak{g}\right)
$$

By Lemma 4.5.12, we therefore have

$$
\begin{equation*}
K_{\operatorname{Bun}_{G} / S}=\mathcal{L}\left(-\sum_{\alpha \in \Phi} \alpha^{2},\left(O_{\operatorname{Pic}_{S}^{0}(E)}^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)\right)^{\otimes \operatorname{dimg}}\right) . \tag{4.6.6}
\end{equation*}
$$

But there is an isomorphism

$$
O_{\mathrm{Pic}_{S}^{0}(E)}^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right) \xrightarrow{\sim} O_{\mathrm{Pic}_{S}^{0}(E)}^{*} K_{\mathrm{Pic}_{S}^{0}(E) / S}=\omega
$$

given by taking exterior derivatives. So (4.6.6) gives

$$
K_{\mathrm{Bun}_{G} / S}=\mathcal{L}\left(-\sum_{\alpha \in \Phi} \alpha^{2}, \omega^{\otimes \operatorname{dim} G}\right)
$$

as required.
For $\widetilde{\operatorname{Bun}}_{G}$, we have

$$
\begin{aligned}
K_{\widetilde{\operatorname{Bun}}_{G} / \mathfrak{D} \operatorname{eg}_{S}(E)} & =\operatorname{det} \mathbb{R}_{\operatorname{pr}}^{\widetilde{\operatorname{Bun}}_{G} *} \\
& \mathbb{R} f_{*}\left(\sigma^{*} \xi_{G}^{u n i} \times{ }^{G} \operatorname{ker}\left(\mathfrak{g} \otimes \mathcal{O}_{F} \rightarrow T_{F}\right)\right) \\
& =\left(\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{G} *} \mathbb{R} f_{*} \mathcal{O}\right)^{\otimes l} \otimes \bigotimes_{\alpha \in \Phi_{+}} \operatorname{det} \mathbb{R p r}_{\widetilde{\operatorname{Bun}}_{G} *} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{-\alpha}\right),
\end{aligned}
$$

where we write $f: \widetilde{\operatorname{Bun}_{G}} \times_{\mathfrak{D e g}_{S}(E)} \mathcal{C} \rightarrow \widetilde{\operatorname{Bun}_{G}} \times_{S} E$ for the pullback of the universal degen-
 flag variety bundle. By Lemma 4.5.11, this is isomorphic to

$$
\begin{aligned}
& \pi_{\operatorname{Bun}_{G}}^{*} \omega^{\otimes l} \otimes \bigotimes_{\alpha \in \Phi_{+}} \operatorname{det} \mathbb{R p r}{\widetilde{\operatorname{Bun}_{G}}} \operatorname{det} \mathbb{R} f_{*}\left(\xi_{T, \mathcal{C}} \times^{T} \mathbb{Z}_{-\alpha}\right) \\
& \otimes \mathcal{O}\left(-\sum_{\mu \in \mathbb{X}_{*}(T)_{+}} \frac{1}{2}\langle-\alpha, \mu\rangle(\langle-\alpha, \mu\rangle+1) D_{\mu}\right) \\
& =\bar{\chi}^{*}\left(\pi_{Y}^{*} \omega^{\otimes l} \otimes \bigotimes_{\alpha \in \Phi_{+}} \alpha^{*} \mathcal{O}\left(-O_{\operatorname{Pic}_{S}^{0}(E)}\right)\right) \otimes \mathcal{O}\left(\sum_{\mu \in \mathbb{X}_{*}(T)_{+}} \sum_{\alpha \in \Phi_{+}} \frac{1}{2}\left(-\langle\alpha, \mu\rangle^{2}+\langle\alpha, \mu\rangle\right) D_{\mu}\right),
\end{aligned}
$$

where we use the fact that

$$
\operatorname{det} \mathbb{R} \mathrm{pr}_{\operatorname{Bun}_{G} *} \mathbb{R} f_{*} \mathcal{O}=\operatorname{det} \mathbb{R} \mathrm{pr}_{\mathrm{Bun}_{G} *} \mathcal{O}=\left(\mathbb{R}^{1} \mathrm{pr}_{\mathrm{Bun}_{G} *} \mathcal{O}\right)^{\vee}=\pi_{\widehat{\operatorname{Bun}}_{G}}^{*} \pi_{E *} K_{E / S}=\pi_{\widehat{\operatorname{Bun}}_{G}}^{*} \omega .
$$

By Lemma 4.6.2, the line bundle

$$
\pi_{Y}^{*} \omega^{\otimes l} \otimes \bigotimes_{\alpha \in \Phi_{+}} \alpha^{*} \mathcal{O}\left(-O_{\mathrm{Pic}_{S}^{0}(E)}\right)
$$

admits a good $W$-linearisation, so corresponds to a line bundle on $\mathrm{Bun}_{G}$, which, examining quadratic classes and the pullback along $O_{Y}$, must be

$$
\mathcal{L}\left(-\sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} B}\right) .
$$

So by Theorem 4.5.8, we get

$$
\begin{aligned}
K_{\widetilde{\operatorname{Bun}}_{G} / \mathfrak{D e g}_{S}(E)} & =\psi^{*} \mathcal{L}\left(-\sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} B}\right) \otimes \mathcal{O}\left(\sum_{\mu \in \mathbb{X}_{*}(T)_{+}} \sum_{\alpha \in \Phi_{+}} \frac{1}{2}\langle\alpha, \mu\rangle D_{\mu}\right) \\
& =\psi^{*} \mathcal{L}\left(-\sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} B}\right) \otimes \mathcal{O}\left(\sum_{\mu \in \mathbb{X}_{*}(T)_{+}}\langle\rho, \mu\rangle D_{\mu}\right)
\end{aligned}
$$

Using Lemma 4.6.3, we therefore get

$$
\begin{aligned}
K_{\mathrm{Bun}_{G} / S} & =K_{\operatorname{Deg}_{S}(E) / S} \otimes K_{\widetilde{\operatorname{Bun}}_{G} / \mathfrak{D e g} \mathrm{eg}_{S}(E)} \\
& =\psi^{*} \mathcal{L}\left(-\sum_{\alpha \in \Phi_{+}} \alpha^{2}, \omega^{\otimes \operatorname{dim} B}\right) \otimes \mathcal{O}\left(\sum_{\mu \in \mathbb{X}_{*}(T)_{+}}(-2+\langle\rho, \mu\rangle) D_{\mu}\right)
\end{aligned}
$$

as claimed.

## Chapter 5

## Slices of Bun ${ }_{G}$

In classical (say, additive) Springer theory, one has the freedom either to work with the stack $\mathfrak{g} / G$, or to pull everything back to the chart $\mathfrak{g}$ and work entirely within the world of affine algebraic varieties. It is also very informative to study transversal slices of $\mathfrak{g}$, which amounts to studying lower dimensional non-surjective charts of $\mathfrak{g} / G$. For example, the section theorem of B. Kostant [K, Theorem 0.10] (and its analogue by R. Steinberg [S5, Theorem 1.4] for the multiplicative case $G / G$ ) shows that slices of minimal dimension give sections of the adjoint quotient map $\mathfrak{g} \rightarrow \mathfrak{g} / / G$, and the work of Brieskorn and Slodowy uses slices of the next lowest dimension to give Lie theoretic constructions of du Val singularities.

In the elliptic context, there is no finite-dimensional chart covering $\mathrm{Bun}_{G}$ to play the role of $\mathfrak{g}$. However, it is still possible to construct incomplete charts, which we can use to destackify the geometry of $\mathrm{Bun}_{G}$ and the elliptic Grothendieck-Springer resolution in low codimension. We will see in this chapter and the next that this construction leads to an analogue of the Kostant and Steinberg section theorems (Theorem 5.4.6) and to interesting simultaneous $\log$ resolutions of families of surfaces.

Given a family of elliptic curves $E \rightarrow S$ with origin $O_{E}: S \rightarrow E$, there is an action of $E$ (endowed with its natural group scheme structure over $S$ ) on itself by translations, and hence on $\operatorname{Bun}_{G}$. Since we were careful in the previous chapter to allow families that do not admit a section, we have an elliptic Grothendieck-Springer resolution (4.5.1) for the family $E^{\prime}:=S \rightarrow \mathbb{B}_{S} E=S^{\prime}$, which is manifestly the quotient of the elliptic Grothendieck-Springer resolution for $E \rightarrow S$ by an action of $E$ compatible with the action on Bun ${ }_{G}$. So in some sense this action adds nothing of interest to the geometry. Rather than working with charts for $\mathrm{Bun}_{G}$, we will therefore work with the following slightly weaker objects.

Definition 5.0.1. Assume that $E \rightarrow S$ has a section $O_{E}: S \rightarrow E$. A slice of Bun ${ }_{G}$ (resp., $\operatorname{Bun}_{G, r i g}$ ) is a morphism $Z \rightarrow \operatorname{Bun}_{G}$ (resp., $Z \rightarrow \operatorname{Bun}_{G, r i g}$ ) of stacks over $S$, such that the composition $Z \rightarrow \operatorname{Bun}_{G} \rightarrow \operatorname{Bun}_{G} / E$ (resp., $Z \rightarrow \operatorname{Bun}_{G, r i g} \rightarrow \operatorname{Bun}_{G, r i g} / E$ ) is smooth.

In this chapter, we give techniques for constructing slices and apply them to give a proof of the Friedman-Morgan section theorem (Theorem 5.4.6). We begin in $\S 5.1$ with a general discussion of equivariant slices, which are slices endowed with some useful extra structure. We then present the parabolic induction construction of Friedman and Morgan [FM2] in $\S 5.2$, which gives a recipe for constructing slices of $\mathrm{Bun}_{G}$ out of slices for a Levi subgroup. In $\S 5.3$ we recall M. Atiyah's classification of stable vector bundles on an elliptic curve $[\mathrm{A}]$ and use it to construct explicit slices for some Levi subgroups. We then apply this machinery in $\S 5.4$ to give a proof of Theorem 5.4.6 using the elliptic Grothendieck-Springer resolution. As in the classical case, the section theorem has a number of important implications for the geometry of the coarse quotient $\chi: \operatorname{Bun}_{G} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ and the Grothendieck-Springer resolution, which we give in §5.5. These implications include the fact (Corollary 5.5.7) that (4.5.1) is a simultaneous $\log$ resolution.

Throughout this chapter, we keep the conventions and notation of Chapter 4. Unless otherwise specified, we will also assume that the elliptic curve $E \rightarrow S$ has a given section
$O_{E}: S \rightarrow E$ and endow $E$ with its natural group scheme structure over $S$ for which $O_{E}$ is the identity.

### 5.1 Equivariant slices

Let $Z \rightarrow \operatorname{Bun}_{G}$ be a slice such that $Z \rightarrow S$ is representable. Then pulling back (4.5.1) gives a commutative diagram

where $\tilde{Z}=\widetilde{\operatorname{Bun}_{G}} \times_{\text {Bun }_{G}} Z$ has finite relative stabilisers over $S$ and maps smoothly to $\widetilde{\operatorname{Bun}}_{G}$. So (5.1.1) gives an approximation to (4.5.1) in which most of the stackiness in the top row has been removed. In this section, we discuss the extra structure that is needed to remove the stackiness in the bottom row of (5.1.1).

Definition 5.1.1. Let $H$ be a torus, and let $\lambda \in \mathbb{X}^{*}(H)$ be a nonzero character. An equivariant slice of $\mathrm{Bun}_{G, \text { rig }}$ with equivariance group $H$ and weight $\lambda$ is a commutative diagram

where $Z$ is a stack with $H$-action over $S$, such that the composition $Z \rightarrow Z / H \rightarrow \operatorname{Bun}_{G, \text { rig }} \rightarrow$ $\operatorname{Bun}_{G, r i g} / E$ is smooth. We will often suppress the group $H$ from the notation and refer to $Z \rightarrow \mathrm{Bun}_{G, r i g}$, or even simply $Z$, as an equivariant slice.

Remark 5.1.2. Unpacking the stacky formalism, the datum of an equivariant slice $Z / H \rightarrow$ $\operatorname{Bun}_{G, \text { rig }}$ of weight $\lambda$ is equivalent to the datum of an $H$-equivariant morphism

$$
\begin{equation*}
Z \longrightarrow\left(\Theta_{\operatorname{Bun}_{G, r i g}}^{-1}\right)^{*}, \tag{5.1.2}
\end{equation*}
$$

where $H$ acts on the complement $\left(\Theta_{\text {Bun }_{G, r i g}}^{-1}\right)^{*}$ of the zero section of $\Theta_{\operatorname{Bun}_{G, \text { rig }}}^{-1}$ through the character $\lambda: H \rightarrow \mathbb{G}_{m}$. Together with the rigidified version of (4.5.1), this gives an $H$-equivariant commutative diagram

where $\tilde{Z}=Z \times_{\text {Bun }_{G, r i g}} \widetilde{\operatorname{Bun}}_{G, \text { rig }}$ and $H$ acts on $\Theta_{Y}^{-1}$ and $\widehat{Y} / / W$ via the character $\lambda$.
Proposition 5.1.3. Let $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be an equivariant slice. Then the composition

$$
\begin{equation*}
Z \longrightarrow\left(\Theta_{\operatorname{Bun}_{G, r i g}}^{-1}\right)^{*} / E \tag{5.1.3}
\end{equation*}
$$

of (5.1.2) with the quotient by $E$ is flat.

Proof. Let $H$ be the equivariance group of $Z$, and $\lambda$ the weight. Since $H$ is a torus, hence smooth over $\operatorname{Spec} \mathbb{Z}$, the morphism $Z / H \rightarrow \operatorname{Bun}_{G, \text { rig }} / E$ is smooth, and in particular flat. Since the morphism

$$
Z / H \times{\operatorname{Bun}_{G, r i g} / E}\left(\left(\Theta_{\operatorname{Bun}_{G, r i g}}^{-1}\right)^{*} / E\right) / H \longrightarrow Z / H
$$

is a gerbe under the flat group scheme $\operatorname{ker}(\lambda) \subseteq H$, the section defined by (5.1.3) is flat, and hence so is the composition

$$
\begin{equation*}
Z / H \longrightarrow Z / H \times_{\operatorname{Bun}_{G, r i g} / E}\left(\left(\Theta_{\operatorname{Bun}_{G, r i g}}^{-1}\right)^{*} / E\right) / H \longrightarrow\left(\left(\Theta_{\operatorname{Bun}_{G, r i g}}^{-1}\right)^{*} / E\right) / H \tag{5.1.4}
\end{equation*}
$$

So the pullback (5.1.3) of (5.1.4) is flat as claimed.
The group action can also be useful for bounding the dimensions of automorphism groups in an equivariant slice.

Proposition 5.1.4. Let $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be an equivariant slice with equivariance group $H$ and suppose that the smooth morphism $Z \rightarrow S$ has finite relative stabilisers and relative dimension d. For any geometric point $z: \operatorname{Spec} k \rightarrow Z$ with corresponding $G$-bundle $\xi_{G, z}$, we have

$$
\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right) \leq d+1-\operatorname{dim} H \cdot z
$$

Proof. Pulling back along Spec $k \rightarrow S$ if necessary, we can assume without loss of generality that $S=\operatorname{Spec} k$. Let $x$ be the image of $z$ in $\operatorname{Bun}_{G, r i g}$ and $x^{\prime}$ its image in $\operatorname{Bun}_{G, r i g} / E$. By $H$-equivariance of $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$, we have

$$
H \cdot z \subseteq Z \times_{\operatorname{Bun}_{G, r i g} / E} \mathbb{B} \operatorname{Aut}\left(x^{\prime}\right)
$$

So

$$
\begin{align*}
\operatorname{dim} \operatorname{Aut}\left(x^{\prime}\right) & =\operatorname{codim}_{\operatorname{Bun}_{G, r i g} / E}\left(\mathbb{B} \operatorname{Aut}\left(x^{\prime}\right)\right)+1 \\
& =\operatorname{codim}_{Z}\left(Z \times_{\operatorname{Bun}_{G, r i g} / E} \mathbb{B} \operatorname{Aut}\left(x^{\prime}\right)\right)+1  \tag{5.1.5}\\
& \leq d+1-\operatorname{dim} H \cdot z
\end{align*}
$$

But we have

$$
\begin{equation*}
\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=\operatorname{dim} \operatorname{Aut}(x) \leq \operatorname{dim} \operatorname{Aut}\left(x^{\prime}\right) \tag{5.1.6}
\end{equation*}
$$

since $\operatorname{Bun}_{G} \rightarrow \operatorname{Bun}_{G, r i g}$ is a $Z(G)$-gerbe and $\operatorname{Bun}_{G, r i g} \rightarrow \operatorname{Bun}_{G, r i g} / E$ is representable. So combining (5.1.5) and (5.1.6) we are done.

Given any slice $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$, equivariant or not, we can consider the pullback $\tilde{Z}=$ $Z \times_{\operatorname{Bun}_{G, r i g}} \widetilde{\operatorname{Bun}}_{\tilde{\sim}, \text { rig }}$, and the divisor with normal crossings $D(Z)=\sum_{\lambda \in \mathbb{X}_{*}(T)_{+}} D_{\lambda}(Z)$, where $D_{\lambda}(Z)=\tilde{Z} \times \widetilde{\text { Bun }_{G, \text { rig }}} D_{\lambda}$. Note that if $Z \rightarrow S$ is of finite type, then all but finitely many $D_{\lambda}(Z)$ will be empty. We remark below on a simple property of the set of nonempty divisors.

Proposition 5.1.5. Let $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be a slice, and $\lambda_{1}, \lambda_{2} \in \mathbb{X}_{*}(T)_{+}$. If $D_{\lambda_{1}+\lambda_{2}}(Z) \neq \emptyset$, then $D_{\lambda_{1}}(Z) \neq \emptyset$ and $D_{\lambda_{2}}(Z) \neq \emptyset$.

Proof. We show that $D_{\lambda_{1}}(Z) \neq \emptyset$; the statement for $D_{\lambda_{2}}(Z)$ follows by symmetry. Choose a geometric point $z$ : Spec $k \rightarrow Z$ over $s: ~ S p e c ~ k \rightarrow S$ with corresponding $G$-bundle $\xi_{G, z} \rightarrow E_{s}$ and a stable map $\sigma: C \rightarrow \xi_{G, z} / B$ corresponding to a point in the interior of $D_{\lambda_{1}+\lambda_{2}}(Z)$. Then by Proposition 3.4.16, $\sigma$ has dual graph

$$
\tau_{\lambda_{1}+\lambda_{2}}^{0}=\stackrel{-\lambda_{1}-\lambda_{2}}{\bullet} \quad \lambda_{1}+\lambda_{2}
$$

Replacing the rational component of $C$ with a chain of two rational curves of degrees $\lambda_{1}$ and $\lambda_{2}$, we therefore also have a stable map $\sigma^{\prime}: C^{\prime} \rightarrow \xi_{G, z} / B$ with dual graph

which lies in $D_{\lambda_{1}}(Z) \cap D_{\lambda_{1}+\lambda_{2}}(Z)$. So $D_{\lambda_{1}}(Z) \neq \emptyset$ as claimed.

### 5.2 Parabolic induction

If $L \subseteq L^{\prime} \subseteq G$ are Levi subgroups and $Z \rightarrow \operatorname{Bun}_{L, \text { rig }}$ is a slice (i.e., a morphism such that $Z \rightarrow \operatorname{Bun}_{L, \text { rig }} / E$ is smooth), then there is a simple procedure for constructing an induced slice $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}$. In this section, we describe this construction, study properties of the induced slices, and show how, when $L^{\prime}=G, \operatorname{Ind}_{L}^{L^{\prime}}(Z)=\operatorname{Ind}_{L}^{G}(Z)$ can be made into an equivariant slice under mild assumptions on $Z \rightarrow \operatorname{Bun}_{L, r i g}$.

Definition 5.2.1. Let $L \subseteq L^{\prime} \subseteq G$ be Levi subgroups, let $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ and let $P \subseteq L^{\prime}$ be the unique parabolic subgroup with Levi factor $L$ such that $-\mu$ is a Harder-Narasimhan vector for $P \subseteq L^{\prime}$ in the sense of Definition 2.5.18. If $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ is a morphism of stacks, the parabolic induction of $Z$ to $L^{\prime}$ is the morphism

$$
\operatorname{Ind}_{L}^{L^{\prime}}(Z)=\operatorname{Bun}_{P, r i g} \times_{\operatorname{Bun}_{L, r i g}} Z \longrightarrow \operatorname{Bun}_{L^{\prime}, r i g}^{\mu^{\prime}}
$$

where $\mu^{\prime} \in \mathbb{X}_{*}\left(Z\left(L^{\prime}\right)^{\circ}\right)_{\mathbb{Q}}$ is the image of $\mu$ under the natural morphism $\mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}} \rightarrow$ $\mathbb{X}_{*}\left(Z\left(L^{\prime}\right)^{\circ}\right)_{\mathbb{Q}}$.

Remark 5.2.2. In the situation of Definition 5.2.1, if the morphism $Z \rightarrow \operatorname{Bun}_{L, r i g}$ factors through $\operatorname{Bun}_{L}$, then we have an isomorphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \cong \operatorname{Bun}_{P} \times_{\operatorname{Bun}_{L}} Z$, and hence a factorisation of $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{L^{\prime}, r i g}^{\mu^{\prime}}$ as $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{L^{\prime}}^{\mu^{\prime}} \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}^{\mu^{\prime}}$.

In the following proposition, we do not assume that $E \rightarrow S$ has a section.
Proposition 5.2.3. Assume that the morphism $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ is smooth. Then so is the morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{L^{\prime}, r i g}^{\mu^{\prime}}$.
Proof. Since $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ is smooth, so is the morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{P, r i g}^{s s, \mu}$. So it suffices to show that the morphism $\mathrm{Bun}_{P, r i g}^{s s, \mu} \rightarrow \operatorname{Bun}_{L^{\prime}, r i g}$ is smooth. By flat descent for smoothness, this is equivalent to showing that $\operatorname{Bun}_{P}^{s s, \mu} \rightarrow \operatorname{Bun}_{L^{\prime}}$ is smooth. The relative tangent complex is

$$
\mathbb{T}=\mathbb{R p r}_{\operatorname{Bun}_{P}^{s,, \mu} *}\left(\xi_{P}^{u n i} \times^{P} \mathfrak{l}^{\prime} / \mathfrak{p}\right)
$$

where $\mathfrak{l}^{\prime}=\operatorname{Lie}\left(L^{\prime}\right), \mathfrak{p}=\operatorname{Lie}(P)$ and $\xi_{P}^{u n i} \rightarrow \operatorname{Bun}_{P}^{s s, \mu} \times{ }_{S} E$ is the universal $P$-bundle. But since $-\mu$ is a Harder-Narasimhan vector for $P$, the vector bundle $\xi_{P} \times{ }^{P} \mathfrak{l}^{\prime} / \mathfrak{p}$ has a filtration whose successive quotients are semistable of positive slope on every fibre of $\operatorname{Bun}_{P}^{s s, \mu} \times{ }_{S} E \rightarrow \operatorname{Bun}_{P}^{s s, \mu}$. So $\mathbb{T}$ is a vector bundle concentrated in degree 0 by Lemma 2.6.3, which proves the claim.

Corollary 5.2.4. Assume that $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ is a slice. Then $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}^{\mu^{\prime}}$ is a slice.

Proof. Apply Proposition 5.2 .3 to the family $E^{\prime}:=S \rightarrow \mathbb{B}_{S} E=: S^{\prime}$.
A key feature of the parabolic induction construction is the existence of a natural action of the torus $Z(L)_{\text {rig }}=Z(L) / Z(G)$ on $\operatorname{Ind}_{L}^{L^{\prime}}(Z)$.

In general, suppose that $X$ is a stack equipped with an action $a: X \times \mathbb{B} H \rightarrow X$ of the classifying stack of some commutative group scheme $H$. Then for any morphism of stacks
$\pi: X^{\prime} \rightarrow X$, there is a canonical action of $H$ on $X^{\prime}$ over $X$ fitting into a commutative diagram

in which both squares are Cartesian, where the morphism $X \rightarrow X \times \mathbb{B} H$ is the quotient by the trivial action of $H$ on $X$. Explicitly, this action can be realised by using the outer square of (5.2.1) to identify $X^{\prime}$ with the stack of tuples ( $x^{\prime} \in X^{\prime}, x \in X, \phi: x \xrightarrow{\sim} \pi\left(x^{\prime}\right)$ ) and setting $\left(x^{\prime}, x, \phi\right) \cdot h=\left(x^{\prime}, x, \phi \circ a(h)\right)$ for $h \in H$.

Now suppose we are in the situation of Definition 5.2.1. Applying the above construction to the action $\operatorname{Bun}_{L, \text { rig }}$ of $\mathbb{B} Z(L)_{\text {rig }}$ on $\mathrm{Bun}_{L, \text { rig }}$ inherited from the action of $\mathbb{B} Z(L)$ on $\mathrm{Bun}_{L}$ gives an action of $Z(L)_{\text {rig }}$ on $\operatorname{Bun}_{P, \text { rig }}$ over $\mathrm{Bun}_{L, \text { rig }}$ and a morphism Bun ${ }_{P, \text { rig }} / Z(L)_{\text {rig }} \rightarrow$ $\operatorname{Bun}_{P, \text { rig }} \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}$. Pulling back along $Z \rightarrow \mathrm{Bun}_{L, \text { rig }}$, we get an action of $Z(L)_{\text {rig }}$ on $\operatorname{Ind}_{L}^{L^{\prime}}(Z)$ over $Z$ and a morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) / Z(L)_{\text {rig }} \rightarrow \operatorname{Bun}_{L^{\prime}, r i g}^{\mu^{\prime}}$.
Remark 5.2.5. If the morphism $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ factors through $\operatorname{Bun}_{L}^{s s, \mu}$, then the $\mathbb{B} Z(L)$ action on $\operatorname{Bun}_{L}$ gives a morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) / Z(L) \rightarrow \operatorname{Bun}_{L^{\prime}}^{\mu^{\prime}}$ and a Cartesian diagram


Note that $\operatorname{Ind}_{L}^{L^{\prime}}(Z) / Z(L) \rightarrow \operatorname{Bun}_{L^{\prime}}^{\mu^{\prime}}$ factors through $\operatorname{Ind}_{L}^{L^{\prime}}(Z) / Z(L)_{\text {rig }}$ if and only if the left hand morphism above admits a section, which holds if and only if $Z(L)=Z(G) \times Z(L)_{\text {rig }}$.

The following proposition describes the structure of the natural morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow Z$ together with the $Z(L)_{\text {rig }}$-action constructed above.

Proposition 5.2.6. Assume we are in the setup of Definition 5.2.1. If $Z$ is an affine scheme, then there exists a (non-canonical) $Z(L)_{\text {rig-equivariant isomorphism of stacks over }}$ $Z$,

$$
\operatorname{Ind}_{L}^{L^{\prime}}(Z) \cong \mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}\right)
$$

where $\xi_{L / Z(G)} \rightarrow Z \times_{S} E$ is the $L / Z(G)$-bundle induced by the morphism $Z \rightarrow \operatorname{Bun}_{L, \text { rig }} \rightarrow$ $\operatorname{Bun}_{L / Z(G)}, \mathfrak{u}$ is the Lie algebra of the unipotent radical $R_{u}(P)$, and $Z(L)_{\text {rig }}$ acts on $\mathfrak{u}$ by right conjugation. Hence, for any $Z$ (not necessarily affine), the morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow Z$ is always an affine space bundle with fibrewise linear $Z(L)_{\text {rig-action }}$.

Proof. Let $\xi_{L / Z(G)}^{u n i} \rightarrow \operatorname{Bun}_{L, \text { rig }} \times_{S} E$ be the $L / Z(G)$-bundle classified by the morphism $\operatorname{Bun}_{L, r i g} \rightarrow \operatorname{Bun}_{L / Z(G)}$, and let $\mathcal{U}^{u n i}=\xi_{L / Z(G)}^{u n i} \times{ }^{L / Z(G)} R_{u}(P)$. It follows directly from the construction that the $\mathbb{B} Z(L)_{\text {rig }}$-action induces the right conjugation action $(x, u) \cdot g=$ $\left(x, g^{-1} u g\right)$ of $Z(L)_{\text {rig }}$ on the group scheme $\mathcal{U}^{u n i}$. Letting $\xi_{L / Z(G)}$ be the pullback of $\xi_{L / Z(G)}^{u n i}$ to $Z$ and $\mathcal{U}=\xi_{L / Z(G)} \times{ }^{L / Z(G)} R_{u}(P)$, it follows that the action of $Z(L)_{\text {rig }}$ on $\operatorname{Ind}_{L}^{L^{\prime}}(Z)=$ $\operatorname{Bun}_{\mathcal{U} / Z}$ is also given by right conjugation.

Let $0<\mu_{1}<\cdots<\mu_{n}$ be the possible positive values of $\langle\alpha,-\mu\rangle$ for $\alpha \in \Phi_{L^{\prime}}$, and let

$$
\{1\}=U_{n+1} \subseteq U_{n} \subseteq \cdots \subseteq U_{1}=R_{u}(P)
$$

be the filtration defined by

$$
U_{i}=\prod_{\substack{\alpha \in \Phi_{L^{\prime}} \\\langle\alpha,-\mu\rangle \geq \mu_{i}}} U_{\alpha}
$$

for $1 \leq i \leq n+1$. Letting

$$
\mathcal{U}_{i}=\xi_{L / Z(G)} \times{ }^{L / Z(G)} U_{i} \subseteq \mathcal{U}
$$

we show by induction on $i$ that each $\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{i}\right) / Z}(E)$ is $Z(L)_{\text {rig }}$-equivariantly isomorphic to the vector bundle

$$
\mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u} / \mathfrak{u}_{i}\right)=\bigoplus_{j \leq i} \mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}_{j-1} / \mathfrak{u}_{j}\right)
$$

where $\mathfrak{u}_{i}$ is the Lie algebra of $R_{u}(P)_{i}$.
For $i=1$, the claim is trivial. For $i>1$, Proposition 2.4.2 implies that each morphism

$$
\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{i}\right) / Z}(E) \longrightarrow \operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{i-1}\right) / Z}(E)
$$

is a $Z(L)_{\text {rig }}$-equivariant $\operatorname{Bun}_{\left(\mathcal{U}_{i-1} / \mathcal{U}_{i}\right) / Z}(E)$-torsor. But $U_{i-1} / U_{i} \cong \mathfrak{u}_{i-1} / \mathfrak{u}_{i}$ as $L / Z(G)$ equivariant group schemes. So by Proposition 2.4.1,

$$
\operatorname{Bun}_{\left(\mathcal{U}_{i-1} / \mathcal{U}_{i}\right) / Z}(E) \cong \operatorname{Bun}_{\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right) / Z}(E)=\mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right)
$$

since

$$
H^{0}\left(E_{s}, \xi_{L / Z(G), z} \times^{L / Z(G)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right)=0
$$

for any geometric point $z$ : Spec $k \rightarrow Z$ over $s$ : Spec $k \rightarrow S$, as this is the space of global sections of a vector bundle all of whose semistable factors have negative degree. By induction, $V_{i}=\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{U}_{i-1}\right) / Z}(E)$ is a vector bundle on $Z$ with linear $Z(L)_{\text {rig }}$-action, so the $Z(L)_{\text {rig }}{ }^{-}$ equivariant torsors on it are classified by the group

$$
\begin{aligned}
H^{1}\left(V_{i} / Z(L)_{\text {rig }}\right. & \left., \mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(L)} \times{ }^{L / Z(L)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right)\right) \\
& =H^{1}\left(\mathbb{B} Z(L)_{\text {rig }}, H^{0}\left(V_{i}, \mathcal{O}_{V_{i}}\right) \otimes_{H^{0}\left(Z, \mathcal{O}_{Z}\right)} H^{1}\left(E, \xi_{L / Z(L)} \times{ }^{L / Z(L)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right)\right) \\
& =0
\end{aligned}
$$

since $Z(L)_{\text {rig }}$ is a torus and $Z$ is affine. So we can trivialise the given torsor $Z(L)_{\text {rig }}$ equivariantly, to give a $Z(L)_{\text {rig }}$-equivariant isomorphism

$$
\begin{aligned}
\operatorname{Bun}_{\left(\mathcal{U} / \mathcal{u}_{i}\right) / Z}(E) & \cong \operatorname{Bun}_{\left(\mathcal{U} / \mathcal{u}_{i-1}\right) / Z}(E) \times{ }_{Z} \mathbb{R}^{1} \mathrm{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}_{i-1} / \mathfrak{u}_{i}\right) \\
& \cong \bigoplus_{j \leq i} \mathbb{R}^{1} \operatorname{pr}_{Z *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} \mathfrak{u}_{j-1} / \mathfrak{u}_{j}\right),
\end{aligned}
$$

as claimed.
The next two propositions give root-theoretic formulas for the $Z(L)_{\text {rig }}$-weights and dimension of the affine space bundle $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow Z$.

Proposition 5.2.7. If $\lambda \in \mathbb{X}^{*}\left(Z(L)_{\text {rig }}\right)$, then the multiplicity of the weight $\lambda$ in a fibre of $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow Z$ is $d_{-\lambda}\langle\lambda, \mu\rangle$, if $\langle\lambda, \mu\rangle>0$ and 0 otherwise, where $d_{-\lambda}$ is the number of $\alpha \in \Phi_{L^{\prime}}$ such that $\left.\alpha\right|_{Z(L)_{r i g}}=-\lambda$.
Proof. By Proposition 5.2.6, the multiplicity of $\lambda$ in $\operatorname{Ind}_{L}^{L^{\prime}}(Z)$ is equal to the multiplicity in $H^{1}\left(E_{s}, \xi_{L} \times^{L} \mathfrak{u}\right)$, where $E_{s}$ is any geometric fibre of $E \rightarrow S, \xi_{L} \rightarrow E_{s}$ is a semistable
$L$-bundle of slope $\mu$, and $Z(L)_{\text {rig }}$ acts by right conjugation on $\mathfrak{u}$. But this is equal to the dimension of $H^{1}\left(E_{s}, \xi_{L} \times{ }^{L} \mathfrak{u}_{-\lambda}\right)$, where

$$
\mathfrak{u}_{-\lambda}= \begin{cases}\bigoplus_{\substack{\left.\alpha \in \Phi_{L^{\prime}} \\ \alpha\right|_{Z(L)_{r i g}=-\lambda}}} \mathfrak{g}_{\alpha}, & \text { if }\langle\lambda, \mu\rangle>0 \\ 0, & \text { otherwise }\end{cases}
$$

is the $\lambda$-weight space of $Z(L)_{\text {rig }}$ acting on $\mathfrak{u}$ by right conjugation. But since $\xi_{L} \times{ }^{L} \mathfrak{u}_{-\lambda}$ is either 0 or a semistable vector bundle of negative slope $-\langle\lambda, \mu\rangle$, it follows that

$$
\operatorname{dim} H^{1}\left(\xi_{L} \times{ }^{L} \mathfrak{u}_{-\lambda}\right)=-\operatorname{deg}\left(\xi_{L} \times{ }^{L} \mathfrak{u}_{-\lambda}\right)= \begin{cases}d_{-\lambda}\langle\lambda, \mu\rangle, & \text { if }\langle\lambda, \mu\rangle>0 \\ 0, & \text { otherwise }\end{cases}
$$

which proves the claim.
Proposition 5.2.8. The morphism $\operatorname{Ind}_{L}^{L^{\prime}}(Z) \rightarrow Z$ has relative dimension $\left\langle 2 \rho_{P}, \mu\right\rangle$, where $-2 \rho_{P}$ is the sum of all roots $\alpha \in \Phi$ such that $U_{\alpha} \subseteq R_{u}(P)$.
Proof. This follows from Proposition 5.2.7 after taking the sum over all $\lambda$.
When $L^{\prime}=G$, it is often the case that the $Z(L)_{r i g}$-action on $\operatorname{Ind}_{L}^{G}(Z)$ can be promoted to the structure of an equivariant slice. The extra structure on the initial slice $Z \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ required to make this happen is the following.

Definition 5.2.9. A $\Theta$-trivial slice of $\operatorname{Bun}_{L, r i g}^{s s, \mu}$ is a slice $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ equipped with a trivialisation of the pullback $\Theta_{\operatorname{Bun}_{L, r i g}^{s s, \mu}}$ of the theta bundle $\Theta_{\mathrm{Bun}_{G, r i g}}$.
Proposition 5.2.10. Let $Z \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ be a $\Theta$-trivial slice. Then there is a natural equivariant slice structure on $\operatorname{Ind}_{L}^{G}(Z) \rightarrow \operatorname{Bun}_{G, \text { rig }}$ with equivariance group $Z(L)_{\text {rig }}$ and weight ( $\mu \mid-$ ).

A key step in the proof of Proposition 5.2.10 is a computation of the action of $Z(L)_{\text {rig }}$ on the pullback of the theta bundle to $Z$. Before we give this computation, it will be useful to introduce the following terminology.

Definition 5.2.11. Let $X$ be a connected stack equipped with an action $a: X \times \mathbb{B} H \rightarrow X$ of the classifying stack of a commutative group scheme $H$. If $\mathcal{L}$ is a line bundle on $X$, then weight of $\mathcal{L}$ is the image of $\mathcal{L} \in \operatorname{Pic}(X)$ under the homomorphism

$$
\operatorname{Pic}(X) \xrightarrow{a^{*}} \operatorname{Pic}(X \times \mathbb{B} H) \cong \operatorname{Pic}(X) \oplus \mathbb{X}^{*}(H) \longrightarrow \mathbb{X}^{*}(H),
$$

where the isomorphism above is given by

$$
\begin{aligned}
\operatorname{Pic}(X) \oplus \mathbb{X}^{*}(H) & \longrightarrow \operatorname{Pic}(X \times \mathbb{B} H) \\
(\mathcal{L}, \lambda) & \longmapsto p^{*} \mathcal{L} \otimes\left(\eta_{H} \times{ }^{H} \mathbb{Z}_{\lambda}\right)
\end{aligned}
$$

for $\eta_{H} \rightarrow \mathbb{B} H$ the universal $H$-torsor.
Remark 5 .2.12. It follows tautologically from the definition that whenever $f: X \rightarrow \mathbb{B} \mathbb{G}_{m}$ classifies a line bundle $\mathcal{L}$ with weight $\lambda$, the diagram

commutes, where the bottom arrow is given by tensor product of line bundles.

Proposition 5.2.13. With respect to the natural $\mathbb{B} Z(L)_{\text {rig }}$-action, the weight of the pullback $\Theta_{\operatorname{Bun}_{L, r i g}^{s s, \mu}}$ of $\Theta_{\operatorname{Bun}_{G, r i g}}$ to $\operatorname{Bun}_{L, r i g}^{s s, \mu}$ is given by

$$
(-\mu \mid-) \in \operatorname{Hom}\left(\mathbb{X}_{*}\left(Z(L)_{\text {rig }}\right), \mathbb{Z}\right)=\mathbb{X}^{*}\left(Z(L)_{\text {rig }}\right)
$$

Proof. We will in fact show that for any $\mathcal{L} \in \operatorname{Pic}\left(\operatorname{Bun}_{G, \text { rig }}\right)$, the pullback of $\mathcal{L}$ to $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ has weight $-Q(\mathcal{L})(\mu,-)$, where $Q(\mathcal{L})$ is the quadratic class of the corresponding $W$-linearised line bundle on $Y$. Since this statement is invariant under tensoring $\mathcal{L}$ with a line bundle on the base stack $S$ and raising $\mathcal{L}$ to a nonzero power, by Corollary 4.4.10, it suffices to show this for a single nontrivial line bundle $\mathcal{L}$.

Choose any nontrivial representation $V$ of $G / Z(G)$, and set

$$
\mathcal{L}=\operatorname{det} \mathbb{R p r}_{\operatorname{Bun}_{G, r i g} *}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} V\right)
$$

where $\xi_{G / Z(G)} \rightarrow \operatorname{Bun}_{G, r i g} \times{ }_{S} E$ is the $G / Z(G)$-bundle classified by the morphism Bun ${ }_{G, r i g} \rightarrow$ $\operatorname{Bun}_{G / Z(G)}$. Then the pullback of $\mathcal{L}$ to $\operatorname{Bun}_{L, r i g}^{s s, \mu}$ is the line bundle

$$
\begin{aligned}
\mathcal{L}_{\operatorname{Bun}_{L, \text { rig }}^{s s, \mu}} & =\operatorname{det} \mathbb{R}_{\operatorname{pr}}^{\operatorname{Bun}_{L, r i g}^{s s, \mu} *} \\
& =\bigotimes_{\lambda \in \mathbb{X}^{*}\left(Z(L)_{\text {rig }}\right)} \operatorname{det} \mathbb{R}_{L / Z(G)} \times_{\operatorname{Bun}_{L, r i g}^{s s, \mu}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} V_{\lambda}\right),
\end{aligned}
$$

where $\xi_{L / Z(G)} \rightarrow \operatorname{Bun}_{L, \text { rig }} \times_{S} E$ is the $L / Z(G)$-bundle classified by the natural morphism $\operatorname{Bun}_{L, r i g} \rightarrow \operatorname{Bun}_{L / Z(G)}$, and $V=\bigoplus_{\lambda \in \mathbb{X}^{*}\left(Z(L)_{r i g}\right)} V_{\lambda}$ is the weight space decomposition of $V$ under the action of the torus $Z(L)_{\text {rig }}=Z(L / Z(G))$. Pulling back along the action morphism $a: \operatorname{Bun}_{L, r i g}^{s s, \mu} \times \mathbb{B} Z(L)_{\text {rig }} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$, we get

$$
\begin{aligned}
a^{*} \mathbb{R p r}_{\operatorname{Bun}_{L, r i g}^{s s, \mu} *}\left(\xi_{L / Z(G)}\right. & \left.\times{ }^{L / Z(G)} V_{\lambda}\right) \\
& ={\mathbb{R} \operatorname{pr}_{\operatorname{Bun}_{L, r i g}^{s s, \mu} \times \mathbb{B} Z(L)_{r i g} *}\left(\left(p^{*} \xi_{L / Z(G)} \otimes q^{*} \eta_{Z(L)_{r i g}}\right) \times{ }^{L / Z(G)} V_{\lambda}\right)}={\mathbb{R} \operatorname{pr}_{\mathrm{Bun}_{L, r i g}^{s s, \mu} \times \mathbb{B} Z(L)_{r i g} *}\left(p^{*}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} V_{\lambda}\right) \otimes q^{*} \lambda\left(\eta_{Z(L)_{r i g}}\right)\right)}=\bar{p}^{*}\left(\mathbb{R p r}_{\operatorname{Bun}_{L, r i g}^{s s, \mu} *}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} V_{\lambda}\right)\right) \otimes \bar{q}^{*} \lambda\left(\eta_{Z(L)_{r i g}}\right)
\end{aligned}
$$

where $\bar{p}: \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} \times \mathbb{B} Z(L)_{\text {rig }} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ and $\bar{q}: \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} \times \mathbb{B} Z(L)_{\text {rig }} \rightarrow \mathbb{B} Z(L)_{\text {rig }}$ are the natural projections, $p$ and $q$ are their respective compositions with the projection ( $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu} \times$ $\left.\mathbb{B} Z(L)_{\text {rig }}\right) \times{ }_{S} E \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} \times \mathbb{B} Z(L)_{\text {rig }}$, and $\eta_{Z(L)_{\text {rig }}}$ is the universal $Z(L)_{\text {rig }}$-bundle on $\mathbb{B} Z(L)_{\text {rig }}$. So the weight of the determinant $\mathcal{L}_{\text {Bun }_{L, r i g}^{s s, \mu}}$ is therefore

$$
\begin{aligned}
& \sum_{\lambda \in \mathbb{X}^{*}\left(Z(L)_{r i g}\right)} \chi\left(\mathbb{R}_{\left.\operatorname{pr}_{\operatorname{Bun}_{L, r i g}^{s s, \mu}}\left(\xi_{L / Z(G)} \times^{L / Z(G)} V_{\lambda}\right)\right) \lambda}=\sum_{\lambda \in \mathbb{X}^{*}\left(Z(L)_{r i g}\right)} \operatorname{dim} V_{\lambda}\langle\lambda, \mu\rangle \lambda\right. \\
&=-Q(\mathcal{L})(\mu,-)
\end{aligned}
$$

as claimed, where the last equality follows from Lemma 4.5.12.
Proof of Proposition 5.2.10. Since $\operatorname{Ind}_{L}^{G}(Z) \rightarrow \operatorname{Bun}_{G, \text { rig }}$ is a slice by Proposition 5.2.3, we just need to construct an isomorphism of the pullback of $\Theta_{\text {Bun }_{G, \text { rig }}}^{-1}$ to $\operatorname{Ind}_{L}^{G}(Z) / Z(L)_{\text {rig }}$ with the line bundle classified by

$$
\operatorname{Ind}_{L}^{G}(Z) / Z(L)_{\text {rig }} \longrightarrow \mathbb{B} Z(L)_{\text {rig }} \xrightarrow{(\mu \mid-)} \mathbb{B} \mathbb{G}_{m}
$$

Since $\operatorname{Ind}_{L}^{G}(Z) / Z(L)_{\text {rig }} \rightarrow Z \times \mathbb{B} Z(L)_{\text {rig }}$ is an affine space bundle by Proposition 5.2.6, the pullback of $\Theta_{\operatorname{Bun}_{G, \text { rig }}}^{-1}$ to $\operatorname{Ind}_{L}^{G}(Z) / Z(L)_{\text {rig }}$ is canonically isomorphic to the pullback of its
restriction $\Theta_{Z \times \mathbb{B} Z(L)_{r i g}}^{-1}$ to the zero section $Z \times \mathbb{B} Z(L)_{\text {rig }}$, i.e., of the pullback of $\Theta_{\operatorname{Bun}}^{-1}$,rig along the morphism

$$
Z \times \mathbb{B} Z(L)_{r i g} \longrightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu} \longrightarrow \operatorname{Bun}_{G, r i g} .
$$

Since the first morphism above is $\mathbb{B} Z(L)_{\text {rig }}$-equivariant, Proposition 5.2.13 implies that $\Theta_{Z \times \mathbb{B} Z(L)_{r i g}}^{-1}$ has weight $(\mu \mid-)$. But the trivialisation of the pullback to $Z$ identifies $\Theta_{Z \times \mathbb{B} Z(L)_{r i g}}^{-1}$ with the pullback of a line bundle on $\mathbb{B} Z(L)_{\text {rig }}$, which must be associated to the character $(\mu \mid-)$, so we are done.

The following technical lemmas will come in handy for some of our explicit computations later on.

Lemma 5.2.14. Assume that $L \subseteq G$ is the Levi factor of a standard parabolic $P^{-}$and that $\mu$ is a Harder-Narasimhan vector for $P^{-}$. Let $Z \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ be a slice such that $Z \rightarrow S$ has finite relative stabilisers and is of relative dimension $d$, and fix a point $z \in \operatorname{Ind}_{L}^{G}(Z)$ with corresponding $G$-bundle $\xi_{G, z}$. If there exists a section of $\xi_{G, z} / Q$ of degree $\nu$, where $Q$ is any standard parabolic with Harder-Narasimhan vector $\nu$ and $(Q, \nu) \neq\left(P^{-}, \mu\right)$, then
(1) there exists $z^{\prime} \in \operatorname{Ind}_{L}^{G}(Z)$ such that the corresponding $G$-bundle $\xi_{G, z^{\prime}}$ has HarderNarasimhan reduction to $Q$ with degree $\nu$, and
(2) $-\langle 2 \rho, \nu\rangle \leq-\langle 2 \rho, \mu\rangle+d-1$.

Proof. The assumptions of the proposition imply that the stack $\operatorname{Ind}_{L}^{G}(Z) \times{ }_{B_{u n}^{G, r i g}} \operatorname{Bun}_{Q, \text { rig }}^{\nu}$ is nonempty. Since $\operatorname{Ind}_{L}^{G}(Z) \rightarrow \operatorname{Bun}_{G, \text { rig }} / E$ is smooth, the preimage $\operatorname{Ind}_{L}^{G}(Z) \times{ }_{\operatorname{Bun}_{G, r i g}}$ $\operatorname{Bun}_{Q, r i g}^{s s, \nu}$ of $\operatorname{Bun}_{Q, \text { rig }}^{s s, \nu} / E$ under the morphism

$$
\operatorname{Ind}_{L}^{G}(Z) \times_{\operatorname{Bun}_{G, r i g}} \operatorname{Bun}_{Q, r i g}^{\nu}=\operatorname{Ind}_{L}^{G}(Z) \times_{\operatorname{Bun}_{G, r i g} / E} \operatorname{Bun}_{Q, \text { rig }} / E \longrightarrow \operatorname{Bun}_{Q, r i g}^{\nu} / E
$$

is dense, hence nonempty. This proves (1). Since $(Q, \nu) \neq\left(P^{\prime}, \mu\right)$, by uniqueness of HarderNarasimhan reductions, the $Z(L)_{\text {rig }}$-invariant locally closed substack $\operatorname{Ind}_{L}^{G}(Z) \times_{\text {Bun }_{G, r i g}}$ $\operatorname{Bun}_{Q, r i g}^{s s, \nu} \subseteq \operatorname{Ind}_{L}^{G}(Z)$ is disjoint from the $Z(L)_{\text {rig }}$-fixed locus $Z \subseteq \operatorname{Ind}_{L}^{G}(Z)$. Since $\operatorname{Ind}_{L}^{G}(Z) \rightarrow$ $S$ has finite relative stabilisers, $\operatorname{Ind}_{L}^{G}(Z) \times{ }_{\text {Bun }_{G, \text { rig }}} \operatorname{Bun}_{Q, \text { rig }}^{\nu} \rightarrow S$ is therefore flat of relative dimension at least 1 , and hence has codimension at most

$$
\operatorname{dim}_{S} \operatorname{Ind}_{L}^{G}(Z)-1=\left\langle 2 \rho_{P}, \mu\right\rangle+d-1 .
$$

But this codimension is equal to the codimension $-\langle 2 \rho, \nu\rangle$ of $\operatorname{Bun}_{Q, \text { rig }}^{s s, \nu} / E$ in $\operatorname{Bun}_{G, \text { rig }} / E$, so we have

$$
-\langle 2 \rho, \nu\rangle \leq\left\langle 2 \rho_{P}, \mu\right\rangle+d-1=-\langle 2 \rho, \mu\rangle+d-1,
$$

which proves (2).
Lemma 5.2.15. Let $Z \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ be a morphism, and assume that $Z$ is connected. Then the set $\operatorname{Ind}_{L}^{G}(Z)^{u}$ of points $z \in \operatorname{Ind}_{L}^{G}(Z)$ such that the corresponding $G$-bundle is unstable is connected.

Proof. It follows easily from Proposition 5.2.6 that for any $z \in \operatorname{Ind}_{L}^{G}(Z)$, the closure of the $Z(L)_{\text {rig }}$-orbit of $z$ has nonempty intersection with $Z$. The claim now follows immediately from the fact that $\operatorname{Ind}_{L}^{G}(Z)^{u}$ is closed and $Z(L)_{\text {rig }}$-invariant and contains $Z$.

### 5.3 Stable bundles under Levi subgroups

In this section, we give an explicit description of the stack of stable $L$-bundles of fixed degree when $L \subseteq G$ is a Levi subgroup all of whose simple factors are of type $A$. These results form the base case for constructing slices of $\mathrm{Bun}_{G}$ and of more complicated Levi subgroups using parabolic induction.

For this section, we will not assume that $E \rightarrow S$ necessarily has a section.
We begin with the following explicit description of the relevant Levi subgroups. Let $t \subseteq \Delta$ be a set of simple roots, and let $L$ be the Levi factor of the standard parabolic subgroup of type $t$. Then $L$ is the split reductive group scheme with root datum $\left(\mathbb{X}^{*}(T), \Phi_{t}, \mathbb{X}_{*}(T), \Phi_{t}^{\vee}\right)$, where $\Phi_{t} \subseteq \Phi$ is the set of roots that are linear combinations of $\alpha_{i}$ for $\alpha_{i} \in \Delta \backslash t$. The Dynkin diagram of $L$ is obtained from the Dynkin diagram of $G$ by deleting the nodes corresponding to elements of $t$. We will assume that the connected components of the Dynkin diagram of $L$ are all of type $A$.

The reductive group $L$ can be described directly in terms of the following data. First, write $\pi_{0}=\pi_{0}(\Delta \backslash t)$ for the set of connected components of the Dynkin diagram of $L$. For each component $c \in \pi_{0}$, write $n_{c}$ for the number of nodes in $c$, and choose a labelling $\alpha_{c, 1}, \ldots, \alpha_{c, n_{c}}$ of the nodes of $c$ so that $\alpha_{c, i}$ is adjacent to $\alpha_{c, i+1}$ for $1 \leq i \leq n_{c}-1$. For each $\alpha_{k} \in t$ adjacent to a node of $c$, let $\alpha_{c, i_{k, c}}$ be the unique node adjacent to $\alpha_{k}$, and for each $\alpha_{k} \in t$ not adjacent to any node of $c$, set $i_{k, c}=n_{c}+1$. Finally, write

$$
m_{k, c}=-\sum_{i=1}^{n_{c}}\left\langle\alpha_{c, i}, \alpha_{k}^{\vee}\right\rangle= \begin{cases}-\left\langle\alpha_{c, i_{k, c}}, \alpha_{k}^{\vee}\right\rangle, & \text { if } i_{k, c} \leq n_{c} \\ 0, & \text { if } i_{k, c}=n_{c}+1\end{cases}
$$

for $c \in \pi_{0}$ and $\alpha_{k} \in t$.
Proposition 5.3.1. Assume we are in the setup above. Then there is an isomorphism

$$
\begin{equation*}
L \xrightarrow{\sim}\left\{\left(\left(A_{c}\right)_{c \in \pi_{0}},\left(\lambda_{k}\right)_{\alpha_{k} \in t}\right) \in \prod_{c \in \pi_{0}} G L_{n_{c}+1} \times \prod_{\alpha_{k} \in t} \mathbb{G}_{m} \mid \operatorname{det} A_{c}=\prod_{\alpha_{k} \in t} \lambda_{k}^{m_{k, c}\left(n_{c}+1-i_{k, c}\right)}\right\} \tag{5.3.1}
\end{equation*}
$$

with the property that for each $\alpha_{k} \in t$, the character $\varpi_{k}$ of $L$ is given by (5.3.1) composed with the projection $\left(\left(A_{c}\right)_{c \in \pi_{0}},\left(\lambda_{j}\right)_{\alpha_{j} \in t}\right) \mapsto \lambda_{k}$.

Proof. Since both sides of (5.3.1) are split reductive groups over $\operatorname{Spec} \mathbb{Z}$, it is enough to specify an isomorphism between their root data.

The root datum $\left(M_{0}, \Psi_{0}, M_{0}^{\vee}, \Psi_{0}^{\vee}\right)$ of $\prod_{c \in \pi_{0}} G L_{n_{c}+1} \times \prod_{\alpha_{k} \in t} \mathbb{G}_{m}$ is specified as follows. The weight lattice is

$$
M_{0}=\bigoplus_{c \in \pi_{0}} \mathbb{Z}^{n_{c}+1} \oplus \bigoplus_{\alpha_{k} \in t} \mathbb{Z} \omega_{k} .
$$

The roots and coroots $\Psi_{0}$ and $\Psi_{0}^{\vee}$ are determined by requiring that

$$
\left\{\beta_{c, j}=e_{c, j}-e_{c, j+1} \mid c \in \pi_{0} \text { and } 1 \leq j \leq n_{c}\right\} \subseteq M_{0}
$$

be a set of positive simple roots for $\Psi_{0}$, and that

$$
\beta_{c, j}^{\vee}=e_{c, j}^{*}-e_{c, j+1}^{*}
$$

where $\left\{e_{c, j} \mid 1 \leq j \leq n_{c}+1\right\}$ is the standard basis for $\mathbb{Z}^{n_{c}+1}$, and $e_{c, j}^{*} \in M_{0}^{\vee}$ satisfies

$$
\left\langle e_{c^{\prime}, j^{\prime}}, e_{c, j}^{*}\right\rangle=\left\{\begin{array}{ll}
1, & \text { if }\left(c^{\prime}, j^{\prime}\right)=(c, j), \\
0, & \text { otherwise },
\end{array} \quad \text { and } \quad\left\langle\omega_{k}, e_{c, j}^{*}\right\rangle=0\right.
$$

The root datum $\left(M, \Psi, M^{\vee}, \Psi^{\vee}\right)$ is given by setting

$$
M=\frac{M_{0}}{\mathbb{Z}-\operatorname{span}\left\{\sum_{j=1}^{n_{c}+1} e_{c, j}-\sum_{\alpha_{k} \in t} m_{k, c}\left(n_{c}+1-i_{k, c}\right) \omega_{k} \mid c \in \pi_{0}\right\}}
$$

setting $\Psi$ to be the image of $\Psi_{0}$ in $M$, and setting $\Psi^{\vee} \subseteq M^{\vee}$ to be the preimage of $\Psi_{0}^{\vee}$ under the injection $M^{\vee} \hookrightarrow M_{0}^{\vee}$. Note that $M$ is indeed a lattice, so this is the root datum of a connected reductive group.

We define an isomorphism of $\left(M, \Psi, M^{\vee}, \Psi^{\vee}\right)$ with the root datum $\left(\mathbb{X}^{*}(T), \Phi_{t}, \mathbb{X}_{*}(T), \Phi_{t}^{\vee}\right)$ via the isomorphism

$$
\begin{aligned}
\phi: \mathbb{X}_{*}(T) & \stackrel{\sim}{\longrightarrow} M^{\vee} \\
\alpha_{c, j}^{\vee} & \longmapsto e_{c, j}^{*}-e_{c, j+1}^{*} \\
\alpha_{k}^{\vee} & \longmapsto \omega_{k}^{*}+\sum_{c \in \pi_{0}} \sum_{j=i_{k, c}+1}^{n_{c}+1} m_{k, c} e_{c, j}^{*},
\end{aligned}
$$

for $c \in \pi_{0}, 1 \leq j \leq n_{c}$ and $\alpha_{k} \in t$, where $\omega_{k}^{*} \in M_{0}^{\vee}$ satisfies $\left\langle e_{c, j}, \omega_{k}^{*}\right\rangle=0$ and $\left\langle\omega_{k^{\prime}}, \omega_{k}^{*}\right\rangle=$ $\delta_{k, k^{\prime}}$. It is clear by inspection that $\phi$ is a well-defined homomorphism of free abelian groups such that the dual is surjective. Since $M^{\vee}$ and $\mathbb{X}_{*}(T)$ have the same rank, $\phi$ is therefore an isomorphism. To prove that $\phi$ defines an isomorphism of root data, it is enough to show that $\phi: \mathbb{X}_{*}(T) \rightarrow M^{\vee}$ sends $\alpha_{c, j}^{\vee}$ to $\beta_{c, j}^{\vee}$ and that $\phi^{*}: M \rightarrow \mathbb{X}^{*}(T)$ sends $\beta_{c, j}$ to $\alpha_{c, j}$ for all $c \in \pi_{0}$ and $1 \leq j \leq n_{c}$. This is easily checked by direct calculation, so we are done.

Next, we state a version of Atiyah's classification [A] of stable vector bundles on an elliptic curve, adapted to our context.

Theorem 5.3.2. Let $r>0$ and $d$ be coprime integers. Then the determinant morphism

$$
\begin{equation*}
\text { det: } \operatorname{Bun}_{G L_{r}}^{s s, d} \longrightarrow \operatorname{Pic}_{S}^{d}(E) \tag{5.3.2}
\end{equation*}
$$

from the stack of semistable vector bundles on $E$ of rank $r$ and degree $d$ to the Picard variety of degree d line bundles on $E$ is a $\mathbb{G}_{m}$-gerbe, where $\mathbb{B} \mathbb{G}_{m}$ acts on $\operatorname{Bun}_{G L_{r}}^{s s, d}$ through the centre $\mathbb{G}_{m}=Z\left(G L_{r}\right)$ in the usual way. If $E \rightarrow S$ has a section, then the gerbe (5.3.2) is trivial.

Remark 5.3.3. If $V \rightarrow E_{s}$ is a vector bundle on a geometric fibre of $E \rightarrow S$ whose rank and degree are coprime, then it is easy to see that semistability, stability, and indecomposability of $V$ are all equivalent. Moreover, if the rank and degree are not coprime, then $V$ is never stable.

Proof of Theorem 5.3.2. Since the claim is local on $S$ for the fppf topology, we can assume without loss of generality that $E \rightarrow S$ has a section $O_{E}: S \rightarrow E$.

We prove the theorem by induction on $r$. We first observe that for $r=1$, the determinant map is a $\mathbb{G}_{m}$-gerbe by definition of the Picard scheme. It is trivial since there is a $\mathbb{B} \mathbb{G}_{m^{-}}$ equivariant morphism

$$
O_{E}^{*}: \operatorname{Bun}_{G L_{1}}^{s s, d}=\operatorname{Bun}_{\mathbb{G}_{m}}^{d} \longrightarrow \mathbb{B}_{S} \mathbb{G}_{m}=S \times \mathbb{B} \mathbb{G}_{m}
$$

Now suppose $r>1$ and that the theorem is true for all smaller $r$. Observe that for any $d \in \mathbb{Z}$, there is a commutative diagram

where the horizontal arrows are the isomorphisms given by tensoring a vector bundle (resp., line bundle) with $\mathcal{O}\left(O_{E}\right)$ (resp., $\mathcal{O}\left(r O_{E}\right)$ ), and the top one is $\mathbb{B} \mathbb{G}_{m}$-equivariant. So we may assume without loss of generality that $0<d<r$.

Let $V \rightarrow \operatorname{Bun}_{G L_{r}}^{s s, d} \times{ }_{S} E$ be the universal vector bundle, and let $p: \operatorname{Bun}_{G L_{r}}^{s s, d} \times{ }_{S} E \rightarrow \operatorname{Bun}_{G L_{r}}^{s s, d}$ denote the projection to the first factor. Consider the canonical exact sequence of coherent sheaves on $\operatorname{Bun}_{G L_{r}}^{s s, d} \times{ }_{S} E$,

$$
0 \longrightarrow p^{*} p_{*}(V) \longrightarrow V \longrightarrow V^{\prime} \longrightarrow 0 .
$$

Since $V$ is a family of semistable vector bundles on $E$ of slope strictly less than 1 , it follows that the cokernel $V^{\prime}$ is itself a vector bundle, which is a family of indecomposable and hence semistable bundles by [A, Lemma II.15]. This construction defines a $\mathbb{B}_{\mathbb{G}_{m}}$-equivariant morphism

$$
\begin{equation*}
\operatorname{Bun}_{G L_{r}}^{s s, d} \longrightarrow \operatorname{Bun}_{G L_{r-d}}^{s s, d} \tag{5.3.4}
\end{equation*}
$$

over $\operatorname{Pic}_{S}^{d}(E)$. By induction on $r$, to complete the proof of the theorem, it suffices to show that (5.3.4) is an isomorphism.

We construct an inverse to (5.3.4) as follows. Let $U \rightarrow \operatorname{Bun}_{G L_{r-d}}^{s s, d} \times{ }_{S} E$ be the universal vector bundle. By Serre duality, there is a canonical morphism in the derived category

$$
\begin{equation*}
U \longrightarrow p^{*} \mathbb{R} p_{*}(U) \otimes q^{*} K_{E / S}[1]=p^{*}\left(p_{*}(U) \otimes \pi_{\operatorname{Bun}_{G L_{r}}^{s s, d}}^{*} \omega\right)[1] \tag{5.3.5}
\end{equation*}
$$

where $q: \operatorname{Bun}_{G L_{r}}^{s s, d} \times{ }_{S} E \rightarrow E$ is the projection to the second factor, $\pi_{\operatorname{Bun}_{G L-}^{s s, d}}: \operatorname{Bun}_{G L_{r}}^{s s, d} \rightarrow S$ is the structure morphism, and $\omega \in \operatorname{Pic}(S)$ is the line bundle defined in $\S 4.6$. The morphism (5.3.5) corresponds to an extension

$$
0 \longrightarrow p^{*}\left(p_{*}(U) \otimes \pi_{\operatorname{Bun}_{G L_{r}}^{s s, d}}^{*} \omega\right) \longrightarrow U^{\prime} \longrightarrow U \longrightarrow 0
$$

such that the induced connecting homomorphism $p_{*}(U) \rightarrow \mathbb{R}^{1} p_{*} p^{*}\left(p_{*}(U) \otimes \pi_{\operatorname{Bun}_{G L L_{r}}^{s, d}}^{*} \omega\right)$ is an isomorphism. So by [A, Lemma II.16], $U^{\prime}$ is a family of semistable vector bundles on $E$, and hence defines a morphism

$$
\operatorname{Bun}_{G L_{r-d}}^{s s, d} \longrightarrow \operatorname{Bun}_{G L_{r}}^{s s, d},
$$

which is manifestly inverse to (5.3.4).
As an aside, we remark that in the case $d=1$, the gerbe (5.3.2) is trivial even when $E \rightarrow S$ does not have a section.

Proposition 5.3.4. For any $r \in \mathbb{Z}_{>0}$, the $\mathbb{G}_{m}$-gerbe

$$
\text { det: } \operatorname{Bun}_{G L_{r}}^{s s, 1} \longrightarrow \operatorname{Pic}_{S}^{1}(E)
$$

is trivial.
Proof. It suffices to construct a $\mathbb{B} \mathbb{G}_{m}$-equivariant morphism

$$
\begin{equation*}
\operatorname{Bun}_{G L_{r}}^{s s, 1} \longrightarrow \mathbb{B} \mathbb{G}_{m} \tag{5.3.6}
\end{equation*}
$$

Since the universal vector bundle $V_{r} \rightarrow \operatorname{Bun}_{G L_{r}}^{s s, 1} \times_{S} E$ is a family of semistable vector bundles of degree 1, by Lemma 2.6.3, it follows that $p_{*}\left(V_{r}\right)$ is a line bundle on $\operatorname{Bun}_{G L_{r}}^{s s, 1}$. This defines the desired morphism (5.3.6). The corresponding section $\operatorname{Pic}_{S}^{1}(E) \rightarrow \operatorname{Bun}_{G L_{r}}^{s s, 1}$ is the unique section such that the pullback of $p_{*}\left(V_{r}\right)$ is trivial.

Combining Proposition 5.3.1 and Theorem 5.3.2 gives the following description of the stack of stable $L$-bundles. In what follows, for $c \in \pi_{0}$, we write

$$
\lambda_{c}=\sum_{\alpha_{k} \in t} m_{k, c}\left(n_{c}+1-i_{k, c}\right) \varpi_{k} \in \mathbb{X}^{*}(L)
$$

Theorem 5.3.5. Let $L \subseteq G$ be a Levi subgroup as above, and let $\mu \in \mathbb{X}_{*}(L /[L, L])$ be such that for all $c \in \pi_{0},\left\langle\lambda_{c}, \mu\right\rangle$ and $n_{c}+1$ are coprime. Then the natural morphism

$$
\begin{equation*}
\left(\varpi_{k}\right)_{\alpha_{k} \in t}: \operatorname{Bun}_{L}^{s s, \mu} \longrightarrow \prod_{\alpha_{k} \in t} \operatorname{Pic}_{S}^{\left\langle\varpi_{k}, \mu\right\rangle}(E) \tag{5.3.7}
\end{equation*}
$$

is a $Z(L)$-gerbe, where the product is taken in the 2-category of stacks over $S$.
Proof. By Proposition 5.3.1, we can identify $\mathrm{Bun}_{L}^{s s, \mu}$ with the fibre product

$$
\begin{gather*}
\operatorname{Bun}_{L}^{s s, \mu} \longrightarrow \prod_{c \in \pi_{0}} \operatorname{Bun}_{G L_{n_{c}+1}}^{s s,\left\langle\lambda_{c}, \mu\right\rangle} \\
\prod_{\alpha_{k} \in t} \operatorname{Bun}_{\mathbb{G}_{m}}^{\left\langle\varpi_{k}, \mu\right\rangle} \longrightarrow \downarrow_{c \in \pi_{0}} \operatorname{Bun}_{\mathbb{G}_{m}}^{\left\langle\lambda_{c}, \mu\right\rangle} . \tag{5.3.8}
\end{gather*}
$$

So, writing

$$
X=\prod_{\alpha_{k} \in t} \operatorname{Pic}_{S}^{\left\langle\varpi_{k}, \mu\right\rangle}(E) \times \prod_{c \in \pi_{0}} \operatorname{Pic}_{S}^{\left\langle\lambda_{c}, \mu\right\rangle}(E) \prod_{c \in \pi_{0}} \operatorname{Bun}_{G L_{n_{c}+1}^{s s,\left\langle\lambda_{c}, \mu\right\rangle}}
$$

and

$$
X^{\prime}=\prod_{\alpha_{k} \in t} \operatorname{Pic}_{S}^{\left\langle\varpi_{k}, \mu\right\rangle}(E) \times \prod_{c \in \pi_{0}} \operatorname{Pic}_{S}^{\left\langle\lambda_{c}, \mu\right\rangle}(E) \prod_{c \in \pi_{0}} \operatorname{Bun}_{\mathbb{G}_{m}}^{\left\langle\lambda_{c}, \mu\right\rangle},
$$

we have a Cartesian diagram

where, by Theorem 5.3.2, $X, X^{\prime}$ and $\prod_{\alpha_{k} \in t} \operatorname{Bun}_{\mathbb{G}_{m}}^{\left\langle\varpi_{k} \mu\right\rangle}$ are $\prod_{c \in \pi_{0}} Z\left(G L_{n_{c}+1}\right), \prod_{c \in \pi_{0}} \mathbb{G}_{m}$ and $\prod_{\alpha_{k} \in t} \mathbb{G}_{m}$-gerbes respectively over $\prod_{\alpha_{k} \in t} \operatorname{Pic}_{S}^{\left\langle\varpi_{k}, \mu\right\rangle}(E)$, and the morphisms are equivariant with respect to the homomorphisms in the Cartesian diagram


It follows that (5.3.7) is a $Z(L)$-gerbe as claimed.

### 5.4 Regular unstable bundles and the Friedman-Morgan section theorem

In this section, we introduce regular unstable bundles, and use them to prove the elliptic analogue of the Kostant and Steinberg section theorems.

Definition 5.4.1. Let $\alpha_{i} \in \Delta$ be a simple root. We say that $\alpha_{i}$ is special if $\alpha_{i}$ is a long root, the connected components of the Dynkin diagram of $\Delta \backslash\left\{\alpha_{i}\right\}$ are all of type $A$, and $\alpha_{i}$ meets each such component at an end vertex. We call a principal bundle $\xi_{P}$ for a parabolic subgroup $P \subseteq G$ special if $P$ is the standard maximal parabolic of type $\left\{\alpha_{i}\right\}$ with $\alpha_{i} \in \Delta$ special, and $\xi_{P}$ has slope $-\varpi_{i}^{\vee} /\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle$.

Proposition 5.4.2. Let $\xi_{G} \rightarrow E_{s}$ be an unstable $G$-bundle on a geometric fibre of $E \rightarrow S$. The following are equivalent.
(1) The Harder-Narasimhan reduction of $\xi_{G}$ is special.
(2) $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l+2$.
(3) $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \leq l+2$.
(4) The Harder-Narasimhan locus of $\xi_{G}$ has codimension $l+1$.
(5) The Harder-Narasimhan locus of $\xi_{G}$ has codimension $\leq l+1$.

Proof. It is clear that $(2) \Rightarrow(3)$ and $(4) \Rightarrow(5)$. We complete the proof by showing that $(3) \Rightarrow(5) \Rightarrow(1) \Rightarrow(4)$ and $(1) \Rightarrow(2)$.

Fix once and for all a Harder-Narasimhan reduction $\xi_{P}$ of $\xi_{G}$, where $P$ is a standard parabolic and let $\mu=\mu\left(\xi_{P}\right)$ be the slope of $\xi_{P}$. Then the codimension of the HarderNarasimhan locus of $\xi_{G}$ is $-\langle 2 \rho, \mu\rangle$ by Proposition 2.6.7, and

$$
\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=\operatorname{dim} \operatorname{Aut}\left(\xi_{P}\right)=\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)-\langle 2 \rho, \mu\rangle
$$

where $\xi_{L}=\xi_{P} \times{ }^{P} L$ is the associated bundle for the Levi factor $L$ of $P$.
Assume (3). Then, since $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right) \geq \operatorname{dim} Z(L) \geq 1$, the codimension of the HarderNarasimhan locus of $\xi_{G}$ is

$$
-\langle 2 \rho, \mu\rangle=\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)-\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right) \leq(l+2)-1=l+1
$$

which proves (5).
Now assume (5). The arguments of Proposition 2.6 .8 show that there exists $\tilde{\mu}=$ $-n \varpi_{i}^{\vee} /\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle$ for some $n \in \mathbb{Z}_{>0}$ and some $\alpha_{i} \in \Delta$ such that $-\langle 2 \rho, \mu\rangle \geq-\langle 2 \rho, \tilde{\mu}\rangle$, with equality if and only if $P$ is of type $\left\{\alpha_{i}\right\}$ and $\mu=\tilde{\mu}$. So we have

$$
n \frac{\left\langle 2 \rho, \varpi_{i}^{\vee}\right\rangle}{\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle}=-\langle 2 \rho, \tilde{\mu}\rangle \leq-\langle 2 \rho, \mu\rangle \leq l+1
$$

But [FM2, Lemma 3.3.2] implies that this is the case if and only if $\alpha_{i}$ is special and $n=1$, and that both inequalities above are in fact equalities in this case. This implies that the Harder-Narasimhan reduction $\xi_{P}$ is special, so (1) holds.

Now assume (1), i.e., that $\xi_{P}$ is special. Then by [FM2, Lemma 3.3.2] again, the codimension of the Harder-Narasimhan locus of $\xi_{G}$ is $-\langle 2 \rho, \mu\rangle=l+1$, so (4) holds. Moreover, by Theorem 5.3.5, $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)=1$, so we have

$$
\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=1-\langle 2 \rho, \mu\rangle=l+2
$$

and (2) holds as well. This completes the proof of the proposition.
Definition 5.4.3. Let $\xi_{G} \rightarrow E_{s}$ be an unstable $G$-bundle on a geometric fibre of $E \rightarrow S$. We say that $\xi_{G}$ is regular if it satisfies the equivalent conditions of Proposition 5.4.2.

The next proposition uses Theorem 5.3.5 to classify regular unstable bundles up to translation.

Proposition 5.4.4. Assume that $E \rightarrow S$ has a section $O_{E}: S \rightarrow E$, fix a special root $\alpha_{i} \in \Delta$ and let $\mu=-\varpi_{i}^{\vee} /\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle$. Then $\operatorname{Bun}_{L, r i g}^{s s, \mu} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ and $\operatorname{Bun}_{L, r i g}^{s s, \mu} / E \rightarrow S$ are $Z(L)_{\text {rig-gerbes, }}$ and there exists a unique section $S \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ lifting the section $O_{E}: S \rightarrow$ $E \cong \operatorname{Pic}_{S}^{-1}(E)$ such that the pullback of the theta bundle $\Theta_{\operatorname{Bun}_{G, r i g}}$ to $S$ is trivial.

Proof. Applying Theorem 5.3.5 to $E \rightarrow S$ and to the family $E^{\prime}:=S \rightarrow \mathbb{B}_{S} E=: S^{\prime}$, we have that

$$
\operatorname{Bun}_{L}^{s s, \mu} \longrightarrow \operatorname{Pic}_{S}^{-1}(E) \cong E
$$

and

$$
\operatorname{Bun}_{L}^{s s, \mu} / E=\operatorname{Bun}_{L / S^{\prime}}^{s s, \mu}\left(E^{\prime}\right) \longrightarrow \operatorname{Pic}_{S^{\prime}}^{-1}\left(E^{\prime}\right) \cong E^{\prime}=S
$$

are $Z(L)$-gerbes. Taking the quotient by $\mathbb{B} Z(G)$, we deduce that $\operatorname{Bun}_{L, r i g}^{s s, \mu} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ and $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E \rightarrow S$ are $Z(L)_{\text {rig }}=Z(L) / Z(G)$-gerbes as claimed.

To construct the section $S \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$, note that since $(-\mu \mid-): Z(L)_{\text {rig }} \rightarrow \mathbb{G}_{m}$ is an isomorphism, the pullback of the theta bundle defines a $\mathbb{B} Z(L)_{\text {rig }}$-equivariant morphism

$$
S \times_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{L, r i g}^{s s, \mu} \longrightarrow \mathbb{B} \mathbb{G}_{m} \cong \mathbb{B} Z(L)_{\text {rig }}
$$

by Proposition 5.2.13. Since the source is a $Z(L)_{\text {rig }}=\mathbb{G}_{m}$-gerbe over $S$, it follows that there is a unique section such that the pullback of $\Theta_{\mathrm{Bun}_{G, r i g}}$ is trivial as claimed.

Fix a special root $\alpha_{i} \in \Delta$ and let $\mu=-\varpi_{i}^{\vee} /\left\langle\varpi_{i}, \varpi_{i}^{\vee}\right\rangle$, as above, and assume that $E \rightarrow S$ has a section $O_{E}$. Then the composition of the section $S \rightarrow \mathrm{Bun}_{L, \text { rig }}^{s s, \mu}$ of Proposition 5.4.4 with $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ is a section of a $Z(L)_{\text {rig }}=\mathbb{G}_{m}$-gerbe, and is hence smooth. Since the pullback of the theta bundle to $S$ is trivial, $Z \rightarrow \operatorname{Bun}_{G, r i g}$ is an equivariant slice with equivariance group $Z(L)_{\text {rig }}$ and weight $(\mu \mid-)$ by Proposition 5.2.10. We therefore have a $\mathbb{G}_{m}$-equivariant commutative diagram

as in Remark 5.1.2.
Remark 5.4.5. Note that if the section $S \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ factors through $\operatorname{Bun}_{L}^{s s, \mu}$, then the slice $Z$ constructed above factors through a morphism $Z \rightarrow \operatorname{Bun}_{G}$. However, even when this happens, this morphism will not necessarily be a slice unless $S \times Z(L) \rightarrow S$ is smooth, i.e., unless $S \rightarrow \operatorname{Spec} \mathbb{Z}$ avoids all primes at which $Z(L)$ is non-reduced.

Theorem 5.4.6 (Friedman-Morgan section theorem). In the setup above, the composition

$$
\chi_{Z}: Z \longrightarrow\left(\Theta_{\mathrm{Bun}_{G, r i g}}^{-1}\right)^{*} \longrightarrow \widehat{Y} / / W
$$

is $a \mathbb{G}_{m}$-equivariant isomorphism. In particular, the rigidified coarse quotient map

$$
\chi: \operatorname{Bun}_{G, \text { rig }} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}
$$

admits a section.

Theorem 5.4.6 is originally due, in a slightly different form, to Friedman and Morgan [FM2, Theorem 5.1.1]. We give a different proof to theirs below, relying on a computation of the pullback $\tilde{Z}=Z \times_{\text {Bun }_{G, \text { rig }}} \widetilde{\operatorname{Bun}}_{G, \text { rig }}$ of the elliptic Grothendieck-Springer resolution to the slice $Z$.

The first step is to identify those $D_{\lambda}(Z) \subseteq \tilde{Z}$ that are nonempty, where we recall the notation $D_{\lambda}(Z)$ from §5.1.

Lemma 5.4.7. Assume $\lambda \in \mathbb{X}_{*}(T)_{+}$and $\lambda \neq \alpha_{i}^{\vee}$. Then $D_{\lambda}(Z)=\emptyset$.
Proof. Assume for a contradiction that there exists $\lambda \in \mathbb{X}_{*}(T)_{+}$with $\lambda \neq \alpha_{i}^{\vee}$ and $D_{\lambda}(Z) \neq \emptyset$. Since $\lambda \neq \alpha_{i}^{\vee}$, there exists $\alpha_{j} \in \Delta$ such that $\left\langle\varpi_{j}, \lambda\right\rangle>0$ and

$$
\mu^{\prime}=-\frac{\left\langle\varpi_{j}, \lambda\right\rangle}{\left\langle\varpi_{j}, \varpi_{j}^{\vee}\right\rangle} \varpi_{j}^{\vee} \neq \mu .
$$

 nonempty, where $P_{j}$ is the standard maximal parabolic of type $\left\{\alpha_{j}\right\}$. Lemma 5.2.14 implies that $\left\langle 2 \rho, \mu^{\prime}\right\rangle<\langle 2 \rho \mu\rangle=l+1$, contradicting [FM2, Lemma 3.3.2], so we are done.

Lemma 5.4.8. The morphism $\tilde{Z} \rightarrow \Theta_{Y}^{-1}$ is representable, separated, of finite type, and flat of relative dimension 0 . For all $\lambda \in \mathbb{X}_{*}(T)$, the morphism $D_{\lambda}(Z) \rightarrow Y$ is representable, separated, and étale.

Proof. First note that Lemma 5.4.7 implies that no point of $\tilde{Z}$ can have nontrivial automorphism group relative to $Z$. So Theorem 3.1.7 implies that $\tilde{Z} \rightarrow Z$ is representable and projective. In particular, $\tilde{Z}, D_{\lambda}(Z), Y$ and $\Theta_{Y}^{-1}$ are all representable, separated and of finite type over $S$, so the morphisms $\tilde{Z} \rightarrow \Theta_{Y}^{-1}$ and $D_{\lambda}(Z) \rightarrow Y$ are necessarily representable, separated and of finite type as well.

Next, observe that since $Y \rightarrow S$ is projective and $\Theta_{Y} \in \operatorname{Pic}(Y)$ is ample relative to $S$, the action of $E$ on the pair $\left(Y, \Theta_{Y}\right)$ is trivial relative to $S$, since it must be trivial on the sheaf of graded algebras $\bigoplus_{d} \pi_{Y *} \Theta_{Y}^{\otimes d}$ as $E$ is an elliptic curve. So we have canonical isomorphisms $Y / E \cong Y \times_{S} \mathbb{B}_{S} E$ and $\Theta_{Y}^{-1} / E \cong \Theta_{Y}^{-1} \times_{S} \mathbb{B}_{S} E$. So the morphism $\tilde{Z} \rightarrow \Theta_{Y}^{-1}$ factors as

$$
\begin{equation*}
\tilde{Z} \longrightarrow\left(\psi^{*} \Theta_{\text {Bun }_{G, r i g}}^{-1}\right)^{*} / E \longrightarrow \Theta_{Y}^{-1} / E=\Theta_{Y}^{-1} \times_{S} \mathbb{B}_{S} E \longrightarrow \Theta_{Y}^{-1} \tag{5.4.1}
\end{equation*}
$$

Since $Z$ is an equivariant slice of $\operatorname{Bun}_{G, \text { rig }}$, Proposition 5.1.3 implies that $Z \rightarrow\left(\Theta_{\text {Bun }_{G, r i g}}^{-1}\right)^{*} / E$ is flat, and hence so is the pullback $\tilde{Z} \rightarrow\left(\psi^{*} \Theta_{\text {Bun }_{G, r i g}}^{-1}\right)^{*} / E$ along $\psi: \widetilde{\operatorname{Bun}}_{G, \text { rig }} \rightarrow \operatorname{Bun}_{G, \text { rig }}$. So by Proposition 4.5.5, the composition (5.4.1) is flat. By Proposition 5.2.8, $Z$ has dimension $\left\langle 2 \rho_{P^{+}}, \mu\right\rangle=-\langle 2 \rho, \mu\rangle=l+1$ relative to $S$, where $P^{+}$is the unique parabolic subgroup with Levi factor $L$ for which $-\mu$ is a Harder-Narasimhan vector. (Note that $P^{+}$contains the Borel subgroup spanned by positive root subgroups.) Since $\widetilde{\operatorname{Bun}}_{G, r i g} \rightarrow \operatorname{Bun}_{G, r i g}$ is generically finite, $\tilde{Z}$ must also have dimension $l+1$ relative to $S$. Since $\Theta_{Y}^{-1}$ has dimension $l+1$ relative to $S$, the flat morphism $\tilde{Z} \rightarrow \Theta_{Y}^{-1}$ must have relative dimension 0 as claimed.

Similarly, the morphism

$$
\tilde{Z} \longrightarrow \widetilde{\operatorname{Bun}}_{G, \text { rig }} / E \longrightarrow\left(Y \times_{S} \mathfrak{D e g}_{S}(E)\right) / E \cong Y \times_{S} \mathfrak{D e g}_{S}(E) / E
$$

is smooth. Since the boundary divisor $D \subseteq \operatorname{Deg}_{S}(E)$ is a reduced divisor with normal crossings relative to $S$, the closed substack $D(Z)=\tilde{Z} \times \mathfrak{D e g}_{S}(E) D=\tilde{Z} \times \times_{S} \mathfrak{D} \operatorname{eg}_{S}(E) / E$ $\left(Y \times_{S}(D / E)\right)$ is a reduced divisor with normal crossings relative to $Y$. Since $\tilde{Z} \rightarrow Y$ has relative dimension 1, the irreducible components $D_{\lambda}(Z)$ of $D(Z)$ are therefore disjoint and smooth of relative dimension 0 , hence étale, over $Y$.

Recall the notation $Y^{\lambda}$ of Definition 4.1.5.
Lemma 5.4.9. There is an isomorphism

$$
\operatorname{Bun}_{L \cap B}^{-\alpha_{i}^{\vee}} \times \operatorname{Bun}_{L}^{\mu} \operatorname{Bun}_{L}^{s s, \mu} \xrightarrow{\sim} Y^{-\alpha_{i}^{\vee}} \times_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{L}^{s s, \mu}
$$

and hence an isomorphism

$$
\operatorname{Bun}_{L \cap B, r i g}^{-\alpha_{i}^{\vee}} \times \times_{\operatorname{Bun}_{L, r i g}^{\mu}}^{\mu} \operatorname{Bun}_{L, r i g}^{s s, \mu} \xrightarrow{\sim} Y^{-\alpha_{i}^{\vee}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{L, r i g}^{s s, \mu}
$$

sending an $L \cap B$-bundle to its associated $T$-bundle and $L$-bundle.
Proof. Using the isomorphism of Proposition 5.3.1, the claim reduces easily to Lemma 5.4.10 below.

In the following lemma, we write

$$
Q_{r}^{n}=\left\{\left(a_{p, q}\right)_{1 \leq p, q \leq n} \in G L_{n} \mid a_{p, q}=0 \text { if } p<\min (q, r)\right\} \subseteq G L_{n}
$$

for $1 \leq r \leq n$. For $1 \leq i \leq n$, we write $e_{i} \in \mathbb{X}^{*}\left(T_{Q_{n}^{n}}\right)$ for the character sending a diagonal matrix to its $i$ th entry, and we write $\left\{e_{1}^{*}, \ldots, e_{n}^{*}\right\} \subseteq \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)$ for the basis dual to $\left\{e_{1}, \ldots, e_{n}\right\}$. If $\lambda \in \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)$, we will also write $\lambda$ for its image in $\mathbb{X}_{*}\left(T_{Q_{r}^{n}}\right)$.

Lemma 5.4.10. Let $n>0$ and $1 \leq r \leq n$. Then the morphism

$$
\begin{equation*}
\operatorname{Bun}_{Q_{r}^{n}}^{-e_{n}^{*}} \times \times_{\operatorname{Bun}_{G L_{n}}^{-1}} \operatorname{Bun}_{G L_{n}}^{s s,-1} \longrightarrow Y_{Q_{r}^{n}}^{-e_{n}^{*}} \times \times_{\operatorname{Pic}_{s}^{-1}(E)} \operatorname{Bun}_{G L_{n}}^{s s,-1} \tag{5.4.2}
\end{equation*}
$$

is an isomorphism, where the morphisms $Y_{Q_{r}^{n}}^{-e_{n}^{*}} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ and $\operatorname{Bun}_{G L_{n}}^{s s,-1} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ are both given by the determinant.

Proof. For the sake of brevity, we will write

$$
\left(\operatorname{Bun}_{Q_{r}^{n}}^{-e_{n}^{*}}\right)^{s s}=\operatorname{Bun}_{Q_{r}^{n}}^{-e_{n}^{*}} \times \operatorname{Bun}_{G L_{n}}^{-1} \operatorname{Bun}_{G L_{n}}^{s s,-1} .
$$

We prove the claim by induction on $r$. For $r=1$, the claim is true since $Q_{1}^{n}=G L_{n}$ and $Y_{Q_{1}^{n}}^{-e_{n}^{*}}=\operatorname{Pic}_{S}^{-1}(E)$.

Next, suppose that $r=2$. In this case, we construct an inverse to (5.4.2) as follows. Let $V \rightarrow Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{G L_{n}}^{s s,-1} \times{ }_{S} E$ be the pullback of the universal vector bundle on $\operatorname{Bun}_{G L_{n}}^{s s,-1} \times_{S} E$ and let $M_{e_{1}} \rightarrow Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{G L_{n}}^{s s,-1} \times{ }_{S} E$ be the pullback of the degree 0 line bundle on $Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times{ }_{S} E$ associated to the character $e_{1}$. Writing $p: Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)}$ $\operatorname{Bun}_{G L_{n}}^{s s,-1} \times{ }_{S} E \rightarrow Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times \times_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{G L_{n}}^{s s,-1}$ for the projection to the first two factors, we have that $N=p_{*}\left(M_{e_{1}} \otimes V^{\vee}\right)$ is a line bundle since $M_{e_{1}} \otimes V^{\vee}$ is a family of semistable vector bundles of degree 1 . By semistability of $V$, we therefore have a natural exact sequence

$$
\begin{equation*}
0 \longrightarrow V^{\prime} \longrightarrow V \longrightarrow M_{e_{1}} \otimes p^{*} N^{-1} \longrightarrow 0 \tag{5.4.3}
\end{equation*}
$$

where $M_{e_{1}} \otimes p^{*} N^{-1}$ is a line bundle fibrewise of degree 0 and $V^{\prime}$ is a vector bundle of rank $n-1$ and degree -1 . The exact sequence (5.4.3) defines a degree $-e_{n}^{*}$ reduction to $Q_{2}^{n}$ of the pullback of the universal $G L_{n}$-bundle, and hence a morphism

$$
Y_{Q_{2}^{n}}^{-e_{n}^{*}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{G L_{n}}^{s s,-1} \longrightarrow\left(\operatorname{Bun}_{Q_{2}^{n}}^{-e_{n}^{*}}\right)^{s s}
$$

which is easily shown to be inverse to (5.4.2). This proves the claim for $r=2$.

Finally, assume that $r>2$ and the lemma holds for all smaller $r$. We need to show that the outer square in the commutative diagram

is Cartesian. Since the rightmost square is Cartesian by induction, it suffices to show that the leftmost suqare is Cartesian.

Observe that there is a surjective homomorphism $Q_{r-1}^{n} \rightarrow G L_{n-r+2}$ given by forgetting the first $r-2$ rows and columns, such that $Q_{r}^{n} \subseteq Q_{r-1}^{n}$ is the preimage of $Q_{2}^{n-r+2} \subseteq G L_{n-r+2}$. Since the morphism

$$
\operatorname{Bun}_{Q_{r-1}^{n}}^{-e_{n}^{*}} \longrightarrow \operatorname{Bun}_{G L_{n-r+2}}^{-1}
$$

sends $\left(\operatorname{Bun}_{Q_{r-1}^{n}}^{-e_{n}^{*}}\right)^{s s}$ to $\operatorname{Bun}_{G L_{n-r+2}}^{s s,-1}$ by [A, Lemma II.15], we therefore have a diagram

in which the leftmost square is Cartesian. Since the rightmost square is also Cartesian by induction, the outer square is also. This is also the outermost square in the diagram


It is easy to see that the rightmost square in this diagram is Cartesian and hence so is the leftmost square. But this coincides with the leftmost square of (5.4.4), so we are done.

Proposition 5.4.11. The morphism $D_{\alpha_{i}^{\vee}}(Z) \rightarrow Y$ is an isomorphism.
Proof. Since the claim is local on $S$, we can assume for convenience that $S \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ factors through $\operatorname{Bun}_{L}^{s s, \mu}$, and hence that $Z \rightarrow \operatorname{Bun}_{G, r i g}$ factors through Bun ${ }_{G}$.

Since every domain curve parametrised by a point in $D_{\alpha_{i}^{\vee}}(Z)$ has dual graph $\tau_{\alpha_{i}^{\vee}}^{0}$, by Proposition 3.4.13 there is an isomorphism

$$
D_{\alpha_{i}^{\vee}}(Z) \cong M_{Z}^{\circ}\left(\xi_{G} / B, \tau_{\alpha_{i}^{\vee}}^{0}\right)=M_{1,1, Z}^{\circ}\left(\xi_{G} / B,\left(-\alpha_{i}^{\vee}, 1\right)\right) \times_{\xi_{G} / B} M_{0,1, Z}^{\circ}\left(\xi_{G} / B,\left(\alpha_{i}^{\vee}, 0\right)\right),
$$

where $\xi_{G} \rightarrow Z \times_{S} E$ is the $G$-bundle classified by $Z \rightarrow \operatorname{Bun}_{G}$. Proposition 3.4.10 implies that the evaluation morphism $M_{0,1, Z}\left(\xi_{G} / B,\left(\alpha_{i}^{\vee}, 0\right)\right) \rightarrow \xi_{G} / B$ is an isomorphism, so we have a canonical identification

$$
\begin{equation*}
D_{\alpha_{i}^{\vee}}(Z) \cong M_{1,1, Z}^{\circ}\left(\xi_{G} / B,\left(-\alpha_{i}^{\vee}, 1\right)\right)=E \times{ }_{S} \operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z . \tag{5.4.5}
\end{equation*}
$$

Observe that Proposition 5.1.4 and 5.4.2 show that the morphism $\operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\text {Bunn }_{G, r i g}} Z \rightarrow$ $Z$ must factor through the natural $\mathbb{G}_{m}$-fixed section $S \rightarrow Z$. So we have

$$
\operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z=\operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} S,
$$

where the morphism $S \rightarrow \operatorname{Bun}_{G, r i g}$ factors through the section $S \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ of Proposition 5.4.4.

Let $P \subseteq G$ be the standard parabolic subgroup with Levi factor $L$, and consider the locally closed Bruhat cells

$$
C^{w}=C_{P, B / S, r i g}^{w,-\alpha_{i}^{\vee}}(E) \times_{\operatorname{Bun}_{P, r i g}} S \subseteq \operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} S,
$$

for $w \in W_{P}^{0}$, where $C_{P, B / S, r i g}^{w,-\alpha_{i}^{\vee}}(E) \subseteq \operatorname{Bun}_{P, r i g} \times_{\text {Bun }_{G, r i g}} \operatorname{Bun}_{B, r i g}$ is the rigidification of the locally closed substack $C_{P, B / S}^{w,-\alpha_{i}^{\vee}}(E) \subseteq \operatorname{Bun}_{P} \times_{\operatorname{Bun}_{G}} \operatorname{Bun}_{B}$ defined in §3.7. Then Proposition 3.7.6 and Lemma 5.4.7 imply that

$$
\operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times \times_{\operatorname{Bun}_{G, r i g}} S=\bigcup_{w \in W_{P}^{0}} C^{w}
$$

Assume that $w \in W_{P}^{0}$ with $C^{w} \neq \emptyset$. Then there exists a geometric point $s: \operatorname{Spec} k \rightarrow S$ with corresponding $L$-bundle $\xi_{L} \rightarrow E_{s}$ and a section $\sigma_{L}: E_{s} \rightarrow \xi_{L} \times{ }^{L} L /(L \cap B)$ of degree $\left[\sigma_{L}\right]=-w \alpha_{i}^{\vee} \in \Phi^{\vee}$. Since $\xi_{L}$ has slope $\mu$, we must have $\left\langle\varpi_{i},\left[\sigma_{L}\right]\right\rangle=-1$ and hence $\left[\sigma_{L}\right] \in \Phi_{-}^{\vee} \subseteq \mathbb{X}_{*}(T)_{-}$. Since $\left[\sigma_{L}\right]$ is the degree of the section

$$
\sigma^{\prime}: E_{s} \xrightarrow{\sigma} \xi_{L} \times^{L} L /(L \cap B) \longleftrightarrow \xi_{L} \times^{L} G / B,
$$

Lemma 5.4.7 implies that we must have $\left[\sigma_{L}\right]=-\alpha_{i}^{\vee}$. So $w \in W_{P}^{0}$ and $w \alpha_{i}^{\vee}=\alpha_{i}^{\vee}$, which implies that $w^{-1}\left(\Phi_{+}^{\vee}\right)=\Phi_{+}^{\vee}$, and hence $w=1$. So $C^{w}=\emptyset$ for $w \neq 1$, and hence the closed immersion

$$
\begin{equation*}
C^{1} \longleftrightarrow \operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} S \tag{5.4.6}
\end{equation*}
$$

is surjective on geometric points. Since $D_{\alpha_{i}^{\vee}}(Z)$ is étale over $Y$, hence reduced, $\operatorname{Bun}_{B, r i g}^{-\alpha_{i}^{\vee}} \times_{\text {Bun }_{G, r i g}}$ $S$ is reduced as well, so (5.4.6) is an isomorphism. But by Lemma 5.4.9,

$$
C^{1}=\operatorname{Bun}_{L \cap B, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}^{\mu}} S \xrightarrow{\sim} Y^{-\alpha_{i}^{\vee}} \times_{\operatorname{Pic}_{S}^{-1}(E)}\left\{\mathcal{O}\left(-O_{E}\right)\right\},
$$

is an isomorphism, where the morphism $Y^{-\alpha_{i}^{\vee}} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ is induced by the character $\varpi_{i}: T \rightarrow \mathbb{G}_{m}$. So by (5.4.5) and Proposition 3.5.5, we can identify the morphism $D_{\alpha_{i}^{\vee}}(Z) \rightarrow$ $Y$ with the isomorphism

$$
\begin{aligned}
E \times{ }_{S} Y^{-\alpha_{i}^{\vee}} \times_{\operatorname{Pic}_{S}^{-1}(E)}\left\{\mathcal{O}\left(-O_{E}\right)\right\} & \longrightarrow Y \\
\left(p, \xi_{T}\right) & \longmapsto \alpha_{i}^{\vee}(\mathcal{O}(p)) \otimes \xi_{T},
\end{aligned}
$$

which completes the proof of the proposition.
Corollary 5.4.12. The morphism $\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right) \rightarrow 0_{\Theta_{Y}^{-1}}=Y$ is an isomorphism, where $0_{\Theta_{Y}^{-1}}$ denotes the zero section of $\Theta_{Y}^{-1}$.

Proof. Since $Z$ is a slice of $\operatorname{Bun}_{G, r i g}$, Corollary 4.5.9 implies that

$$
\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)=\sum_{\lambda \in \mathbb{X}_{*}(T)_{+}} \frac{1}{2}(\lambda \mid \lambda) D_{\lambda}(Z) .
$$

Applying Lemma 5.4.7, this simplifies to

$$
\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)=\frac{1}{2}\left(\alpha_{i}^{\vee} \mid \alpha_{i}^{\vee}\right) D_{\alpha_{i}^{\vee}}(Z)=D_{\alpha_{i}^{\vee}}(Z) .
$$

The claim now follows from Proposition 5.4.11.

Proposition 5.4.13. The morphism $\tilde{Z} \rightarrow \Theta_{Y}^{-1}$ is an isomorphism.
The idea behind the proof of Proposition 5.4.13 is to use the $\mathbb{G}_{m}$-action on $\tilde{Z}$ and $\Theta_{Y}^{-1}$ to reduce to Proposition 5.4.11. The key tool is the following technical lemma.

Lemma 5.4.14. Suppose that $f: X \rightarrow X^{\prime}$ is a proper representable $\mathbb{G}_{m}$-equivariant morphism of stacks and that $X_{0}^{\prime} \subseteq X^{\prime}$ and $X_{0} \subseteq f^{-1}\left(X_{0}^{\prime}\right)$ are closed substacks satisfying the following conditions.
(1) $X_{0}=f^{-1}\left(X_{0}^{\prime}\right)$ set-theoretically.
(2) $\mathbb{G}_{m}$ acts trivially on the closed substacks $X_{0}^{\prime}$ and $X_{0}$.
(3) There exists a $\mathbb{G}_{m}$-equivariant retraction $X^{\prime} \rightarrow X_{0}^{\prime}$ so that $X^{\prime}$ is an affine space bundle over $X_{0}^{\prime}$ on which $\mathbb{G}_{m}$ acts with positive weights.
(4) The induced action of $\mathbb{G}_{m}$ on the normal cone $C_{X_{0} / X}$ has a single nonzero weight.

Then there is a unique $\mathbb{G}_{m}$-equivariant isomorphism $X \cong C_{X_{0} / X}$ over $X_{0}^{\prime}$ sending $X_{0} \subseteq X$ to the zero section via the identity and inducing the identity on normal cones.

Proof. We first remark that since $\mathbb{G}_{m}$ acts on $C_{X_{0} / X}$ with a single nonzero weight, every $\mathbb{G}_{m}$-equivariant automorphism of $C_{X_{0} / X}$ that acts as the identity on $X_{0}$ and the normal cone of $X_{0}$ in $C_{X_{0} / X}$ is (canonically 2-isomorphic to) the identity. So uniqueness follows.

The idea behind the proof of existence is to show that the deformation to the normal cone is trivial. We do this by first compactifying, so that we are in a position to apply the Grothendieck existence theorem, and then showing that the deformation is trivial infinitesimally.

First note that by the uniqueness just shown, we can reduce the proof of existence by descent to the case where $X_{0}^{\prime}=\operatorname{Spec} A$ for some Noetherian ring $A$. Again using uniqueness and fpqc descent for morphisms of separated algebraic spaces, it suffices to show that the desired isomorphism exists after base change along the fpqc morphism Spec $A \llbracket t \rrbracket\left[t^{-1}\right] \rightarrow$ $\operatorname{Spec} A$.

Let $C \rightarrow \operatorname{Spec} A[t]$ and $C^{\prime} \rightarrow \operatorname{Spec} A[t]$ denote the deformations to the normal cone of $X_{0}$ in $X$ and $X_{0}^{\prime}$ in $X^{\prime}$ respectively. Then there are canonical inclusions

$$
X_{0} \times_{\operatorname{Spec} A} \operatorname{Spec} A[t] \longleftrightarrow C, \quad \text { and } \quad X_{0}^{\prime} \times \operatorname{Spec} A \operatorname{Spec} A[t]=\operatorname{Spec} A[t] \hookrightarrow C^{\prime} .
$$

Define compactifications
$\bar{C}=\left(\left(C \times \mathbb{A}^{1}\right) \backslash\left(X_{0} \times_{\operatorname{Spec} A} \operatorname{Spec} A[t] \times\{0\}\right)\right) / \mathbb{G}_{m} \quad$ and $\quad \bar{C}^{\prime}=\left(\left(C^{\prime} \times \mathbb{A}^{1}\right) \backslash(\operatorname{Spec} A[t] \times\{0\})\right) / \mathbb{G}_{m}$,
where $\mathbb{G}_{m}$ acts on $C$ and $C^{\prime}$ over Spec $A \llbracket t \rrbracket$ via the action induced from the action on $X$ and $X^{\prime}$, and $\mathbb{G}_{m}$ acts on $\mathbb{A}^{1}$ via the usual weight 1 action. Then $C^{\prime} \rightarrow \operatorname{Spec} A[t]$ is an affine space bundle on which $\mathbb{G}_{m}$ acts with positive weights, and hence $\bar{C}^{\prime} \rightarrow \operatorname{Spec} A[t]$ is a bundle of weighted projective spaces, and in particular proper. We also have that $C \rightarrow C^{\prime}$ factors as

$$
C \hookrightarrow C^{\prime} \times_{X^{\prime}} X \longrightarrow C^{\prime},
$$

where the first morphism is a closed immersion, hence proper, and the second morphism is proper by assumption on $f$. So $C \rightarrow C^{\prime}$ is proper, and hence so are $\bar{C} \rightarrow \bar{C}^{\prime}$ and $\bar{C} \rightarrow \operatorname{Spec} A[t]$. We also write

$$
\bar{C}_{X_{0} / X}=\left(\left(C_{X_{0} / X} \times \mathbb{A}^{1}\right) \backslash\left(X_{0} \times\{0\}\right)\right) / \mathbb{G}_{m}=\bar{C} \times_{\operatorname{Spec} A[t], t \mapsto 0} \operatorname{Spec} A,
$$

and observe that $\bar{C}_{X_{0} / X} \rightarrow X_{0}^{\prime}=\operatorname{Spec} A$ is also proper. Note also that we have divisors at infinity $\left(C_{X_{0} / X} \backslash X_{0}\right) / \mathbb{G}_{m} \subseteq \bar{C}_{X_{0} / X}$ and $\left(C \backslash X_{0} \times_{\text {Spec } A} \operatorname{Spec} A[t]\right) / \mathbb{G}_{m} \subseteq \bar{C}$ whose complements are canonically isomorphic to $C_{X_{0} / X}$ and $C$ respectively.

We claim that there is a unique $\mathbb{G}_{m}$-equivariant isomorphism $C^{\wedge} \cong C_{X_{0} / X} \times{ }_{\operatorname{Spec} A}$ $\operatorname{Spf} A \llbracket t \rrbracket$ of formal stacks over $\operatorname{Spf} A \llbracket t \rrbracket$ acting as the identity on $X_{0} \times_{\operatorname{Spec} A} \operatorname{Spf} A \llbracket t \rrbracket$ and on the normal cone of $X_{0} \times_{\operatorname{Spec} A} \operatorname{Spf} A \llbracket t \rrbracket$ in both sides. Given the claim, this extends to an isomorphism between proper formal stacks $\bar{C}^{\wedge} \cong \bar{C}_{X_{0} / X} \times_{\text {Spec } A} \operatorname{Spf} A \llbracket t \rrbracket$, and hence an isomorphism $\bar{C} \times_{\text {Spec } A[t]} \operatorname{Spec} A \llbracket t \rrbracket \cong \bar{C}_{X_{0} / X} \times_{\text {Spec } A} \operatorname{Spec} A \llbracket t \rrbracket$ by the Grothendieck existence theorem. Since this isomorphism identifies the divisors at infinity and since the restricted deformation to the normal cone $C \rightarrow \operatorname{Spec} A\left[t, t^{-1}\right]$ is canonically trivial, it restricts to give the desired isomorphism

$$
X \times_{\operatorname{Spec} A} \operatorname{Spec} A \llbracket t \rrbracket\left[t^{-1}\right] \cong C_{X_{0} / X} \times_{\operatorname{Spec} A} \operatorname{Spec} A \llbracket t \rrbracket\left[t^{-1}\right]
$$

To prove the claim, it is enough to prove existence and uniqueness of isomorphisms $C_{n} \cong C_{X_{0} / X} \times_{\operatorname{Spec} A} \operatorname{Spec} A[t] /\left(t^{n}\right)$ for all $n \geq 0$ with the desired properties, where $C_{n}=$ $C \times_{\text {Spec } A[t]} \operatorname{Spec} A[t] /\left(t^{n}\right)$. Uniqueness is clear. Letting $U=\operatorname{Spec} R_{0}$ be any affine étale chart for $X_{0}$ (note that $X_{0}$ is an algebraic space since $f$ is representable), we have an affine étale chart

$$
C_{X_{0} / X} \times_{X_{0}} U=\operatorname{Spec} \bigoplus_{d \geq 0} R_{d}
$$

for $C_{X_{0} / X}$, which lifts to a canonical $\mathbb{G}_{m}$-equivariant affine étale chart

$$
V_{n}=\operatorname{Spec} \bigoplus_{d \in \mathbb{Z}} R_{n, d}
$$

for $C_{n}$ since $C_{n}$ is a nilpotent thickening of $C_{X_{0} / X}$, where the gradings are induced by the $\mathbb{G}_{m}$-action. From the flatness properties of the deformation to the normal cone, we deduce that the map $U \times_{\text {Spec } A} \operatorname{Spec} A[t] /\left(t^{n}\right)=\operatorname{Spec} R_{0}[t] /\left(t^{n}\right) \rightarrow V_{n}$ induces an isomorphism $R_{n, 0} \cong R_{0}[t] /\left(t^{n}\right)$, that $R_{n, d}=0$ for all $d<0$, and that $\bigoplus_{d} R_{n, d}$ is generated by $R_{n, 0}$ and $R_{n, d_{0}}$, where $d_{0}=\min \left\{d>0 \mid R_{d} \neq 0\right\}$ is the single weight of $\mathbb{G}_{m}$ acting on $C_{X_{0} / X}$. So $V_{n}$ is canonically identified with the normal cone of $U \times_{\operatorname{Spec} A} \operatorname{Spec} A[t] /\left(t^{n}\right)$ in $V_{n}$. But this is in turn canonically isomorphic to $C_{X_{0} / X} \times_{X_{0}} U$ since the normal cone is constant in the deformation to the normal cone. By uniqueness of this identification, it glues over all étale affine charts of $X_{0}$ to give the desired isomorphism $C_{n} \cong C_{X_{0} / X} \times{ }_{\operatorname{Spec} A} \operatorname{Spec} A[t] /\left(t^{n}\right)$.

Proof of Proposition 5.4.13. Applying Lemma 5.4.14 to the morphism $\tilde{Z} \rightarrow Z$, we deduce that $\tilde{Z}$ is $\mathbb{G}_{m}$-equivariantly isomorphic to a line bundle over $D_{\alpha_{i}^{\vee}}(Z)=Y$. So by Corollary 5.4.12, $\tilde{\chi}_{Z}: \tilde{Z} \rightarrow \Theta_{Y}^{-1}$ is a morphism of line bundles over $Y$ such that the preimage of the zero section is the zero section, and is therefore an isomorphism.

Proof of Theorem 5.4.6. Let $Z^{s s, \text { reg }} \subseteq Z$ and $\tilde{Z}^{\text {ss,reg }} \subseteq \tilde{Z}$ denote the preimages of $\operatorname{Bun}_{G, \text { rig }}^{\text {ss,reg }}$ in $Z$ and $\tilde{Z}$ respectively. Since the morphism $Z \rightarrow \operatorname{Bun}_{G, \text { rig }} / E$ is smooth and $Z \rightarrow S$ is surjective, Propositions 4.3 .14 and 4.3.15 imply that $\tilde{Z}^{s s, r e g} \rightarrow Z^{s s, \text { reg }}$ is a ramified Galois cover relative to $S$ with Galois group $W$, and that $Z^{s s, \text { reg }} \subseteq Z$ and $\tilde{Z}^{\text {ss,reg }} \subseteq Z^{\text {ss }}=$ $\tilde{\chi}_{Z}^{-1}\left(\left(\Theta_{Y}^{-1}\right)^{*}\right)$ are big relative to $S$. So, since $Z$ and $\widehat{Y} / / W$ are affine over $S$, there is a
commutative diagram

where the vertical arrows are isomorphisms by Proposition 5.4.13, and the horizontal arrows marked are isomorphisms either by construction or by ramified Galois descent for regular functions. So $\chi_{Z}$ is an isomorphism, which completes the proof of the theorem.

### 5.5 Applications of the Friedman-Morgan section theorem

In this section, we give some applications of the Friedman-Morgan section theorem.
Corollary 5.5.1 (cf., [L2, Theorem 3.4]). The quotient $\widehat{Y} / / W$ is an affine space bundle over $S$, on which $\mathbb{G}_{m}$ acts linearly and with positive weights.

Proof. This is immediate from Theorem 5.4.6 since the claim holds for the slice $Z \rightarrow S$ by Proposition 5.2.6.

Corollary 5.5.2. The coarse quotient map $\chi: \operatorname{Bun}_{G} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ is flat.
Proof. Since $(\widehat{Y} / / W) / \mathbb{G}_{m}$ is smooth over $S$, hence regular, this follows from Proposition 4.5.5.

Theorem 5.4.6 also has applications to the theory of regular semistable bundles. We first note the following properties of the $G$-bundles arising from regular slices.

Proposition 5.5.3. Let $Z \rightarrow \operatorname{Bun}_{G, r i g}$ be as in Theorem 5.4.6, let $z$ : Spec $k \rightarrow Z$ be a geometric point not fixed under the $\mathbb{G}_{m}$-action, and let $\xi_{G, z} \rightarrow E_{s}$ be the corresponding $G$ bundle. Then the $G$-bundle $\xi_{G, z}$ is regular semistable in the sense of Definition 4.3.7, and $\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right)=l$.

Proof. We can assume for simplicity that $S=\operatorname{Spec} k$.
Since $z$ does not map to the image 0 of the cone point in $\widehat{Y} / / W, \xi_{G, z}$ is semistable by Proposition 4.5.4. Since the morphism $\tilde{Z} \rightarrow Z$ can be identified with $\Theta_{Y}^{-1} \rightarrow \widehat{Y} / / W$, it is finite over $z$, so $\operatorname{dim} \psi^{-1}\left(\xi_{G, z}\right)=0$ and $\xi_{G, z}$ is regular.

To show that $\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right)=l$, let $x$ be the image of $z$ in $\operatorname{Bun}_{G, r i g}$, and let $x^{\prime}$ be its image in $\operatorname{Bun}_{G, \text { rig }} / E$. By Lemma 5.5.4 below, any translate of $x$ is isomorphic to $x$, so the $E$-action on $\operatorname{Bun}_{G, r i g}$ restricts to an action on $\mathbb{B} \operatorname{Aut}(x)$ with quotient $\mathbb{B} \operatorname{Aut}\left(x^{\prime}\right)$. So we have

$$
\operatorname{dim} \operatorname{Aut}\left(x^{\prime}\right)=\operatorname{dim} \operatorname{Aut}(x)+1=\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right)+1
$$

Since the morphism $Z \rightarrow \operatorname{Bun}_{G, r i g} / E$ is smooth and $\operatorname{Bun}_{G, r i g} / E$ has dimension -1 , the locally closed substack

$$
\mathbb{B} \operatorname{Aut}\left(x^{\prime}\right) \times_{\operatorname{Bun}_{G, r i g} / E} Z \longleftrightarrow Z
$$

has codimension $\operatorname{dim} \operatorname{Aut}\left(x^{\prime}\right)-1=\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right)$. But it is clear from Theorem 5.4.6 that this is simply the $\mathbb{G}_{m}$-orbit of $z$, which has codimension $l$, so we are done.

Given a point $y: \operatorname{Spec} k \rightarrow Y$ over $s: \operatorname{Spec} k \rightarrow S$, we write

$$
U_{y}=\prod_{\substack{\alpha \in \Phi_{-} \\ \alpha(y)=0}} U_{\alpha} \subseteq R_{u}(B) .
$$

Note that the group scheme $\xi_{T} \times{ }^{T} U_{y}$ on $E_{s}$ is canonically isomorphic to $U_{y} \times E_{s}$ once we fix a trivialisation of the associated $T_{y}$-bundle, where $T_{y}$ is the torus with character group $\mathbb{Z} \Phi_{y}, \Phi_{y}=\{\alpha \in \Phi \mid \alpha(y)=0\}$ and $\xi_{T}$ is the $T$-bundle corresponding to $y$. We also write $U_{y} /\left[U_{y}, U_{y}\right]=\prod_{\alpha \in \Delta_{y}} U_{-\alpha}$.

Lemma 5.5.4. Fix a geometric point $s$ : $\operatorname{Spec} k \rightarrow S$ and a semistable $G$-bundle $\xi_{G} \rightarrow E_{s}$. Then any translate of $\xi_{G}$ is isomorphic to $\xi_{G}$.

Proof. For ease of notation, we may as well assume that $S=$ Spec $k$. We need to show that for any $x: \operatorname{Spec} k \rightarrow E$, we have $t_{x}^{*} \xi_{G} \cong \xi_{G}$, where $t_{x}: E \rightarrow E$ is the translation by $x$. Since $\xi_{G}$ is semistable, there exists a $B$-reduction $\xi_{B}$ of $\xi_{G}$ of degree 0 . We will show that $t_{x}^{*} \xi_{B} \cong \xi_{B}$ as $B$-bundles.

Writing $\xi_{T}=\xi_{B} \times{ }^{B} T$ for the associated $T$-bundle, and $y \in Y$ for the point classifying $\xi_{T}$, by Lemma 4.3.11, we have that $\xi_{B}$ reduces canonically to a $T U_{y}$-bundle $\xi_{T U_{y}}$. Moreover, we have $t_{x}^{*} \xi_{T} \cong \xi_{T}$ since $\xi_{T}$ has degree 0 (this follows from translation invariance for degree 0 line bundles). Fixing such an isomorphism and a trivialisation of the associated $T_{y}$-bundle, and hence an isomorphism $\xi_{T} \times{ }^{T} U_{y} \cong E \times U_{y}$ as above, we have that the $U_{y}$-bundle $t_{x}^{*} \xi_{T U_{y}} / T$ is the image of the $U_{y}$-bundle $\xi_{T U_{y}} / T$ under the homomorphism

$$
H^{1}\left(E, U_{y}\right) \xrightarrow{t_{x}^{*}} H^{1}\left(E, U_{y}\right) \xrightarrow{t} H^{1}\left(E, U_{y}\right),
$$

where the second morphism is induced by some element $t$ of $T$ acting on $U_{y}$ determined by our choice of isomorphism $t_{x}^{*} \xi_{T} \cong \xi_{T}$. Since the translation action of $E$ on $H^{1}\left(E, U_{y}\right)$ is trivial, since $H^{1}\left(E, U_{y}\right)$ is an affine variety, the morphism $t_{x}^{*}$ above is the identity. It follows that $t_{x}^{*} \xi_{T U_{y}} \cong \xi_{T U_{y}}$, and hence $t_{x}^{*} \xi_{G} \cong \xi_{G}$ as claimed.

The next result is the analogue of $[\mathrm{S} 6, \S 3.7$, Theorem 2] for elliptic Springer theory.
Proposition 5.5.5. Let $\xi_{G} \in \operatorname{Bun}_{G}^{s s}$ be a semistable $G$-bundle on a geometric fibre $E_{s}$ of $E \rightarrow S$. Then $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \geq l$, and the following are equivalent.
(1) The bundle $\xi_{G}$ is regular.
(2) $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l$.
(3) For any degree 0 reduction $\xi_{B}$ of $\xi_{G}$ to $B$ with associated $T$-bundle $\xi_{T}$ corresponding to $y \in Y$, the associated $\xi_{T} \times{ }^{T} R_{u}(B)$-bundle $\xi_{B} / T$ is induced from a $U_{y}$-bundle with nontrivial associated $U_{-\alpha}$-bundles for $\alpha \in \Delta_{y}$.
(4) For some degree 0 reduction $\xi_{B}$ of $\xi_{G}$ to $B$ with associated $T$-bundle $\xi_{T}$ corresponding to $y \in Y$, the associated $\xi_{T} \times{ }^{T} R_{u}(B)$-bundle $\xi_{B} / T$ is induced from a $U_{y}$-bundle with nontrivial associated $U_{-\alpha}$-bundles for $\alpha \in \Delta_{y}$.

Moreover, there is a unique G-bundle satisfying the above equivalent conditions in every geometric fibre of $\chi^{s s}: \operatorname{Bun}_{G}^{s s} \rightarrow Y / / W$.

Proof. Since the statement only concerns individual $G$-bundles on geometric fibres of $E \rightarrow S$, we may assume for simplicity that $S=\operatorname{Spec} k$ for $k$ some algebraically closed field, and that $\xi_{G} \rightarrow E_{s}=E$ is defined over $k$.

To show that $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \geq l$, fix any reduction $\xi_{B} \in \widetilde{\operatorname{Bun}}_{G}^{s s}$ of $\xi_{G}$ to a $B$-bundle of degree 0 , and write $\xi_{T}=\xi_{B} \times{ }^{B} T$. Note that $\operatorname{Aut}\left(\xi_{B}\right)$ is a closed subgroup of $\operatorname{Aut}\left(\xi_{G}\right)$, so
$\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \geq \operatorname{dim} \operatorname{Aut}\left(\xi_{B}\right)$. Moerover, there is a commutative diagram

where the top and bottom horizontal arrows are locally closed immersions of codimension $\operatorname{dim} \operatorname{Aut}\left(\xi_{B}\right)$ and $\operatorname{dim} \operatorname{Aut}\left(\xi_{T}\right)$ respectively (since both target stacks have dimension 0 ), and the right hand vertical arrow is smooth. It follows that $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \geq \operatorname{dim} \operatorname{Aut}\left(\xi_{B}\right) \geq$ $\operatorname{dim} \operatorname{Aut}\left(\xi_{T}\right)=l$.

To prove the equivalence of (1), (2), (3) and (4), we first remark that by Lemma 4.3.11, for every degree $0 B$-bundle $\xi_{B}$ with image $y \in Y$, the associated $\xi_{T} \times{ }^{T} R_{u}(B)$-bundle $\xi_{B} / T$ reduces canonically to $U_{y}$.

It is clear that $(3) \Rightarrow(4)$. We show that $(4) \Rightarrow(2) \Rightarrow(3)$ and $(4) \Rightarrow(1) \Rightarrow(3)$.
Assume (4) holds, and let $\xi_{U_{y}}$ be the reduction of $\xi_{B} / T$ to $U_{y}$. Observe that the set of $U_{y}$-bundles $\eta_{U_{y}}$ such that the induced $G$-bundle $\eta_{G}$ is regular with $\operatorname{dim} \operatorname{Aut}\left(\eta_{G}\right)=l$ is open, and nonempty by Proposition 5.5.3 and the existence of reductions to $U_{y}$ remarked above. So we can find $\eta_{U_{y}}$ with these properties such that all the associated $U_{-\alpha}$-bundles are nontrivial for $\alpha \in \Delta_{y}$. We will show that $\xi_{G} \cong \eta_{G}$, from which we can deduce (1) and (2), as well as the uniqueness statement of the proposition.

First, observe that $\Delta_{y}$ is a set of positive simple roots for the root system $\Phi_{y}=\{\alpha \in$ $\Phi \mid \alpha(y)=0\}$. In particular, the homomorphism

$$
\begin{aligned}
T & \mathbb{G}_{m}^{\Delta_{y}}=T_{y} \\
t & \longmapsto(\alpha(t))_{\alpha \in \Delta_{y}}
\end{aligned}
$$

is surjective, with kernel $K_{y}$ of dimension $l-\left|\Delta_{y}\right|$. So $T=\operatorname{Aut}\left(\xi_{T}\right)$ acts transitively on the subset of points in $\prod_{\alpha \in \Delta_{y}} H^{1}\left(E, U_{-\alpha}\right)$ with nonzero projection to each factor. So, acting by automorphisms of $\xi_{T}$ if necessary, we may assume that $\eta_{U_{y}} \times{ }^{U_{y}} U_{y} /\left[U_{y}, U_{y}\right] \cong$ $\xi_{U_{y}} \times{ }^{U_{y}} U_{y} /\left[U_{y}, U_{y}\right]$. To prove $\eta_{U_{y}} \cong \xi_{U_{y}}$, and hence $\eta_{G} \cong \xi_{G}$, it will suffice to show that the diagram

is a pullback.
Observe that, since $K_{y}$ acts trivially on $U_{y}$ by definition, it also acts trivially on $\mathrm{Bun}_{U_{y}}$, so $\operatorname{Bun}_{U_{y}} / T$ is a $K_{y}$-gerbe over $\operatorname{Bun}_{U_{y}} / T_{y}$. Since $\operatorname{Bun}_{U_{y}} / T$ embeds into $\operatorname{Bun}_{B}^{0}$ as the fibre over $y \in Y$, we therefore have

$$
l=\operatorname{dim} \operatorname{Aut}\left(\eta_{G}\right) \geq \operatorname{dim} \operatorname{Aut}\left(\eta_{B}\right) \geq \operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}}\right)+l-\left|\Delta_{y}\right|
$$

and hence $\operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}}\right) \leq\left|\Delta_{y}\right|$. But since the top and bottom arrows of (5.5.1) are locally closed immersions of codimensions $\operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}}\right)$ and $\operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}} \times{ }^{U_{y}} U_{y} /\left[U_{y}, U_{y}\right]\right)=\left|\Delta_{y}\right|$ respectively and the right vertical morphism is smooth, we have that $\operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}}\right) \geq\left|\Delta_{y}\right|$, and hence $\operatorname{dim} \operatorname{Aut}\left(\eta_{U_{y}}\right)=\left|\Delta_{y}\right|$. So the locally closed immersion of $\mathbb{B} \operatorname{Aut}\left(\eta_{U_{y}}\right)$ into the pullback induced by (5.5.1) has codimension 0 . But since $U_{y}$ and $U_{y} /\left[U_{y}, U_{y}\right]$ are unipotent,
this is actually a closed immersion by Proposition 2.4.6, hence an isomorphism since the pullback is smooth and connected. This completes the proof that $(4) \Rightarrow(1)$ and $(4) \Rightarrow(2)$.

We prove $(1) \Rightarrow(3)$ via the contrapositive. Assume that (3) is false, and choose a degree 0 reduction $\xi_{B}$ with with associated $T$-bundle $\xi_{T}$ corresponding to $y \in Y$ induced from $\xi_{T U_{y}}$, and $\alpha \in \Delta_{y}$ such that the associated $U_{-\alpha}$-bundle $\xi_{U_{-\alpha}}$ is trivial. The space $C^{s_{\alpha}}$ of sections of

$$
\xi_{T U_{y}} \times{ }^{T U_{y}} B s_{\alpha} B / B=\xi_{T U_{y}} \times{ }^{T U_{y}} R_{u}(B) /\left(R_{u}(B) \cap s_{\alpha} R_{u}(B) s_{\alpha}\right)
$$

embeds as a locally closed subscheme of $\psi^{-1}\left(\xi_{G}\right)$. But the image of $U_{-\alpha}$ in $R_{u}(B) /\left(R_{u}(B) \cap\right.$ $\left.s_{\alpha} R_{u}(B) s_{\alpha}\right)$ is $T U_{y}$-invariant, so gives a closed immersion

$$
\xi_{U_{-\alpha}}=E \times U_{-\alpha} \longleftrightarrow \xi_{T U_{y}} \times^{T U_{y}} R_{u}(B) /\left(R_{u}(B) \cap s_{\alpha} R_{u}(B) s_{\alpha}\right)
$$

and hence a locally closed immersion $\mathbb{A}_{k}^{1} \hookrightarrow C^{s_{\alpha}} \hookrightarrow \psi^{-1}\left(\xi_{G}\right)$, from which we deduce $\operatorname{dim} \psi^{-1}\left(\xi_{G}\right)>0$. So $\xi_{G}$ is not regular.

Finally, to prove that $(2) \Rightarrow(3)$, note that any reduction $\xi_{T U_{y}} \in \operatorname{Bun}_{U_{y}} / T \subseteq \operatorname{Bun}_{B}^{0}$ of a bundle $\xi_{G}$ with $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l$ must satisfy $\operatorname{dim} \operatorname{Aut}\left(\xi_{T U_{y}}\right) \leq l$. So

$$
\mathbb{B A u t}\left(\xi_{T U_{y}}\right) \longleftrightarrow \operatorname{Bun}_{U_{y}} / T
$$

is a locally closed immersion of codimension $\leq 0$, hence an open immersion. Since $\operatorname{Bun}_{U_{y}} / T$ is irreducible, this open substack meets the open substack of points with nontrivial associated $U_{-\alpha}$-bundles for all $\alpha \in \Delta_{y}$, so $\xi_{T U_{y}}$ itself must have nontrivial associated $U_{-\alpha}$-bundle, and we are done.

As the terminology suggests, the regular semistable and regular unstable $G$-bundles can be grouped together into a single open substack of $\mathrm{Bun}_{G}$. In what follows, we define $\operatorname{Bun}_{G}^{r e g} \subseteq \operatorname{Bun}_{G}$ to be the union over all open substacks $U \subseteq$ Bun $_{G}$ such that the morphism

$$
\begin{equation*}
\psi^{-1}(U) \longrightarrow U \times_{(\widehat{Y} / / W) / \mathbb{G}_{m}} \Theta_{Y}^{-1} / \mathbb{G}_{m} \tag{5.5.2}
\end{equation*}
$$

is an isomorphism.
Proposition 5.5.6. The open substack $\operatorname{Bun}_{G}^{r e g} \subseteq \operatorname{Bun}_{G}$ is dense in every geometric fibre of $\chi: \operatorname{Bun}_{G} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$, and big relative to $S$.

Proof. Let $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\} \subseteq \Delta$ denote the set of special roots, and let $Z_{1}, \ldots, Z_{n}$ be the corresponding regular slices of $\mathrm{Bun}_{G, r i g}$. Let $U \subseteq \mathrm{Bun}_{G}$ be the preimage in $\mathrm{Bun}_{G}$ of the union of the images of $Z_{i} \rightarrow \operatorname{Bun}_{G, r i g} / E$. Note that this is open since each $Z_{i} \rightarrow$ $\operatorname{Bun}_{G, r i g} / E$ is smooth. By Proposition 5.4.13 and Theorem 5.4.6, it is clear that (5.5.2) is an isomorphism, so $U \subseteq \operatorname{Bun}_{G}^{r e g}$. Note that $U$ contains all regular unstable bundles by construction and that Propositions 5.5.3 and 5.5 .5 imply that $U$ also contains all regular semistable bundles, so the same is true for $\mathrm{Bun}_{G}^{r e g}$.

We first show that for every $x \in(\widehat{Y} / / W) / \mathbb{G}_{m}, \chi^{-1}(x) \cap \operatorname{Bun}_{G}^{r e g}$ is dense in $\chi^{-1}(x)$. For $x$ not in the zero section of $\widehat{Y} / / W \rightarrow S$, this is clear since Proposition 5.5.5 implies that $\chi^{-1}(x) \cap \operatorname{Bun}_{G}^{r e g}$ is open and nonempty, and that $\chi^{-1}(x)$ is irreducible. For $x$ in the zero section, note that since $\widehat{Y} / / W$ is regular, the inclusion $\{x\} \hookrightarrow \widehat{Y} / / W$ is a local complete intersection morphism. So by Corollary 5.5.2, $\chi^{-1}(x)$ is a local complete intersection stack, hence of pure dimension. Since $\chi^{-1}(x)$ is the locus of unstable bundles on some fibre of $E \rightarrow$ $S$, $\operatorname{Bun}_{G}^{r e g}$ meets every irreducible component of $\chi^{-1}(x)$ by construction, so $\operatorname{Bun}_{G}^{r e g} \cap \chi^{-1}(x)$ is dense in $\chi^{-1}(x)$ as claimed.

Finally, notice that $\operatorname{Bun}_{G}^{s s, r e g} \subseteq \operatorname{Bun}_{G}^{r e g}$, and the complement of $\operatorname{Bun}_{G}^{s s}$ in $\operatorname{Bun}_{G}$ has codimension at least 2, so $\operatorname{Bun}_{G}^{r e g} \subseteq \operatorname{Bun}_{G}$ is big by Proposition 4.3.15.

Corollary 5.5.7. The elliptic Grothendieck-Springer resolution

and its rigidification are simultaneous $\log$ resolutions in the sense of Definition 1.0.2.
Proof. It is enough to prove the claim for the non-rigidified diagram: the statement for rigidification follows immediately by descent along the gerbe $\operatorname{Bun}_{G} \rightarrow \operatorname{Bun}_{G, \text { rig }}$. For the non-rigidified diagram, condition (1) of Definition 1.0.2 holds by Propositions 4.1.1 and 4.5.5 and Corollary 5.5.2, condition (2) holds by Proposition 5.5.6, and (3) holds by Corollaries 3.5.4, 4.1.3 and 4.5.9.

We also have the following useful result on connectedness of elliptic Springer fibres.
Proposition 5.5.8. Let $f$ denote the morphism

$$
f: \widehat{\operatorname{Bun}}_{G} \longrightarrow \operatorname{Bun}_{G} \times_{(\widehat{Y} / / W) / \mathbb{G}_{m}} \Theta_{Y}^{-1} / \mathbb{G}_{m}
$$

Then $f_{*} \mathcal{O}=\mathcal{O}$ and $f$ has connected fibres.
Proof. We first remark that since the morphism $\Theta_{Y}^{-1} / \mathbb{G}_{m} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$ is a finite type morphism between regular stacks, it is necessarily a local complete intersection morphism. Since $\chi$ is flat, we deduce that the stack

$$
\operatorname{Bun}_{G} \times(\widehat{Y} / / W) / \mathbb{G}_{m} \Theta_{Y}^{-1} / \mathbb{G}_{m}
$$

is a local complete intersection stack, hence Cohen-Macaulay. Proposition 5.5.6 implies that the open substack of the target $\operatorname{Bun}_{G}^{r e g} \times_{(\widehat{Y} / / W) / \mathbb{G}_{m}} \Theta_{Y}^{-1} / \mathbb{G}_{m}$ is big, and necessarily regular since $\widetilde{\operatorname{Bun}}_{G}$ is. So $\operatorname{Bun}_{G} \times_{(\widehat{Y} / / W) / \mathbb{G}_{m}} \Theta_{Y}^{-1} / \mathbb{G}_{m}$ is normal by Serre's criterion and we must have $f_{*} \mathcal{O}=\mathcal{O}$. So by Zariski's connectedness theorem [O1, Theorem 11.3], $f$ has connected fibres, so we are done.

Corollary 5.5.9. Let $P \subseteq G$ be a standard parabolic subgroup with Levi factor $L$, let $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ be a Harder-Narasimhan vector for $P$, let $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ be a $\Theta$-trivial slice such that $Z_{0} \rightarrow S$ has connected fibres, and let $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right) \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be the corresponding equivariant slice. Then the morphism $D(Z) \rightarrow Y$ has connected fibres.

Proof. Since the diagram

is a pullback modulo non-reducedness and since Proposition 5.5.8 implies that the bottom arrow has connected fibres, the top arrow

$$
\begin{equation*}
D(Z) \longrightarrow Z \times_{\widehat{Y} / / W} 0_{\Theta_{Y}^{-1}} \tag{5.5.3}
\end{equation*}
$$

also has connected fibres. But

$$
\begin{equation*}
Z \times_{\widehat{Y} / / W} 0_{\Theta_{Y}^{-1}}=\chi_{Z}^{-1}(0) \times_{S} 0_{\Theta_{Y}^{-1}} \longrightarrow \chi_{Z}^{-1}(0) \tag{5.5.4}
\end{equation*}
$$

manifestly has connected fibres. Since both morphisms (5.5.3) and (5.5.4) are proper and surjective, it follows that the composition $D(Z) \rightarrow Y$ has connected fibres as claimed.

## Chapter 6

## Subregular unstable bundles

In the previous chapter, we saw how slices through regular unstable $G$-bundles could be used to construct sections of the coarse quotient map $\chi: \operatorname{Bun}_{G, \text { rig }} \rightarrow(\widehat{Y} / / W) / \mathbb{G}_{m}$. In this chapter, we use slices through slightly more unstable bundles to probe the geometry of the unstable locus $\chi^{-1}(0)$ and the elliptic Grothendieck-Springer resolution $\widetilde{\operatorname{Bun}}_{G, \text { rig }}$ in codimension 2. More precisely, given a subregular unstable bundle $\xi_{G}$ (see Definition 6.1.1), we construct an explicit equivariant slice $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$ meeting the orbit of $\xi_{G}$ under translations in a single point, such that $\chi_{Z}: Z \rightarrow \widehat{Y} / / W$ is a family of surfaces over $S$ with a simultaneous $\log$ resolution


We give explicit descriptions of the normal crossings varieties $\tilde{\chi}_{Z}^{-1}(y)$ for $y \in 0_{\Theta_{Y}^{-1}}$ in all cases, and deduce descriptions of the variety $\chi_{Z}^{-1}(0)$ and its singularities.

Remark 6.0.1. Although we have used the word "variety" in the above discussion, it must be confessed that in type $B$ the slice $Z$ is not representable over $S$, but has finite relative stabilisers. In all other cases, however, $Z$ and $\tilde{Z}$ are honest varieties over $S$.

The results presented here extend the work of Helmke and Slodowy [HS2], who computed the codimension 2 singularities of $\chi^{-1}(0)$ in types $A, D$ and $E$, and of Grojnowski and Shepherd-Barron [GSB], who gave a less explicit description of the sliced elliptic GrothendieckSpringer resolution (6.0.1) in type $E$ only.

The outline of this chapter is as follows. In $\S 6.1$, we review the definition and classification of subregular unstable $G$-bundles, and state our main general theorems on existence of wellbehaved subregular slices (Theorem 6.1.5) and the behaviour of the associated simultaneous log resolutions (6.0.1) (Theorem 6.1.9). In $\S 6.2$, we write down some computations of certain Bruhat cells for parabolic subgroups of $G L_{n}$, which we use in the proof of Theorem 6.1.9 in $\S \S 6.3-6.4$. We give the proof of Theorem 6.1 .5 in $\S 6.5$. In $\S 6.6$ we give case by case descriptions of $\tilde{\chi}_{Z}^{-1}(y)$ for $y \in 0_{\Theta_{Y}^{-1}}$, which refine Theorem 6.1.9. Finally, in $\S 6.7$, we illustrate in some examples how to use Theorem 6.1.9 and the results of $\S 6.6$ to identify the singularities of $\chi_{Z}^{-1}(0)$.

### 6.1 Classification and overview

In this section, we review Helmke and Slodowy's classification of subregular unstable bundles, and summarise our main results about the behaviour of the Grothendieck-Springer resolution near them.

Definition 6.1.1. Let $s: \operatorname{Spec} k \rightarrow S$ be a geometric point and let $\xi_{G} \rightarrow E_{s}$ be an unstable $G$-bundle. We say that $\xi_{G}$ is subregular if $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l+4$.

In the following theorem, if $s: \operatorname{Spec} k \rightarrow S$ is a geometric point, $L \subseteq G$ is a Levi subgroup, and $\xi_{L}$ is a semistable $L$-bundle on $E_{s}$ of slope $\mu \in \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$, then we say that $\xi_{L}$ is regular if its automorphism group has minimal dimension among all automorphism groups of semistable $L$-bundles on $E_{s}$ of slope $\mu$.

Theorem 6.1.2. Let $s: \operatorname{Spec} k \rightarrow S$ be a geometric point and let $\xi_{G} \rightarrow E_{s}$ be an unstable $G$-bundle. Then either $\xi_{G}$ is regular and $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l+2$, or $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right) \geq l+4$. If $\xi_{G}$ has Harder-Narasimhan reduction $\xi_{P}$ to a standard parabolic $P$ with Levi factor L, and associated L-bundle $\xi_{L}$ of slope $\mu$, then $\xi_{G}$ is subregular if and only if $\xi_{L}$ is regular semistable and $(G, P, \mu)$ satisfies one of the following conditions.
(Type $A_{1}$ ) $G$ is of type $A_{1}, t(P)=\left\{\alpha_{1}\right\}$ and $\left\langle\varpi_{1}, \mu\right\rangle=-2$.
(Type $A$ ) $G$ is of type $A_{l}$ for $l>1, t(P)=\left\{\alpha_{i}, \alpha_{i+1}\right\}$ for some $i$ with $1 \leq i<l$, and $\left\langle\varpi_{i}, \mu\right\rangle=\left\langle\varpi_{i+1}, \mu\right\rangle=-1$.
(Type $B$ ) $G$ is of type $B_{l}$ for $l \geq 3, t(P)=\left\{\alpha_{l-2}\right\}$ and $\left\langle\varpi_{l-2}, \mu\right\rangle=-1$.
(Type $C$ ) $G$ is of type $C_{l}$ for $l \geq 2, t(P)=\left\{\alpha_{l-1}\right\}$ and $\left\langle\varpi_{l-1}, \mu\right\rangle=-1$.
(Type $D$ ) $G$ is of type $D_{l}$ for $l \geq 4, t(P)=\left\{\alpha_{i}\right\}$ and $\left\langle\varpi_{i}, \mu\right\rangle=-1$, where $i \in\{1,3,4\}$ if $l=4$ and $i=l-3$ otherwise.
(Type $E$ ) $G$ is of type $D_{5}, E_{6}, E_{7}$ or $E_{8}, t(P)=\left\{\alpha_{i}\right\}$ and $\left\langle\varpi_{i}, \mu\right\rangle=-1$, where $i \in\{4,5\}$ if $G$ is of type $D_{5}, i \in\{2,5\}$ if $G$ is of type $E_{6}$, and $i=5$ if $G$ is of type $E_{7}$ or $E_{8}$.
(Type $F$ ) $G$ is of type $B_{3}$ or $F_{4}, t(P)=\left\{\alpha_{3}\right\}$ and $\left\langle\varpi_{3}, \mu\right\rangle=-1$.
(Type $G$ ) $G$ is of type $G_{2}, t(P)=\left\{\alpha_{1}\right\}$ and $\left\langle\varpi_{1}, \mu\right\rangle=-1$.
Here the labelling of the Dynkin diagrams is as in Table 6.1.


Table 6.1: Labelling of the Dynkin diagrams

Proof. The theorem is a selection of statements from [HS1, Theorems 5.1 and 5.12], which are proved there when $S=\operatorname{Spec} \mathbb{C}$. To deduce the theorem in general, note that by specialisation we have

$$
\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=-\langle 2 \rho, \mu\rangle+\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right) \geq-\langle 2 \rho, \mu\rangle+d(L, \mu)
$$

where $d(L, \mu)$ is the dimension of the automorphism group of a regular semistable $L$-bundle with slope $\mu$ over $\mathbb{C}$. So Proposition 5.4.2 and the statement of the theorem over $\mathbb{C}$ imply that there are no unstable bundles with $\operatorname{dim} \operatorname{Aut}\left(\xi_{G}\right)=l+3$ and that the Harder-Narasimhan reduction of any subregular unstable bundle must appear on the list above. A priori, there may be an elliptic curve $E_{s}$ over a field of positive characteristic such that regular semistable $L$-bundles $\xi_{L}$ on $E_{s}$ of slope $\mu$ have $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)>d(L, \mu)$, and hence $G$-bundles with Harder-Narasimhan reductions on the list above that are not subregular. However, in case (Type $A_{1}$ ) this cannot happen since $L=T$, and the proof of Theorem 6.1 .5 shows that this does not happen for the other Levis and slopes on the list (see Remark 6.1.6). So the theorem holds in all characteristics.

Definition 6.1.3. We will say that a tuple $(G, P, \mu)$ consisting of a simply connected simple group $G$, a standard parabolic $P$ with Levi factor $L$, and a Harder-Narasimhan vector $\mu$ for $P$ is a subregular Harder-Narasimhan class if $\xi_{L} \times{ }^{L} G$ is subregular unstable for $\xi_{L}$ a regular semistable $L$-bundle of slope $\mu$. We will say that $(G, P, \mu)$ is of type $A_{1}$ (resp., type $A$, type $B$, etc.) if it satisfies (Type $A_{1}$ ) (resp., (Type $A$ ), (Type B), etc.) of Theorem 6.1.2.

Remark 6.1.4. We stress that the type of a subregular Harder-Narasimhan class $(G, P, \mu)$ is often, but not always, the type of the group $G$. For example, for $G$ of type $B_{3}$, there are subregular Harder-Narasimhan classes of types $B$ and $F$, and for $G$ of type $D_{5}$, there are subregular Harder-Narasimhan classes of types $D$ and $E$.

For the rest of this chapter, unless otherwise specified we will assume that $G$ does not have type $A_{1}$. In the following theorem, we write

$$
d= \begin{cases}1, & \text { if }(G, P, \mu) \text { is of type } A, B, D \text { or } E, \\ 2, & \text { if }(G, P, \mu) \text { is of type } C \text { or } F \\ 3, & \text { if }(G, P, \mu) \text { is of type } G\end{cases}
$$

Theorem 6.1.5. Let $(G, P, \mu)$ be a subregular Harder-Narasimhan class not of type $A_{1}$. Then there is a $\mu_{d}$-gerbe $\mathfrak{G}^{\text {uni }}$ on the stack $M_{1,1}$ of elliptic curves such that if the pullback $\mathfrak{G}$ of $\mathfrak{G}^{\text {uni }}$ to $S$ is trivial then there exists a $\Theta$-trivial slice $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ with the following properties.
(1) The morphism $Z_{0} \rightarrow S$ is smooth and proper with finite and generically trivial relative stabilisers.
(2) The morphism $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ is smooth with connected fibres.
(3) The image of $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ is equal to the locus of regular semistable bundles.
(4) The induced equivariant slice $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right) \rightarrow \operatorname{Bun}_{G, \text { rig }}$ has relative dimension $l+3$ over $S$.

We prove Theorem 6.1.5 in $\S 6.5$ by writing down explicit slices in each case of Theorem 6.1.2. Although a classification-free proof is probably possible, the explicit slices also help us to describe interesting case-dependent features of the Grothendieck-Springer resolution near the locus of subregular $G$-bundles.

Remark 6.1.6. The proof will show that Theorem 6.1.5 holds for every tuple ( $G, P, \mu$ ) on the list of Theorem 6.1.2, excluding (Type $A_{1}$ ). In the notation of the proof of Theorem 6.1.2, this shows that in each case we have a slice $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ with relative dimension
$l+3+\langle 2 \rho, \mu\rangle=d(L, \mu)-1$ over $S$, and hence relative dimension $d(L, \mu)$ over Bun $_{L, \text { rig }}^{s s, \mu} / E$. Since $Z_{0} \rightarrow S$ has finite relative stabilisers, this shows that $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right) \leq d(L, \mu)$ for a regular semistable $L$-bundle in all characteristics.

Remark 6.1.7. As promised in the introduction to this chapter, the slices $Z \rightarrow \operatorname{Bun}_{G, r i g}$ of Theorem 6.1.2 meet the translation orbit of a subregular unstable bundle with HarderNarasimhan reduction $\xi_{P}$ to $P$ of slope $\mu$ in a single point. To see this, note that Remark 6.1.6 and the proof of Theorem 6.1.2 show that the automorphism group of the image $x$ of $\xi_{L}=\xi_{P} \times{ }^{P} L$ in $\operatorname{Bun}_{L, r i g}^{s s, \mu} / E$ is equal to the dimension of the fibre $\left(Z_{0}\right)_{x}$ of $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ over $x$. So $\left(Z_{0}\right)_{x} / \operatorname{Aut}(x) \subseteq\left(Z_{0}\right)_{s} \subseteq Z_{s}$ is a closed connected substack of dimension 0 , where $s$ is the image of $x$ in $S$, and is hence a single point since $\left(Z_{0}\right)_{s}$ has finite stabilisers.

Remark 6.1.8. We have deliberately excluded the subregular Harder-Narasimhan class of type $A_{1}$ from Theorem 6.1.5. In this case, we have $L=T \cong \mathbb{G}_{m}$ and $\operatorname{Bun}_{L}^{s s, \mu}=\operatorname{Bun}_{\mathbb{G}_{m}}^{-2}$, and one can try to construct the desired slice $Z_{0}=S \rightarrow \operatorname{Bun}_{L}^{s s, \mu}$ by lifting the natural section $\mathcal{O}\left(-2 O_{E}\right): S \rightarrow \operatorname{Pic}_{S}^{-2}(E)$. The resulting map $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ will be a slice as long as 2 is invertible in $\mathcal{O}_{S}$ (so that the stabiliser $E[2]$ of a point in $\operatorname{Pic}_{S}^{-2}(E)$ is smooth), and can be taken to be $\Theta$-trivial after passing to some smooth cover of $S$ if necessary. The resulting slice satisfies (1), (3) and (4), but the map $Z_{0} \rightarrow \mathrm{Bun}_{L, \text { rig }}^{s s,-2} / E$ is a torsor under an extension of $E[2]$ by $\mathbb{G}_{m}$ and hence has disconnected fibres.

In the next theorem, we describe the main case-independent features of the elliptic Grothendieck-Springer resolution near the subregular Harder-Narasimhan locus defined by $(G, P, \mu)$. For a clean statement, we will introduce the following notation.

If $(G, P, \mu)$ is of type $A$, then we set $\left\{\alpha_{i}, \alpha_{j}\right\}=\left\{\alpha_{i}, \alpha_{i+1}\right\}=t(P)$. Otherwise, we let $\left\{\alpha_{i}\right\}=t(P)$ and let $\alpha_{j} \in \Delta$ be the unique special root. Theorem 6.1.2 shows that in each case, $\alpha_{i}$ is adjacent to $\alpha_{j}$. Deleting the edge joining $\alpha_{i}$ and $\alpha_{j}$ breaks the Dynkin diagram of $G$ into two connected components; we write $c_{0}$ (resp., $c_{1}$ ) for the component containing $\alpha_{i}$ (resp., $\alpha_{j}$ ) and $n_{0}$ (resp., $n_{1}$ ) for the number of vertices in $c_{0}$ (resp., $c_{1}$ ). Since $\alpha_{j}$ is special, the Dynkin diagram of $c_{0}$ is of type $A_{n_{0}}$. We write $\left\{\alpha_{c_{0}, 1}, \ldots, \alpha_{c_{0}, n_{0}}, \alpha_{c_{0}, n_{0}}\right\} \subseteq \Delta$ for the vertices of $c_{0}$, labelled so that $\alpha_{c_{0}, k}$ is adjacent to $\alpha_{c_{0}, k+1}$ for all $k<n_{0}$ and $\alpha_{c_{0}, n_{0}}=\alpha_{i}$. For $k \leq n_{0}$, we also write $\varpi_{c_{0}, k} \in \mathbb{X}^{*}(T)$ for the fundamental dominant weight associated to $\alpha_{c_{0}, k} \in \Delta$, and for $k_{0} \leq n_{0}+1$, we write $\theta_{k}$ for the section

$$
\begin{array}{rl}
\theta_{k}: Y & Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \\
y & \longmapsto \begin{cases}\left(y, \varpi_{j}(y)-\varpi_{i}(y)-\varpi_{c_{0}, 1}(y)\right), & \text { if } k=1, \\
\left(y, \varpi_{j}(y)-\varpi_{i}(y)-\varpi_{c_{0}, k}(y)+\varpi_{c_{0}, k-1}(y)\right), & \text { if } 1<k \leq n_{0}, \\
(y, 0), & \text { if } k=n_{0}+1\end{cases}
\end{array}
$$

Theorem 6.1.9. Assume that $(G, P, \mu)$ is not of type $A_{1}$, let $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ be any $\Theta$ trivial slice satisfying the conditions of Theorem 6.1.5, and let $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right) \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be the induced equivariant slice. Then we have the following.
(1) We have

$$
\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)=d D_{\alpha_{i}^{\vee}}(Z)+D_{\alpha_{j}^{\vee}}(Z)+D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z),
$$

where each divisor is connected and smooth over $Y$, and $d=\frac{1}{2}\left(\alpha_{i}^{\vee} \mid \alpha_{i}^{\vee}\right)=-\left\langle\alpha_{j}, \alpha_{i}^{\vee}\right\rangle$ is the number defined before Theorem 6.1.5.
(2) Each fibre of the morphism $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z) \rightarrow Y$ is isomorphic to the Hirzebruch surface $\mathbb{F}_{d-1}$.
(3) There is a sequence of $n_{0}+1$ morphisms

$$
D_{\alpha_{j}^{\vee}}(Z)=D_{n_{0}+2} \longrightarrow D_{n_{0}+1} \longrightarrow \cdots \longrightarrow D_{1}
$$

over $Y \times_{S} Z$ such that $D_{1}$ is a line bundle over $Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$ and $D_{k+1} \rightarrow D_{k}$ is the blowup along the section $\theta_{k}: Y \rightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \subseteq D_{k}$ of the proper transform of the zero section of $D_{1}$.
(4) The divisors $D_{\alpha_{i}^{\vee}}(Z)$ and $D_{\alpha_{j}^{\vee}}(Z)$ intersect along the proper transform of the zero section $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E) \hookrightarrow D_{\alpha_{j}^{\vee}}(Z)$, and the induced map

$$
D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z) \longrightarrow \operatorname{Pic}_{S}^{0}(E)
$$

sends a stable map with two rational components of degrees $\alpha_{i}^{\vee}$ and $\alpha_{j}^{\vee}$ meeting $E$ in points $x^{\prime}$ and $x$ to the difference $x-x^{\prime} \in \operatorname{Pic}_{S}^{0}(E)$.
(5) The divisors $D_{\alpha_{i}^{\vee}}(Z)$ and $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$ intersect along a ruling of $\mathbb{F}_{d-1}$.
(6) The divisors $D_{\alpha_{j}^{\vee}}(Z)$ and $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$ intersect along the exceptional divisor of the final blowup in $D_{\alpha_{j}^{\vee}}(Z)$, which appears in each fibre $\mathbb{F}_{d-1}$ of $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z) \rightarrow Y$ as a curve of self-intersection $1-d$.

We give the proof of Theorem 6.1.9 in $\S 6.3$ and $\S 6.4$.
Remark 6.1.10. The statement of Theorem 6.1 .9 leaves completely open the identity of the family of surfaces $D_{\alpha_{i}^{\vee}}(Z) \rightarrow Y$, and the behaviour of the contraction $\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right) \rightarrow \chi_{Z}^{-1}(0)$. In fact, these both depend drastically on the subregular Harder-Narasimhan class. We give case-by-case descriptions in Theorem 6.6.1 and Theorem 6.7.3.

Remark 6.1.11. In [GSB, Theorem 6.7], it is argued that in type $E$, the fibre over $0 \in Y$ of $D_{\alpha_{i}^{\vee}}(Z) \rightarrow Y$ contains a chain of $t+1$ curves $\beta, \epsilon_{1}, \cdots, \epsilon_{t}$ isomorphic to $\mathbb{P}^{1}$ that are contracted under the morphism to $Z$, where $\beta=D_{\alpha_{i}^{\vee}}(Z)_{0} \cap D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{0}$ and $0 \leq t \leq l-1$. Using Theorem 6.1.9, we can identify these curves with strict transforms of the exceptional divisors of the blowups $D_{\alpha_{i}^{\vee}}(Z)_{0} \rightarrow\left(D_{n_{0}+1}\right)_{0} \rightarrow \cdots \rightarrow\left(D_{1}\right)_{0}$. This shows in particular that $t=n_{0}=l-4$, resolving the ambiguity in [GSB]. Theorem 6.1.9 also gives an explicit description of how the configuration of curves changes as we vary the point in $Y$.

Remark 6.1.12. As for Theorem 6.1.5, we have excluded type $A_{1}$ from Theorem 6.1.9 because the elliptic Grothendieck-Springer resolution in this case behaves very differently to the other types, as we now explain. Assume for simplicity that $S=\operatorname{Spec} k$ for $k$ some algebraically closed field of characteristic not 2. Then the slice $Z \rightarrow \operatorname{Bun}_{G, r i g}=\operatorname{Bun}_{S L_{2}, \text { rig }}$ of Remark 6.1.8 is the space $Z=\operatorname{Ext}^{1}\left(\mathcal{O}\left(2 O_{E}\right), \mathcal{O}\left(-2 O_{E}\right)\right)$ of extensions

$$
0 \longrightarrow \mathcal{O}\left(-2 O_{E}\right) \longrightarrow V \longrightarrow \mathcal{O}\left(2 O_{E}\right) \longrightarrow 0
$$

A negative degree reduction of such an $S L_{2}$-bundle to $B$ corresponds to a subbundle $L \subseteq V$ where $L$ is a line bundle of positive degree, necessarily 1 or 2 , so we deduce that the unstable locus $\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$ decomposes as a divisor with normal crossings

$$
\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)=D_{\alpha_{1}^{\vee}}(Z)+4 D_{2 \alpha_{1}^{\vee}}(Z)
$$

by Corollary 4.5.9. By Proposition 3.4.16, the section of degree $-2 \alpha_{1}^{\vee}$ corresponding to $\mathcal{O}\left(2 O_{E}\right) \subseteq V=\mathcal{O}\left(-2 O_{E}\right) \oplus \mathcal{O}\left(2 O_{E}\right)$ lifts to points in the self-intersection of $D_{\alpha_{1}^{\vee}}(Z)$, so
the divisor $D_{\alpha_{1}^{\vee}}(Z)$ is not smooth over $S$. Even worse, we can lift this section to a stable map with dual graph

and automorphism group $\mathbb{Z} / 2$, so $\tilde{Z} \rightarrow Z$ is not even representable in this case.

### 6.2 Some Bruhat cells for unstable vector bundles

In this section, we write down some auxiliary results on Bruhat cells for parabolic reductions of certain unstable vector bundles. These results form the basis for identifying the blowups in Theorem 6.1.9 and for identifying the divisor $D_{\alpha_{i}^{\vee}}(Z)$ in many examples.

Fix an integer $n>0$, and let $R_{n} \subseteq G L_{n}$ be the standard parabolic subgroup

$$
\begin{aligned}
R_{n} & =\left\{\left(a_{p, q}\right)_{1 \leq p, q \leq n} \in G L_{n} \mid a_{p, q}=0 \text { for } q>\max (p, n-1)\right\} \\
& =\left\{\left(\begin{array}{ccccc}
* & * & \cdots & * & 0 \\
* & * & \cdots & * & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
* & * & \cdots & * & 0 \\
* & * & \cdots & * & *
\end{array}\right)\right\}
\end{aligned}
$$

of type $\left\{\beta_{n-1}\right\}$, where, in the notation given just before Lemma 5.4.10, we write $\beta_{i}=$ $e_{i}-e_{i+1} \in \mathbb{X}^{*}\left(T_{Q_{n}^{n}}\right)=\mathbb{Z}^{n}, 1 \leq i \leq n-1$, for the simple roots of $G L_{n}$. Note that $-e_{1}^{*}$ is a Harder-Narasimhan vector for $R_{n}$. For $1 \leq k \leq n$, we consider the stack

$$
X_{k}^{n}=Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}} \mathrm{KM}_{Q_{k}^{n}, G L_{n}}^{-e_{n}^{*}} \times{\times \operatorname{Bun}_{G L_{n}}^{-1}}^{\operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}} .}
$$

where we recall that

$$
Q_{k}^{n}=\left\{\left(a_{p, q}\right)_{1 \leq p, q \leq n} \in G L_{n} \mid a_{p, q}=0 \text { for } p<\min (q, k)\right\}
$$

is the standard parabolic subgroup of $G L_{n}$ of type $\left\{\beta_{1}, \ldots, \beta_{k-1}\right\}$. Note that since $-e_{1}^{*}$ is a Harder-Narasimhan vector for $R_{n}, X_{k}^{n}$ is a locally closed substack of $Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}} \mathrm{KM}_{Q_{k}^{n}, G L_{n}}^{-e_{n}^{*}}$ by Proposition 2.6.5.

The aim of this section is to decompose $X_{k}^{n}$ into Bruhat cells (Proposition 6.2.1) and to describe the behaviour of the cells under the natural morphisms $X_{k+1}^{n} \rightarrow X_{k}^{n}$ (Proposition 6.2.7).

For $1 \leq k \leq n, w \in W_{R_{n}, Q_{k}^{n}}^{0}$ and $\lambda \in \mathbb{X}_{*}\left(T_{Q_{k}^{n}}\right)$, we write

$$
C_{k}^{w, \lambda}=C_{R_{n}, Q_{k}^{n} / S}^{w, \lambda}(E) \times_{\operatorname{Bun}_{R_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}
$$

where $C_{R_{n}, Q_{k}^{n} / S}^{w, \lambda}(E)$ is the Bruhat cell of Definition 3.7.3. For $1 \leq p \leq n-1$, let $w_{p} \in$ $W_{G L_{n}}=S_{n}$ be the cyclic permutation

$$
w_{p}=(n, n-1, \ldots, p+1, p)=s_{n-1} s_{n-2} \cdots s_{p}
$$

and let $w_{n}=1$ be the identity, where $W_{G L_{n}}$ is the Weyl group of $G L_{n}$, and $s_{i}=(i, i+1)$ is the reflection in the root $\beta_{i}$. We write

$$
C_{k, p}^{G L_{n}}= \begin{cases}Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}} C_{k}^{w_{p},-e_{n}^{*}}, & \text { if }(k, p) \neq(n, n), \\ C_{n}^{1,-e_{n-1}^{*}} \times_{S} E, & \text { if }(k, p)=(n, n),\end{cases}
$$

for $1 \leq k \leq n$ and $1 \leq p<k$ or $p=n$.
Proposition 6.2.1. For $1 \leq k \leq n$, there is a decomposition

$$
X_{k}^{n}=\bigcup_{1 \leq p<k} C_{k, p}^{G L_{n}} \cup C_{k, n}^{G L_{n}}
$$

into disjoint locally closed substacks.
We break the proof of Proposition 6.2.1 into several lemmas.
Lemma 6.2.2. Assume that $\xi_{R_{n}} \rightarrow E_{s}$ is a semistable $R_{n}$-bundle on a geometric fibre of $E \rightarrow S$ of degree $-e_{1}^{*}$ and that $\sigma: E_{s} \rightarrow \xi_{R_{n}} \times{ }^{R_{n}} G L_{n} / Q_{n}^{n}$ is a section of degree $\lambda \leq-e_{n}^{*}$. Then $\lambda \in\left\{-e_{n}^{*},-e_{n-1}^{*}\right\}$.

Proof. The section $\sigma$ corresponds to a complete flag

$$
0=V_{n} \subsetneq V_{n-1} \subsetneq \cdots \subsetneq V_{0}=V
$$

where $V$ is the vector bundle associated to the $G L_{n}$-bundle $\xi_{G L_{n}}=\xi_{R_{n}} \times{ }^{R_{n}} G L_{n}$, such that $V_{i-1} / V_{i}$ is a line bundle of degree $\left\langle e_{i}, \lambda\right\rangle$ for $i=1, \ldots, n$. Since $\xi_{R_{n}}$ is the HarderNarasimhan parabolic of $\xi_{G L_{n}}, V$ has Harder-Narasimhan decomposition $V=M \oplus U$, where $U$ is a semistable vector bundle of rank $n-1$ and degree -1 and $M$ is a line bundle of degree 0 . In particular, any quotient bundle of $V$ has slope $\geq-1 /(n-1)$, so we deduce that

$$
\begin{equation*}
\left\langle e_{1}+\cdots+e_{i}, \lambda\right\rangle=\operatorname{deg} V / V_{i} \geq \frac{-i}{n-1} \tag{6.2.1}
\end{equation*}
$$

for $i=1, \ldots, n-1$.
Since $\lambda \leq-e_{n}^{*}$ by assumption, we have

$$
\lambda=-e_{n}^{*}-\sum_{i=1}^{n-1} d_{i} \beta_{i}^{\vee}
$$

for some $d_{i} \in \mathbb{Z}_{\geq 0}$, where $\beta_{i}^{\bigvee}=e_{i}^{*}-e_{i+1}^{*}$. Applying (6.2.1), we have $d_{i}=0$ for $1 \leq i \leq n-2$ and $d_{n-1} \in\{0,1\}$, which implies the lemma.

Lemma 6.2.3. Assume that $w \in W_{R_{n}, Q_{n}^{n}}^{0}$ and $\lambda \in \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)$ with $C_{n}^{w, \lambda} \neq \emptyset$ and $\lambda \leq-e_{n}^{*}$. Then

$$
(w, \lambda) \in\left\{\left(1,-e_{n-1}^{*}\right)\right\} \cup\left\{\left(w_{p},-e_{n}^{*}\right) \mid 1 \leq p<n\right\}
$$

Proof. First note that by Lemma 6.2.2, we know that $\lambda \in\left\{-e_{n}^{*},-e_{n-1}^{*}\right\}$. Moreover, we have from the definition (3.7.2) that

$$
W_{R_{n}, Q_{n}^{n}}^{0}=\left\{w \in S_{n} \mid w^{-1}(i)<w^{-1}(i+1) \text { for } 1 \leq i<n-1\right\}=\left\{w_{p} \mid 1 \leq p \leq n\right\}
$$

Since $Q_{n}^{n} \subseteq G L_{n}$ is the standard Borel subgroup, the homomorphism

$$
j_{w}: \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)=\mathbb{X}_{*}\left(T_{R_{n} \cap Q_{n}^{n}}\right)=\mathbb{X}_{*}\left(T_{R_{n} \cap w Q_{n}^{n} w^{-1}}\right) \longrightarrow \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)
$$

is the isomorphism given by $w^{-1}$. So by Proposition 3.7.4 there exists a semistable $L_{n^{-}}$ bundle $\xi_{L_{n}} \rightarrow E_{s}$ on a geometric fibre of $E \rightarrow S$ of degree $-e_{1}^{*}$, where $L_{n} \cong G L_{n-1} \times \mathbb{G}_{m}$ is the standard Levi factor of $R_{n}$ and a section $\sigma_{L}: E_{s} \rightarrow \xi_{L_{n}} /\left(L_{n} \cap Q_{n}^{n}\right)$ of degree $w \lambda$. In particular, since $e_{n} \in \mathbb{X}^{*}\left(L_{n}\right),\left\langle e_{n}, w \lambda\right\rangle=\left\langle e_{n},-e_{1}^{*}\right\rangle=0$ and $w \lambda$ is the degree of a section

$$
E_{s} \xrightarrow{\sigma_{L}} \xi_{L_{n}} /\left(L_{n} \cap Q_{n}^{n}\right) \longleftrightarrow \xi_{L_{n}} \times{ }^{L_{n}} G L_{n} / Q_{n}^{n}
$$

If $\lambda=-e_{n}^{*}$ and $w=w_{p}$, then

$$
w \lambda= \begin{cases}-e_{n-1}^{*}, & \text { if } p<n \\ -e_{n}^{*}, & \text { if } p=n\end{cases}
$$

so from the above discussion we must have $p \in\{1, \ldots, n-1\}$. If $\lambda=-e_{n-1}^{*}$, on the other hand, then

$$
w \lambda= \begin{cases}-e_{n-2}^{*}, & \text { if } p<n-1 \\ -e_{n}^{*}, & \text { if } p=n-1 \\ -e_{n-1}^{*}, & \text { if } p=n\end{cases}
$$

so the above discussion and Lemma 6.2.2 imply that $p=n$. Combining these two cases gives that $(w, \lambda)$ is in the desired set.

Lemma 6.2.4. For all $\lambda \in \mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)$ with $\lambda \leq-e_{n}^{*}$, we have

$$
\bigcup_{w \in W_{R_{n}, Q_{n}^{n}}^{0}} C_{n}^{w, \lambda}=\operatorname{Bun}_{Q_{n}^{n}}^{\lambda} \times_{\operatorname{Bun}_{G L_{n}}^{-1}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}} .
$$

Proof. Assume for a contradiction that this fails for some $\lambda \leq-e_{n}^{*}$. Then by Proposition 3.7.6 there exist $w \in W_{R_{n}, Q_{n}^{n}}^{0} \backslash\{1\}$ and $\lambda^{\prime}<\lambda$ such that $C_{n}^{w, \lambda^{\prime}} \neq \emptyset$. So Lemmas 6.2.2 and 6.2.3 imply that $\lambda^{\prime}=-e_{n}^{*}$ and $\lambda \in\left\{-e_{n}^{*},-e_{n-1}^{*}\right\}$. But this contradicts $\lambda^{\prime}<\lambda$ so we are done.

Lemma 6.2.5. Let $1 \leq k<n$. Then

$$
W_{R_{n}, Q_{k}^{n}}^{0}=\left\{w_{p} \mid 1 \leq p<k\right\} \cup\left\{w_{n}\right\}
$$

and

$$
\begin{equation*}
\operatorname{Bun}_{Q_{k}^{n}}^{-e_{n}^{*}} \times \operatorname{Bun}_{G L_{n}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}=\bigcup_{w \in W_{R_{n}, Q_{k}^{n}}^{0}} C_{k}^{w,-e_{n}^{*}} \tag{6.2.2}
\end{equation*}
$$

Proof. From the definition,

$$
W_{R_{n}, Q_{k}^{n}}^{0}=\left\{w \in W_{R_{n}, Q_{n}^{n}}^{0} \mid w(i)<w(i+1) \text { for } k \leq i \leq n-1\right\}=\left\{w_{p} \mid 1 \leq p<k\right\} \cup\left\{w_{n}\right\}
$$

as claimed. Next, note that by Proposition 3.6.4, the natural morphism

$$
\mathrm{KM}_{Q_{n}^{n}, G L_{n}}^{-e_{n}^{*}} \longrightarrow \mathrm{KM}_{Q_{k}^{n}, G L_{n}}^{-e_{n}^{*}}
$$

is surjective. So any geometric point of $\operatorname{Bun}_{Q_{k}^{n}}^{-e_{n}^{*}} \times_{\operatorname{Bun}_{G L_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}$ lifts to a point of $\operatorname{Bun}_{Q_{n}^{n}}^{\lambda} \times \operatorname{Bun}_{G L_{n}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}$ for some $\lambda \leq-e_{n}^{*}$, and hence $\lambda \in\left\{-e_{n}^{*},-e_{n-1}^{*}\right\}$ by Lemma 6.2.2. So by Lemma 6.2.4, the morphism

$$
\coprod_{\substack{w \in W_{R_{n}, Q_{n}^{n}}^{n} \\ \lambda \in\left\{-e_{n}^{*},-e_{n-1}^{*}\right\}}} C_{n}^{w, \lambda} \longrightarrow \coprod_{w \in W_{R_{n}, Q_{k}^{n}}^{0}} C_{k}^{w,-e_{n}^{*}} \longrightarrow \operatorname{Bun}_{Q_{k}^{n}}^{-e_{n}^{*}} \times_{\operatorname{Bun}_{G L_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}
$$

is surjective, which proves (6.2.2).
Proof of Proposition 6.2.1. Suppose first that $k<n$. Then Proposition 3.6.4 and Lemma 6.2.2 imply that

$$
\mathrm{KM}_{Q_{k}^{n}, G L_{n}}^{-e_{n}^{*}} \times \times_{\operatorname{Bun}_{G L_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}=\operatorname{Bun}_{Q_{k}^{n}}^{-e_{n}^{*}} \times \operatorname{Bun}_{G L_{n}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}
$$

since $-e_{n}^{*}$ and $-e_{n-1}^{*}$ have the same image in $\mathbb{X}_{*}\left(T_{Q_{k}^{n}}\right)$. So we have the desired decomposition of $X_{k}^{n}$ into locally closed substacks by Lemma 6.2.5.

On the other hand, if $k=n$, then Proposition 3.2.18 implies that we have a decomposition

$$
X_{n}^{n}=M_{1,0, \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}}^{\circ}\left(\xi_{R_{n}}^{u n i} \times{ }^{R_{n}} G L_{n} / Q_{n}^{n},\left(-e_{n}^{*}, 1\right)\right) \cup M_{\operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}}^{\circ}\left(\xi_{R_{n}}^{u n i} \times{ }^{R_{n}} G L_{n} / Q_{n}^{n}, \tau_{\beta_{n-1}^{\vee}}^{-e_{n}^{*}}\right)
$$

since, by Lemma 6.2.2, $\tau_{\beta_{n-1}^{-}}^{-e_{n}^{*}}$ is the only stable $\mathbb{X}^{*}\left(T_{Q_{n}^{n}}\right) \oplus \mathbb{Z}$-graph of the correct degree and genus such that the corresponding space of marked stable maps is nonempty. By Proposition 3.4.10 and the definition of $\tau_{\beta_{n-1}}^{-e_{n}^{*}}$-marked stable maps we can rewrite this as

$$
X_{n}^{n}=\left(\operatorname{Bun}_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{\operatorname{Bun}_{G L_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}\right) \cup\left(\operatorname{Bun}_{Q_{n}^{n}}^{-e_{n-1}^{*}} \times_{\operatorname{Bun}_{G L_{n}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}} \times{ }_{S} E\right),
$$

which decomposes further as the desired decomposition

$$
X_{n}^{n}=\bigcup_{1 \leq p<n} C_{n, p}^{G L_{n}} \cup C_{n, n}^{G L_{n}}
$$

by Lemmas 6.2.3 and 6.2.4.
From the proof of Proposition 6.2.1, it is clear that $C_{n, n}^{G L_{n}} \subseteq X_{n}^{n}$ is the locus of stable maps with a single rational component of degree $\beta_{n-1}^{\vee}$ in the relevant fibre of the flag variety bundle, and that the natural projection

$$
\begin{equation*}
C_{n, n}^{G L_{n}}=C_{n}^{1,-e_{n-1}^{*}} \times_{S} E \longrightarrow E \tag{6.2.3}
\end{equation*}
$$

takes such a stable map to the point on $E$ meeting the rational curve. There is also a morphism

$$
\begin{array}{r}
C_{1, n}^{G L_{n}}=Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}} \longrightarrow Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{\operatorname{Pic}_{S}^{-1}(E)} Y_{R_{n}}^{-e_{1}^{*}} \longrightarrow \operatorname{Pic}_{S}^{1}(E)=E  \tag{6.2.4}\\
\left(y, y^{\prime}\right)
\end{array}>e_{n}\left(y^{\prime}\right)-e_{n}(y) .
$$

Lemma 6.2.6. The natural projection (6.2.3) is equal to the composition of (6.2.4) with the natural morphism $C_{n, n}^{G L_{n}} \rightarrow C_{1, n}^{G L_{n}}$.

Proof. This follows by direct calculation.
For $1 \leq p<n$, we let

$$
M_{p}^{G L_{n}} \subseteq C_{1, n}^{G L_{n}}
$$

be the closed substack given by the fibre product

where the morphism $C_{1, n} \rightarrow E$ is (6.2.4), and the morphism $Y_{Q_{n}^{n}}^{-e_{n}^{*}} \rightarrow Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{S} \operatorname{Pic}_{S}^{1}(E)$ is given by

$$
\begin{aligned}
\theta_{p}^{G L_{n}}: Y_{Q_{n}^{n}}^{-e_{n}^{*}} & \longrightarrow Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{S} \operatorname{Pic}_{S}^{1}(E)=Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times_{S} E \\
y & \longmapsto\left(y, e_{p}(y)-e_{n}(y)\right) .
\end{aligned}
$$

Proposition 6.2.7. For all $1 \leq k<n$, the morphism $X_{k+1}^{n} \rightarrow X_{k}^{n}$ restricts to isomorphisms

$$
C_{k+1, n}^{G L_{n}} \xrightarrow{\sim} C_{k, n}^{G L_{n}} \quad \text { and } \quad C_{k+1, p}^{G L_{n}} \xrightarrow{\sim} C_{k, p}^{G L_{n}}
$$

for $1 \leq p<k$, and a morphism

$$
C_{k+1, k}^{G L_{n}} \longrightarrow M_{k}^{G L_{n}} \subseteq C_{k, n}^{G L_{n}} \cong C_{1, n}^{G L_{n}}
$$

that identifies $C_{k+1, k}^{G L_{n}}$ with the total space of a line bundle over $M_{k}^{G L_{n}}$.
We will prove Proposition 6.2.7 at the end of this section using Propositions 3.7.4 and 3.7.5 to compute the relevant Bruhat cells in terms of reductions of $L_{n}$-bundles to $L_{n} \cap$ $w_{p} Q_{k}^{n} w_{p}^{-1}$ and sections of the associated $R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right)$-bundles. The first step is to identify the parabolics $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$.

Lemma 6.2.8. Suppose that $1 \leq k \leq n$ and $1 \leq p<k$. Then $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1} \subseteq L_{n}$ is the standard parabolic with type

$$
t\left(L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right)=\left\{\beta_{1}, \ldots, \beta_{k-2}\right\}
$$

so $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$ is identified with $Q_{k-1}^{n-1} \times \mathbb{G}_{m}$ under the natural identification $L_{n} \cong$ $G L_{n-1} \times \mathbb{G}_{m}$.

Proof. First observe that since $w_{p} \in W_{R_{n}, Q_{n}^{n}}^{0}$, we have

$$
L_{n} \cap Q_{n}^{n}=L_{n} \cap w_{p} Q_{n}^{n} w_{p}^{-1} \subseteq L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}
$$

so $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$ is indeed a standard parabolic subgroup of $L_{n}$. The type follows by direct computation.

In order to compute the degrees of the $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$-bundles of interest, we need to describe the homomorphism

$$
\begin{equation*}
T_{L \cap w_{p} Q_{k}^{n} w_{p}^{-1}} \longrightarrow T_{Q_{k}^{n}} \times_{\mathbb{G}_{m}} T_{R_{n}} \tag{6.2.5}
\end{equation*}
$$

induced by $j_{w_{p}}: w_{p}^{-1}(-) w_{p}: L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1} \rightarrow Q_{k}^{n}$ on the first factor and the inclusion $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1} \subseteq L_{n} \subseteq R_{n}$ on the second. (Here the maps to $\mathbb{G}_{m}$ in the fibre product on the right hand side are given by the determinant $e_{1}+\cdots+e_{n}$.)

Note that if $1 \leq p<k$, then by Lemma 6.2 .8 , the character lattices are given by

$$
\begin{gathered}
\mathbb{X}^{*}\left(T_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}\right)=\bigoplus_{1 \leq i \leq k-2} \mathbb{Z} e_{i} \oplus \mathbb{Z}\left(e_{k-1}+\cdots+e_{n-1}\right) \oplus \mathbb{Z} e_{n} \\
\mathbb{X}^{*}\left(T_{Q_{k}^{n}}\right)=\bigoplus_{1 \leq i \leq k-1} \mathbb{Z} e_{i} \oplus \mathbb{Z}\left(e_{k}+\cdots+e_{n}\right) \quad \text { and } \quad \mathbb{X}^{*}\left(T_{R_{n}}\right)=\mathbb{Z}\left(e_{1}+\cdots+e_{n-1}\right) \oplus \mathbb{Z} e_{n} .
\end{gathered}
$$

The cocharacter lattices are therefore given by

$$
\mathbb{X}_{*}\left(T_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}\right)=\bigoplus_{1 \leq i \leq k-1} \mathbb{Z} e_{i}^{*} \oplus \mathbb{Z} e_{n}^{*}, \quad \mathbb{X}_{*}\left(T_{Q_{k}^{n}}\right)=\bigoplus_{1 \leq i \leq k-1} \mathbb{Z} e_{i}^{*} \oplus \mathbb{Z} e_{n}^{*}
$$

and

$$
\mathbb{X}^{*}\left(T_{R_{n}}\right)=\mathbb{Z} e_{1}^{*} \oplus \mathbb{Z} e_{n}^{*}
$$

as quotients of $\mathbb{X}_{*}\left(T_{Q_{n}^{n}}\right)$.

Lemma 6.2.9. If $p<k$, then (6.2.5) factors as an isomorphism onto $T_{Q_{k}^{n}}$ sending $e_{n-1}^{*}$ to $e_{n}^{*}$ followed by the section (id, $\gamma_{k, p}$ ): $T_{Q_{k}^{n}} \rightarrow T_{Q_{k}^{n}} \times_{\mathbb{G}_{m}} T_{R_{n}}$, where $\gamma_{k, p}: T_{Q_{k}^{n}} \rightarrow T_{R_{n}}$ is the homomorphism given on cocharacters by

$$
\gamma_{k, p}\left(e_{i}^{*}\right)= \begin{cases}e_{1}^{*}, & \text { for } i \neq p \\ e_{n}^{*}, & \text { for } i=p\end{cases}
$$

Proof. Using the fact that the natural diagram

commutes, the claim follows by direct computation.
The situation for $p=n$ is also very simple.
Lemma 6.2.10. The canonical morphism $T_{L_{n} \cap Q_{k}^{n}} \rightarrow T_{Q_{k}^{n}} \times_{\mathbb{G}_{m}} T_{R_{n}}$ is an isomorphism.
Proof. The character lattice of $T_{L_{n} \cap Q_{k}^{n}}$ is given by

$$
\mathbb{X}^{*}\left(T_{L_{n} \cap Q_{k}^{n}}\right)=\bigoplus_{1 \leq i \leq k-1} \mathbb{Z} e_{i} \oplus \mathbb{Z}\left(e_{k}+\cdots+e_{n-1}\right) \oplus \mathbb{Z} e_{n}
$$

and hence the cocharacter lattice is

$$
\mathbb{X}_{*}\left(T_{L_{n} \cap Q_{k}^{n}}\right)=\bigoplus_{1 \leq i \leq k-1} \mathbb{Z} e_{i}^{*} \oplus \mathbb{Z} e_{k}^{*} \oplus \mathbb{Z} e_{n}^{*}
$$

The claim now follows by inspection.
We deduce that the degrees of the parabolic reductions appearing in the Bruhat cells are given as follows.

Lemma 6.2.11. If $p<k$ or $p=n$ then the morphism

$$
\begin{equation*}
C_{k, p}^{G L_{n}} \longrightarrow \operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}} \tag{6.2.6}
\end{equation*}
$$

factors through

$$
\operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n-1}^{*}} \subseteq \operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}
$$

Proof. First suppose that $(k, p) \neq(n, n)$. By Proposition 3.7.4 and our restriction on the degrees of the $L_{n}$-bundles, the bundles in the image of (6.2.6) have degrees mapping to $\left(-e_{n}^{*},-e_{1}^{*}\right) \in \mathbb{X}_{*}\left(T_{Q_{k}^{n}} \times_{\mathbb{G}_{m}} T_{L_{n}}\right)$ under (6.2.5). By Lemmas 6.2.9 and 6.2.10, this homomorphism is injective and sends $-e_{n-1}^{*}$ to $\left(-e_{n}^{*},-e_{1}^{*}\right)$, so the claim follows.

On the other hand, suppose that $(k, p)=(n, n)$. Then the degrees of bundles in the image of (6.2.6) must map to $\left(-e_{n-1}^{*},-e_{1}^{*}\right) \in \mathbb{X}_{*}\left(T_{Q_{n}^{n}} \times_{\mathbb{G}_{m}} T_{L_{n}}\right)$ under (6.2.5). Since this is an isomorphism by Lemma 6.2.10 and sends $-e_{n-1}^{*}$ to $\left(-e_{n-1}^{*},-e_{1}^{*}\right)$, the claim follows in this case as well.

Lemma 6.2.12. If $p<k$ or $p=n$, then the natural morphism

$$
\begin{equation*}
\operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n-1}^{*}} \times_{\operatorname{Bun}_{L_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{L_{n}}^{s s,-e_{1}^{*}} \longrightarrow Y_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n}^{*}} \times_{Y_{L_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{L_{n}}^{s s,-e_{1}^{*}} \tag{6.2.7}
\end{equation*}
$$

is an isomorphism.

Proof. Lemmas 6.2.8 and 6.2.13 show that we have a pullback

so the claim follows immediately from Lemma 5.4.10.
Lemma 6.2.13. Let $\rho: H \rightarrow H^{\prime}$ be a surjective homomorphism of reductive groups and let $P_{1} \subseteq P_{2} \subseteq H$ be parabolic subgroups. For any $\lambda \in \mathbb{X}_{*}\left(T_{\rho^{-1}\left(P_{1}\right)}\right)$, the natural diagrams

and

are pullbacks, where we also write $\lambda$ for its images in $\mathbb{X}_{*}\left(T_{\rho^{-1}\left(P_{2}\right)}\right), \mathbb{X}_{*}\left(T_{P_{1}}\right)$ and $\mathbb{X}_{*}\left(T_{P_{2}}\right)$.
Proof. The statement follows easily from the fact that

$$
\operatorname{Bun}_{\rho^{-1}\left(P_{1}\right)}^{\lambda}=\operatorname{Bun}_{P_{1}}^{\lambda} \times \times_{\operatorname{Bun}_{P_{2}}^{\lambda}} \operatorname{Bun}_{\rho^{-1}\left(P_{2}\right)}^{\lambda}, \quad Y_{\rho^{-1}\left(P_{1}\right)}^{\lambda}=Y_{P_{1}}^{\lambda} \times_{Y_{P_{2}}^{\lambda}} Y_{\rho^{-1}\left(P_{2}\right)}^{\lambda}
$$

and

$$
\mathrm{KM}_{\rho^{-1}\left(P_{1}\right), H}^{\lambda}=\mathrm{KM}_{P_{1}, H^{\prime}}^{\lambda} \times_{\mathrm{KM}_{P_{2}, H^{\prime}}^{\lambda}} \mathrm{KM}_{\rho^{-1}\left(P_{2}\right), H}^{\lambda}
$$

Lemma 6.2.14. If $p<k \leq n$, then the morphism
$Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}}\left(\operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n}^{*}} \times_{\operatorname{Bun}_{L_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}\right) \longrightarrow Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}=C_{1, n}^{G L_{n}}=X_{1}^{n}$
induced by the inclusion $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1} \subseteq L_{n}$ factors through an isomorphism onto $M_{p}^{G L_{n}}$. Here the morphisms to $\operatorname{Pic}_{S}^{-1}(E)$ in the fibre product in the right hand side of (6.2.8) are both given by the determinant.

Proof. First note that by Lemma 6.2.12 and Lemma 6.2.9, we have isomorphisms

$$
\begin{aligned}
Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}}\left(\operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n-1}^{*}}\right. & \left.\times_{\operatorname{Bun}_{L_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}\right) \\
& \xrightarrow{\sim} Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times \times_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}}\left(Y_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e^{*}} \times{ }_{Y_{R_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}}\right), \\
& \xrightarrow{\sim} Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times \times_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}}\left(Y_{Q_{k}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{R_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{R_{n}}^{s,-e_{1}^{*}}\right),
\end{aligned}
$$

where $Y_{Q_{k}^{n}}^{-e_{n}^{*}}$ maps to $Y_{R_{n}}^{-e_{1}^{*}}$ via the homomorphism $\gamma_{k, p}$. So we can identify (6.2.8) with the closed immersion

$$
Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times{ }_{Y_{R_{n}}^{-e_{1}^{*}}} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}} \longrightarrow Y_{Q_{n}^{n}}^{-e_{n}^{*}} \times \times_{\operatorname{Pic}_{S}^{-1}(E)} \operatorname{Bun}_{R_{n}}^{s s,-e_{1}^{*}},
$$

where the morphism $Y_{Q_{n}^{n}}^{-e_{n}^{*}} \rightarrow Y_{R_{n}}^{-e_{1}^{*}}$ is the composition of $Y_{Q_{n}^{n}}^{-e_{n}^{*}} \rightarrow Y_{Q_{k}^{n}}^{-e_{n}^{*}}$ with $\gamma_{k, p}$. Chasing through the various definitions now shows that the source of this morphism is precisely $M_{p}^{G L_{n}}$, so we are done.

We next identify the $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$-variety $R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right) . \quad$ In the following lemma, we write $U_{k, p}$ for the $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$-representation induced by the homomorphism

$$
L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}=Q_{k-1}^{n-1} \times \mathbb{G}_{m} \longrightarrow Q_{k-1}^{n-1} \longrightarrow G L_{n-p}
$$

given by deleting the last row and column and the first $p-1$ rows and columns.
Lemma 6.2.15. If $p<k$, then there is an $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$-equivariant isomorphism

$$
\begin{equation*}
R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right) \xrightarrow{\sim} U_{k, p}^{\vee} \otimes \mathbb{Z}_{e_{n}} . \tag{6.2.9}
\end{equation*}
$$

Proof. If $\beta$ is a root of $R_{u}\left(R_{n}\right)$, then $U_{\beta} \subseteq R_{u}\left(R_{n}\right)$ maps injectively into $R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap\right.$ $w_{p} Q_{k}^{n} w_{p}^{-1}$ ) if and only if $w_{p}^{-1} \beta$ is not a root of $Q_{k}^{n}$. In particular, this implies that $\beta$ is a negative root and $w_{p}^{-1} \beta$ is a positive root, and hence that

$$
\beta \in \Sigma=\left\{-\beta_{n-1},-\beta_{n-1}-\beta_{n-2}, \ldots,-\beta_{n-1}-\beta_{n-2}-\cdots-\beta_{p}\right\},
$$

and

$$
w_{p}^{-1} \beta \in\left\{\beta_{n-1}+\beta_{n-2}+\cdots+\beta_{p}, \beta_{n-2}+\cdots+\beta_{p}, \ldots, \beta_{p}\right\} .
$$

Note that if $\beta \in \Sigma$, then $U_{\beta} \subseteq R_{u}(P)$, and $w_{p}^{-1} \beta$ is not a root of $Q_{k}^{n}$, so $\Sigma$ is precisely the set of roots appearing in $R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right)$.

It is clear from the above that $R_{u}\left(R_{n}\right) /\left(R_{u}\left(R_{n}\right) \cap w_{p} Q_{k}^{n} w_{p}^{-1}\right)$ is isomorphic to an $L_{n} \cap$ $w_{p} Q_{k}^{n} w_{p}^{-1}$-representation. The isomorphism (6.2.9) follows by inspection of the weights of this representation.

Proof of Proposition 6.2.7. If $k<n-1$, then there is a pullback


But since

$$
Y_{Q_{k+1}^{n}}^{-e_{n}^{*}} \times_{Y_{Q_{k}^{n}}^{-e_{n}^{*}}} Y_{L_{n} \cap Q_{k}^{n}}^{-e_{n}^{*}}=Y_{L \cap Q_{k+1}^{n}}^{-e^{*}}
$$

this implies that $C_{k+1, n}^{G L_{n}} \rightarrow C_{k, n}^{G L_{n}}$ is an isomorphism by Lemma 6.2.12. If $k=n-1$, then since $L_{n} \cap Q_{n}^{n}=L \cap Q_{n-1}^{n}$, there is instead a pullback

where the vertical arrow on the right is induced by the isomorphism of tori

$$
\begin{aligned}
T_{Q_{n}^{n}} \times \mathbb{G}_{m} & \longrightarrow T_{Q_{n}^{n}} \times_{T_{Q_{n-1}^{n}}} T_{Q_{n}^{n}} \\
\left(t_{1}, t_{2}\right) & \longmapsto\left(t_{1} \beta_{n-1}^{\vee}\left(t_{2}\right), t_{1}\right) .
\end{aligned}
$$

So $C_{n, n}^{G L_{n}} \rightarrow C_{n-1, n}^{G L_{n}}$ is also an isomorphism.
If $k \leq n$ and $1 \leq p<k$, then Proposition 3.7.5 and Lemma 6.2.14 show that $C_{k, p}^{G L_{n}}$ is the relative space of sections of

$$
\eta_{k, p} \times{ }^{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}} \frac{R_{u}(P)}{R_{u}(P) \cap w_{p} Q_{k}^{n} w_{p}^{-1}} \longrightarrow M_{p}^{G L_{n}} \times_{S} E
$$

where $\eta_{k, p}$ is the pullback of the universal $L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}$-bundle under the map

$$
M_{p}^{G L_{n}} \longrightarrow \operatorname{Bun}_{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}}^{-e_{n-1}^{*}}
$$

coming from Lemma 6.2 .14 . If $k \leq n-1$, then by Lemma 6.2.15, we can therefore identify the morphism $C_{k+1, p}^{G L_{n}} \rightarrow C_{k, p}^{G L_{n}}$ with the morphism

$$
\pi_{p *}\left(\eta_{k+1, p} \times^{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}} U_{k+1, p}^{\vee} \otimes \mathbb{Z}_{e_{n}}\right) \longrightarrow \pi_{p *}\left(\eta_{k, p} \times^{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}} U_{k, p}^{\vee} \otimes \mathbb{Z}_{e_{n}}\right),
$$

where $\pi_{p}: M_{p}^{G L_{n}} \times_{S} E \rightarrow M_{p}^{G L_{n}}$ is the natural projection. But this is the pushforward of a surjective morphism between families of stable vector bundles of degree 1 (since both vector bundles are naturally degree 1 quotients of stable vector bundles of degree 1 ), and is therefore an isomorphism as claimed. We can also identify the morphism $C_{k+1, k}^{G L_{n}} \rightarrow C_{k, n}$ with the morphism

$$
\pi_{p *}\left(\eta_{k+1, p} \times{ }^{L_{n} \cap w_{p} Q_{k}^{n} w_{p}^{-1}} U_{k+1, p}^{\vee} \otimes \mathbb{Z}_{e_{n}}\right) \longrightarrow M_{p}^{G L_{n}} \longleftrightarrow C_{1, n}^{G L_{n}} \cong C_{k, n}^{G L_{n}},
$$

which factors as a line bundle over $M_{p}^{G L_{n}}$ as claimed.

### 6.3 Computing the divisor $D_{\alpha_{j}^{\vee}}(Z)$

In this section, we prove parts (3) and (4) of Theorem 6.1.9.
Throughout this section, we will suppose that we are in the setup of Theorem 6.1.9, i.e., that $(G, P, \mu)$ is a subregular Harder-Narasimhan class not of type $A_{1}$, and that we are given a $\Theta$-trivial slice $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ satisfying the conditions of Theorem 6.1.5, for $L \subseteq P$ the standard Levi subgroup. Writing $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right)$, note that Proposition 5.4.2 implies that $-\langle 2 \rho, \mu\rangle \geq l+2$, and Proposition 5.2.8 implies that $\operatorname{dim}_{S} Z_{0}-\langle 2 \rho, \mu\rangle=\operatorname{dim}_{S} Z=l+3$, so in particular $\operatorname{dim}_{S} Z_{0} \leq 1$.

Lemma 6.3.1. Assume that $z \in Z \backslash Z_{0}$ and that the corresponding $G$-bundle $\xi_{G, z}$ is unstable. Then $\xi_{G, z}$ is regular unstable.

Proof. Since $z \in Z \backslash Z_{0}$ is not fixed under the $Z(L)_{\text {rig }}$-action, Proposition 5.1.4 implies that

$$
\operatorname{dim} \operatorname{Aut}\left(\xi_{G, z}\right) \leq l+3
$$

So $\xi_{G, z}$ is regular as claimed by Theorem 6.1.2.
Lemma 6.3.2. For $\lambda \in \mathbb{X}_{*}(T)_{+}$, we have $D_{\lambda}(Z) \neq \emptyset$ if and only if $\lambda \in\left\{\alpha_{i}^{\vee}, \alpha_{j}^{\vee}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right\}$.

Proof. For simplicity, we can assume without loss of generality that $S=\operatorname{Spec} k$ for $k$ an algebraically closed field and hence that $Z_{0}$ is connected. We first show that $D_{\alpha_{i}^{\vee}}(Z) \neq \emptyset$ and $D_{\alpha_{j}^{\vee}}(Z) \neq \emptyset$.

If $(G, P, \mu)$ has type $A$, then $\mu$ is the image of $-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}$ under the homomorphism $\mathbb{X}_{*}(T) \rightarrow \mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$ and $\left\langle\alpha, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right\rangle \leq 0$ for all $\alpha \in \Phi_{+}$a root of $P$. So by Proposition 3.6.4, the morphism

$$
\mathrm{KM}_{B, G}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \longrightarrow \mathrm{KM}_{P, G / S}^{\mu}
$$

is surjective. In particular, for every $z \in Z_{0}$, there exists a section of $\xi_{L, z} \times{ }^{L} P / B \subseteq$ $\xi_{L, z} \times{ }^{L} G / B$ with degree $-\lambda_{0} \leq-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}$. So we must have $D_{\lambda_{0}}(Z) \neq \emptyset$, and hence $D_{\alpha_{i}^{\vee}}(Z) \neq \emptyset$ and $D_{\alpha_{j}^{\vee}}(Z) \neq \emptyset$ by Proposition 5.1.5.

On the other hand, if $(G, P, \mu)$ does not have type $A$, then $\mu$ is the image of $-\alpha_{i}^{\vee}$ in $\mathbb{X}_{*}\left(Z(L)^{\circ}\right)_{\mathbb{Q}}$, and $\left\langle\alpha, \alpha_{i}^{\vee}\right\rangle \leq 0$ for $\alpha \in \Phi_{+}$a root of $P$. So

$$
\mathrm{KM}_{B, G}^{-\alpha_{i}^{\vee}} \longrightarrow \mathrm{KM}_{P, G}^{\mu}
$$

is surjective by Proposition 3.6.4, so we deduce that $D_{\alpha_{i}^{\vee}}(Z) \neq \emptyset$. For $D_{\alpha_{j}^{\vee}}(Z)$, note that since $\alpha_{j} \in \Delta$ is the unique special root, Proposition 5.4.2 implies that the HarderNarasimhan locus $\operatorname{Bun}_{Q}^{s s,-\alpha_{j}^{\vee}} \subseteq \operatorname{Bun}_{G}$ is dense in the locus of unstable $G$-bundles, where $Q$ is the standard parabolic with $t(Q)=\left\{\alpha_{j}^{\vee}\right\}$. So $\operatorname{Bun}_{Q, \text { rig }}^{s s,-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{G, \text { rig }}} Z \neq \emptyset$, and hence $D_{\alpha_{j}^{\vee}}(Z) \neq \emptyset$ by Proposition 5.4.11.

Conversely, suppose that $\lambda \in \mathbb{X}_{*}(T)$ and that $D_{\lambda}(Z) \neq \emptyset$. Then for any $\alpha_{k} \in \Delta$ with corresponding maximal parabolic $P_{k}$, there exists a point in $Z$ and a section of the corresponding $G / P_{k}$-bundle with degree $\nu_{k}=-\left\langle\varpi_{k}, \lambda\right\rangle /\left\langle\varpi_{k}, \varpi_{k}^{\vee}\right\rangle \varpi_{k}^{\vee}$ (the image of $\lambda$ in $\left.\mathbb{X}_{*}\left(T_{P_{k}}\right)\right)$. So by Lemma 5.2.14 and [FM2, Lemma 3.3.2], we must have

$$
(l+1)\left\langle\varpi_{k}, \lambda\right\rangle \leq \frac{\left\langle 2 \rho, \varpi_{k}^{\vee}\right\rangle}{\left\langle\varpi_{k}, \varpi_{k}^{\vee}\right\rangle}\left\langle\varpi_{k}, \lambda\right\rangle=-\left\langle 2 \rho, \nu_{k}\right\rangle \leq-\langle 2 \rho, \mu\rangle \leq l+3 .
$$

So

$$
\left\langle\varpi_{k}, \lambda\right\rangle \leq \frac{l+3}{l+1}<2
$$

since $l>1$. So $\left\langle\varpi_{k}, \lambda\right\rangle=0$ or 1 for all $k$.
Now assume for a contradiction that there exists $\lambda \in \mathbb{X}_{*}(T)_{+} \backslash\left\{\alpha_{i}^{\vee}, \alpha_{j}^{\vee}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right\}$ such that $D_{\lambda}(Z) \neq \emptyset$. Since the divisor $D(Z)=\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$ is connected by Corollary 5.5.9, we can choose $\lambda$ so that $D_{\lambda}(Z)$ has nonempty intersection with one of $D_{\alpha_{i}^{\vee}}(Z), D_{\alpha_{j}^{\vee}}(Z)$ or $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$. Choose a point in such an intersection over $z \in Z$, and let $-\lambda^{\prime} \in \mathbb{X}_{*}(T)_{-}$ denote the degree of the corresponding stable map restricted to the irreducible component of genus 1. Then we have $D_{\lambda^{\prime}}(Z) \neq \emptyset, \lambda^{\prime} \geq \lambda$ and $\lambda^{\prime} \geq \alpha_{r}^{\vee}$ for some $\alpha_{r} \in\left\{\alpha_{i}, \alpha_{j}\right\}$. By the bound proved above, we must have $\left\langle\varpi_{k}, \lambda\right\rangle=1$ for some $\alpha_{k} \in \Delta \backslash\left\{\alpha_{i}, \alpha_{j}\right\}$, and hence $\lambda^{\prime} \geq \alpha_{r}^{\vee}+\alpha_{k}^{\vee}$. So by Proposition 5.1.5, we have $D_{\alpha_{r}^{\vee}+\alpha_{k}^{\vee}}(Z) \neq \emptyset$ and $D_{\alpha_{k}^{\vee}}(Z) \neq \emptyset$.

Assume first that $G$ is not of type $A$. Since $D_{\alpha_{k}^{\vee}}(Z) \neq \emptyset$, there exists $z \in Z$ and a section of $\xi_{G, z} / B$ with degree $-\alpha_{k}^{\vee}$, and hence a section of $\xi_{G, z} / P_{k}$ with slope $-\varpi_{k}^{\vee} /\left\langle\varpi_{k}, \varpi_{k}^{\vee}\right\rangle$. So by Lemma 5.2.14, there exists $z^{\prime} \in Z$ such that $\xi_{G, z^{\prime}}$ has Harder-Narasimhan reduction to $P_{k}$ with slope $-\varpi_{k}^{\vee} /\left\langle\varpi_{k}, \varpi_{k}^{\vee}\right\rangle$. Since $P_{k} \neq P$, we have $z^{\prime} \in Z \backslash Z_{0}$. So $\xi_{G, z^{\prime}}$ is regular by Lemma 6.3.1, which contradicts Proposition 5.4.2 since $\alpha_{k}$ is not special.

Assume on the other hand that $G$ is of type $A$. We have $k \notin\{i, i+1\}$ and $r \in\{i, i+1\}$ such that $D_{\alpha_{r}^{\vee}+\alpha_{k}^{\vee}}(Z) \neq \emptyset$. So there exists $z \in Z$ and a section of $\xi_{G, z} / P_{r, k}$ of slope $\nu \in$ $\mathbb{X}_{*}\left(Z\left(L_{r, k}\right)^{\circ}\right)_{\mathbb{Q}}$ satisfying $\left\langle\varpi_{r}, \nu\right\rangle=\left\langle\varpi_{k}, \nu\right\rangle=-1$, where $P_{r, k} \subseteq G$ is the standard parabolic of type $\left\{\alpha_{r}, \alpha_{k}\right\}$ and $L_{r, k}$ its standard Levi factor. But $\nu$ is a Harder-Narasimhan vector for $P_{r, k}$, so by Lemma 5.2.14, there exists $z^{\prime} \in Z$ such that $\xi_{G, z^{\prime}}$ has Harder-Narasimhan
reduction to $P_{r, k}$ with slope $\nu$. Since $P_{r, k} \neq P$, we have $z \in Z \backslash Z_{0}$, so $\xi_{G, z^{\prime}}$ is regular unstable by Lemma 6.3.1, which again contradicts Proposition 5.4.2.

So $D_{\lambda}(Z)=\emptyset$ for $\lambda \notin\left\{\alpha_{i}^{\vee}, \alpha_{j}^{\vee}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right\}$, and $D_{\alpha_{i}^{\vee}}(Z), D_{\alpha_{j}^{\vee}}(Z) \neq \emptyset$. This implies that $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z) \neq \emptyset$, for if this were not the case, we would have $D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z)=\emptyset$ by Proposition 3.4.16 and hence $\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$ would be disconnected, contradicting Corollary 5.5.9.

Given a torus $T^{\prime}$ and a cocharacter $\lambda \in \mathbb{X}_{*}\left(T^{\prime}\right)$, there is a natural morphism

$$
\begin{align*}
Y_{T^{\prime}}^{-\lambda} \times_{S} E & \longrightarrow Y_{T^{\prime}}=Y_{T^{\prime}}^{0}  \tag{6.3.1}\\
\left(\xi_{T^{\prime}}, x\right) & \longmapsto \xi_{T^{\prime}} \otimes \lambda(\mathcal{O}(x)) .
\end{align*}
$$

We will repeatedly make use of this in what follows.
Lemma 6.3.3. There are isomorphisms

$$
\begin{equation*}
D_{\alpha_{i}^{\vee}}(Z) \cong M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right) \tag{6.3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{\alpha_{j}^{\vee}}(Z) \cong M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{j}^{\vee}, 1\right)\right), \tag{6.3.3}
\end{equation*}
$$

where $\xi_{G / Z(G)} \rightarrow Z \times_{S} E$ is the $G / Z(G)$-bundle classified by the morphism $Z \rightarrow \operatorname{Bun}_{G, \text { rig }} \rightarrow$ $\operatorname{Bun}_{G / Z(G)}$, and we use the degree datum of Lemma 3.4.4 in the notation for spaces of stable maps. Moreover, the isomorphisms commute with the maps to $Y$ given by

$$
M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right) \longrightarrow Y^{-\alpha_{i}^{\vee}} \times_{S} E \longrightarrow Y
$$

and

$$
M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{j}^{\vee}, 1\right)\right) \longrightarrow Y^{-\alpha_{j}^{\vee}} \times_{S} E \longrightarrow Y
$$

where the first morphism in each composition is given on the first factor by forgetting the marked point and applying $\mathrm{Bl}_{B}$ and on the second factor by evaluation at the marked point followed by projection to $E$, and the second morphisms are given by (6.3.1).

Proof. We prove the claims for $D_{\alpha_{i}^{\vee}}(Z)$; the proofs for $D_{\alpha_{j}^{\vee}}(Z)$ are identical.
By Propositions 3.2.18 and 3.4.13, $D_{\alpha_{i}^{\vee}}(Z)$ is the image of the gluing morphism

$$
\begin{equation*}
M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau_{\alpha_{i}^{\vee}}^{0}\right) \longrightarrow \tilde{Z}=M_{1,0, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,(0,1)\right) \tag{6.3.4}
\end{equation*}
$$

By Lemma 6.3.2, the stable $\mathbb{X}^{*}(T) \oplus \mathbb{Z}$-graphs $\tau$ admitting contractions $\tau \rightarrow \tau_{\alpha_{i}^{\vee}}^{0}$ such that $M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau\right) \neq \emptyset$ are as follows.


In particular, any such $\tau$ has a unique contraction onto $\tau_{\alpha_{i}^{\vee}}^{0}$, so Corollary 3.2.21 implies that (6.3.4) is a closed immersion, and hence that

$$
D_{\alpha_{i}^{\vee}}(Z) \cong M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau_{\alpha_{i}^{\vee}}^{0}\right)
$$

By definition of $\tau_{\alpha_{i}^{v}}^{0}$-marked stable maps, there is therefore a Cartesian diagram


But the vertical arrow on the right can be identified with the morphism

$$
\xi_{G / Z(G)} \times{ }^{G / Z(G)} M_{0,1}\left(G / B, \alpha_{i}^{\vee}\right) \longrightarrow \xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B
$$

and is therefore an isomorphism by Proposition 3.4.10. So the vertical arrow on the left gives the desired isomorphism (6.3.2). Commutativity with the maps to $Y$ follows from the construction and Proposition 3.5.5.

We can now construct the sequence of blow downs of $D_{\alpha_{j}^{\vee}}(Z)$ promised in Theorem 6.1.9 (3). For $1 \leq k \leq n_{0}$, let $P_{k} \subseteq G$ be the standard parabolic with type $t\left(P_{k}\right)=$ $\Delta \backslash\left\{\alpha_{c_{0}, k}, \ldots, \alpha_{c_{0}, n_{0}}\right\}=\Delta \backslash\left\{\alpha_{c_{0}, k}, \ldots, \alpha_{c_{0}, n_{0}-1}, \alpha_{i}\right\}$, and let $P_{n_{0}+1}=B$. Then for $1 \leq k \leq$ $n_{0}+1$, we define

$$
\begin{aligned}
D_{k} & =Y_{B}^{-\alpha_{j}^{\vee}} \times{ }_{Y_{P_{k}}^{-\alpha_{j}^{\vee}}} \mathrm{KM}_{P_{k}, G, r i g}^{-\alpha_{j}^{\vee}} \times \text { Bun }_{G, r i g} Z \times_{S} E \\
& \cong Y \times_{Y_{P_{k}}}\left(\mathrm{KM}_{P_{k}, G, r i g}^{-\alpha_{j}^{\vee}} \times{ }_{\operatorname{Bun}_{G, r i g}} Z \times_{S} E\right),
\end{aligned}
$$

where the morphism to $Y_{P_{k}}$ in the last fibre product is given by the composition

$$
\mathrm{KM}_{P_{k}, G / S, r i g}^{-\alpha_{j}^{\vee}} \times_{\text {Bun }_{G, r i g}} Z \times_{S} E \xrightarrow{\mathrm{Bl}_{P_{k}}} Y_{P_{k}}^{-\alpha_{j}^{\curlyvee}} \times_{S} E \xrightarrow{(6.3 .1)} Y_{P_{k}} .
$$

There is a morphism

$$
\begin{equation*}
D_{\alpha_{j}^{\vee}}(Z) \longrightarrow D_{n_{0}+1}=M_{1,0, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / P_{k},\left(-\alpha_{j}^{\vee}, 1\right)\right) \times_{S} E \tag{6.3.5}
\end{equation*}
$$

over $Y$, given in terms of the isomorphism of Lemma 6.3 .3 by forgetting the marked point and stabilising on the first factor, and by evaluating at the marked point and composing with the projection to $E$ on the second factor. For $1 \leq k \leq n_{0}$, the projection $G / P_{k+1} \rightarrow G / P_{k}$ also induces a morphism

$$
\begin{equation*}
D_{k+1} \longrightarrow D_{k} \tag{6.3.6}
\end{equation*}
$$

over $Y$. We show later (Propositions 6.3.11 and 6.3.17) that the morphisms (6.3.5) and (6.3.6) are blowups along explicit loci. The first step towards formulating and proving these propositions is Proposition 6.3.4, which shows that $D_{1} \times_{Z} Z_{0}$ is controlled entirely by the subgroup $P_{1}$.

In what follows, for $1 \leq k \leq n_{0}+1, w \in W_{P, P_{k}}^{0}$ and $\lambda \in \mathbb{X}_{*}\left(T_{P_{k}}\right)$, we write

$$
C_{P_{k}}^{w, \lambda}\left(Z_{0}\right)=C_{P, P_{k} / S}^{w, \lambda}(E)_{r i g} \times_{\operatorname{Bun}_{P, r i g}} Z_{0} \subseteq \operatorname{Bun}_{P_{k}, r i g}^{\lambda} \times_{\operatorname{Bun}_{G, r i g}} Z_{0}
$$

and $C^{w, \lambda}\left(Z_{0}\right)=C_{P_{n_{0}+1}}^{w, \lambda}\left(Z_{0}\right)$. (Recall that $\left.P_{n_{0}+1}=B.\right)$
Proposition 6.3.4. The natural inclusion $L /\left(L \cap P_{1}\right)=P /\left(P \cap P_{1}\right) \rightarrow G / P_{1}$ induces an isomorphism

$$
\begin{equation*}
\operatorname{Bun}_{L \cap P_{1}, r i g}^{-\alpha_{j}^{\vee}-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}} Z_{0}=\operatorname{Bun}_{P \cap P_{1}, r i g}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{P, r i g}} Z_{0} \xrightarrow{\sim} \mathrm{KM}_{P_{1}, G, r i g}^{-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z_{0}, \tag{6.3.7}
\end{equation*}
$$

and hence isomorphisms

$$
\begin{equation*}
\mathrm{KM}_{L_{1} \cap P_{k}, L_{1} / S, r i g}^{-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{L_{1}, r i g}} \operatorname{Bun}_{P \cap P_{1}, r i g}^{-\alpha_{j}^{\vee}-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{P, r i g}} Z_{0} \xrightarrow{\sim} \mathrm{KM}_{P_{k}, G, r i g}^{-\alpha_{j}^{\vee}} \times \text { Bun }_{G, r i g} Z_{0}, \tag{6.3.8}
\end{equation*}
$$

for $1 \leq k \leq n_{0}+1$, where $L_{1} \subseteq P_{1}$ is the standard Levi subgroup.

Proof. The isomorphism (6.3.8) is obtained from (6.3.7) by noting that for $\xi_{P \cap P_{1}} \rightarrow E_{s}$ a $P \cap P_{1}$-bundle over a geometric fibre of $E \rightarrow S$, the preimage of the canonical section of $\xi_{P \cap P_{1}} \times{ }^{P \cap P_{1}} G / P_{1} \rightarrow E_{s}$ under $\xi_{P \cap P_{1}} \times{ }^{P \cap P_{1}} G / P_{k} \rightarrow \xi_{P \cap P_{1}} \times{ }^{P \cap P_{1}} G / P_{1}$ is

$$
\xi_{P \cap P_{1}} \times{ }^{P \cap P_{1}} P_{1} / P_{k} \cong \xi_{P \cap P_{1}} \times{ }^{P \cap P_{1}} L_{1} /\left(L_{1} \cap P_{k}\right) .
$$

To prove that (6.3.7) is an isomorphism, first note that it can be identified with the locally closed immersion

$$
\begin{equation*}
C_{P_{1}}^{1,-\alpha_{j}^{\vee}}\left(Z_{0}\right) \longleftrightarrow \operatorname{Bun}_{P_{1}, r i g}^{-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z_{0} \longleftrightarrow \mathrm{KM}_{P_{1}, G, r i g}^{-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z_{0} . \tag{6.3.9}
\end{equation*}
$$

Since both sides are reduced, it is therefore enough to show that (6.3.9) is surjective.
To see this, note that Lemma 6.3.5 below and Proposition 3.7.6 imply that the morphism

$$
\coprod_{\substack{w \in W_{P}^{0} \cap W_{L_{1}} \\ \lambda=-w^{-1}\left(\alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)}} C^{w, \lambda}\left(Z_{0}\right) \longrightarrow \operatorname{Bun}_{B, \text { rig }}^{\lambda} \times_{\text {Bun }_{G, r i g}} Z_{0}
$$

is surjective for all $\lambda \leq-\alpha_{j}^{\vee}$. Since the morphism $\mathrm{KM}_{B, G}^{-\alpha_{j}^{\vee}} \rightarrow \mathrm{KM}_{P_{1}, G}^{-\alpha_{j}^{\vee}}$ is also surjective by Proposition 3.6.4, and maps sections coming from $C^{w, \lambda}\left(Z_{0}\right)$ to $C_{P_{1}}^{1,-\alpha_{j}^{\vee}}\left(Z_{0}\right)$, surjectivity of (6.3.9) now follows.

Lemma 6.3.5. Assume that $w \in W_{P, B}^{0}, \lambda \leq-\alpha_{j}^{\vee}$ and $C^{w, \lambda}\left(Z_{0}\right) \neq \emptyset$. Then $w \in W_{L_{1}}$ and $\lambda=-w^{-1}\left(\alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right) \in\left\{-\alpha_{j}^{\vee},-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}\right\}$, where $L_{1} \subseteq P_{1}$ is the standard Levi subgroup.

Proof. It is immediate from Lemma 6.3.2 that $\lambda \in\left\{-\alpha_{j}^{\vee},-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}\right\}$. By Proposition 3.7.4, if $C^{w, \lambda}\left(Z_{0}\right) \neq \emptyset$, then there exists a geometric point $z$ : Spec $k \rightarrow Z_{0}$ over $s: ~ S p e c k \rightarrow S$ and a section $\sigma_{L}: E_{s} \rightarrow \xi_{L, z} /(L \cap B)$ of degree $w \lambda \in \mathbb{X}_{*}(T)$. Since $\xi_{L, z}$ has slope $\mu$, we must have

$$
\left\langle\varpi_{i}, w \lambda\right\rangle=\left\langle\varpi_{i}, \mu\right\rangle=-1 .
$$

Since $\lambda$ and hence $w \lambda$ is a coroot, we therefore have $w \lambda \in \Phi_{-}^{\vee} \subseteq \mathbb{X}_{*}(T)_{-}$. Since composing $\sigma_{L}$ with the inclusion $\xi_{L, z} /(L \cap B) \rightarrow \xi_{G, z} / B$ defines a section of degree $w \lambda$, we deduce that $D_{-w \lambda}(Z) \neq \emptyset$, and hence that $w \lambda \in\left\{-\alpha_{i}^{\vee},-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}\right\}$.

If $w \lambda=-\alpha_{i}^{\vee}$, then $w^{-1} \alpha_{i}^{\vee} \in \Phi_{+}^{\vee}$, so $w=1$ since $w \in W_{P, B}^{0}$. So $\lambda=-\alpha_{i}^{\vee}$, contradicting $\lambda \leq-\alpha_{j}^{\vee}$. So we must have $w \lambda=-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}$, and in particular $w^{-1}\left(\alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right) \in \Phi_{+}^{\vee}$.

If $(G, P, \mu)$ is not of type $A$, then $w^{-1}\left(\alpha_{k}^{\vee}\right) \in \Phi_{+}^{\vee}$ for $\alpha_{k} \neq \alpha_{i}$ (since $w \in W_{P, B}^{0}$ and $t(P)=$ $\left.\left\{\alpha_{i}\right\}\right)$ so Lemma 6.3.6 implies that $w \in W_{L_{1}}$. If $(G, P, \mu)$ is of type $A$, then $w^{-1}\left(\alpha_{k}^{\vee}\right) \in \Phi_{+}^{\vee}$ for $\alpha_{k} \neq \alpha_{i}, \alpha_{j}$. If $w^{-1}\left(\alpha_{j}^{\vee}\right) \in \Phi_{+}^{\vee}$ then $w \in W_{L_{1}}$ by Lemma 6.3.6 again. Otherwise, we must have $w^{-1}\left(\alpha_{i}^{\vee}\right) \in \Phi_{+}^{\vee}$ and hence

$$
w \in\left\{s_{i+1} s_{i+2} \cdots s_{k} \mid i<k \leq l\right\}
$$

by Lemma 6.3.6. But this implies that $\lambda=w^{-1}\left(-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}\right)=w^{-1}\left(-\alpha_{i}^{\vee}-\alpha_{i+1}^{\vee}\right)=-\alpha_{i}^{\vee}$, contradicting $\lambda \leq-\alpha_{j}^{\vee}$, so we are done.

Lemma 6.3.6. Let $\left(M, \Psi, M^{\vee}, \Psi^{\vee}\right)$ be a root datum with Weyl group $W(\Psi)$, and let $\Gamma \subseteq \Psi$ be a complete set of positive simple roots. Let $\beta_{j} \in \Gamma$ be a simple root, and let $c \in \pi_{0}\left(\Gamma \backslash\left\{\beta_{j}\right\}\right)$ be a connected component of the Dynkin diagram of $\Gamma \backslash\left\{\beta_{j}\right\}$ of type $A_{n}$ such that $\beta_{j}$ is adjacent to one end of $c$. Let $\beta_{c, 1}, \ldots, \beta_{c, n} \in \Gamma$ denote the nodes of $c$, labelled so that $\beta_{c, k}$ is adjacent to $\beta_{c, k+1}$ for all $k$ and $\beta_{c, n}$ is adjacent to $\beta_{j}$, and let

$$
\Sigma=\left\{w \in W(\Psi) \mid w^{-1} \beta_{k}^{\vee} \in \Psi_{+}^{\vee} \text { for all } \beta_{k} \in \Gamma \backslash\left\{\beta_{c, n}\right\} \text { and } w^{-1}\left(\beta_{c, n}^{\vee}+\beta_{j}^{\vee}\right) \in \Psi_{+}^{\vee}\right\}
$$

Then

$$
\Sigma=\{1\} \cup\left\{s_{c, n} s_{c, n-1} \cdots s_{c, k} \mid 1 \leq k \leq n\right\}
$$

where $s_{c, k} \in W(\Psi)$ is the reflection in the root $\beta_{c, k}$
Proof. First note that an easy inspection shows that

$$
\{1\} \cup\left\{s_{c, n} s_{c, n-1} \cdots s_{c, k} \mid 1 \leq k \leq n\right\} \subseteq \Sigma
$$

so it suffices to prove the reverse inclusion.
We prove the claim by induction on $n \geq 1$. Suppose that $w \in \Sigma$. Then either $w=1$ or $w^{-1} \beta_{c, n} \in \Psi_{-}$. In the second case, we see that $\left(s_{c, n} w\right)^{-1} \beta_{k}^{\vee} \in \Psi_{+}^{\vee}$ for $\beta_{k} \in \Gamma \backslash\left\{\beta_{c, n-1}\right\}$ and $\left(s_{c, n} w\right)^{-1}\left(\beta_{c, n-1}^{\vee}+\beta_{c, n}^{\vee}\right) \in \Psi_{+}^{\vee}$ if $n>1$. So either $n=1$ and $w \in\left\{1, s_{c, n}\right\}$, or $n>1$ and by induction we have

$$
s_{c, n} w \in\left\{s_{c, n-1} \cdots s_{c, k} \mid 1 \leq k \leq n-1\right\},
$$

and hence

$$
w \in\{1\} \cup\left\{s_{c, n} s_{c, n-1} \cdots s_{c, k} \mid 1 \leq k \leq n\right\} .
$$

This proves the lemma.

Proposition 6.3.7. There exists a surjective homomorphism

$$
\rho_{P_{1}}: P_{1} \longrightarrow G L_{n_{0}+1}
$$

such that $\rho_{P_{1}}^{-1}\left(R_{n_{0}+1}\right)=P \cap P_{1}$ and $\rho_{P_{1}}^{-1}\left(Q_{k}^{n_{0}+1}\right)=P_{k}$ for $1 \leq k \leq n_{0}+1$, and such that the induced map $T=T_{P_{n_{0}+1}} \rightarrow Q_{n_{0}+1}^{n_{0}+1}$ is given on cocharacters by

$$
\begin{aligned}
\mathbb{X}_{*}(T) & \longrightarrow \mathbb{X}_{*}\left(T_{Q_{n_{0}+1}^{n_{0+1}}}\right) \\
\alpha_{c_{0}, k}^{\vee} & \longmapsto e_{k}^{*}-e_{k+1}^{*} \\
\alpha_{j}^{\vee} & \longmapsto e_{n_{0}+1}^{*} \\
\alpha_{p}^{\vee} & \longmapsto 0, \quad \text { if } \alpha_{p} \notin\left\{\alpha_{c_{0}, 1}, \ldots, \alpha_{c_{0}, n_{0}}, \alpha_{j}\right\} .
\end{aligned}
$$

Proof. Since the Dynkin diagram $\Delta \backslash t\left(P_{1}\right)$ has exactly one connected component of type $A_{n_{0}}$, Proposition 5.3.1 gives an embedding

$$
\begin{equation*}
L_{1} \hookrightarrow G L_{n_{0}+1} \times \mathbb{G}_{m}^{n_{1}} . \tag{6.3.10}
\end{equation*}
$$

Let $\rho_{L_{1}}$ be the composition of (6.3.10) with the projection to the first factor, and let $\rho_{P_{1}}$ be the composition of $\rho_{L_{1}}$ with the quotient $P_{1} \rightarrow L_{1}$. The remaining claims can now be checked routinely using the explicit isomorphism of Proposition 5.3.1.

Returning to the study of the divisors $D_{k}$, Propositions 6.3 .4 and 6.3 .7 give a morphism

$$
\begin{aligned}
& D_{k} \times{ }_{Z} Z_{0} \longrightarrow Y^{-\alpha_{j}^{\vee}} \times{ }_{Y_{P_{k}}^{-\alpha_{j}^{\vee}}} \mathrm{KM}_{L_{1} \cap P_{k}, L_{1}, r i g}^{-\alpha_{j}^{\vee}} \times \operatorname{Bun}_{L_{1}, r i g} \operatorname{Bun}_{P \cap P_{1}, r i g}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \\
& \longrightarrow Y_{Q_{n_{0}+1}^{n_{n}+1}}^{-e_{n_{0}+1}^{*}} \times{ }_{\substack{Y_{Q_{k}^{n_{0}+1}}^{-e_{n_{0}+1}^{*}}}} \mathrm{KM}_{Q_{k}^{n_{0}+1}, G L_{n_{0}+1}, r i g} \times{\operatorname{Bun}{ }_{G L_{n_{0}+1}}}^{\operatorname{Bun}_{R_{n_{0}+1}}^{s s,-e_{1}^{*}}=X_{k, r i g}^{n_{0}+1}}
\end{aligned}
$$

where, in the notation of $\S 6.2, X_{k, \text { rig }}^{n_{0}+1}$ is the rigidification of $X_{k}^{n_{0}+1}$ with respect to the image of $Z(G)$ in $Z\left(G L_{n_{0}+1}\right)$ under $\rho_{P_{1}}$. Moreover, Lemma 6.2 .13 shows that there is a sequence
of pullback squares

for $1 \leq k \leq n_{0}$. For $1 \leq k \leq n_{0}+1$ and $1 \leq p<k$ or $p=n_{0}+1$, we define

$$
C_{k, p}=\left(D_{k} \times{ }_{Z} Z_{0}\right) \times_{X_{k, r i g}^{n_{0}+1}} C_{k, p, r i g}^{G L_{n_{0}+1}} .
$$

Proposition 6.3.8. For $1 \leq k \leq n_{0}+1$, there is a decomposition

$$
D_{k}=\left(D_{k} \times_{Z}\left(Z \backslash Z_{0}\right)\right) \cup \bigcup_{1 \leq p<k} C_{k, p} \cup C_{k, n_{0}+1}
$$

into disjoint locally closed substacks.
Proof. This follows immediately from the definitions and Proposition 6.2.1.
By construction, the morphism $C_{1, n_{0}+1} \rightarrow C_{1, n_{0}+1}^{G L_{n_{0}+1}}$ factors through a morphism

$$
C_{1, n_{0}+1} \longrightarrow Y^{-\alpha_{j}^{\vee}} \times_{\substack{Y_{Q_{Q_{0}}^{n_{n_{0}+1}}}^{-e_{0}^{*}+1}}}\left(C_{1, n_{0}+1}^{G L_{n_{0}+1}} \times_{S} E\right)=Y \times_{Y_{Q_{n_{0}+1}^{n_{0}+1}}}\left(C_{1, n_{0}+1}^{G L_{n_{0}+1}} \times_{S} E\right),
$$

where the morphism $C_{1, n_{0}+1}^{G L_{n_{0}+1}} \times_{S} E \rightarrow Y_{Q_{n_{0}+1}^{n_{0}+1}}$ is given by the natural morphism to $Y_{Q_{n_{0}+1}^{n_{0}+1}}^{-e_{n_{0}}^{*}} \times_{S}$ $E$ composed with (6.3.1). Composing with the morphism (6.2.4) gives a morphism

$$
\begin{align*}
C_{1, n_{0}+1} \longrightarrow Y \times_{S} E \times_{S} E & \longrightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E)  \tag{6.3.11}\\
\left(y, x_{1}, x_{2}\right) & \longmapsto\left(y, x_{2}-x_{1}\right)
\end{align*}
$$

over $Y$.
Remark 6.3.9. From the definitions, $C_{n_{0}+1, n_{0}+1} \subseteq D_{n_{0}+1}$ can be identified with the locus of stable maps with one rational component of degree $\alpha_{i}^{\vee}$. By Lemma 6.2.6, the composition of (6.3.11) with $C_{n_{0}+1, n_{0}+1} \rightarrow C_{1, n_{0}+1}$ sends a point in $C_{n_{0}+1, n_{0}+1}$ over $x \in E$ to $x-x^{\prime} \in$ $\operatorname{Pic}_{S}^{0}(E)$, where $x^{\prime} \in E$ is the point where $E$ meets the rational component.

For $1 \leq p \leq n_{0}+1$, we let

$$
M_{p} \subseteq C_{1, n_{0}+1}
$$

be the closed substack given by the fibre product

where $\theta_{p}$ is defined as in $\S 6.1$.
Proposition 6.3.10. For all $1 \leq k \leq n_{0}$, the morphism (6.3.6) restricts to isomorphisms

$$
D_{k+1} \times_{Z}\left(Z \backslash Z_{0}\right) \xrightarrow{\sim} D_{k} \times_{Z}\left(Z \backslash Z_{0}\right), \quad C_{k+1, n_{0}+1} \xrightarrow{\sim} C_{k, n_{0}+1} \quad \text { and } \quad C_{k+1, p} \xrightarrow{\sim} C_{k, p}
$$

for $1 \leq p<k$, and a morphism

$$
C_{k+1, k} \longrightarrow M_{k} \subseteq C_{k, n_{0}+1} \cong C_{1, n_{0}+1}
$$

that identifies $C_{k+1, k}$ with the total space of a line bundle over $M_{k}$.

Proof. Chasing through the definitions, we have

$$
M_{k}=C_{1, n_{0}+1} \times \underset{C_{1, n_{0}+1}^{G L_{n_{0}+1}}}{ } M_{k}^{G L_{n_{0}+1}}
$$

So by Proposition 6.2.7, it remains to show that

$$
\begin{equation*}
D_{k} \times_{Z}\left(Z \backslash Z_{0}\right) \longrightarrow D_{k} \times_{Z}\left(Z \backslash Z_{0}\right) \tag{6.3.12}
\end{equation*}
$$

is an isomorphism. By Lemma 6.3.1, every $G$-bundle in the image of $D_{k} \times_{Z}\left(Z \backslash Z_{0}\right)$ is regular unstable, necessarily with Harder-Narasimhan reduction to the parabolic $Q$ of type $t(Q)=\left\{\alpha_{j}\right\}$ by Lemma 5.4.7. So the morphism to Bun $_{G, \text { rig }}$ factors through

$$
D_{k} \times_{Z}\left(Z \backslash Z_{0}\right) \longrightarrow \operatorname{Bun}_{Q, r i g}^{s s,-\alpha_{j}^{\vee}} \longleftrightarrow \operatorname{Bun}_{G, r i g}
$$

The argument of the proof of Proposition 5.4.11, together with the observation that $\mathrm{KM}_{B, G}^{-\alpha_{j}^{\vee}} \rightarrow$ $\mathrm{KM}_{P_{k}, G}^{-\alpha_{j}^{\vee}}$ is surjective for all $k$ by Proposition 3.6.4, shows that we have isomorphisms

$$
D_{k} \times_{Z}\left(Z \backslash Z_{0}\right) \cong Y \times_{Y_{P_{k}}}\left(\operatorname{Bun}_{M \cap P_{k}, r i g}^{-\alpha_{j}^{\vee}} \times \times_{\operatorname{Bun}_{M, r i g}^{-\alpha_{j}^{\vee}}} \operatorname{Bun}_{Q, r i g}^{s s,-\alpha_{j}^{\vee}} \times \times_{\operatorname{Bun}_{G, r i g}}\left(Z \backslash Z_{0}\right) \times{ }_{S} E\right)
$$

for all $k$, where $M$ is the Levi factor of $Q$. So Proposition 5.3.1 and Lemma 5.4.10 show that (6.3.12) is an isomorphism as claimed.

Proposition 6.3.11. The morphism (6.3.5) is the blowup of $D_{n_{0}+1}$ along the closed substack $M_{n_{0}+1} \subseteq C_{n_{0}+1, n_{0}+1} \subseteq D_{n_{0}+1}$.

Proof. First notice that by Proposition 3.1.13, we can identify (6.3.5) with the pullback of morphism

$$
\begin{equation*}
\mathrm{KM}_{B, G, \text { rig }}^{-\alpha_{j}^{\vee}} \times_{\mathfrak{D e g}_{S}(E)} \mathcal{C} \longrightarrow \mathrm{KM}_{B, G, \text { rig }}^{-\alpha_{j}^{\vee}} \times_{S} E \tag{6.3.13}
\end{equation*}
$$

along the map $D_{n_{0}+1} \rightarrow \mathrm{KM}_{B, G, \text { rig }}^{-\alpha_{j}^{\vee}} \times_{S} E$. Since every stable map parametrised by a point in $D_{n_{0}+1}$ has a domain curve with at most 1 node, it therefore follows from Proposition 3.3.8 (4) that (6.3.13) is the blowup at the image of the locus of points where stabilisation is not an isomorphism. But from Remark 6.3 .9 it is clear that this locus is $M_{n_{0}+1}$, so we are done.

Lemma 6.3.12. The stacks $D_{k}$ are all smooth of relative dimension 2 over $Y$.
Proof. Since $\mathrm{KM}_{P_{k}, G, r i g}^{-\alpha_{j}^{\vee}} \times \times_{\text {Bun }_{G, r i g}} Z \times_{S} E$ is smooth over $Y_{P_{k}}^{-\alpha_{j}^{\vee}} \times{ }_{S} E$, and hence over $Y_{P_{k}}$, the stacks $D_{k}$ are all smooth over $Y$. Moreover, Propositions 6.3.10 and 6.3.11 imply that $D_{\alpha_{j}^{\vee}}(Z) \rightarrow D_{k}$ is birational for all $k$, so $D_{k}$ has relative dimension 2 over $Y$ since $D_{\alpha_{j}^{\vee}}(Z)$ does. (Note that the flat morphism $\chi: Z \rightarrow \widehat{Y} / / W$ has relative dimension 2 by construction.)

Lemma 6.3.13. The morphism (6.3.11) is smooth with connected fibres.
Proof. From the construction and Lemma 6.2.13, we have

$$
C_{1, n_{0}+1}=D_{1} \times{ }_{Z} Z_{0}=Y^{-\alpha_{j}^{\vee}} \times{ }_{Y_{P_{1}}^{-\alpha \vee}} \operatorname{Bun}_{L \cap P_{1}, r i g}^{-\alpha_{j}^{\vee}-\alpha_{\curlyvee}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}} Z_{0} \times{ }_{S} E .
$$

There is an isomorphism

$$
\begin{align*}
Y \times_{Y_{P_{1}}} Y_{L \cap P_{1}} & \xrightarrow{\sim} Y \times_{S} \operatorname{Pic}_{S}^{0}(E)  \tag{6.3.14}\\
\left(y_{1}, y_{2}\right) & \longmapsto\left(y_{1}, \varpi_{i}\left(y_{2}\right)-\varpi_{i}\left(y_{1}\right)\right) .
\end{align*}
$$

Chasing through the definitions of the various morphisms involved, we deduce that there is a pullback

where the morphisms $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E) \rightarrow Y_{L \cap P_{1}}$ is the composition of the inverse to (6.3.14) with the natural projection.

It therefore suffices to show that the composition $f$ of the first two morphisms in the bottom row is smooth with connected fibres. Note that the morphism

$$
Y_{L \cap P_{1}}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \times{ }_{S} E \longrightarrow Y_{L \cap P_{1}}
$$

naturally identifies $Y_{L \cap P_{1}}$ with the quotient $\left(Y_{L \cap P_{1}}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \times{ }_{S} E\right) / E$ by the diagonal action of $E$ by translations. So we can identify $f$ with the composition of the middle vertical arrows in the diagram


The vertical arrow on the left in (6.3.15) is smooth, and has connected fibres since the semisimple part of $L \cap P_{1}$ is simply connected. The vertical arrow on the right in (6.3.15) is smooth with connected fibres by assumption. Since both squares are Cartesian, and the horizontal arrows in the square on the left are faithfully flat, it follows that both vertical arrows in the middle are smooth with connected fibres, and hence so is their composition $f$.

Lemma 6.3.14. The morphism (6.3.11) is an isomorphism.
Proof. Observe that the cell

$$
C_{n_{0}+1, n_{0}+1} \subseteq D_{n_{0}+1}=\mathrm{KM}_{B, G, r i g}^{-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{G, r i g}} Z \times_{S} E
$$

is equal to the locus of singular domain curves, and is therefore a divisor in $D_{n_{0}+1}$ flat over $Y$. Since $D_{n_{0}+1} \rightarrow Y$ has relative dimension 2 by Lemma 6.3.12, $C_{n_{0}+1, n_{0}+1} \rightarrow Y$ therefore has relative dimension 1. So by Lemma 6.3.13, (6.3.11) is a smooth proper morphism with connected fibres and finite relative stabilisers between smooth stacks of the same dimension over $S$. Since $C_{n_{0}+1, n_{0}+1} \rightarrow S$ is representable over the dense open substack where $Z_{0} \rightarrow S$ is representable, so is $C_{n_{0}+1, n_{0}+1} \rightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$. Since $Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \rightarrow S$ has irreducible fibres, (6.3.11) is therefore surjective, so by Lemma 6.3 .15 below, it is an isomorphism as claimed.

Lemma 6.3.15. Let $X$ and $X^{\prime}$ be stacks that are smooth and of the same dimension over $S$, and let $f: X \rightarrow X^{\prime}$ be a smooth surjective proper morphism with connected fibres and finite relative stabilisers. Assume that there exists some open set $U \subseteq X$ that is dense in every fibre of $X \rightarrow S$ such that $\left.f\right|_{U}$ is representable. Then $f$ is an isomorphism.

Proof. First note that $\left.f\right|_{U}: U \rightarrow X^{\prime}$ is étale and representable with connected fibres, and hence an open immersion. Moreover, the morphism $X \times_{X^{\prime}} X \rightarrow X$ is smooth with connected fibres, so the preimage of $U$ under either projection is dense. So the diagonal $X \rightarrow X \times{ }_{X^{\prime}} X$, which is finite by assumption, is an isomorphism over the dense open subset $U$, and hence surjective. Since $X \times{ }_{X^{\prime}} X$ is smooth over $S$, and hence normal, it follows that $X \rightarrow X \times{ }_{X^{\prime}} X$ is an isomorphism. Since $f$ is smooth and surjective, by flat descent it follows that $f: X \rightarrow$ $X^{\prime}$ is also an isomorphism as claimed.

Proposition 6.3.16. The stack $D_{1}$ is isomorphic to a line bundle over

$$
C_{1, n_{0}+1} \cong Y \times_{S} \operatorname{Pic}_{S}^{0}(E)
$$

Proof. Propositions 6.3.8, 6.3.10 and Lemma 6.3.14 together imply that $C_{1, n_{0}+1}=D_{1} \times{ }_{Z}$ $Z_{0}$ is a Cartier divisor on $D_{1}$. Moreover, choosing any cocharacter of the torus $Z(L)_{\text {rig }}$ whose negative is a Harder-Narasimhan vector for the parabolic $P^{+}$opposite to $P$, we get compatible actions of $\mathbb{G}_{m}$ on $Z$ and $D_{1}$ acting trivially on $Z_{0}$ and $D_{1} \times_{Z} Z_{0}$, such that $\mathbb{G}_{m}$ acts on the fibres of the affine space bundle $Z \rightarrow Z_{0}$ with positive weights. Since the normal cone of $D_{1} \times{ }_{Z} Z_{0}$ in $D_{1}$ is a line bundle and $\mathbb{G}_{m}$ acts nontrivially on it, $\mathbb{G}_{m}$ acts on it with a single nonzero weight. So the proposition follows from Lemma 5.4.14.

Proposition 6.3.17. For $1 \leq k \leq n_{0}$, the natural morphism $D_{k+1} \rightarrow D_{k}$ is the blowup along $M_{k}$.

Proof. Propositions 6.3.8 and 6.3.10, and Lemmas 6.3.14 and 6.3.12 imply that $D_{k+1} \rightarrow D_{k}$ is a projective birational morphism between smooth stacks of relative dimension 2 over $Y$ that is an isomorphism outside the section $Y \cong M_{k} \subseteq D_{k}$, such that the fibres of $D_{k+1} \rightarrow D_{k}$ over points in $M_{k}$ are irreducible curves. Moreover, Proposition 6.3.16 implies that $D_{1} \rightarrow Y$ is representable, and hence so is $D_{k} \rightarrow Y$. So by Lemma 6.3 .18 below, $D_{k+1} \rightarrow D_{k}$ is the blowup along $M_{k}$ as claimed.

Lemma 6.3.18. Let $U$ be a regular stack, let $X \rightarrow U$ and $X^{\prime} \rightarrow U$ be smooth representable morphisms of relative dimension 2, and let $f: X \rightarrow X^{\prime}$ be a projective morphism over $U$. Suppose that there exists a section $g: U \rightarrow X^{\prime}$ such that $f^{-1}\left(X^{\prime} \backslash g(U)\right) \rightarrow X^{\prime} \backslash g(U)$ is an isomorphism, and such that every fibre of $f$ over a point in $g(U)$ is an irreducible curve. Then $f$ is the blowup of $X^{\prime}$ along $g(U)$.

Proof. Since the claim is local in the smooth topology on $U$ and in the étale topology on $X^{\prime}$, we can reduce to the case where $X^{\prime} \rightarrow U$ is a smooth morphism of schemes with $U$ connected and regular.

First note that the underlying reduced scheme $D$ of the exceptional locus $f^{-1}(g(U))$ is an integral closed subscheme of codimension 1 in a regular scheme, and hence a Cartier divisor. Since $X$ and $X^{\prime}$ are smooth over $U$ and $f$ is an isomorphism outside $D$, we therefore have $K_{X / U}=f^{*} K_{X^{\prime} / U}(n D)$ for some $n>0$. If $k$ is any field and $u$ : Spec $k \rightarrow U$ is a $k$-point, we have $\left.D\right|_{X_{u}}=m_{u} C_{u}$ for some $m_{u}>0$, where $C_{u} \subseteq X_{u}$ is the irreducible curve contracted under $f$, and hence, by adjunction

$$
-2 \leq \operatorname{deg} K_{C_{u}}=\left(m_{u} n+1\right) C_{u}^{2}
$$

Since $C_{u}^{2}<0$, we deduce that $m_{u}=n=1, C_{u}^{2}=-1$, $\operatorname{deg} K_{C_{u}}=-2$, and hence that $C_{u}$ is a smooth rational curve. In particular, by Castelnuovo's theorem, $f_{u}: X_{u} \rightarrow X_{u}^{\prime}$ is the blowup at $g(u)$.

We next prove the claim in the case where $U=\operatorname{Spec} R$ for some discrete valuation ring $R$. If $\eta$ : Spec $K \rightarrow U$ is the generic point and $u: \operatorname{Spec} k \rightarrow U$ the closed point, we have shown that on the open generic fibre, $f_{\eta}: X_{\eta} \rightarrow X_{\eta}^{\prime}$ is the blowup along $g(\eta)$, and hence we get an isomorphism

$$
h: X \backslash f^{-1}(g(u)) \xrightarrow{\sim} \tilde{X}^{\prime} \backslash \pi^{-1}(g(u))
$$

over $X^{\prime}$, where $\pi: \tilde{X}^{\prime} \rightarrow X^{\prime}$ is the blowup of $X^{\prime}$ along $g(U)$. Since $f$ is projective and is an isomorphism outside $D$, it follows that either $D$ or $-D$ is $f$-ample. Since $D \cdot C_{u}=$ $\left(C_{u}^{2}\right)_{X_{u}}=-1$, it follows that $-D$ is $f$-ample. But $h$ is an isomorphism in codimension 1 between regular schemes projective over $X^{\prime}, h\left(D \backslash f^{-1}(g(u))\right)=\pi^{-1}(g(U)) \backslash \pi^{-1}(g(u))$, and $-\pi^{-1}(g(U))$ is $f$-ample, so

$$
X \xrightarrow{\sim} \operatorname{Proj}_{X^{\prime}} \bigoplus_{d \geq 0} f_{*} \mathcal{O}(-d D) \cong \operatorname{Proj}_{X^{\prime}} \bigoplus_{d \geq 0} \pi_{*} \mathcal{O}\left(-d \pi^{-1}(g(U))\right) \stackrel{\sim}{\sim} \tilde{X}^{\prime},
$$

which proves that $X$ is the blowup as claimed.
Now consider a general connected regular $U$, let $\pi: \tilde{X}^{\prime} \rightarrow X^{\prime}$ be the blowup along $g(U)$ as before, and let $\tilde{X} \subseteq X \times_{X^{\prime}} \tilde{X}^{\prime}$ be the closure of $X^{\prime} \backslash g(U)$. We claim that $\tilde{X} \rightarrow X$ is an isomorphism. To see this, it suffices to show that $\tilde{X} \rightarrow X$ is quasi-finite, since it is proper and birational and $X$ is normal. If not, then there exists a curve $\tilde{C}$ in $\tilde{X}_{u}$ for some $u$ : $\operatorname{Spec} k \rightarrow U$, say with $k$ algebraically closed, that is contracted under the map to $X$. Since $X_{u} \cong \tilde{X}_{u}^{\prime}$ over $X_{u}^{\prime}$, it follows that $\tilde{C}$ cannot be contained in the closure of $X_{u}^{\prime} \backslash g(u)$. Choose some $k$-point $x: \operatorname{Spec} k \rightarrow \tilde{C}_{u}$ that does not lie in this closure. Since $\tilde{C}$ is in the closure of $X^{\prime} \backslash g(U)$, we can find a discrete valuation ring $R$ with residue field $k$ and a morphism $\operatorname{Spec} R \rightarrow X \times_{X^{\prime}} \tilde{X}^{\prime}$ sending the closed point to $x$ and the generic point to $X^{\prime} \backslash g(U)$. Pulling everything back along $\operatorname{Spec} R \rightarrow U$, we deduce that there is a point in the closure of $X_{R}^{\prime} \backslash g(U)_{R}$ in $X_{R} \times_{X_{R}^{\prime}} \tilde{X}_{R}^{\prime}$ over the closed point of Spec $R$ that is not in the closure of $X_{u}^{\prime} \backslash g(u)$. But since we have shown that $X_{R}=\tilde{X}_{R}^{\prime}$ above, the closure of $X_{R}^{\prime} \backslash g(U)_{R}$ is isomorphic to $X_{R}$, so this is a contradiction. So we must have $\tilde{X} \cong X$ as claimed.

So the morphism $f: X \rightarrow X^{\prime}$ factors through a morphism $X \rightarrow \tilde{X}^{\prime}$. But since this map is an isomorphism on every fibre over $U$, it is therefore an isomorphism globally, and we are done.

Proof of Theorem 6.1.9, (3) and (4). To prove (3), apply Propositions 6.3.11, 6.3.16 and 6.3.17. To prove (4), observe that the proper transform of the zero section is the closure of the locus of stable maps with dual graph

$$
\stackrel{\circ}{\alpha_{j}^{\vee}}-\underset{\alpha_{i}^{\vee}-\alpha_{j}^{\vee}}{\bullet} \alpha_{i}^{\vee}
$$

and is hence equal to the intersection with $D_{\alpha_{j}^{\vee}}(Z)$ by Propositions 3.4.13 and 3.4.16. The map to $\operatorname{Pic}_{S}^{0}(E)$ is given as in the statement of the theorem by Remark 6.3.9.

### 6.4 Computing the divisor $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$

In this section, we complete the proof of Theorem 6.1 .9 by computing the divisor $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$. We will assume for this section that the hypotheses of Theorem 6.1.9 are satisfied.

In the following proposition, for $\lambda \in \mathbb{X}_{*}(T)_{+}$, we write

$$
M_{0,1}^{+}(G / B, \lambda)=M_{0,1}(G / B, \lambda) \times_{G / B} \operatorname{Spec} \mathbb{Z}
$$

for the stack of 1-pointed genus 0 stable maps of degree $\lambda$ sending the marked point to the base point $B / B \in G / B$.

Proposition 6.4.1. Let $y: \operatorname{Spec} k \rightarrow Y$ be a geometric point. Then there is an isomorphism

$$
D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \cong M_{0,1}^{+}\left(G / B, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)_{k}
$$

Proof. Since Lemma 6.3.2 implies that all stable $\mathbb{X}^{*}(T) \oplus \mathbb{Z}$-graphs $\tau$ admitting contractions $\tau \rightarrow \tau_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}^{0}$ with $M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau\right) \neq \emptyset$ have a unique such contraction, the gluing map

$$
\begin{equation*}
M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}^{0}\right) \longrightarrow D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z) \tag{6.4.1}
\end{equation*}
$$

is an isomorphism by Corollary 3.2.21 and Proposition 3.4.13, where $\xi_{G / Z(G)} \rightarrow Z \times_{S} E$ is the $G / Z(G)$-bundle classified by $Z \rightarrow \operatorname{Bun}_{G, \text { rig }} \rightarrow \operatorname{Bun}_{G / Z(G)}$. Moreover, every fibre of the morphism

$$
\begin{equation*}
M_{Z}\left(\xi_{G / Z(G)} \times^{G / Z(G)} G / B, \tau_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}^{0}\right) \longrightarrow M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}, 1\right)\right) \tag{6.4.2}
\end{equation*}
$$

over a $k$-point is isomorphic to $M_{0,1}^{+}\left(G / B, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)_{k}$. But by the isomorphism (6.4.1) and Lemma 6.4.2 below, we can identify (6.4.2) with the morphism $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z) \rightarrow Y$, so the result now follows.

Lemma 6.4.2. The morphism

$$
\begin{equation*}
M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}, 1\right)\right) \longrightarrow Y^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \times_{S} E \longrightarrow Y \tag{6.4.3}
\end{equation*}
$$

is an isomorphism, where the first morphism is the usual (blow down) map to $Y^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}}$ on the first factor and the map evaluating at the marked point and projecting to $E$ on the second factor, and the second morphism is (6.3.1).

Proof. Using Proposition 3.4.10 and the fact that each of the evalation maps $M_{0,3}(G / B, 0) \rightarrow$ $G / B$ is an isomorphism, we deduce that the canonical morphism

$$
M_{Z_{0}}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau\right) \longrightarrow M_{1,1, Z_{0}}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}, 1\right)\right)
$$

is an isomorphism, where $\tau$ is the $\mathbb{X}^{*}(T) \oplus \mathbb{Z}$-graph below.


But $M_{Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B, \tau\right)$ is equal to the intersection of the proper transform of $C_{n_{0}+1, n_{0}+1}=Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \subseteq D_{n_{0}+1}$ with the exceptional divisor of the blowup $D_{\alpha_{j}^{\vee}}(Z) \rightarrow$ $D_{n_{0}+1}$, and therefore maps isomorphically to $Y$ under $D_{\alpha_{j}^{\vee}}(Z) \rightarrow Y$. Since this map to $Y$ agrees with the one in the statement of the lemma by Proposition 3.5.5, this completes the proof.

Proposition 6.4.3. Let $d=-\left\langle\alpha_{j}, \alpha_{i}^{\vee}\right\rangle-1$. Then there is an isomorphism

$$
M_{0,1}^{+}\left(G / B, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right) \cong \mathbb{F}_{d-1}
$$

such that the closure of the locus of stable maps with dual graph

is a fibre of $\mathbb{F}_{d-1} \rightarrow \mathbb{P}^{1}$, and the closure of the locus of stable maps with dual graph

is a section $\mathbb{P}^{1} \rightarrow \mathbb{F}_{d-1}$ with self-intersection $1-d$.
An important role in the proof of Proposition 6.4.3 is played by the Schubert varieties in $G / B$. Given $w \in W$, recall that the Schubert variety associated to $w$ is the closed subvariety

$$
X_{w}=\overline{B w B / B} \subseteq G / B
$$

In what follows, we write $Q_{i}, Q_{j} \subseteq G$ for the standard minimal parabolics of types $t\left(Q_{i}\right)=$ $\Delta \backslash\left\{\alpha_{i}\right\}$ and $t\left(Q_{j}\right)=\Delta \backslash\left\{\alpha_{j}\right\}$.

Lemma 6.4.4. There are isomorphisms

$$
X_{s_{i} s_{j}} \cong \mathbb{F}_{d}, \quad\left(\text { resp } . \quad X_{s_{j} s_{i}} \cong \mathbb{F}_{1}\right)
$$

such that $X_{s_{j}}$ is identified with a fibre of $\mathbb{F}_{d} \rightarrow \mathbb{P}^{1}$ (resp., the unique section $\mathbb{P}^{1} \rightarrow \mathbb{F}_{1}$ of selfintersection -1) and $X_{s_{i}}$ is identified with the unique section $\mathbb{P}^{1} \rightarrow \mathbb{F}_{d}$ of self-intersection $-d$ (resp., a fibre of $\mathbb{F}_{1} \rightarrow \mathbb{P}^{1}$ ).

Proof. We prove the claim for $X_{s_{i} s_{j}}$; the proof for $X_{s_{j} s_{i}}$ is identical after noting that $\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle=-1$.

There is an isomorphism

$$
S L_{2} \times{ }^{B_{S L_{2}}, \rho_{\alpha_{i}}} Q_{j} / B=Q_{i} \times{ }^{B} Q_{j} / B \xrightarrow{\sim} X_{s_{i} s_{j}},
$$

given by multiplication, where $B_{S L_{2}} \subseteq S L_{2}$ is the Borel subgroup of lower triangular matrices, and $\rho_{\alpha_{i}}: S L_{2} \rightarrow G$ is the root homomorphism corresponding to $\alpha_{i}$. We also have an isomorphism of $Q_{j}$-varieties $Q_{j} / B \cong \mathbb{P}\left(V^{\vee}\right)$, where $V$ is the $Q_{j}$-representation $V=\operatorname{Ind}_{B}^{Q_{j}}\left(\mathbb{Z}_{\varpi_{j}}\right)$, and an exact sequence

$$
0 \longrightarrow \mathbb{Z}_{\varpi_{j}-\alpha_{j}} \longrightarrow V \longrightarrow \mathbb{Z}_{\varpi_{j}} \longrightarrow 0
$$

of $B$-representations, which splits uniquely as an exact sequence of $B_{S L_{2}}$-representations. So we have
$\left.X_{s_{i} s_{j}}=S L_{2} \times{ }^{B_{S L_{2}}} \mathbb{P}\left(V^{\vee}\right)=\mathbb{P}_{\mathbb{P}^{1}}\left(\mathcal{O}\left(-\left\langle\varpi_{j}, \alpha_{i}^{\vee}\right\rangle\right) \oplus \mathcal{O}\left(-\left\langle\varpi_{j}-\alpha_{j}, \alpha_{i}^{\vee}\right\rangle\right)\right)=\mathbb{P}_{\mathbb{P}^{1}}(\mathcal{O} \oplus \mathcal{O}(-d))\right)=\mathbb{F}_{d}$.
The identifications of $X_{s_{i}}=Q_{i} / B$ and $X_{s_{j}}=Q_{j} / B$ under this isomorphism follow immediately.

Lemma 6.4.5. The partial Schubert variety $X_{s_{i} s_{j}} / Q_{i}=\overline{B s_{i} s_{j} Q_{i} / Q_{i}} \subseteq G / Q_{i}$ is isomorphic to the projective cone $\widehat{\mathbb{P}}_{d}^{1}$ on $\mathbb{P}^{1}$ of degree $d$, and the morphism

$$
\begin{equation*}
X_{s_{i} s_{j}} \longrightarrow X_{s_{i} s_{j}} / Q_{i} \tag{6.4.4}
\end{equation*}
$$

is the blowup of $X_{s_{i} s_{j}} / Q_{i}$ at the origin $Q_{i} / Q_{i}$.

Proof. First note that the morphisms $B s_{i} s_{j} B / B \rightarrow B s_{i} s_{j} Q_{i} / Q_{i}$ and $B s_{j} B / B \rightarrow B s_{j} Q_{i} / Q_{i}$ are isomorphisms. So (6.4.4) is birational and finite outside $Q_{i} / Q_{i}$, and hence an isomorphism outside $Q_{i} / Q_{i}$ since partial Schubert varieties are always normal. Since the preimage of $Q_{i} / Q_{i}$ under (6.4.4) is $Q_{i} / B=X_{s_{i}}$, using normality of $X_{s_{i} s_{j}} / Q_{i}$ and of $\widehat{\mathbb{P}}_{d}^{1}$ (note that $d \leq 3$ ), we can conclude from Lemma 6.4.4 that (6.4.4) can be identified with the morphism

$$
\mathbb{F}_{d} \longrightarrow \widehat{\mathbb{P}}_{d}^{1}
$$

contracting the curve of self-intersection $-d$. But this is indeed the blowup at the cone point, so we are done.

Lemma 6.4.6. There is a $Q_{i}$-equivariant isomorphism

$$
M_{0,1}^{+}\left(X_{s_{i} s_{j}} / Q_{i}, \alpha_{j}^{\vee}\right) \cong Q_{i} / B \cong \mathbb{P}^{1}
$$

identifying the universal stable map with

$$
\begin{equation*}
Q_{i} \times{ }^{B} Q_{j} / B \longrightarrow X_{s_{i} s_{j}} \longrightarrow X_{s_{i} s_{j}} / Q_{i} . \tag{6.4.5}
\end{equation*}
$$

Proof. Assume that $U$ is a scheme and $\left(f: C \rightarrow X_{s_{i} s_{j}} / Q_{i}, x: U \rightarrow C\right)$ is a 1-pointed stable map over $U$ of degree $\alpha_{j}^{\vee}$ sending $x$ to the base point. We need to show that there is a unique morphism $U \rightarrow Q_{i} / B$ such that $(f, x)$ is the pullback of (6.4.5) and the canonical section $Q_{i} / B=Q_{i} \times{ }^{B} B / B \rightarrow Q_{i} \times{ }^{B} Q_{j} / B$.

We first claim that $C \rightarrow U$ is smooth and that every geometric fibre of $f^{-1}\left(Q_{i} / Q_{i}\right) \rightarrow U$ is a reduced point. Since $f^{-1}\left(Q_{i} / Q_{i}\right) \rightarrow U$ has a section $x$, it then follows that it is an isomorphism.

To prove the claim, fix a geometric point $u: \operatorname{Spec} k \rightarrow U$, and consider the stable map $f_{u}: C_{u} \rightarrow\left(X_{s_{i} s_{j}} / Q_{i}\right)_{k}$. Since $\alpha_{j}^{\vee}$ is not the sum of two nonzero effective curve classes, it follows that $C_{u}$ is irreducible, hence smooth over Spec $k$, and hence that $f_{u}^{-1}\left(Q_{i} / Q_{i}\right)$ is a Cartier divisor on $C_{u}$. So by Lemmas 6.4.4 and 6.4.5, $f_{u}$ lifts to a morphism $\bar{f}_{u}: C_{u} \rightarrow$ $\left(X_{s_{i} s_{j}}\right)_{k} \cong\left(\mathbb{F}_{d}\right)_{k}$ such that $C_{u} \cdot X_{s_{i}}>0$ and $C_{u} \cdot\left(d X_{s_{j}}+X_{s_{i}}\right)=1$. Since $d>0$, it follows that $C_{u} \cdot X_{s_{i}}=1$ and $C_{u} \cdot X_{s_{j}}=0$. In particular, $f_{u}^{-1}\left(Q_{i} / Q_{i}\right)=C_{u} \cap X_{s_{i}}$ is a reduced closed point on $C_{u}$, so $f_{u}^{-1}\left(Q_{i} / Q_{i}\right) \cong \operatorname{Spec} k$ as claimed.

Since $f^{-1}\left(Q_{i} / Q_{i}\right) \subseteq C$ is a section of the smooth curve $C \rightarrow U$, it is a Cartier divisor, so by Lemma $6.4 .5, f$ lifts uniquely to a morphism $\bar{f}: C \rightarrow X_{s_{i} s_{j}}$. Since the above argument shows that the composition $\bar{f}: C \rightarrow X_{s_{i} s_{j}}=Q_{i} \times{ }^{B} Q_{j} / B \rightarrow Q_{i} / B$ has degree 0 on every fibre, this descends to a unique morphism $U \rightarrow Q_{i} / B$. The induced morphism

$$
\begin{equation*}
C \longrightarrow U \times_{Q_{i} / B}\left(Q_{i} \times^{B} Q_{j} / B\right) \tag{6.4.6}
\end{equation*}
$$

has degree 1 on every fibre and is therefore an isomorphism. Since (6.4.6) sends the section $x$ to the section $Q_{i} / B \rightarrow Q_{i} \times{ }^{B} Q_{j} / B$ (as both are the preimage of $Q_{i} / Q_{i} \subseteq X_{s_{i} s_{j}} / Q_{i}$ ), this proves the lemma.

Proof of Proposition 6.4.3. For the sake of brevity, write

$$
M=M_{0,1}^{+}\left(G / B, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)
$$

We first claim that $M$ is connected. To see this, observe that $B$ acts on $M$, that any $B$-fixed point corresponds to a stable map factoring through $X_{s_{i}} \cup X_{s_{j}} \subseteq G / B$, and that there is a unique such pointed stable map of class $\alpha_{i}^{\vee}+\alpha_{j}^{\vee}$ defined over $k$ for any algebraically closed field $k$. Since every connected component of $M$ must have at least one $B$-fixed point over every algebraically closed field, connectedness of $M$ follows immediately.

We now compute the closed subscheme

$$
M^{\prime}=M_{0,1}^{+}\left(X_{s_{i} s_{j} s_{i}}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right) \subseteq M
$$

consisting of stable maps factoring through the Schubert variety $X_{s_{i} s_{j} s_{i}}$. We will show that $M^{\prime} \cong \mathbb{F}_{d}$ is smooth and projective of relative dimension 2 over Spec $\mathbb{Z}$, from which it follows from connectedness of $M$ and Proposition 3.4.9 that $M^{\prime}=M$.

Since $X_{s_{i} s_{j} s_{i}} / Q_{i}=X_{s_{i} s_{j}} / Q_{i}$, by Lemma 6.4 .6 we have a morphism

$$
M^{\prime} \longrightarrow M_{0,1}^{+}\left(X_{s_{i} s_{j}} / Q_{i}, \alpha_{j}^{\vee}\right) \cong Q_{i} / B=\mathbb{P}^{1}
$$

sending a stable map to the stabilisation of its composition with $G / B \rightarrow G / Q_{i}$. The pullback of the universal domain curve of $M_{0,1}^{+}\left(X_{s_{i} s_{j}} / Q_{i}, \alpha_{j}^{\vee}\right)$ along $X_{s_{i} s_{j} s_{i}} \rightarrow X_{s_{i} s_{j}} / Q_{i}$ is

$$
X_{s_{i} s_{j} s_{i}} \times{ }_{X_{s_{i} s_{j}} / Q_{i}}\left(Q_{i} \times{ }^{B} Q_{j} / B\right)=G / B \times_{G / Q_{i}}\left(Q_{i} \times{ }^{B} Q_{j} / B\right),
$$

which is identified with the Bott-Samelson variety $\tilde{X}_{s_{i} s_{j} s_{i}}$ via

$$
\begin{aligned}
\tilde{X}_{s_{i} s_{j} s_{i}}=Q_{i} \times{ }^{B} Q_{j} \times{ }^{B} Q_{i} / B & \sim \\
\left(g_{1}, g_{2}, g_{3} B\right) & \longmapsto\left(g_{1} g_{2} g_{3} B,\left(g_{1}, g_{2} B\right)\right) .
\end{aligned}
$$

So we can identify $M^{\prime}$ with the relative space of stable maps

$$
M^{\prime} \cong M_{0,1, Q_{i} / B}^{+}\left(\tilde{X}_{s_{i} s_{j} s_{i}}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)
$$

where $M_{0,1, Q_{i} / B}^{+}\left(\tilde{X}_{s_{i} s_{j} s_{i}}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)$ is the fibre product


Here $\sigma$ is the section defined by $Q_{i} / B \cong m^{-1}(B / B) \rightarrow \tilde{X}_{s_{i} s_{j} s_{i}}$, for $m: \tilde{X}_{s_{i} s_{j} s_{i}} \rightarrow G / B$ the natural morphism given by multiplication. By Proposition 3.1.13, we therefore have a fibre product

where $C$ is the universal domain curve over $M_{0, Q_{i} / B}\left(\tilde{X}_{s_{i} s_{j} s_{i}}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)$.
By Lemma 6.4.4, every fibre of $\tilde{X}_{s_{i} s_{j} s_{j}} \rightarrow Q_{i} / B$ is isomorphic to $\mathbb{F}_{1}=X_{s_{j} s_{i}}=Q_{j} \times{ }^{B}$ $Q_{i} / B$, and $\alpha_{i}^{\vee}+\alpha_{j}^{\vee}$ is the class $X_{s_{i}}+X_{s_{j}}$ of the $(-1)$-curve plus a fibre of $\mathbb{F}_{1} \rightarrow \mathbb{P}^{1}$. Unpointed stable maps of class $\alpha_{i}^{\vee}+\alpha_{j}^{\vee}$ are the same things as closed subschemes with ideal sheaf $\mathcal{O}\left(-X_{s_{i}}-X_{s_{j}}\right)=m^{*} \mathcal{L}_{-\varpi_{i}}$. So we can identify $M_{0, Q_{i} / B}\left(\tilde{X}_{s_{i} s_{j} s_{i}}, \alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)$ with the Hilbert scheme $\mathbb{P}_{Q_{i} / B}\left(\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}}\right)$ and $M^{\prime}$ with the closed subscheme

$$
M^{\prime}=\mathbb{P}_{Q_{i} / B}\left(\operatorname{ker} \pi_{*} m^{*} \mathcal{L}_{\varpi_{i}} \rightarrow \sigma^{*} m^{*} \mathcal{L}_{\varpi_{2}}\right),
$$

where $\pi: \tilde{X}_{s_{i} s_{j} s_{i}} \rightarrow Q_{i} / B$ is the natural projection.
It therefore remains to identify the vector bundle $\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}}$ on $Q_{i} / B \cong \mathbb{P}^{1}$ and the morphism $\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}} \rightarrow \sigma^{*} m^{*} \mathcal{L}_{\varpi_{i}}=\mathcal{O}$. It is clear from the identification $\tilde{X}_{s_{i} s_{j} s_{i}}=$
$Q_{i} \times{ }^{B} Q_{j} \times{ }^{B} Q_{i} / B$ that $\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}}$ is the $Q_{i}$-linearised vector bundle associated to the $B$-representation

$$
V=\operatorname{Ind}_{B}^{Q_{j}} \operatorname{Ind}_{B}^{Q_{i}} \mathbb{Z}_{\varpi_{i}}
$$

The representation $V$ has rank 3 , with weights $\varpi_{i}, \varpi_{i}-\alpha_{i}$ and $\varpi_{i}-\alpha_{i}-\alpha_{j}$, and restricting $V$ to a $B_{S L_{2}}$-representation via the root homomorphism $\rho_{\alpha_{i}}: S L_{2} \rightarrow Q_{i} \subseteq G$, we have

$$
V=U \oplus \mathbb{Z}_{\left\langle\varpi_{i}-\alpha_{i}-\alpha_{j}, \alpha_{i}^{\vee}\right\rangle}=U \oplus \mathbb{Z}_{d-1}
$$

where $U$ is the standard representation of $S L_{2}$ and $\mathbb{Z}_{d-1}$ is the rank $1 B_{S L_{2}}$-module of weight $d-1$. So we get

$$
\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}}=U \otimes \mathcal{O}_{\mathbb{P}^{1}} \oplus \mathcal{O}(d-1)
$$

Since $d>0$, the kernel of

$$
\pi_{*} m^{*} \mathcal{L}_{\varpi_{i}}=\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(d-1) \longrightarrow \mathcal{O}=\sigma^{*} m^{*} \mathcal{L}_{\varpi_{i}}
$$

must be isomorphic to $\mathcal{O} \oplus \mathcal{O}(d-1)$, which gives the desired isomorphism $M=M^{\prime} \cong \mathbb{F}_{d-1}$.
Finally, to identify the loci of stable maps with given dual graphs in the statement of the proposition, notice that Proposition 3.4.10 implies that each closure is isomorphic to $\mathbb{P}^{1}$, and that the closure of curves with dual graph

is contracted under the map to $M_{0,1}^{+}\left(G / Q_{i}, \alpha_{j}^{\vee}\right)$, and is hence a fibre of $\mathbb{F}_{d-1} \rightarrow \mathbb{P}^{1}$ as claimed. Moreover, the canonical section of $\mathbb{F}_{d-1}$ of degree $1-d$ is the subscheme

$$
\mathbb{P}_{Q_{i} / B}\left(\operatorname{ker} \pi_{*} m^{*} \mathcal{L}_{\varpi_{i}} \rightarrow \pi_{*}^{\prime}\left(\sigma^{\prime}\right)^{*} m^{*} \mathcal{L}_{\varpi_{i}}\right) \subseteq \mathbb{P}_{Q_{i} / B}\left(\operatorname{ker} \pi_{*} m^{*} \mathcal{L}_{\varpi_{i}} \rightarrow \sigma^{*} m^{*} \mathcal{L}_{\varpi_{i}}\right)=M^{\prime}
$$

where $\sigma^{\prime}$ is the morphism $Q_{i} \times{ }^{B} Q_{i} / B=Q_{i} \times{ }^{B} B \times{ }^{B} Q_{i} / B \rightarrow \tilde{X}_{s_{i} s_{j} s_{i}}$ and $\pi^{\prime}: Q_{i} \times{ }^{B} Q_{i} / B \rightarrow$ $Q_{i} / B$ is the natural projection onto the first factor. But this parametrises curves of class $\alpha_{i}^{\vee}+\alpha_{j}^{\vee}$ containing some curve of class $\alpha_{i}^{\vee}$, so this must be the closure of the locus of curves with dual graph

as claimed.
Proof of Theorem 6.1.9 (1), (2), (5) and (6). First note that (2), (5) and (6) follow immediately from Propositions 6.4.1 and 6.4.3.

To prove (1), by Corollary 4.5.9 and Lemma 6.3.2, the only thing left to show is that $D_{\alpha_{i}^{\vee}}(Z), D_{\alpha_{j}^{\vee}}(Z)$ and $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$ are connected. We have shown that $D_{\alpha_{j}^{\vee}}(Z)$ and $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$ are both connected, and that the union of their intersections with $D_{\alpha_{i}^{\vee}}(Z)$ is connected. Since the normal crossings divisor $D(Z)=\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$ is connected by Corollary 5.5.9, this implies that $D_{\alpha_{i}^{\vee}}(Z)$ must also be connected, so we are done.

### 6.5 Constructing slices

In this section, we give the proof of Theorem 6.1.5. The proof we give here is somewhat ad hoc, and relies on directly understanding the structure of the Levi subgroup $L$ in each case.

We first give the construction in type $A$.

Proof of Theorem 6.1 .5 in type $A$. First note that since $d=1$ in this case, the $\mu_{d}=\mu_{1}$ gerbe $\mathfrak{G}^{\text {uni }}$ must be the trivial one, so we need to construct a slice without passing to a smooth cover.

Proposition 5.3.1 gives an identification $L \cong G L_{i} \times G L_{l-i}$ so that the characters $\varpi_{i}$ and $\varpi_{i+1} \in \mathbb{X}^{*}(L)$ are identified with the determinants of the first and second factors respectively. (Explicitly, the isomorphism is given by

$$
\begin{aligned}
G L_{i} \times G L_{l-i} & \stackrel{\sim}{\longrightarrow} L \subseteq S L_{l+1} \\
(A, B) & \longmapsto\left(\begin{array}{ccc}
A & 0 & 0 \\
0 & (\operatorname{det} A)^{-1} \operatorname{det} B & 0 \\
0 & 0 & v\left(B^{t}\right)^{-1} v^{-1}
\end{array}\right),
\end{aligned}
$$

where $v \in S_{l-i}$ is the matrix of the permutation of $\{1, \ldots, l-i\}$ sending $j$ to $l-i-j+1$.) Theorem 5.3.5 and Proposition 5.3.4 therefore imply that the morphism

$$
\left(\varpi_{i}, \varpi_{i+1}\right): \operatorname{Bun}_{L, r i g}^{s s, \mu} \longrightarrow \operatorname{Pic}_{S}^{-1}(E) \times_{S} \operatorname{Pic}_{S}^{-1}(E)
$$

is a trivial $Z(L)_{\text {rig }}$-gerbe. Note that in particular, all semistable $L$-bundles of slope $\mu$ are regular in this case.

By Proposition 5.2.13, the pullback of $\Theta$ to $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ has $Z(L)_{\text {rig }}$-weight $(-\mu \mid-) \in$ $\mathbb{X}^{*}\left(Z(L)_{\text {rig }}\right)$. Since the corresponding homomorphism $\mathbb{X}^{*}\left(Z(L)_{\text {rig }}\right) \rightarrow \mathbb{Z}$ is surjective, it follows that there exists a section

$$
\operatorname{Pic}_{S}^{-1}(E) \times{ }_{S} \operatorname{Pic}_{S}^{-1}(E) \longrightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}
$$

such that the pullback of $\Theta_{\operatorname{Bun}_{G, r i g}}$ is trivial. Since such a section is necessarily smooth, composing it with any choice of section of

$$
\operatorname{Pic}_{S}^{-1}(E) \times_{S} \operatorname{Pic}_{S}^{-1}(E) \longrightarrow \operatorname{Pic}_{S}^{-1}(E) \times_{S} \operatorname{Pic}_{S}^{-1}(E) / E \cong E
$$

gives a $\Theta$-trivial slice $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ with $Z_{0}=E$, such that $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu} / E$ is surjective with fibres isomorphic to $Z(L)_{\text {rig }}$, hence connected. So (1), (2) and (3) are satisfied. A simple root-theoretic calculation shows that $-\langle 2 \rho, \mu\rangle=l+2$, so (4) follows from Proposition 5.2.8. So this proves the theorem in this case.

The construction in the exceptional types $E, F$ and $G$ is also fairly straightforward.
Proof of Theorem 6.1 .5 in types $E, F$ and $G$. In these cases, Theorem 5.3.5 shows that the morphism

$$
\begin{equation*}
\varpi_{i}: \operatorname{Bun}_{L, r i g}^{s s, \mu} \longrightarrow \operatorname{Pic}_{S}^{-1}(E) \tag{6.5.1}
\end{equation*}
$$

is a $\mathbb{G}_{m}=Z(L)_{\text {rig }}$-gerbe. Let $Z_{0}=S$, and let $\mathfrak{G}^{\prime}$ be the $Z(L)_{\text {rig }}$-gerbe given by the pullback along $\mathcal{O}\left(-O_{E}\right): Z_{0} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$. By Proposition 5.2.13, the pullback of the theta bundle defines a $\mathbb{B} Z(L)_{\text {rig }}$-equivariant morphism $\mathfrak{G}^{\prime} \rightarrow \mathbb{B} \mathbb{G}_{m}$, where $\mathbb{B} Z(L)_{\text {rig }}$ acts on $\mathbb{B} \mathbb{G}_{m}$ through the homomorphism $-(\mu \mid-): Z(L)_{\text {rig }} \rightarrow \mathbb{G}_{m}$. So a section of $\mathfrak{G}^{\prime}$ such that the pullback of $\Theta_{\operatorname{Bun}_{G, r i g}}$ is trivial is the same thing as a section of the $\mu_{d}=\operatorname{ker}(\mu \mid-)$-gerbe $\mathfrak{G}=\mathfrak{G}^{\prime} \times_{\mathbb{B G}_{m}} \operatorname{Spec} \mathbb{Z}$. The $\mu_{d}$-gerbe is by construction pulled back from one $\mathfrak{G}^{\text {uni }}$ on $M_{1,1}$, defined in the same way, and if it is trivial then there is a morphism $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ lifting the section $\mathcal{O}\left(-O_{E}\right): Z_{0} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ such that the pullback of $\Theta_{\text {Bun }_{G, r i g}}$ is trivial.

It is immediately clear that (1) is satisfied. Letting $\left(\operatorname{Bun}_{L, r i g}^{s s, \mu}\right)_{0}$ be the fibre of (6.5.1) over $\mathcal{O}\left(-O_{E}\right): S \rightarrow \operatorname{Pic}_{S}^{-1}(E)$, we have that $\left(\operatorname{Bun}_{L, r i g}^{s s, \mu}\right)_{0} \cong \operatorname{Bun}_{L, r i g}^{s s, \mu} / E$ is a $Z(L)_{r i g}$-gerbe over $S=Z_{0}$ and the map $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu} / E$ is a section. In particular, it is smooth with
connected fibres, so (2) is satisfied, and surjective, so (3) is also satisfied. Finally, to prove (4), simply note that Proposition 5.2 .8 and a simple root-theoretic calculation shows that $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right) \rightarrow Z_{0}=S$ is an affine space bundle of relative dimension $l+3$.

The construction of the slice $Z_{0}$ for $(G, P, \mu)$ of types $B, C$ and $D$ is somewhat more complicated, and involves parabolic induction from a smaller Levi $L^{\prime} \subseteq L$. To prove that this construction works, we will need to describe the Levi subgroup $L$ in some detail. For future reference, we do this now in all types.

Assume first that we are in types $C, D, E$ or $F$. Then the connected component $c_{1}$ containing $\alpha_{j}$ of the Dynkin diagram with the edge joining $\alpha_{i}$ and $\alpha_{j}$ deleted is of type $A_{n_{1}}$, and we can choose a labelling $\alpha_{c_{1}, 1}, \ldots, \alpha_{c_{1}, n_{1}}$ such that $\alpha_{c_{1}, p}$ is adjacent to $\alpha_{c_{1}, p+1}$ for each $p$ and $\alpha_{j}$ is either $\alpha_{c_{1}, n_{1}}\left(\right.$ in types $C$ and $F$ ) or $\alpha_{c_{1}, n_{1}-1}($ in types $D$ and $E$ ).

Lemma 6.5.1. In the setup above, there is an isomorphism

$$
L \cong\left\{(A, B) \in G L_{n_{0}} \times G L_{n_{1}+1} \mid \operatorname{det} B=(\operatorname{det} A)^{2}\right\}
$$

such that $\varpi_{i}$ is identified with the character $(A, B) \mapsto \operatorname{det} A$ and $L \cap B$ is the preimage of the Borel subgroup $Q_{n_{0}}^{n_{0}} \times Q_{n_{1}+1}^{n_{1}+1} \subseteq G L_{n_{0}} \times G L_{n_{1}+1}$. Moreover, the induced map

$$
\mathbb{X}^{*}\left(Q_{n_{1}+1}^{n_{1}+1}\right) \longrightarrow \mathbb{X}^{*}(L \cap B)=\mathbb{X}^{*}(T)
$$

is given in types $D$ and $E$ by

$$
e_{k} \longmapsto \begin{cases}\varpi_{1, c_{1}} & \text { if } k=1, \\ \varpi_{k, c_{1}}-\varpi_{k-1, c_{1}} & \text { if } 1<k<n_{1}, \\ \varpi_{n_{1}, c_{1}}-\varpi_{n_{1}-1, c_{1}}+\varpi_{i}, & \text { if } k=n_{1}, \\ -\varpi_{n_{1}, c_{1}}+\varpi_{i}, & \text { if } k=n_{1}+1,\end{cases}
$$

and in types $C$ and $F$ by

$$
e_{k} \longmapsto \begin{cases}\varpi_{1, c_{1}} \longmapsto & \text { if } k=1, \\ \varpi_{k, c_{1}}-\varpi_{k-1, c_{1}} & \text { if } 1<k \leq n_{1} \\ -\varpi_{n_{1}, c_{1}}+2 \varpi_{i}, & \text { if } k=n_{1}+1\end{cases}
$$

Proof. Apply Proposition 5.3.1; the expressions for $\mathbb{X}^{*}\left(Q_{n_{1}+1}^{n_{1}+1}\right) \rightarrow \mathbb{X}^{*}(T)$ follow by examining the specific isomorphism given in the proof.

In type $G$, the Levi $L$ has a similarly simple form.
Lemma 6.5.2. Assume that $(G, P, \mu)$ is of type $G$. Then there is an isomorphism

$$
\begin{equation*}
L \xrightarrow{\sim}\left\{(\lambda, A) \in \mathbb{G}_{m} \times G L_{2} \mid \operatorname{det} A=\lambda^{3}\right\} \tag{6.5.2}
\end{equation*}
$$

such that $\varpi_{1}$ is identified with the character $(\lambda, A) \mapsto \lambda$ and $L \cap B$ is the preimage of the Borel subgroup $\mathbb{G}_{m} \times Q_{2}^{2} \subseteq \mathbb{G}_{m} \times G L_{2}$. Moreover, the induced morphism

$$
\mathbb{X}^{*}\left(Q_{2}^{2}\right) \longrightarrow \mathbb{X}^{*}(L \cap B)=\mathbb{X}^{*}(T)
$$

sends the characters $e_{1}$ and $e_{2}$ to $\varpi_{2}$ and $3 \varpi_{1}-\varpi_{2}$ respectively.
Proof. Apply Proposition 5.3 .1 again and inspect the explicit isomorphism given in the proof to compute $\mathbb{X}^{*}\left(Q_{2}^{2}\right) \rightarrow \mathbb{X}^{*}(T)$.

The case of type $B$ is somewhat more complicated, as the Levi subgroup $L$ is not of type $A$. In what follows, we let

$$
G S p_{4}=\left\{B \in G L_{4} \mid B^{t} J B=\chi(B) J \text { for some } \chi(B) \in \mathbb{G}_{m}\right\}
$$

where

$$
J=\left(\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right)
$$

Note that $G S p_{4}$ is a reductive group with weight lattice $\mathbb{X}^{*}\left(G S p_{4} \cap Q_{4}^{4}\right)=\bigoplus_{k=1}^{4} \mathbb{Z} f_{k} / \mathbb{Z}\left(f_{1}-\right.$ $f_{2}-f_{3}+f_{4}$ ), where $f_{k}$ is the character sending a matrix to its $k$ th diagonal entry, simple roots $\beta_{1}=f_{2}-f_{3}$ and $\beta_{2}=f_{1}-f_{2}$, and simple coroots $\beta_{1}^{\vee}=f_{2}^{*}-f_{3}^{*}$ and $\beta_{2}^{\vee}=f_{1}^{*}-f_{2}^{*}+f_{3}^{*}-f_{4}^{*}$. In this description, $\chi$ is the character $\chi=f_{1}+f_{4}=f_{2}+f_{3}$.

Lemma 6.5.3. If $(G, P, \mu)$ is of type $B$, then there is an isomorphism

$$
L \xrightarrow{\sim}\left\{(A, B) \in G L_{l-2} \times G S p_{4} \mid \operatorname{det}(A)=\chi(B)\right\}
$$

such that $\varpi_{i}=\varpi_{l-2}$ is identified with the character $(A, B) \mapsto \operatorname{det}(A)=\chi(B)$ and $L \cap B$ is the preimage of the Borel subgroup $Q_{l-2}^{l-2} \times\left(G S p_{4} \cap Q_{4}^{4}\right) \subseteq G L_{l-2} \times G S p_{4}$. Moreover, the induced morphism

$$
\mathbb{X}^{*}\left(G S p_{4} \cap Q_{4}^{4}\right)=\bigoplus_{k=1}^{4} \mathbb{Z} f_{k} \longrightarrow \mathbb{X}^{*}(L \cap B)=\mathbb{X}^{*}(T)
$$

sends $f_{1}, f_{2}, f_{3}$ and $f_{4}$ to $\varpi_{l}, \varpi_{l-1}-\varpi_{l}, \varpi_{l-2}-\varpi_{l-1}+\varpi_{l}$ and $\varpi_{l-2}-\varpi_{l}$ respectively.
Proof. We describe the isomorphism at the level of root data.
Write

$$
L_{0}=\left\{(A, B) \in G L_{l-2} \times G S p_{4} \mid \operatorname{det}(A)=\chi(B)\right\} \subseteq G L_{l-2} \times G S p_{4}
$$

The root datum $\left(M, \Psi, M^{\vee}, \Psi^{\vee}\right)$ of $L_{0}$ is specified as follows. The weight lattice is

$$
M=\frac{\bigoplus_{i=1}^{l-2} \mathbb{Z} e_{i} \oplus \bigoplus_{j=1}^{4} \mathbb{Z} f_{j}}{\left\langle f_{1}-f_{2}-f_{3}+f_{4}, f_{1}+f_{4}-\sum_{i=1}^{l-2} e_{i}\right\rangle}
$$

and the coweight lattice is therefore

$$
M^{\vee}=\left\{\lambda \in \bigoplus_{i=1}^{l-2} \mathbb{Z} e_{i}^{*} \oplus \bigoplus_{j=1}^{4} \mathbb{Z} f_{j}^{*} \mid\left\langle f_{1}+f_{4}, \lambda\right\rangle=\left\langle f_{2}+f_{3}, \lambda\right\rangle=\sum_{i=1}^{l-2}\left\langle e_{i}, \lambda\right\rangle\right\}
$$

The roots $\Psi$ and coroots $\Psi^{\vee}$ and the bijection $\Psi \rightarrow \Psi^{\vee}$ are determined by requiring that

$$
\left\{\gamma_{i} \mid 1 \leq i \leq l, i \neq l-2\right\}
$$

be a set of simple roots, where

$$
\gamma_{i}=\left\{\begin{array}{ll}
e_{i}-e_{i+1}, & \text { if } i<l-2, \\
f_{2}-f_{3}, & \text { if } i=l-1, \\
f_{1}-f_{2}, & \text { if } i=l,
\end{array} \quad \text { and } \quad \gamma_{i}^{\vee}= \begin{cases}e_{i}^{*}-e_{i+1}^{*}, & \text { if } i<l-2, \\
f_{2}^{*}-f_{3}^{*}, & \text { if } i=l-1, \\
f_{1}^{*}-f_{2}^{*}+f_{3}^{*}-f_{4}^{*}, & \text { if } i=l\end{cases}\right.
$$

There is an isomorphism

$$
\phi: \mathbb{X}_{*}(T)=\bigoplus_{i=1}^{l} \mathbb{Z} \alpha_{i}^{\vee} \xrightarrow{\sim} M^{\vee}
$$

sending $\alpha_{i}^{\vee}$ to $\gamma_{i}^{\vee}$ for $i \neq l-2$, and $\alpha_{l-2}^{\vee}$ to $e_{l-2}^{*}+f_{3}^{*}+f_{4}^{*}$, such that the dual $\phi^{*}: M \rightarrow \mathbb{X}^{*}(T)$ sends $\beta_{i}$ to $\alpha_{i}$ for $i \neq l-2$. So $\phi$ defines an isomorphism of root data, which has the desired properties by inspection.

In view of Lemma 6.5.3, it will be useful to have a description of $G S p_{4}$-bundles in terms of vector bundles.

Definition 6.5.4. A conformally symplectic vector bundle is a tuple ( $W, M, \omega$ ), where $W$ is a vector bundle, $M$ is a line bundle, and $\omega: \wedge^{2} W \rightarrow M$ is a morphism such that the induced morphism $W \rightarrow W^{\vee} \otimes M$ is an isomorphism.
Lemma 6.5.5. There is an isomorphism of $\operatorname{Bun}_{G S p_{4}}$ with the relative stack of conformally symplectic vector bundles $(W, M, \omega)$ on $E$ over $S$, which identifies $\chi: \operatorname{Bun}_{G S p_{4}} \rightarrow \operatorname{Bun}_{\mathbb{G}_{m}}$ with the map $(W, M, \omega) \mapsto M$.

Proof. Let $V$ be the standard representation of $G S p_{4}$ coming from the inclusion $G S p_{4} \subseteq$ $G L_{4}$. Then $J$ defines a homomorphism of $G S p_{4}$-representations $J: \wedge^{2} V \rightarrow \mathbb{Z}_{\chi}$. The isomorphism from $\operatorname{Bun}_{G S p_{4}}$ to the stack of conformally symplectic vector bundles sends a $G S p_{4}$-bundle $\xi$ to $\left(\xi \times{ }^{G S p_{4}} V, \xi \times{ }^{G S p_{4}} \mathbb{Z}_{\chi}, \xi \times{ }^{G S p_{4}} J\right)$.

Returning to the problem of constructing slices in types $B, C$ and $D$, let $P^{\prime} \subseteq L$ be the standard parabolic of type $t\left(P^{\prime}\right)=\left\{\alpha_{l}\right\}$, and $L^{\prime} \subseteq P^{\prime}$ its standard Levi subgroup. In types $C$ and $D$, let $\rho_{L}: L \rightarrow G L_{n_{1}+1}$ be the composition of the isomorphism of Lemma 6.5.1 with the projection to the second factor (where for concreteness we choose the labelling so that $\alpha_{c_{1}, n_{1}}=\alpha_{l}$ ), and in type $B$ let $\rho_{L}: L \rightarrow G L_{4}$ be the composition of the isomorphism of Lemma 6.5.3 with the projection to the second factor and the inclusion $G S p_{4} \subseteq G L_{4}$.
Lemma 6.5.6. Assume we are in types $B, C$ or $D$. Then there is an isomorphism of $\mathrm{Bun}_{P^{\prime}}$ with the stack of pairs $\left(\xi_{L}, M \subseteq W\right)$, where $\xi_{L} \in \operatorname{Bun}_{L}$ and $M \subseteq W$ is a line subbundle of the vector bundle $W$ associated to $\xi_{L}$ under the representation $\rho_{L}$, such that the morphism

$$
\varpi_{l}: \operatorname{Bun}_{P^{\prime}} \longrightarrow \operatorname{Bun}_{\mathbb{G}_{m}}
$$

is identified with the morphism

$$
\left(\xi_{L}, M \subseteq W\right) \longmapsto \begin{cases}\varpi_{i}\left(\xi_{L}\right) \otimes M^{-1}, & \text { in types } B \text { and } D \\ \varpi_{i}\left(\xi_{L}\right)^{\otimes 2} \otimes M^{-1}, & \text { in type } C .\end{cases}
$$

In types $C$ and $D$ (resp., type $B$ ), if $\xi_{P^{\prime}}$ corresponds to $\left(\xi_{L}, M \subseteq W\right)$ and $V$ is the vector bundle induced by $\xi_{L}$ under the projection $L \rightarrow G L_{n_{0}}$ coming from Lemma 6.5.1 (resp., 6.5.3), then the $L^{\prime}$-bundle $\xi_{P^{\prime}} \times{ }^{P^{\prime}} L^{\prime}$ is semistable if and only if the vector bundles $V$ and $W / M$ (resp., $\operatorname{ker}\left(\omega: W / M \rightarrow \operatorname{det} V \otimes M^{\vee}\right)$ ) are semistable.

Proof. In types $C$ and $D$ this is clear from Lemma 6.5.1. In type $B$, using Lemma 6.5.3 we have an $L$-equivariant identification $L / P^{\prime} \cong G S p_{4} /\left(G S p_{4} \cap R_{4}\right) \cong G L_{4} / R_{4} \cong \mathbb{P}^{4}$ with the space of lines in the representation $\rho_{L}$, where we recall that

$$
R_{4}=\left\{\left(\begin{array}{cccc}
* & * & * & 0 \\
* & * & * & 0 \\
* & * & * & 0 \\
* & * & * & *
\end{array}\right)\right\} \subseteq G L_{4}
$$

The claimed isomorphism in this case now follows. To get the desired identification of the semistable bundles, notice that the Levi factor of $G S p_{4} \cap R_{4}$ is

$$
\left\{\left.\left(\begin{array}{c|cc|c}
\lambda^{-1} \operatorname{det} A & 0 & 0 & 0 \\
\hline 0 & & A & 0 \\
0 & & 0 \\
\hline 0 & 0 & 0 & \lambda
\end{array}\right) \right\rvert\, A \in G L_{2}, \lambda \in \mathbb{G}_{m}\right\} \cong G L_{2} \times \mathbb{G}_{m}
$$

so we have an isomorphism

$$
\operatorname{Bun}_{L^{\prime}} \cong \operatorname{Bun}_{G L_{n_{0}}} \times{\times \operatorname{Bun}_{\mathbb{G}_{m}}}^{\operatorname{Bun}_{G L_{2}} \times{ }_{S} \operatorname{Bun}_{\mathbb{G}_{m}},}
$$

such that the map $\operatorname{Bun}_{P^{\prime}} \rightarrow \operatorname{Bun}_{L^{\prime}}$ is identified with

$$
\left(\xi_{L}, M \subseteq W\right) \longmapsto\left(V, \operatorname{ker}\left(W / M \rightarrow \operatorname{det} V \otimes M^{\vee}\right), M\right)
$$

This now implies the claim.
Lemma 6.5.7. Let $(G, P, \mu)$ be of type $B, C$ or $D$, and assume that $\xi_{L} \rightarrow E_{s}$ is a semistable $L$-bundle of slope $\mu$ over a geometric fibre of $E \rightarrow S$. Then $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right) \geq 2$.

Proof. By Lemmas 6.5.1, 6.5.3 and 6.5.5 and Theorem 5.3.2, it suffices to show that if $W$ is a semistable vector bundle of degree -2 and rank $2 r$ (resp., $(W, M, \omega)$ is a conformally symplectic vector bundle with $W$ semistable and $\operatorname{deg} M=-1$ ), then $\operatorname{dim} \operatorname{Aut}(W) \geq 2$ (resp., $\operatorname{dim} \operatorname{Aut}(W, M, \omega) \geq 2)$.

In the first case, observe that if $U$ and $U^{\prime}$ are nonisomorphic semistable vector bundles of degree -1 and rank $r$, then $U \otimes\left(U^{\prime}\right)^{\vee}$ is a vector bundle of degree 0 with $H^{0}\left(E, U \otimes\left(U^{\prime}\right)^{\vee}\right)=$ 0 , and hence $H^{1}\left(E, U \otimes\left(U^{\prime}\right)^{\vee}\right)=0$ also. It follows that the morphism

$$
\begin{aligned}
\operatorname{Bun}_{G L_{r}}^{s s,-1} \times \operatorname{Bun}_{G L_{r}}^{s s,-1} & \longrightarrow \operatorname{Bun}_{G L_{2 r}}^{s s,-2} \\
\left(U, U^{\prime}\right) & \longmapsto U \oplus U^{\prime}
\end{aligned}
$$

is étale at $\left(U, U^{\prime}\right)$ if $U \not \approx U^{\prime}$. Since the locus of vector bundles $W$ in $\operatorname{Bun}_{G L_{2 d}}^{s s,-2}$ with $\operatorname{dim} \operatorname{Aut}(W)<2$ is open, it is either empty or dense. So by openness of étale morphisms, if it is nonempty, then there exists such a bundle $W=U \oplus U^{\prime}$ with $U \not \approx U^{\prime}$. $\operatorname{But} \operatorname{Aut}(W)=\operatorname{Aut}(U) \times \operatorname{Aut}\left(U^{\prime}\right)=\mathbb{G}_{m} \times \mathbb{G}_{m}$ for such bundles, so this is a contradiction and we are done in this case.

The proof for conformally symplectic bundles is similar. Consider the Levi subgroup

$$
G L_{2} \times \mathbb{G}_{m} \cong L^{\prime \prime}=\left\{\left.\left(\begin{array}{c|c}
\lambda J_{0}\left(A^{t}\right)^{-1} J_{0} & 0 \\
\hline 0 & \lambda A
\end{array}\right) \right\rvert\, A \in G L_{2}, \lambda \in \mathbb{G}_{m}\right\},
$$

where

$$
J_{0}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

Given $(U, M) \in \operatorname{Bun}_{G L_{2}}^{s s,-1} \times{ }_{S} \operatorname{Bun}_{\mathbb{G}_{m}}^{-1}$ corresponding to an $L^{\prime \prime}$-bundle $\xi_{L^{\prime \prime}}$, with $U \not \not 二 U^{\vee} \otimes M$, we have that
is a degree 0 vector bundle on $E_{s}$ with $H^{0}\left(E_{s}, \xi_{L^{\prime \prime}} \times L^{L^{\prime \prime}} \mathfrak{g s p}_{4} / \mathfrak{l}^{\prime \prime}\right)=0$ and hence $H^{1}\left(E_{s}, \xi_{L^{\prime \prime}} \times{ }^{L^{\prime \prime}}\right.$ $\left.\mathfrak{g s p}_{4} / \mathfrak{l}^{\prime \prime}\right)=0$ also, where $\mathfrak{g s p}_{4}=\operatorname{Lie}\left(G S p_{4}\right)$ and $\mathfrak{l}^{\prime \prime}=\operatorname{Lie}\left(L^{\prime \prime}\right)$. So we conclude that the morphism

$$
\operatorname{Bun}_{L^{\prime \prime}} \longrightarrow \operatorname{Bun}_{G S p_{4}}^{-1}
$$

is étale at $(U, M)$.
Since the locus of conformally symplectic vector bundles $(W, M, \omega)$ in $\operatorname{Bun}_{G S p_{4}}^{s s,-1}$ with automorphism group of dimension $<1$ is open, it is either empty or dense. If it is nonempty, then by openness of étale morphisms we can find such a bundle of the form $W=U \oplus U^{\vee} \otimes M$ as above. But $\operatorname{dim} \operatorname{Aut}_{G S p_{4}}(W)=\operatorname{dim} \operatorname{Aut}(U) \times \operatorname{dim} \operatorname{Aut}(M)=2$, so this is a contradiction and the lemma is proved.

Proof of Theorem 6.1 .5 in types $B, C$ and $D$. Let $\mu^{\prime} \in \mathbb{X}_{*}\left(Z\left(L^{\prime}\right)\right)_{\mathbb{Q}}$ be the unique vector with $\left\langle\varpi_{i}, \mu^{\prime}\right\rangle=-1$ and $\left\langle\varpi_{l}, \mu^{\prime}\right\rangle=0$. Then Theorem 5.3.5 shows that the morphism

$$
\left(\varpi_{i}, \varpi_{l}\right): \operatorname{Bun}_{L^{\prime}, r i g}^{s s, \mu^{\prime}} \longrightarrow \operatorname{Pic}_{S}^{-1}(E) \times_{S} \operatorname{Pic}_{S}^{0}(E)
$$

is a $Z\left(L^{\prime}\right)_{\text {rig }}$-gerbe. Let $\mathfrak{G}^{\prime \prime}$ be the $Z\left(L^{\prime}\right)_{\text {rig }}$-gerbe on $S$ given by pulling back along the section

$$
\left(\mathcal{O}\left(-O_{E}\right), \mathcal{O}\right): S \rightarrow \operatorname{Pic}_{S}^{-1}(E) \times_{S} \operatorname{Pic}_{S}^{0}(E)
$$

The pullback of the theta bundle gives a $\mathbb{B} Z\left(L^{\prime}\right)_{\text {rig }}$-equivariant morphism $\mathfrak{G}^{\prime \prime} \rightarrow \mathbb{B} \mathbb{G}_{m}$ where $\mathbb{B} Z\left(L^{\prime}\right)_{\text {rig }}$ acts through the homomorphism $\left(-\mu^{\prime} \mid-\right)$, so we get a $\operatorname{ker}\left(\mu^{\prime} \mid-\right)$-gerbe $\mathfrak{G}^{\prime}=\mathfrak{G}^{\prime \prime} \times_{\mathbb{B G}_{m}}$ Spec $\mathbb{Z}$. Let $\mathfrak{G}$ be the rigidification of $\mathfrak{G}^{\prime}$ with respect to $\varpi_{l}^{\vee}\left(\mathbb{G}_{m}\right)$. Then $\mathfrak{G}$ is a $\operatorname{ker}\left(\mu^{\prime} \mid-\right) / \varpi_{l}^{\vee}\left(\mathbb{G}_{m}\right) \cong \mu_{d}$-gerbe, pulled back from a gerbe $\mathfrak{G}^{u n i}$ on $M_{1,1}$, and if it is trivial then we have a $\mathbb{B} \mathbb{G}_{m}$-equivariant morphism $\mathbb{B}_{S} \mathbb{G}_{m} \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}^{s s, \mu^{\prime}}$ (with $\mathbb{B} \mathbb{G}_{m}$ acting through $\left.\varpi_{l}^{\vee}\right)$ lifting the section $\left(\mathcal{O}\left(-O_{E}\right), \mathcal{O}\right)$ such that the pullback of the theta bundle is trivial. Define

$$
Z_{0}=\operatorname{Ind}_{L^{\prime}}^{L}\left(\mathbb{B}_{S} \mathbb{G}_{m}\right) \backslash \mathbb{B}_{S} \mathbb{G}_{m} \longrightarrow \operatorname{Bun}_{L, r i g}^{\mu}
$$

and observe that the pullback of $\Theta_{\operatorname{Bun}_{G, r i g}}$ to $Z_{0}$ is also trivial since $Z_{0} \rightarrow \mathbb{B}_{S} \mathbb{G}_{m}$ is an affine space bundle.

| Type | $\alpha \in \Phi_{L}$ with $\left\langle\alpha, \mu^{\prime}\right\rangle<0$ | $\left\langle\alpha, \mu^{\prime}\right\rangle$ | $\left\langle\alpha, \varpi_{l}^{\vee}\right\rangle$ |
| :---: | :---: | :---: | :---: |
| $B$ | $-\alpha_{l}$ | $-\frac{1}{2}$ | -1 |
|  | $-\alpha_{l-1}-\alpha_{l}$ | $-\frac{1}{2}$ | -1 |
|  | $-\alpha_{l-1}-2 \alpha_{l}$ | -1 | -2 |
| $C$ | $-\alpha_{l}$ | -2 | -1 |
| $D$ | $-\alpha_{l}$ | $-\frac{2}{3}$ | -1 |
|  | $-\alpha_{l-2}-\alpha_{l}$ | $-\frac{2}{3}$ | -1 |
|  | $-\alpha_{l-2}-\alpha_{l-1}-\alpha_{l}$ | $-\frac{2}{3}$ | -1 |

Table 6.2: Roots of $L$ with $\left\langle\alpha, \mu^{\prime}\right\rangle<0$
We now check that $Z_{0}$ satisfies the conditions of Theorem 6.1.5. Since the claims are local on $S$, we can assume for convenience that the section $\mathbb{B}_{S} \mathbb{G}_{m} \rightarrow \operatorname{Bun}_{L^{\prime}, \text { rig }}^{s s, \mu^{\prime}}$ lifts to a morphism $S \rightarrow \operatorname{Bun}_{L^{\prime}}^{s s, \mu^{\prime}}$ and that the line bundle on $E$ associated to this section via the character $\varpi_{l}$ is trivial. Note that in this case, we have a natural identification

$$
Z_{0} \cong\left(\operatorname{Ind}_{L^{\prime}}^{L}(S) \backslash S\right) / \mathbb{G}_{m}
$$

First, the roots $\alpha \in \Phi_{L}$ with $\left\langle\alpha, \mu^{\prime}\right\rangle<0$ are given in Table 6.2, along with the values of $\left\langle\alpha, \mu^{\prime}\right\rangle$ and $\left\langle\alpha, \varpi_{l}^{\vee}\right\rangle$. Using Propositions 5.2.6 and 5.2.7, it follows that $\operatorname{Ind}_{L^{\prime}}^{L}(S) \rightarrow S$ is an $\mathbb{A}^{2}$-bundle on which $\mathbb{G}_{m}$ acts with weight 1 in types $C$ and $D$, and weights 1 and 2 in type $B$. So $Z_{0} \rightarrow S$ is a $\mathbb{P}(1,2)$-bundle in type $B$ and a $\mathbb{P}^{1}$-bundle in types $C$ and $D$. In particular, (1) is satisfied.

We next show that $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{\mu}$ factors through $\operatorname{Bun}_{L, r i g}^{s s, \mu}$. Note that Table 6.2 shows that $-\mu^{\prime}$ is a Harder-Narasimhan vector for $P^{\prime} \subseteq L$, so $\operatorname{Ind}_{L^{\prime}}^{L}(S)=\operatorname{Bun}_{P^{\prime}, r i g}^{s s, \mu^{\prime}} \times_{\operatorname{Bun}_{L^{\prime}, \text { rig }}^{s s, \mu^{\prime}}} S$. So Lemma 6.5.6 shows that $\xi_{L}$ is in the image of $\operatorname{Ind}_{L^{\prime}}^{L}(S)$ if and only if $V$ is semistable of determinant $\mathcal{O}\left(-O_{E}\right)$ and there exists a nonvanishing section of $W \otimes \mathcal{O}\left(d O_{E}\right)$ such that the vector bundle

$$
U= \begin{cases}W / \mathcal{O}\left(-d O_{E}\right), & \text { in types } C \text { and } D \\ \operatorname{ker}\left(W / \mathcal{O}\left(-\left(d O_{E}\right) \rightarrow \mathcal{O}\right),\right. & \text { in type } B\end{cases}
$$

is semistable. Here $V$ and $W$ are as in the statement of Lemma 6.5.6, and

$$
d= \begin{cases}1, & \text { in types } B \text { and } D \\ 2, & \text { in type } C\end{cases}
$$

is as in the statement of the theorem. The bundle $\xi_{L}$ is in the image of $\operatorname{Ind}_{L^{\prime}}^{L}(S) \backslash S$ if and only if $\mathcal{O}\left(-d O_{E}\right) \rightarrow W$ can be chosen not to admit a retraction. Suppose that $\xi_{L}$ is such a bundle and that $\xi_{L}$ is unstable; we deduce a contradiction in each type.

In type $B, W$ is an unstable conformally symplectic vector bundle of rank 4 and degree -2 , so there exists a quotient $W \rightarrow N$ where $N$ has slope $<-1 / 2$. Replacing $N$ with $\operatorname{coker}\left(N^{\vee} \otimes \mathcal{O}\left(-O_{E}\right) \rightarrow W\right)$ if necessary, we can assume that $N$ has rank $\leq 2$. Since any vector bundle of rank 2 and slope $<-1 / 2$ surjects onto some line bundle of negative degree, we can therefore assume without loss of generality that $N$ is a line bundle. Examining slopes, we see from semistability of $U$ that $W \rightarrow N$ does not factor through $W / \mathcal{O}\left(-O_{E}\right)$, and hence that $\mathcal{O}\left(-O_{E}\right) \rightarrow N$ is nonzero. So $\mathcal{O}\left(-O_{E}\right) \rightarrow N$ must be an isomorphism since $\operatorname{deg} N \leq \operatorname{deg} \mathcal{O}\left(-O_{E}\right)$, and we therefore have a retraction $W \rightarrow \mathcal{O}\left(-O_{E}\right)=N$. Since this is a contradiction, we are done in this case.

In type $C, W$ is an unstable vector bundle of rank 2 and degree -2 , so there exists a quotient $W \rightarrow N$ where $N$ is a a line bundle of degree $<-1$. Examining slopes, we see that $W \rightarrow N$ does not factor through $W / \mathcal{O}\left(-2 O_{E}\right)$ and hence that $\mathcal{O}\left(-2 O_{E}\right) \rightarrow N$ is nonzero. So $\mathcal{O}\left(-2 O_{E}\right) \rightarrow N$ must be an isomorphism since $\operatorname{deg} N \leq \operatorname{deg} \mathcal{O}\left(-2 O_{E}\right)$, and we therefore have a retraction $W \rightarrow \mathcal{O}\left(-2 O_{E}\right)=N$. Since this is a contradiction, we are done in this case as well.

Finally, in type $D, W$ is an unstable vector bundle of rank 4 and degree -2 , so there exists a quotient $W \rightarrow N$ where $N$ is a semistable vector bundle of slope $<-1 / 2$. Examining slopes and using semistability of $W / \mathcal{O}\left(-O_{E}\right)$ and of $N$, we see that $W \rightarrow N$ does not factor through $W / \mathcal{O}\left(-O_{E}\right)$ and we again get a retraction $W \rightarrow N \cong \mathcal{O}\left(-O_{E}\right)$. So we have shown that $\xi_{L}$ must be semistable in all cases.

We next show that the morphism $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ is smooth with connected fibres, which proves (2) and that $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$ is a $\Theta$-trivial slice. Write $\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$ for the fibre of $\varpi_{i}: \operatorname{Bun}_{L}^{s s, \mu} \rightarrow \operatorname{Pic}_{S}^{-1}(E)$ over $\mathcal{O}\left(-O_{E}\right)$ and $\left(\operatorname{Bun}_{P^{\prime}}^{\mu^{\prime}}\right)_{0}^{s s}=\operatorname{Bun}_{P^{\prime}}^{\mu^{\prime}} \times \operatorname{Bun}_{L}^{\mu}\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$. Then Lemma 6.5.6 gives an open immersion

$$
\left(\operatorname{Bun}_{P^{\prime}}^{\mu^{\prime}}\right)_{0}^{s s} \subseteq \mathbb{P}_{\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}} \pi_{*}\left(W^{u n i} \otimes \mathcal{O}\left(d O_{E}\right)\right)
$$

where we write $W^{u n i}$ for the universal bundle on $\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0} \times_{S} E$ induced by the representation $\rho_{L}$ of $L$ and $\pi:\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0} \times_{S} E \rightarrow\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$ for the natural projection. Moreover,

$$
Z_{0} \times{ }_{\left(\operatorname{Bun}_{L, r i g}^{s s, \mu}\right)_{0}}\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0} \longrightarrow\left(\operatorname{Bun}_{P^{\prime}}^{\mu^{\prime}}\right)_{0}^{s s}
$$

is a $\mathbb{G}_{m}=Z(L)_{\text {rig }} / \varpi_{l}^{\vee}\left(\mathbb{G}_{m}\right)$-torsor over the open substack where the associated $L^{\prime}$-bundle is semistable. This shows in particular that $Z_{0} \times{ }_{\left(\operatorname{Bun}_{L, r i g}^{s s, \mu}\right)_{0}}\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0} \rightarrow\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$ is smooth
with connected fibres of dimension 2 , and hence that the same is true for $Z_{0} \rightarrow\left(\operatorname{Bun}_{L, \text { rig }}^{s s, \mu}\right)_{0} \cong$ $\operatorname{Bun}_{L, \text { rig }}^{s s, \mu} / E$ as claimed.

To prove (3), first observe that since $Z_{0} \rightarrow S$ has finite relative stabilisers, any $L$-bundle in the image of $Z_{0} \rightarrow\left(\operatorname{Bun}_{L, r i g}^{s s, \mu}\right)_{0} \subseteq \operatorname{Bun}_{L, r i g}^{s s, \mu}$ can have automorphism group of dimension at most 2 , and is hence regular by Lemma 6.5.7. For the converse, note that since every regular semistable $L$-bundle is a translate of one in $\left(\mathrm{Bun}_{L}^{s s, \mu}\right)_{0}$, it suffices to show that any regular semistable bundle in $\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$ is in the image of $\left(\operatorname{Bun}_{P^{\prime}}^{s s, \mu}\right)_{0} \rightarrow \operatorname{Bun}_{L}^{\mu}$, and hence in the image of $Z_{0} \rightarrow \operatorname{Bun}_{L, \text { rig }}^{s s, \mu}$.

Suppose then that $\xi_{L} \rightarrow E_{s}$ is a semistable $L$-bundle in $\left(\operatorname{Bun}_{L}^{s s, \mu}\right)_{0}$ over $s: \operatorname{Spec} k \rightarrow S$ that is not in the image of $\left(\operatorname{Bun}_{P \prime}^{s s, \mu^{\prime}}\right)_{0}$. We show in each type that $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)>2$ so $\xi_{L}$ is not regular.

In type $B$, in the notation of Lemma 6.5.6, we have that for every nonzero morphism $\gamma: \mathcal{O}\left(-O_{E}\right) \rightarrow W$, the vector bundle $U_{\gamma}=\operatorname{ker}\left(W / \mathcal{O}\left(-O_{E}\right) \rightarrow \mathcal{O}\right)$ is unstable. (Note that $W$ is semistable of rank 4 and degree -2 , so any such morphism is a subbundle.) Using semistability of $W$, the Harder-Narasimhan decomposition of $U_{\gamma}$ must be of the form $U_{\gamma}=N_{\gamma} \oplus N_{\gamma}^{\vee} \otimes \mathcal{O}\left(-O_{E}\right)$, where $N_{\gamma}$ is a line bundle of degree 0 on $E_{s}$ and the preimage of $N_{\gamma}$ in $W$ is the unique non-split extension $N_{\gamma}^{\prime}$ of $N_{\gamma}$ by $\mathcal{O}\left(-O_{E}\right)$. By Proposition 2.6.5 it follows that we have a morphism $\mathbb{P}_{k}^{1}=\mathbb{P} H^{0}\left(E_{s}, W \otimes \mathcal{O}\left(O_{E}\right)\right) \rightarrow \operatorname{Pic}^{0}\left(E_{s}\right)$ sending $\gamma$ to the isomorphism class of $N_{\gamma}$. Since there are no non-constant morphisms from $\mathbb{P}_{k}^{1}$ to any elliptic curve over $k$, we deduce that $N_{\gamma}=N$ and $N_{\gamma}^{\prime}=N^{\prime}$ are independent of $\gamma$. So every nonzero morphism $\mathcal{O}\left(-O_{E}\right) \rightarrow W$ factors through a Lagrangian subbundle $N^{\prime} \subseteq W$. Choosing any such morohism gives an exact sequence

$$
0 \longrightarrow N^{\prime} \longrightarrow W \longrightarrow\left(N^{\prime}\right)^{\vee} \otimes \mathcal{O}\left(-O_{E}\right) \longrightarrow 0
$$

Since $\operatorname{dim} \operatorname{Hom}\left(\mathcal{O}\left(-O_{E}\right), N^{\prime}\right)=1$ and $\operatorname{dim} \operatorname{Hom}\left(\mathcal{O}\left(-O_{E}\right), W\right)=2$, we can choose another homomorphism $\mathcal{O}\left(-O_{E}\right)$ not factoring through the given $N^{\prime}$, and hence get another Lagrangian morphism $N^{\prime} \hookrightarrow W$, which must map $N^{\prime}$ isomorphically onto $\left(N^{\prime}\right)^{\vee} \otimes \mathcal{O}\left(-O_{E}\right)$. So the above exact sequence splits, and we have

$$
W \cong N^{\prime} \oplus N^{\prime}
$$

where both summands are Lagrangian. In particular, $W$ and hence $\xi_{L}$ carries a faithful action of $S p_{2}$, so $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)>2$ as claimed.

In type $C$, we have that every nonzero morphism $\gamma: \mathcal{O}\left(-2 O_{E}\right) \rightarrow W$ must vanish at some unique point $x_{\gamma} \in E_{s}$. So again we have a morphism $\mathbb{P}_{k}^{1}=\mathbb{P} H^{0}\left(E_{s}, W \otimes \mathcal{O}\left(-2 O_{E}\right)\right) \rightarrow E_{s}$ sending $\gamma$ to $x_{\gamma}$, which must be constant. So $x_{\gamma}=x$ is independent of $\gamma$, and every morphism $\mathcal{O}\left(-2 O_{E}\right) \rightarrow W$ therefore factors through a subbundle $\mathcal{O}\left(x-2 O_{E}\right) \subseteq W$. Since $W$ is semistable of trivial determinant, choosing any two linearly independent morphisms gives an isomorphism $W \cong \mathcal{O}\left(x-2 O_{E}\right) \oplus \mathcal{O}\left(x-2 O_{E}\right)$. So $S L_{2}$ acts faithfully on $W$ and hence on $\xi_{L}$ and $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)>2$ as claimed.

In type $D$, we have that $U_{\gamma}=W / \mathcal{O}\left(-O_{E}\right)$ is unstable for every nonzero morphism $\gamma: \mathcal{O}\left(-O_{E}\right) \rightarrow W$. (Note that again any such $\gamma$ must be a subbundle since $W$ is semistable of slope $-1 / 2$.) Since $W$ is semistable, one sees that the Harder-Narasimhan decomposition of $U_{\gamma}$ must be of the form $U_{\gamma}=N_{\gamma} \oplus \operatorname{det}\left(N_{\gamma}\right)^{\vee} \otimes \mathcal{O}(-O E)$, where $N_{\gamma}$ is a rank 2 semistable vector bundle of degree -1 . Again by Proposition 2.6.5, we get a morphism $\mathbb{P}_{k}^{1}=\mathbb{P} H^{0}\left(E_{s}, W \otimes \mathcal{O}\left(O_{E}\right)\right) \rightarrow \operatorname{Pic}^{-1}\left(E_{s}\right)$ sending $\gamma$ to the isomorphism class of $\operatorname{det}\left(N_{\gamma}\right)$, which again must be constant. So $\operatorname{det}\left(N_{\gamma}\right)$, and hence $N_{\gamma}=N$ are independent of $\gamma$, and every nonzero morphism $\mathcal{O}\left(-O_{E}\right) \rightarrow W$ factors through the kernel of some surjection
$W \rightarrow N$. Choosing two linearly independent morphisms $\mathcal{O}\left(-O_{E}\right) \rightarrow W$ therefore gives a map $W \rightarrow N \oplus N$, which one easily sees must be an isomorphism. So again $S L_{2}$ acts faithfully on $W$ fixing the determinant, and hence on $\xi_{L}$, which proves that $\operatorname{dim} \operatorname{Aut}\left(\xi_{L}\right)>2$ in this case as well.

Finally, to prove (4), simply note that Proposition 5.2.8 implies that $Z \rightarrow Z_{0}$ is an affine space bundle of relative dimension $-\langle 2 \rho, \mu\rangle=l+2$, so $Z \rightarrow S$ has relative dimension $l+3$ as required.

### 6.6 Computing the divisor $D_{\alpha_{i}^{\vee}}(Z)$

In this section, we give case-by-case computations of the divisors $D_{\alpha_{i}^{\vee}}(Z)$ for the slices $Z=\operatorname{Ind}_{L}^{G}\left(Z_{0}\right)$ constructed in the previous section. We summarise the results of these calculations in Theorem 6.6 .1 below.

For the statement of the theorem, we let

$$
N= \begin{cases}n_{1}+1, & \text { in type } A \\ n_{1}-1, & \text { in type } F \\ n_{1}, & \text { otherwise }\end{cases}
$$

We let $\theta_{N}^{\prime}: Y \rightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$ be the section $\theta_{N}^{\prime}(y)=(y, 0)$, and for $1 \leq k<N$, we let $\theta_{k}^{\prime}: Y \rightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$ be the section given in type $A$ by
$\theta_{k}^{\prime}(y)= \begin{cases}\left(y,-\varpi_{i}(y)+\varpi_{i+1}(y)+\varpi_{l}(y)\right), & \text { if } k=1, \\ \left(y,-\varpi_{i}(y)+\varpi_{i+1}(y)+\varpi_{l-k+1}(y)-\varpi_{l-k+2}(y)\right), & \text { if } 1<k \leq l-i+1=N-1,\end{cases}$
and in types $B, D$ and $E$ by

$$
\theta_{k}^{\prime}(y)= \begin{cases}\left(y, \alpha_{l-1}(y)\right), & \text { in type } B \\ \left(y, \alpha_{l-2}(y)+\cdots+\alpha_{l-k}(y)\right), & \text { in type } D \\ \left(y, \alpha_{k}(y)+\alpha_{k+1}(y)+\cdots+\alpha_{3}(y)\right), & \text { in type } E\end{cases}
$$

(Note that $N=1$ in types $C, F$ and $G$.)
Theorem 6.6.1. Assume that $(G, P, \mu)$ is not of type $A_{1}$. Then we have the following.
(1) There is a sequence of $N$ morphisms

$$
D_{\alpha_{i}^{\vee}}(Z)=D_{N+1}^{\prime} \longrightarrow D_{N}^{\prime} \longrightarrow \cdots \longrightarrow D_{1}^{\prime}
$$

over $Y \times{ }_{S} Z$ such that $D_{1}^{\prime}$ is a family of smooth surfaces over $Y$ containing $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)$ as a closed substack, and $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ is the blowup along the section $\theta_{k}^{\prime}: Y \rightarrow Y \times_{S}$ $\operatorname{Pic}_{S}^{0}(E) \subseteq D_{k}^{\prime}$ of the proper transform of $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E) \subseteq D_{1}^{\prime}$.
(2) The intersection $D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)$ is the exceptional divisor of the final blowup.
(3) The intersection $D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z)$ is the proper transform of $Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$, and the identification of $D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z)$ with $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)$ given here agrees with the identification given by Theorem 6.1.9.
(4) The stack $D_{1}^{\prime}$ is a line bundle over $Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$ in type $A$, and the fibres of $D_{1}^{\prime} \rightarrow Y$ are as given in Proposition 6.6.8 in the other types.

Remark 6.6.2. Using the methods of [GSB, Corollary 6.29], one can use Theorems 6.1.9 and 6.6 .1 to deduce descriptions of all fibres of $\tilde{\chi}_{Z}: \tilde{Z} \rightarrow \Theta_{Y}^{-1}$ in types $A, B, D$ and $E$ as follows. Assume for simplicity that $S=\operatorname{Spec} k$, let $y \in Y$ be any $k$-point with fibre $\mathbb{A}_{k}^{1} \subseteq \Theta_{Y}^{-1}$, and let $\tilde{X}=\tilde{\chi}_{Z}^{-1}\left(\mathbb{A}_{k}^{1}\right)$. By Theorem 6.1.9 and Theorem 4.6.1, we have

$$
\tilde{X}_{0}=D_{\alpha_{i}^{\vee}}(Z)_{y}+D_{\alpha_{j}^{\vee}}(Z)_{y}+D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \quad \text { and } \quad K_{\tilde{Z}}=\psi_{Z}^{*} M \otimes \mathcal{O}\left(-D_{\alpha_{i}^{\vee}}(Z)-D_{\alpha_{j}^{\vee}}(Z)\right),
$$

for some line bundle $M$ on $Z$. Writing $\beta=D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \cap D_{\alpha_{j}^{\vee}}(Z)_{y}$, we deduce that $K_{\tilde{X}} \cdot \beta=-1$. Since $\beta$ is a ruling of the divisor $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \cong \mathbb{P}^{1} \times \mathbb{P}^{1}$, which is contracted to a point in $Z$, it follows that there is a morphism $\tilde{X} \rightarrow \tilde{X}^{+}$over $Z \times \mathbb{A}^{1}$ contracting the ruling $\beta$ of $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y}$, where $\tilde{X}^{+}$is again smooth. We can then flop the strict transforms of the exceptional divisors of $\left(D_{k+1}\right)_{y} \rightarrow\left(D_{k}\right)_{y}$ in sequence to produce a rational map $\tilde{X}^{+} \xrightarrow{-} \tilde{X}^{-}$over $Z \times \mathbb{A}^{1}$, such that the fibre of $\tilde{X}^{-}$over $0 \in \mathbb{A}^{1}$ is now a normal crossings divisor

$$
\tilde{X}_{0}^{-}=\left(D_{1}\right)_{y}+\left(D_{N+n_{0}}^{\prime}\right)_{y},
$$

where $\left(D_{N+n_{0}}^{\prime}\right)_{y}$ is the iterated blowup of $\left(D_{N}^{\prime}\right)_{y}$ at the points $\theta_{n_{0}}(y), \theta_{n_{0}-1}(y), \ldots, \theta_{1}(y) \in$ $\{y\} \times \operatorname{Pic}^{0}(E) \subseteq\left(D_{N}^{\prime}\right)_{y}$.

The map $\tilde{X} \rightarrow \mathbb{A}_{k}^{1}$ is $\mathbb{G}_{m}$-equivariant, where $\mathbb{G}_{m}=-\varpi_{\tilde{i}}^{\vee}\left(\mathbb{G}_{m}\right)$ acts on $\mathbb{A}_{k}^{1}$ with weight 1 by Proposition 5.2.10, and hence the same is true for $\tilde{X}^{-} \rightarrow \mathbb{A}_{k}^{1}$. Moreover, the action on $\tilde{X}^{-}$is trivial on the preimage $\left(D_{N+n_{0}}^{\prime}\right)_{y}$ of 0 in $Z$, so Lemma 5.4 .14 implies that $\tilde{X}^{-}$is a line bundle on $\left(D_{N+n_{0}}^{\prime}\right)_{y}$, and the map $\tilde{X}^{-} \rightarrow \mathbb{A}_{k}^{1}$ is given by a section of the dual vanishing along $\{y\} \times \operatorname{Pic}^{0}(E)$. So the fibres of $\tilde{X}^{-}$over nonzero points in $\mathbb{A}^{1}$ are isomorphic to the complement of $\{y\} \times \operatorname{Pic}^{0}(E)$ in $\left(D_{N+n_{0}}^{\prime}\right)_{y}$. Since $\tilde{X} \rightarrow \tilde{X}^{-}$is an isomorphism outside $\tilde{X}_{0}$, these are exactly fibres of $\tilde{\chi}_{Z}$ over nonzero points in $\Theta_{Y}^{-1}$ over $y \in Y$.

In type $A$, since $\alpha_{i}$ and $\alpha_{j}=\alpha_{i+1}$ play completely symmetric roles, we can use Theorem 6.1.9 to deduce a Theorem 6.6.1.

Proof of Theorem 6.6.1 in type A. Applying Theorem 6.1.9 with the vertices of the Dynkin diagram $A_{l}$ labelled in reverse order gives contractions

$$
D_{\alpha_{i}^{\vee}}(Z)=D_{l-i+2}^{\prime} \longrightarrow D_{l-i+1}^{\prime} \longrightarrow \cdots \longrightarrow D_{1}^{\prime}
$$

with the desired properties (1), (2) and (4), where to get the correct blowup loci we have composed the identification of $D_{1}^{\prime}$ with a line bundle over $Y \times_{S} \operatorname{Pic}_{S}^{0}(E)$ given by Theorem 6.1.9 with the isomorphism

$$
\begin{aligned}
Y \times_{S} \operatorname{Pic}_{S}^{0}(E) & \xrightarrow{\sim} Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \\
(y, x) & \longmapsto(y,-x) .
\end{aligned}
$$

The two identifications of the intersection

$$
D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{i+1}^{\vee}}(Z) \cong M_{0,2, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}-\alpha_{i+1}^{\vee}, 1\right)\right),
$$

with $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)$ are both given by taking the difference between the images in $E$ of the two marked points, so they agree, proving (3).

From now on, we will assume that $(G, P, \mu)$ is not of type $A$. In this case, we have the following description of $D_{\alpha_{i}^{\vee}}(Z)$ as a space of stable maps, from which we can construct the first blow down just as for $D_{\alpha_{j}^{\vee}}(Z)$.

Lemma 6.6.3. The natural morphism
$M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /(L \cap B),\left(-\alpha_{i}^{\vee}, 1\right)\right) \longrightarrow M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right)$
is an isomorphism.
Proof. Observe that since $\alpha_{i}$ is not a special root, Lemma 5.4.7 implies that the closed immersion

$$
\begin{equation*}
M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right) \longleftrightarrow M_{1,1, Z}\left(\xi_{G / Z(G)} \times{ }^{G / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right) \tag{6.6.2}
\end{equation*}
$$

is an isomorphism, since the right hand side is smooth over $S$ hence reduced, and for $z \in$ $Z \backslash Z_{0}, \xi_{G, z}$ is either semistable or regular unstable, and hence $\xi_{G, z} / B$ has no sections of the given degree. But Lemma 6.6.4 below and Proposition 3.7.6 imply that every stable map in the left hand side of (6.6.2) factors through $\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /(L \cap B)$, so the closed immersion
$M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /(L \cap B),\left(-\alpha_{i}^{\vee}, 1\right)\right) \longleftrightarrow M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} G / B,\left(-\alpha_{i}^{\vee}, 1\right)\right)$ is also an isomorphism, and we are done.

Lemma 6.6.4. Assume $(G, P, \mu)$ is not of type $A, w \in W_{P, B}^{0}, \lambda \leq-\alpha_{i}^{\vee}$ and $C^{w, \lambda}\left(Z_{0}\right) \neq \emptyset$. Then $w=1$ and $\lambda \in\left\{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}\right\}$.

Proof. From the proof of Lemma 6.3.5, we have either $w \lambda=-\alpha_{i}^{\vee}$ and $w=1$, or $w \lambda=$ $-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}$ and

$$
w \in\{1\} \cup\left\{s_{c_{0}, n_{0}} s_{c_{0}, n_{0}-1} \cdots s_{c_{0}, k} \mid 1 \leq k \leq n_{0}\right\}
$$

If $w \neq 1$, then this implies that $\lambda=-w^{-1}\left(\alpha_{i}^{\vee}+\alpha_{j}^{\vee}\right)=-\alpha_{j}^{\vee}$, contradicting $\lambda \leq-\alpha_{i}^{\vee}$. So this proves the lemma.

Lemma 6.6.5. The projection $L /(L \cap B) \rightarrow L /\left(L \cap P_{1}\right)$ and the isomorphisms of Lemmas 6.3.3 and 6.6.3 induce an isomorphism

$$
D_{\alpha_{i}^{\vee}}(Z) \xrightarrow{\sim} Y \times_{Y_{L \cap P_{1}}} M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right),
$$

where $P_{1} \subseteq G$ is the parabolic subgroup of type $t\left(P_{1}\right)=\Delta \backslash\left\{\alpha_{c_{0}, 1}, \ldots, \alpha_{c_{0}, n_{0}}\right\}$.
Proof. First notice that the flag variety $L /(L \cap B)$ splits as a product

$$
L /(L \cap B) \cong G L_{n_{0}} / Q_{n_{0}}^{n_{0}} \times L /\left(L \cap P_{1}\right)
$$

and that $-\alpha_{i}^{\vee}$ restricts to the cocharacter $-e_{n_{0}}^{*}$ on the Borel $Q_{n_{0}}^{n_{0}} \subseteq G L_{n_{0}}$. Since every stable section of $\xi_{L / Z(G)} \times{ }^{L / Z(G)} G L_{n_{0}} / Q_{n_{0}}^{n_{0}}$ of degree $-e_{n_{0}}^{*}$ is a genuine section, it follows from Lemmas 5.4.10, 6.3.3 and 6.6.3 that there are isomorphisms

$$
\begin{aligned}
D_{\alpha_{i}^{\vee}}(Z) & \xrightarrow{\sim} M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /(L \cap B),\left(-\alpha_{i}^{\vee}, 1\right)\right) \\
& \xrightarrow{\sim} M_{1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} G L_{n_{0}} / Q_{n_{0}}^{n_{0}},\left(-e_{n}^{*}, 1\right)\right) \times Z_{0} \\
& M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right) \\
& \xrightarrow{\sim} Y_{Q_{n_{0}}^{n_{0}}}^{-e_{0}^{*}} \times \times_{P_{P_{i}^{-1}(E)}^{-1}} M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right) \\
& \xrightarrow{\sim} Y^{-\alpha_{i}^{\vee}} \times{ }_{Y_{L \cap P_{1}}^{-\alpha_{i}^{\vee}}} M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right), \\
& \xrightarrow{\sim} Y \times_{Y_{L \cap P_{1}}} M_{1,1, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right),
\end{aligned}
$$

which proves the lemma.

We can now construct the first blow down of $D_{\alpha_{i}^{\vee}}(Z)$ as follows. Observe that we have a natural morphism

$$
\begin{aligned}
D_{\alpha_{i}^{\vee}}(Z) \xrightarrow{\left(\mathrm{Bl}_{B}, f, g\right)} D_{N}^{\prime} & =Y \times_{Y_{L \cap P_{1}}}\left(M_{1,0, Z_{0}}\left(\xi_{L / Z(G)} \times{ }^{L / Z(G)} L /\left(L \cap P_{1}\right),\left(-\alpha_{i}^{\vee}, 1\right)\right) \times{ }_{S} E\right) \\
& =Y \times_{Y_{L \cap P_{1}}}\left(\mathrm{KM}_{L \cap P_{1}, L, r i g}^{-\alpha_{i}^{\vee}} \times{ }_{\text {Bun }_{L, r i g}^{\mu}} Z_{0} \times{ }_{S} E\right)
\end{aligned}
$$

where $f$ is given in terms of the isomorphism of Lemma 6.6 .5 by the morphism forgetting the marked point and stabilising and $g$ is given by evaluation at the marked point composed with the projection to $E$ on the second factor.

Let

$$
\left(D_{N}^{\prime}\right)_{0}=Y \times_{Y_{L \cap P_{1}}}\left(\operatorname{Bun}_{L \cap P_{1}, \text { rig }}^{-\alpha_{V}^{\vee}} \times_{\operatorname{Bun}_{L, \text { rig }}^{\mu}}^{\mu} Z_{0} \times_{S} E\right) \subseteq D_{N}^{\prime}
$$

and let $\left(D_{N}^{\prime}\right)_{1}=D_{N}^{\prime} \backslash\left(D_{N}^{\prime}\right)_{0}$. Then Propositions 3.4.10 and 3.4.13 and Lemma 6.3.2 imply that $\left(D_{N}^{\prime}\right)_{1}$ is a smooth divisor in $D_{N}^{\prime}$ isomorphic to

$$
\left(D_{N}^{\prime}\right)_{1} \cong Y \times_{Y_{L \cap P_{1}}}\left(\operatorname{Bun}_{L \cap P_{1}, \text { rig }}^{-\alpha_{i}^{\vee}-\alpha_{j}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}^{\mu}} Z_{0} \times_{S} E \times{ }_{S} E\right),
$$

where the first factor of $E$ above keeps track of the point of attachment of an $\alpha_{j}^{\vee}$ curve. There is a morphism

$$
\begin{equation*}
\left(D_{N}^{\prime}\right)_{1} \longrightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \tag{6.6.3}
\end{equation*}
$$

given on the first factor by the morphism $\left(D_{N}^{\prime}\right)_{1} \rightarrow D_{N}^{\prime} \rightarrow Y$ and on the second by the morphism

$$
\begin{aligned}
\left(D_{N}^{\prime}\right)_{1} \longrightarrow E \times_{S} E & \longrightarrow \operatorname{Pic}_{S}^{0}(E) \\
\left(x, x^{\prime}\right) & \longmapsto x-x^{\prime} .
\end{aligned}
$$

Proposition 6.6.6. The morphism $D_{\alpha_{i}^{\vee}}(Z) \rightarrow D_{N}^{\prime}$ is the blowup along the preimage of the zero section $\theta_{N}^{\prime}$ of $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E) \rightarrow Y$ under the morphism (6.6.3).

Proof. The proof is identical to the proof of Proposition 6.3.11.
Lemma 6.6.7. The morphism (6.6.3) is an isomorphism.
Proof. It is clear from Proposition 6.6.6 and Proposition 3.4.13 that the morphism $D_{\alpha_{i}^{\vee}}(Z) \rightarrow$ $D_{N}^{\prime}$ restricts to an isomorphism

$$
D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z) \xrightarrow{\sim}\left(D_{N}^{\prime}\right)_{1}
$$

and by Remark 6.3.9 that the composition of this isomorphism with (6.6.3) agrees with the morphism $D_{\alpha_{i}^{\vee}}(Z) \cap D_{\alpha_{j}^{\vee}}(Z) \rightarrow Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)$ coming from (6.3.11). But this composition is an isomorphism by Lemma 6.3.14 so we are done.

The next step is to construct the family of surfaces $D_{1}^{\prime}$ over $Y$. Let $P_{1}^{\prime} \subseteq L$ be the standard parabolic subgroup of type

$$
t\left(P_{1}^{\prime}\right)= \begin{cases}\left\{\alpha_{l}\right\}, & \text { in types } B, C, D \\ \left\{\alpha_{4}\right\}, & \text { in type } E \\ \left\{\alpha_{1}, \alpha_{2}\right\}, & \text { in type } F \\ \left\{\alpha_{1}\right\}, & \text { in type } G\end{cases}
$$

and define

$$
D_{1}^{\prime}=Y \times_{Y_{P_{1}^{\prime}}}\left(\mathrm{KM}_{P_{1}^{\prime}, L, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}^{\mu}} Z_{0} \times_{S} E\right)
$$

Proposition 6.6.8. We have the following descriptions of $D_{1}^{\prime}$ in each type.
(1) In type $B$, the morphism $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}$ is a $\mathbb{P}^{1}$-bundle such that the fibre of $D_{1}^{\prime} \rightarrow Y$ over a point $y \in Y$ is isomorphic to the stacky Hirzebruch surface

$$
\left(D_{1}^{\prime}\right)_{y} \cong \begin{cases}\mathbb{P}_{\mathbb{P}(1,2)}(\mathcal{O} \oplus \mathcal{O}(1)), & \text { if } \varpi_{l}(y) \neq 0, \\ \mathbb{P}_{\mathbb{P}(1,2)}(\mathcal{O} \oplus \mathcal{O}(3)), & \text { if } \varpi_{l}(y)=0\end{cases}
$$

(2) In types $C$ and $D$, the morphism $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}$ is a $\mathbb{P}^{1}$-bundle such that the fibre of $D_{1}^{\prime} \rightarrow Y$ over a point $y \in Y$ is isomorphic to the Hirzebruch surface

$$
\left(D_{1}^{\prime}\right)_{y} \cong \begin{cases}\mathbb{F}_{0}, & \text { if } \varpi_{l}(y) \neq 0 \\ \mathbb{F}_{2}, & \text { if } \varpi_{l}(y)=0\end{cases}
$$

(3) In types $E$ and $G$, the morphism $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}=Y$ is a $\mathbb{P}^{2}$-bundle.
(4) In type $F$, the morphism $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}=Y$ factors as a sequence of two $\mathbb{P}^{1}$-bundles $D_{1}^{\prime} \rightarrow D_{1}^{\prime \prime} \rightarrow Y$, and the fibre over a point $y \in Y$ is isomorphic to the Hirzebruch surface

$$
\left(D_{1}^{\prime}\right)_{y} \cong \begin{cases}\mathbb{F}_{0}, & \text { if } \alpha_{1}(y) \neq 0 \\ \mathbb{F}_{2}, & \text { if } \alpha_{1}(y)=0\end{cases}
$$

Since Proposition 6.6 .8 is local on $S$, we will assume for the proof and Lemmas 6.6.9 and 6.6.10 below that the initial section $S \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ (resp., $\mathbb{B}_{S} \mathbb{G}_{m} \rightarrow \operatorname{Bun}_{L^{\prime}, r i g}^{s s, \mu^{\prime}}$ ) used in the construction of the slice $Z_{0}$ in types $E, F$ and $G$ (resp., $B, C$ and $D$ ) lifts to a section $S \rightarrow \operatorname{Bun}_{L}^{s s, \mu}$ (resp., $S \rightarrow \operatorname{Bun}_{L^{\prime}}^{s s, \mu^{\prime}}$ ). We will also write $Z_{1}=Z_{0}=S$ in types $E, F$ and $G$ and $Z_{1}=\operatorname{Ind}_{L^{\prime}}^{L}(S) \backslash S$ in types $B, C$ and $D$; our assumption implies that $Z_{0} \rightarrow \operatorname{Bun}_{L, r i g}^{s s, \mu}$ lifts to $Z_{1} \rightarrow \operatorname{Bun}_{L}^{s s, \mu}$.

The first step in the proof of Proposition 6.6.8 is to relate $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}$ to the projectivisation of a vector bundle. Let $\rho_{L}$ be the representation of $L$ given by the isomorphism of Lemmas 6.5.1 and 6.5.2 composed with the projection to the second factor in types $C, D$, $E, F$ and $G$, and given by the isomorphism of Lemma 6.5 .3 composed with the projection to the second factor and the inclusion $G S p_{4} \subseteq G L_{4}$ in type $B$. We will write $W$ for the vector bundle on $Z_{1} \times_{S} E$ induced by $Z_{1} \rightarrow \operatorname{Bun}_{L}^{s s, \mu}$ and $\rho_{L}$. We will also write $\lambda \in \mathbb{X}^{*}(T)$ for the character

$$
\lambda= \begin{cases}\varpi_{l}, & \text { in types } B, C, D \\ \varpi_{4}, & \text { in type } E \\ \varpi_{2}, & \text { in type } G\end{cases}
$$

and

$$
d= \begin{cases}1, & \text { in types } B, D, E \\ 2, & \text { in type } C \\ 3, & \text { in type } G\end{cases}
$$

Lemma 6.6.9. In types $B, C, D, E$ and $G$, there is an isomorphism

$$
D_{1}^{\prime} \times{ }_{Z_{0}} Z_{1} \cong \mathbb{P}_{Y \times_{S} Z_{1}} \pi_{*}\left(M_{\lambda} \otimes \mathcal{O}\left(d O_{E}\right) \otimes W\right),
$$

where $\pi: Y \times_{S} Z_{1} \times_{S} E \rightarrow Y \times_{S} Z_{1}$ is the natural projection and $M_{\lambda}$ is the line bundle on $Y \times{ }_{S} Z_{1} \times{ }_{S} E$ classified by the morphism

$$
Y \times_{S} Z_{1} \longrightarrow Y \xrightarrow{\lambda} \operatorname{Pic}_{S}^{0}(E) .
$$

Proof. We first prove the lemma in types $B, D$ and $E$. Let

$$
X=Y \times_{Y_{P_{1}^{\prime}}}\left(\operatorname{Bun}_{P_{1}^{\prime}}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{L}} Z_{1} \times_{S} E\right) \subseteq D_{1}^{\prime} \times_{Z_{0}} Z_{1}
$$

Then Lemmas 6.5.1 and 6.5.6 show that $\mathfrak{X}$ is the stack of tuples $\left(y, z, M_{\lambda, y}^{-1} \otimes \mathcal{O}\left(-O_{E}\right) \subseteq W_{z}\right)$, where $y \in Y, z \in Z_{1}, M_{\lambda, y}$ is the line bundle on $E$ corresponding to $\lambda(y) \in \operatorname{Pic}_{S}^{0}(E)$, and $W_{z}$ is the restriction of $W$ to the fibre over $z \in Z_{1}$. Since the vector bundle $W_{z}$ is semistable of slope $<0$, any nonzero morphism $M_{\lambda, y}^{-1} \otimes \mathcal{O}\left(-O_{E}\right) \rightarrow W_{z}$ must be a subbundle, so we have an isomorphism

$$
X \cong \mathbb{P}_{Y \times{ }_{S} Z_{1}} \pi_{*}\left(M_{\lambda} \otimes \mathcal{O}\left(O_{E}\right) \otimes W\right)
$$

Since this implies in particular that $X$ is already proper over $Y \times{ }_{S} Z_{1}=Y \times_{Y_{P_{1}^{\prime}}}\left(Z_{1} \times{ }_{S} E\right)$, we conclude that $X=D_{1}^{\prime} \times{ }_{Z_{0}} Z_{1}$ and the claim is proved.

In types $C$ and $G$, we argue instead as follows. Observe that there is a pullback

where the bottom morphism is given by

$$
(y, z) \longmapsto\left(M_{\lambda, y}^{-1} \otimes \mathcal{O}\left(-d O_{E}\right), W_{z}\right)
$$

and the right morphism is given on the first factor by the blow down to $T_{Q_{2}^{2}}$-bundles composed with the character $e_{2}$. If $(y, z) \in Y \times_{S} Z_{1}$ lies over a geometric point $s: \operatorname{Spec} k \rightarrow S$, then any stable map to the $G L_{2}$ flag variety bundle $\mathbb{P}\left(W_{z}^{\vee}\right)$ corresponding to a point in $D_{1}^{\prime} \times{ }_{Z_{0}} Z_{1}$ over $(y, z)$ is a closed immersion with ideal sheaf $\left.p^{*}\left(M_{\lambda, y}^{-1} \otimes \mathcal{O}(-d) O_{E}\right)\right) \otimes \mathcal{O}(-1)$, where $p: \mathbb{P}\left(W_{z}^{\vee}\right) \rightarrow E_{s}$ is the structure morphism. So we deduce that

$$
D_{1}^{\prime} \times_{Z_{0}} Z_{1}=\mathbb{P}_{Y \times_{S} Z_{1}} \pi_{*} p_{*}\left(p^{*}\left(M_{\lambda} \otimes \mathcal{O}\left(d O_{E}\right)\right) \otimes \mathcal{O}(1)\right)=\mathbb{P}_{Y \times_{S} Z_{1}} \pi_{*}\left(M_{\lambda} \otimes \mathcal{O}\left(d O_{E}\right) \otimes W_{z}\right)
$$

as claimed.
The situation in type $F$ is similar. In this case, we let $P_{1}^{\prime \prime} \subseteq L$ be the standard parabolic subgroup of type $t\left(P_{1}^{\prime \prime}\right)=\left\{\alpha_{1}\right\}$, and define

$$
D_{1}^{\prime \prime}=Y \times_{Y_{P_{1}^{\prime \prime}}^{-\alpha_{i}^{\vee}}}\left(\mathrm{KM}_{P_{1}^{\prime \prime}, L, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{L, r i g}^{\mu}} Z_{0} \times_{S} E\right) .
$$

Lemma 6.6.10. In type $F$, there are isomorphisms

$$
D_{1}^{\prime \prime} \cong \mathbb{P}_{Y \times_{S} Z_{1}} \pi_{*}\left(M_{\varpi_{1}} \otimes W^{\vee}\right)
$$

and

$$
D_{1}^{\prime} \cong \mathbb{P}_{D_{1}^{\prime \prime}} \pi_{*}^{\prime}\left(p^{*} M_{\varpi_{2}} \otimes \mathcal{O}\left(2 O_{E}\right) \otimes \operatorname{ker}\left(p^{*} W \rightarrow p^{*} M_{\varpi_{1}} \otimes \mathcal{O}_{D_{1}^{\prime \prime}}(1)\right)\right)
$$

where $\pi: Y \times_{S} Z_{1} \times_{S} E \rightarrow Y \times_{S} Z_{1}$ and $\pi^{\prime}: D_{1}^{\prime} \times_{S} Z_{1} \times_{S} E \rightarrow Y \times_{S} Z_{1}$ are the natural projections, and $p: D_{1}^{\prime \prime} \rightarrow Y \times_{S} Z_{1}$ is the structure morphism.

Proof. Recall that $\alpha_{i}=\alpha_{3}$ and $Z_{1}=S$ in this case and let

$$
X=Y \times_{Y_{P_{1}^{\prime \prime}}^{-\alpha \vee}}\left(\operatorname{Bun}_{P_{1}^{\prime \prime}}^{-\alpha_{3}^{\vee}} \times_{\operatorname{Bun}_{L}^{\mu}} Z_{1} \times_{S} E\right) \subseteq D_{1}^{\prime \prime}
$$

Then Lemma 6.5 .1 shows that $X$ is the stack of tuples $\left(y, z, W_{z} \rightarrow M_{\varpi_{1}, y}\right)$, where $y \in Y$ and $z \in Z_{1}$. Since the vector bundle $W_{z}$ is semistable of slope $>-1$, any nonzero morphism $W_{z} \rightarrow M_{\varpi_{1}, y}$ is surjective, so we have an isomorphism

$$
X \cong \mathbb{P}_{Y \times_{S} Z_{1}}\left(\pi_{*}\left(M_{\varpi_{1}} \otimes W^{\vee}\right)\right)
$$

Since this shows that $X$ is already proper over $Y \times_{S} Z_{1}=Y \times_{Y_{P^{\prime}}}\left(Z_{1} \times_{S} E\right)$, it follows that $X=D_{1}^{\prime \prime}$, so this gives the first of the desired isomorphisms.

For the second isomorphism, there is a pullback

where the bottom horizontal morphism is classified by the pair $\left(p^{*} M_{\varpi_{2}}^{-1} \otimes \mathcal{O}\left(-2 O_{E}\right), \operatorname{ker}\left(p^{*} W \rightarrow\right.\right.$ $\left.p^{*} M_{\varpi_{1}} \otimes \mathcal{O}_{D_{1}^{\prime \prime}}(1)\right)$ ) of line bundle and vector bundle on $D_{1}^{\prime \prime} \times_{S} E$. Since any stable map to the associated flag variety bundle appearing in $D_{1}^{\prime}$ is again a closed immersion, the argument used in the proof of Lemma 6.6.9 for types $C$ and $G$ gives the desired isomorphism

$$
D_{1}^{\prime} \cong \mathbb{P}_{D_{1}^{\prime \prime}} \pi_{*}^{\prime}\left(p^{*} M_{\varpi_{2}} \otimes \mathcal{O}\left(2 O_{E}\right) \otimes \operatorname{ker}\left(p^{*} W \rightarrow p^{*} M_{\varpi_{1}} \otimes \mathcal{O}_{D_{1}^{\prime \prime}}(1)\right)\right)
$$

Proof of Proposition 6.6.8. First observe that in types $E$ and $G, M_{\lambda} \otimes \mathcal{O}\left(d O_{E}\right) \otimes W$ is a family of semistable vector bundles of degree 3 , so Lemma 6.6.9 shows that $D_{1}^{\prime} \rightarrow Y \times{ }_{S} Z_{1}=$ $Y$ is a $\mathbb{P}^{2}$-bundle, which proves (3).

In types $B, C$ and $D, M_{\lambda} \otimes \mathcal{O}\left((d+1) O_{E}\right) \otimes W$ is a family of semistable vector bundles of degree 2, so Lemma 6.6 .9 shows that $D_{1}^{\prime} \times{ }_{Z_{0}} Z_{1} \rightarrow Y \times_{S} Z_{1}$ is a $\mathbb{P}^{1}$-bundle, and hence that $D_{1}^{\prime} \rightarrow Y \times_{S} Z_{0}$ is also.

To complete the proof of (1), note that in type $B$, we have a canonical $Z\left(L^{\prime}\right)$-invariant subbundle $\mathcal{O}\left(-O_{E}\right) \subseteq W$ and a $Z\left(L^{\prime}\right)$-equivariant exact sequence

$$
0 \longrightarrow U \longrightarrow W / \mathcal{O}\left(-O_{E}\right) \longrightarrow \mathcal{O} \longrightarrow 0
$$

where $U$ is a family of stable vector bundles on $E$ of rank 2 and determinant $\mathcal{O}\left(-O_{E}\right)$. So if we fix a geometric point $y: \operatorname{Spec} k \rightarrow Y$ over $s: \operatorname{Spec} k \rightarrow S$, we have $Z\left(L^{\prime}\right)$-equivariant exact sequences

$$
\begin{align*}
0 \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y}\right) \longrightarrow \pi_{*} & \left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes W_{s}\right)  \tag{6.6.5}\\
& \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes\left(W_{s} / \mathcal{O}\left(-O_{E}\right)\right)\right) \longrightarrow \mathbb{R}^{1} \pi_{*}\left(M_{\varpi_{l}, y}\right) \longrightarrow 0,
\end{align*}
$$

and

$$
\begin{align*}
& 0 \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes U_{s}\right) \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes\left(W_{s} / \mathcal{O}\left(-O_{E}\right)\right)\right)  \tag{6.6.6}\\
& \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right)\right) \longrightarrow 0
\end{align*}
$$

of $Z\left(L^{\prime}\right)$-linearised vector bundles on $\left(Z_{1}\right)_{s}$. Note that $\pi_{*}\left(M_{\varpi_{l}, y}\right), \mathbb{R}^{1} \pi_{*}\left(M_{\varpi_{l}, y}\right), \pi_{*}\left(M_{\varpi_{l}, y} \otimes\right.$ $\left.\mathcal{O}\left(O_{E}\right) \otimes U_{s}\right)$ and $\pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right)\right)$ are each either a trivial line bundle or zero, with $Z\left(L^{\prime}\right)$ weights $f_{4}, f_{4}, f_{2}=f_{3}$ and $f_{1}$ respectively, where we use the notation of the proof of Lemma 6.5.3. So after tensoring with the character $-f_{1}$ of $Z\left(L^{\prime}\right), Z(G)$ acts trivially on (6.6.5) and
(6.6.6), so they descend to exact sequences of vector bundles on $\left(Z_{0}\right)_{s}=\left(Z_{1}\right)_{s} / \mathbb{G}_{m} \cong \mathbb{P}(1,2)$. Examining the $\mathbb{G}_{m}$-weights, the sequence (6.6.6) descends to a sequence of the form

$$
0 \longrightarrow \mathcal{O}(1) \longrightarrow W^{\prime} \longrightarrow \mathcal{O} \longrightarrow 0
$$

Since any such sequence splits, we must have $W^{\prime} \cong \mathcal{O} \oplus \mathcal{O}(1)$ as vector bundles on $\mathbb{P}(1,2)$.
If $\varpi_{l}(y) \neq 0$, then $\pi_{*}\left(M_{\varpi_{l}, y}\right)=\mathbb{R}^{1} \pi_{*}\left(M_{\varpi_{l}, y}\right)=0$, so we have

$$
\pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes W_{s}\right) \cong \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(O_{E}\right) \otimes W_{s} / \mathcal{O}\left(-O_{E}\right)\right)
$$

and hence $\left(D_{1}^{\prime}\right)_{y}=\mathbb{P}_{\mathbb{P}(1,2)}\left(W^{\prime}\right)=\mathbb{P}_{\mathbb{P}(1,2)}(\mathcal{O} \oplus \mathcal{O}(1))$. Otherwise, (6.6.5) tensored with $-f_{1}$ descends to an exact sequence

$$
0 \longrightarrow \mathcal{O}(2) \longrightarrow W^{\prime \prime} \longrightarrow W^{\prime}=\mathcal{O} \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0
$$

such that $\left(D_{1}^{\prime}\right)_{y}=\mathbb{P}_{\mathbb{P}(1,2)}\left(W^{\prime \prime}\right)$. But since the kernel of any surjection $\mathcal{O} \oplus \mathcal{O}(1) \rightarrow \mathcal{O}(2)$ on $\mathbb{P}(1,2)$ must be isomorphic to $\mathcal{O}(-1)$, this means that we must have $W^{\prime \prime}=\mathcal{O}(-1) \oplus \mathcal{O}(2)$, so

$$
\left(D_{1}^{\prime}\right)_{y}=\mathbb{P}_{\mathbb{P}(1,2)}(\mathcal{O}(-1) \oplus \mathcal{O}(2))=\mathbb{P}_{\mathbb{P}(1,2)}(\mathcal{O} \oplus \mathcal{O}(3))
$$

This proves (1).
Similarly, to prove (2), note that in types $C$ and $D$ we have a canonical $Z\left(L^{\prime}\right)$-equivariant exact sequence

$$
0 \longrightarrow \mathcal{O}\left(-d O_{E}\right) \longrightarrow W \longrightarrow U \longrightarrow 0
$$

where $U$ is semistable and $Z\left(L^{\prime}\right)$ acts on $\mathcal{O}\left(-d O_{E}\right)$ and $\mathcal{O}$ respectively with weights $e_{n_{1}+1}=-\varpi_{l}+(d+1) \varpi_{i}=\left\{\begin{array}{ll}-\varpi_{l}+2 \varpi_{l-1}, & \text { in type } C, \\ -\varpi_{l}+\varpi_{l-3}, & \text { in type } D,\end{array} \quad\right.$ and $\quad e_{1}= \begin{cases}\varpi_{l}, & \text { in type } C, \\ \varpi_{l-1}, & \text { in type } D .\end{cases}$

So over any geometric point $y: \operatorname{Spec} k \rightarrow Y$ over $s: \operatorname{Spec} k \rightarrow S$, we have an exact sequence

$$
\begin{align*}
0 \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y}\right) \longrightarrow \pi_{*} & \left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(d O_{E}\right) \otimes W_{s}\right)  \tag{6.6.7}\\
& \longrightarrow \pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left(d O_{E}\right) \otimes U_{s}\right) \longrightarrow \mathbb{R}^{1} \pi_{*}\left(M_{\varpi_{l}, y}\right) \longrightarrow 0,
\end{align*}
$$

of $Z\left(L^{\prime}\right)$-linearised vector bundles on $\left(Z_{1}\right)_{s}$, which descends to an exact sequence of vector bundles on $\mathbb{P}^{1}=\left(Z_{0}\right)_{s}=\left(Z_{1}\right)_{s} / \mathbb{G}_{m}$ after tensoring with $-e_{1}$. Note that in both cases $M_{\varpi_{l}, y} \otimes \mathcal{O}\left((d+1) O_{E}\right) \otimes U_{s}$ is a semistable vector bundle of degree 2 on which $Z\left(L^{\prime}\right)$ acts with the single weight $e_{1}$, so $\pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left((d+1) O_{E}\right) \otimes U_{s}\right) \otimes \mathbb{Z}_{-e_{1}}$ descends to a trivial rank 2 vector bundle $\mathcal{O} \oplus \mathcal{O}$ on $\mathbb{P}^{1}$.

If $\varpi_{l}(y) \neq 0$, then $\pi_{*}\left(M_{\varpi_{l}, y}\right)=\mathbb{R}^{1} \pi_{*}\left(M_{\varpi_{l}, y}\right)=0$, so

$$
\pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left((d+1) O_{E}\right) \otimes W_{s}\right) \otimes \mathbb{Z}_{-e_{1}}=\pi_{*}\left(M_{\varpi_{l}, y} \otimes \mathcal{O}\left((d+1) O_{E}\right) \otimes U_{s}\right) \otimes \mathbb{Z}_{-e_{1}}
$$

descends to $\mathcal{O} \oplus \mathcal{O}$ on $\mathbb{P}^{1}$, which together with Lemma 6.6.9 shows that $\left(D_{1}^{\prime}\right)_{y}=\mathbb{P}_{\mathbb{P}^{1}}(\mathcal{O} \oplus \mathcal{O})=$ $\mathbb{F}_{0}$. Otherwise, (6.6.7) descends to an exact sequence

$$
0 \longrightarrow \mathcal{O}(1) \longrightarrow W^{\prime} \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0
$$

such that $\left(D_{1}^{\prime}\right)_{y} \cong \mathbb{P}_{\mathbb{P}^{1}}\left(W^{\prime}\right)$. Since the kernel of any surjection $\mathcal{O} \oplus \mathcal{O} \rightarrow \mathcal{O}(1)$ must be isomorphic to $\mathcal{O}(-1)$, this implies that $W^{\prime} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and hence that

$$
\left(D_{1}^{\prime}\right)_{y} \cong \mathbb{P}_{\mathbb{P}^{1}}(\mathcal{O}(-1) \oplus \mathcal{O}(1)) \cong \mathbb{F}_{2}
$$

This proves (2).
Finally, in type $F$, we have already constructed the morphisms $D_{1}^{\prime} \rightarrow D_{1}^{\prime \prime} \rightarrow Y=Y \times{ }_{S} Z_{0}$. Since $M_{\varpi_{1}} \otimes W^{\vee}$ is a family of semistable vector bundles of degree 2, Lemma 6.6.10 shows that $D_{1}^{\prime \prime} \rightarrow Y$ is a $\mathbb{P}^{1}$-bundle as claimed. Moreover, any rank 2 degree -2 subbundle of $W$ is necessarily also semistable, so Lemma 6.6 .10 also shows that $D_{1}^{\prime} \rightarrow D_{1}^{\prime \prime}$ is a $\mathbb{P}^{1}$-bundle.

If $y: \operatorname{Spec} k \rightarrow Y$ is a geometric point over $s: \operatorname{Spec} k \rightarrow S$, then by Lemma 6.6.10 we have an exact sequence

$$
\begin{equation*}
0 \longrightarrow U \longrightarrow q^{*}\left(M_{\varpi_{2}, y} \otimes \mathcal{O}\left(2 O_{E}\right) \otimes W_{s}\right) \longrightarrow q^{*}\left(M_{\varpi_{1}+\varpi_{2}, y} \otimes \mathcal{O}\left(2 O_{E}\right)\right) \otimes\left(\pi^{\prime}\right)^{*} \mathcal{O}(1) \longrightarrow 0 \tag{6.6.8}
\end{equation*}
$$

of vector bundles on $\mathbb{P}^{1} \times E_{s}$ such that $\left(D_{1}^{\prime}\right)_{y}=\mathbb{P} \pi_{*}^{\prime} U$, where $\pi^{\prime}$ and $q$ are the projections to the first and second factors respectively. Since $U$ is a vector bundle of rank 2 and determinant $q^{*}\left(M_{-\varpi_{1}+2 \varpi_{2}, y} \otimes \mathcal{O}\left(2 O_{E}\right)\right) \otimes\left(\pi^{\prime}\right)^{*} \mathcal{O}(-1)$, it follows that we have an isomorphism

$$
U \xrightarrow{\sim} U^{\vee} \otimes \operatorname{det} U=U^{\vee} \otimes q^{*}\left(M_{-\varpi_{1}+2 \varpi_{2}, y} \otimes \mathcal{O}\left(2 O_{E}\right)\right) \otimes\left(\pi^{\prime}\right)^{*} \mathcal{O}(-1) .
$$

So the dual of (6.6.8) gives an exact sequence

$$
0 \longrightarrow q^{*} M_{-2 \varpi_{1}+\varpi_{2}, y} \otimes\left(\pi^{\prime}\right)^{*} \mathcal{O}(-2) \longrightarrow q^{*}\left(M_{-\varpi_{1}+\varpi_{2}, y} \otimes W_{s}^{\vee}\right) \otimes\left(\pi^{\prime}\right)^{*} \mathcal{O}(-1) \longrightarrow U \longrightarrow 0
$$

and hence an exact sequence

$$
\begin{align*}
0 \longrightarrow H^{0}\left(E_{s}, M_{-2 \varpi_{1}+\varpi_{2}, y}\right) \otimes \mathcal{O}(-2) & \longrightarrow H^{0}\left(E_{s}, M_{-\varpi_{1}+\varpi_{2}, y} \otimes W_{s}^{\vee}\right) \otimes \mathcal{O}(-1) \\
& \longrightarrow\left(\pi^{\prime}\right)_{*} U \longrightarrow H^{1}\left(E_{s}, M_{-2 \varpi_{1}+\varpi_{2}, y}\right) \otimes \mathcal{O}(-2) \longrightarrow 0 \tag{6.6.9}
\end{align*}
$$

If $\alpha_{1}(y)=2 \varpi_{1}(y)-\varpi_{2}(y) \neq 0$, then $H^{0}\left(E_{s}, M_{-2 \varpi_{1}+\varpi_{2}, y}\right)=H^{1}\left(E_{s}, M_{-2 \varpi_{1}+\varpi_{2}, y}\right)=0$, so (6.6.9) gives an isomorphism

$$
\left(\pi^{\prime}\right)_{*} U \cong H^{0}\left(E_{s}, M_{-\varpi_{1}+\varpi_{2}, y} \otimes W_{s}^{\vee}\right) \otimes \mathcal{O}(-1)=\mathcal{O}(-1) \oplus \mathcal{O}(-1)
$$

so $\left(D_{1}^{\prime}\right)_{y} \cong \mathbb{P}_{\mathbb{P}^{1}}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))=\mathbb{F}_{0}$. Otherwise, (6.6.9) gives an exact sequence

$$
0 \longrightarrow \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \longrightarrow\left(\pi^{\prime}\right)_{*} U \longrightarrow \mathcal{O}(-2) \longrightarrow 0
$$

Since the cokernel of the injective morphism $\mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ must be isomorphic to $\mathcal{O}$, we get $\left(\pi^{\prime}\right)_{*} U \cong \mathcal{O}(-2) \oplus \mathcal{O}$ and hence $\left(D_{1}^{\prime}\right)_{y} \cong \mathbb{F}_{2}$. This completes the proof of (4) and of the proposition.

We can now prove Theorem 6.6.1 in types $C, F$ and $G$.
Proof of Theorem 6.6.1 in types $C, F$ and $G$. Since $L \cap P_{1}=P_{1}^{\prime}$ and $N=1$ in these cases, Proposition 6.6.6 and Lemma 6.6.7 prove (1), and (2) and (3) are clear from the construction. Proposition 6.6 .8 shows (4), so the theorem is proved in this case.

In types $B, D$ and $E$, we still have $L \cap P_{1} \subseteq P_{1}^{\prime}$, so we get a morphism $D_{N}^{\prime} \rightarrow D_{1}^{\prime}$. In type $D$, let $P_{2}^{\prime} \subseteq L$ be the standard parabolic of type $t\left(P_{2}^{\prime}\right)=\left\{\alpha_{l-1}, \alpha_{l}\right\}$ and in type $E$, let $P_{2}^{\prime}, P_{3}^{\prime} \subseteq L$ be the standard parabolics of type $t\left(P_{2}^{\prime}\right)=\left\{\alpha_{1}, \alpha_{4}\right\}$ and $t\left(P_{3}^{\prime}\right)=\left\{\alpha_{1}, \alpha_{2}, \alpha_{4}\right\}$. Set

$$
D_{k}^{\prime}=Y \times_{Y_{P_{k}^{\prime}}}\left(\mathrm{KM}_{P_{k}^{\prime}, L, r i g}^{-\alpha_{i}^{\vee}} \times_{\operatorname{Bun}_{P_{k}^{\prime}}^{\mu}} Z_{0} \times{ }_{S} E\right)
$$

for $1 \leq k<N$. Note that in each case we have a sequence of morphisms

$$
D_{N}^{\prime} \longrightarrow D_{N-1}^{\prime} \longrightarrow \cdots \longrightarrow D_{1}^{\prime}
$$

as desired.
In type $B$, we let $\rho_{P_{1}^{\prime}}: P_{1}^{\prime} \rightarrow G L_{n_{1}}=G L_{2}$ be the representation given by the restriction $P_{1}^{\prime} \rightarrow G S p_{4} \cap R_{4}$ of $\rho_{L}$ to $P_{1}^{\prime}$ composed with the homomorphism

\[

\]

In types $D$ and $E$, we let $\rho_{P_{1}^{\prime}}: P_{1}^{\prime} \rightarrow G L_{n_{1}}$ be the composition

$$
\rho_{P_{1}^{\prime}}: P_{1}^{\prime} \xrightarrow{\rho_{L}} R_{n_{1}+1} \longrightarrow G L_{n_{1}},
$$

where the second homomorphism is given by deleting the last row and column. In each of types $B, D$ and $E$, we then have $P_{k}^{\prime}=\left(\rho_{P_{1}^{\prime}}\right)^{-1}\left(Q_{k}^{n_{1}}\right)$ for $1 \leq k \leq n_{1}=N$ and hence a sequence of pullback squares
by Lemma 6.2.13, where the subscript $(-)_{\text {rig }}$ denotes the rigidification with respect to the image of $Z(G)$ in $Z\left(G L_{n_{1}}\right)$. Note that, in the notation of $\S 6.2$, the rigidification $X_{k, r i g}^{n_{1}}$ of $X_{k}^{n_{1}}$ is naturally a locally closed substack of $Y_{Q_{k}^{n_{1}}}^{-e_{n_{1}}^{*}} \times_{Y_{Q_{n_{1}}}^{-e_{1}^{*}}} \mathrm{KM}_{Q_{k}^{n_{1}, G L_{n_{1}}, r i g}}^{-e_{n_{1}}^{*}}$. For $1 \leq k \leq n_{1}$ and $1 \leq p<k$ or $p=n_{1}$, we write $C_{k, p}^{\prime} \subseteq D_{k}^{\prime}$ for the preimage of $C_{k, p, r i g}^{G L_{n_{1}}} \subseteq X_{k, r i g}^{n_{1}}$ in $D_{k}^{\prime}$.

Lemma 6.6.11. In types $B, D$ and $E$, for $1 \leq k \leq n_{1}$, there is a decomposition

$$
D_{k}^{\prime}=\left(D_{k}^{\prime} \times_{\operatorname{Bun}_{G L_{n_{1}}, r i g}^{-1}} \operatorname{Bun}_{G L_{n_{1}}, r i g}^{s s,-1}\right) \cup \bigcup_{1 \leq p<k} C_{k, p}^{\prime} \cup C_{k, n_{1}}^{\prime}
$$

into disjoint locally closed substacks.
Proof. Using Proposition 6.2.1, we can reduce to showing that any unstable $G L_{n_{1}}$-bundle in the image of $D_{1}^{\prime} \rightarrow \operatorname{Bun}_{G L_{n_{1}}, \text { rig }}^{-1}$ has Harder-Narasimhan reduction to $R_{n_{1}}$ with degree $-e_{1}^{*}$, i.e., that the Harder-Narasimhan decomposition of the corresponding vector bundle $U$ is $U=M_{1} \oplus M_{2}$ with $M_{1}$ a line bundle of degree 0 and $M_{2}$ a semistable vector bundle of rank $n_{1}-1$ and degree -1 .

In type $B$, we argue as follows. Since $U$ is a vector bundle of rank $n_{1}=2$ and degree -1 , we know that the Harder-Narasimhan decomposition of $U$ is $U=M_{1} \oplus M_{2}$ for some line bundles $M_{1}$ and $M_{2}$ with $\operatorname{deg} M_{1} \leq-1$ and $\operatorname{deg} M_{2}=-1-\operatorname{deg} M_{1} \geq 0$. Moreover, since $U$ is in the image of $D_{1}^{\prime} \rightarrow \operatorname{Bun}_{G L_{2}, \text { rig }}^{-1}$, we know that there exist exact sequences

$$
0 \longrightarrow U \longrightarrow U^{\prime} \longrightarrow N_{1} \longrightarrow 0
$$

and

$$
0 \longrightarrow N_{2} \longrightarrow W \longrightarrow U^{\prime} \longrightarrow 0
$$

where $N_{1}$ and $N_{2}$ are line bundles of degree 0 and -1 respectively and $W$ is a semistable vector bundle of rank 4 and degree -2 . So we have an exact sequence

$$
0 \longrightarrow N_{2} \longrightarrow U^{\prime \prime} \longrightarrow M_{2} \longrightarrow 0
$$

for $U^{\prime \prime} \subseteq W$ a subbundle of rank 2 . In particular, by semistability of $W$, we have $\mu\left(U^{\prime \prime}\right) \leq$ $-1 / 2$, so $\operatorname{deg} U^{\prime \prime}=\operatorname{deg} M_{2}-1 \leq-1$. So we must have $\operatorname{deg} M_{2}=0$ and hence $U$ has the desired Harder-Narasimhan decomposition.

In types $D$ and $E$, we instead have an exact sequence

$$
0 \longrightarrow N \longrightarrow W \longrightarrow U \longrightarrow 0
$$

where $N$ is a line bundle of degree -1 and $W$ is semistable rank $n_{1}+1$ and degree -2 . Since $U$ is unstable, there exists a semistable quotient $M_{2}$ of $U$ with slope $\mu\left(M_{2}\right)<-1 / n_{1}=\mu(U)$. Since $M_{2}$ is also a quotient of the semistable vector bundle $W$, it follows that

$$
\mu\left(M_{2}\right) \geq \mu(W)=\frac{-2}{n_{1}+1} .
$$

This implies that

$$
0<\frac{\operatorname{rank} M_{2}}{n_{1}}<-\operatorname{deg} M_{2} \leq \frac{2 \operatorname{rank} M_{2}}{n_{1}+1}<2,
$$

so $\operatorname{deg} M_{2}=-1$. So we have

$$
\mu(W)=\frac{-2}{n_{1}+1} \leq \mu\left(M_{2}\right)=\frac{-1}{\operatorname{rank} M_{2}}<\mu(U)=\frac{-1}{n_{1}}
$$

and hence

$$
\frac{n_{1}+1}{2} \leq \operatorname{rank} M_{2}<n_{1}
$$

But since $n_{1} \leq 4$, we have $n_{1}-2<\frac{n_{1}+1}{2}$, so it follows that rank $M_{2}=n_{1}-1$. So we have an exact sequence

$$
0 \longrightarrow M_{1} \longrightarrow U \longrightarrow M_{2} \longrightarrow 0,
$$

with $M_{1}$ a line bundle of degree 0 , which shows that $U$ has the Harder-Narasimhan decomposition $U=M_{1} \oplus M_{2}$ as claimed

Next, observe that the morphism (6.2.3) gives a morphism

$$
\begin{equation*}
C_{n_{1}, n_{1}}^{\prime} \longrightarrow Y \times_{S} \operatorname{Pic}_{S}^{0}(E) \tag{6.6.11}
\end{equation*}
$$

given by the composition

$$
\begin{aligned}
C_{n_{1}, n_{1}}^{\prime} & \longrightarrow Y \times_{Y_{L \cap P_{1}}}\left(Y_{L \cap P_{1}}^{-\alpha_{i}^{\vee}} \times_{Y_{Q_{n_{1}}^{n}}^{-e_{1}^{*}}} C_{n_{1}, n_{1}, r i g}^{G L_{n_{1}}} \times_{S} E\right) \\
& \longrightarrow Y \times_{Y_{L \cap P_{1}}}\left(Y_{L \cap P_{1}}^{-\alpha_{i}^{\vee}} \times_{S} E \times_{S} E\right) \\
& \longrightarrow Y \times_{Y_{L \cap P_{1}}}\left(Y_{L \cap P_{1}} \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)\right)=Y \times_{S} \operatorname{Pic}_{S}^{0}(E),
\end{aligned}
$$

where the last morphism is the pullback of

$$
\begin{aligned}
Y_{L \cap P_{1}}^{-\alpha_{1}^{\vee}} \times_{S} E \times_{S} E & \longrightarrow Y_{L \cap P_{1}} \times_{S} \operatorname{Pic}_{S}^{0}(E) \\
\left(y, x_{1}, x_{2}\right) & \longmapsto\left(y+\alpha_{i}^{\vee}\left(x_{2}\right), x_{1}-x_{2}\right) .
\end{aligned}
$$

Lemma 6.6.12. The closed substack $C_{n_{1}, n_{1}}^{\prime} \subseteq D_{n_{1}}^{\prime}=D_{N}^{\prime}$ coincides with $\left(D_{N}^{\prime}\right)_{1}$ and the morphism (6.6.11) agrees with (6.6.3).

Proof. This follows directly from the definitions and Lemma 6.2.6.
Lemma 6.6.13. For $1 \leq k<n_{1}$, the morphism $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ restricts to isomorphisms
$\left(D_{k+1}^{\prime} \times\right.$ Bun $\left._{G L_{n_{1}}, r i g}^{-1} \operatorname{Bun}_{G L_{n_{1}}, r i g}^{s s,-1}\right) \xrightarrow{\sim}\left(D_{k}^{\prime} \times \times_{\operatorname{Bun}_{G L_{n_{1}}, r i g}^{-1}} \operatorname{Bun}_{G L_{n_{1}}, r i g}^{s s,-1}\right)$ and $\quad C_{k+1, p}^{\prime} \xrightarrow{\sim} C_{k, p}^{\prime}$
for $1 \leq p<k$, and a morphism $C_{k+1, k}^{\prime} \rightarrow C_{k, n_{1}}^{\prime}$ identifying $C_{k+1, k}^{\prime}$ with the total space of a line bundle over the section $\theta_{k}^{\prime}$ of $Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E)=C_{k, n_{1}}^{\prime}$.

Proof. Using the natural Cartesian diagram (6.6.10), the claim follows easily from Lemma 5.4.10 and Proposition 6.2.7.

Proof of Theorem 6.6.1 in types B, D and E. By Propositions 6.6.6, 6.6.8 and Lemma 6.6.7, the only thing left to show is that $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ is the blowup along $\theta_{k}^{\prime}$ for $1 \leq k<n_{1}=N$. To see this, note that Lemmas 6.6 .11 and 6.6 .7 imply that $D_{k+1}^{\prime} \rightarrow Y$ is a family of smooth surfaces, that $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ is an isomorphism outside $\theta_{k}^{\prime}: Y \rightarrow Y \times{ }_{S} \operatorname{Pic}_{S}^{0}(E) \hookrightarrow D_{k}^{\prime}$ and that every fibre of $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ over that section is an irreducible curve. So Lemma 6.3.18 then shows that $D_{k+1}^{\prime} \rightarrow D_{k}^{\prime}$ is the blowup along the given section as claimed, and the theorem is proved.

### 6.7 Singularities

Theorems 6.1.9 and 6.6 .1 give very explicit descriptions of the families of normal crossings surfaces $\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right) \rightarrow Y$. We show in this section how these results can be used to identify the singularities of the unstable loci $\chi_{Z}^{-1}(0)$. For the sake of simplicity, we will assume always that $S=\operatorname{Spec} k$ for some algebraically closed field $k$.

Definition 6.7.1. Let $k$ be an algebraically closed field, and let $R$ be a 2-dimensional complete local $k$-algebra with residue field $k$.
(1) We say that $R$ has a singularity of type $A_{\infty}$ if

$$
R \cong \frac{k \llbracket x, y, z \rrbracket}{(x y)}
$$

(2) If the characteristic of $k$ is not 2 , we say that $R$ has a singularity of type $D_{\infty}$ if

$$
R \cong \frac{k \llbracket x, y, z \rrbracket}{\left(x^{2} y-z^{2}\right)} .
$$

(3) If $5 \leq l \leq 8$, we say that $R$ has a singularity of type $\tilde{E}_{l}$, or a simply elliptic singularity of degree $9-l$ if there exists a smooth elliptic curve $X$ over $k$, a line bundle $L$ on $X$ of degree $9-l$, and an isomorphism

$$
R \cong \prod_{n \geq 0} H^{0}\left(X, L^{\otimes n}\right)
$$

We say that a stack $X$ over $k$ has a singularity of type $A_{\infty}$ (resp., $D_{\infty}, \tilde{E}_{l}$ ) at a point $x$ : Spec $k \rightarrow X$ if there is a ring $R$ as above and a formally smooth morphism $\operatorname{Spec} R \rightarrow X$ sending the closed point to $x$, such that $R$ has a singularity of type $A_{\infty}\left(\right.$ resp., $\left.D_{\infty}, \tilde{E}_{l}\right)$.

Remark 6.7.2. Note that the singularities $A_{\infty}$ and $D_{\infty}$ are not isolated, whereas the singularities $\tilde{E}_{l}$ are isolated.

Theorem 6.7.3. Assume that $S=\operatorname{Spec} k$ for $k$ an algebraically closed field, let $(G, P, \mu)$ be a subregular Harder-Narasimhan class, not of type $A_{1}$, and let $Z \rightarrow \operatorname{Bun}_{G, \text { rig }}$ be the equivariant slice constructed in the proof of Theorem 6.1.5. Then the stack $\chi_{Z}^{-1}(0) \subseteq Z$ has the following singularities.
(1) If $(G, P, \mu)$ is of type $A$ (but not $A_{1}$ ), then then $\chi_{Z}^{-1}(0)$ is a union of two line bundles on $E$ meeting along the zero section with singularities of type $A_{\infty}$.
(2) Assume that the characteristic of $k$ is not 2 . If $(G, P, \mu)$ is of type $B$ (resp., $C, D)$, then $\chi_{Z}^{-1}(0)$ is obtained by contracting the zero section of a line bundle on $E$ along a degree 2 map $E \rightarrow \mathbb{P}(1,2)$ (resp., $E \rightarrow \mathbb{P}^{1}$ ) branched over 3 (resp., 4) points. The singularities are of type $A_{\infty}$ at the non-branch points of $\mathbb{P}(1,2)$ (resp., $\mathbb{P}^{1}$ ) and of type $D_{\infty}$ at the branch points.
(3) If $(G, P, \mu)$ is of type $E$ (resp., $F, G$ ), then $\chi_{Z}^{-1}(0)$ is obtained by contracting the zero section of a line bundle on $E$ of degree $l-9$ (resp., $l-5, l-3=-1$ ) to a point. The singularity is simply elliptic of degree $9-l$ (resp., $5-l, 3-l$ ).

Remark 6.7.4. The restriction on the characteristic in types $B, C$ and $D$ in Theorem 6.7.3 is not essential: it will be clear from the proof that the general description of $\chi_{Z}^{-1}(0)$ as a contraction of a line bundle on $E$ is still correct in characteristic 2 , and the techniques of the proof can still be used to compute local equations for the singularities. However, in characteristic 2 the maps $E \rightarrow \mathbb{P}(1,2)$ and $E \rightarrow \mathbb{P}^{1}$ have more complicated local equations than in other characteristics, which depend on the precise elliptic curve $E$, and hence the same is true for the singularities of $\chi_{Z}^{-1}(0)$.

To prove Theorem 6.7.3, we first compute the degrees of the line bundles $D_{1}$ appearing in Theorem 6.1.9 in types $E, F$ and $G$.

Lemma 6.7.5. Assume we are in the setup of Theorem 6.1 .9 with $(G, P, \mu)$ of type $E$ (resp., $F, G)$, and fix a geometric point $y$ : Spec $k \rightarrow Y$. Then the fibre of $D_{1} \rightarrow Y$ over $y$ is a line bundle over $\operatorname{Pic}^{0}(E)$ of degree $l-9$ (resp., $l-5, l-3=-1$ ).

Proof. To simplify the notation, identify $\operatorname{Pic}^{0}(E) \subseteq\left(D_{1}\right)_{y}$ with $E$. The desired degree is equal to the self-intersection number $\left(E^{2}\right)_{\left(D_{1}\right)_{y}}$ of $E$ on the surface $\left(D_{1}\right)_{y}$.

First note that by Theorem 6.1.9, $D_{\alpha_{j}^{\vee}}(Z)_{y}$ is the iterated blowup of $\left(D_{1}\right)_{y}$ at $n_{0}+1$ points on $E$, so we have

$$
\begin{equation*}
\left(E^{2}\right)_{\left(D_{1}\right)_{y}}=\left(E^{2}\right)_{D_{\alpha_{j}^{\vee}}(Z)_{y}}+n_{0}+1 . \tag{6.7.1}
\end{equation*}
$$

Next, observe that we have
$0=\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right) \cdot E=\left(d D_{\alpha_{i}^{\vee}}(Z)+D_{\alpha_{j}^{\vee}}(Z)+D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)\right) \cdot E=d\left(E^{2}\right)_{D_{\alpha_{j}^{\vee}}(Z)_{y}}+\left(E^{2}\right)_{D_{\alpha_{i}^{\vee}}(Z)_{y}}+1$,
where $d=\frac{1}{2}\left(\alpha_{i}^{\vee} \mid \alpha_{i}^{\vee}\right)$ and we have used the fact that $D_{\alpha_{i}^{\vee}}(Z)_{y} \cap D_{\alpha_{j}^{\vee}}(Z)_{y}=E$ and that the exceptional curve of the final blowup $D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \cap D_{\alpha_{j}^{\vee}}(Z)_{y}$ meets $E$ transversely in a single point. Since $D_{\alpha_{j}^{\vee}}(Z)_{y}$ is the iterated blowup of the smooth surface $\left(D_{1}^{\prime}\right)_{y}$ of Proposition 6.6.8 at $N$ points on $E$, we have

$$
\left(E^{2}\right)_{D_{\alpha_{i}}(Z)}=\left(E^{2}\right)_{\left(D_{1}^{\prime}\right)_{y}}-N,
$$

and hence (6.7.1) and (6.7.2) give

$$
\begin{equation*}
\left(E^{2}\right)_{\left(D_{1}\right)_{y}}=\frac{1}{d}\left(N-\left(E^{2}\right)_{\left(D_{1}^{\prime}\right)_{y}}-1\right)+n_{0}+1 . \tag{6.7.3}
\end{equation*}
$$

To compute the self-intersection number $\left(E^{2}\right)_{\left(D_{1}^{\prime}\right)_{y}}$, note that by Theorem 4.6.1, we have

$$
K_{\tilde{Z} / Z}=f^{*} K_{\widetilde{\operatorname{Bun}}_{G, r i g} / \operatorname{Bun}_{G, r i g}} \cong \psi_{Z}^{*} M \otimes \mathcal{O}\left(-D_{\alpha_{i}^{\vee}}(Z)-D_{\alpha_{j}^{\vee}}(Z)\right)
$$

for some line bundle on $M$ on $Z$, where $f: \tilde{Z} \rightarrow \widetilde{\operatorname{Bun}}_{G, \text { rig }}$ is the natural morphism. Since $Z \cong \mathbb{A}^{l+3}$ is an affine space, every line bundle on $Z$ is trivial, so

$$
K_{\tilde{Z}}=K_{\tilde{Z} / Z} \otimes \psi_{Z}^{*} K_{Z} \cong \mathcal{O}\left(-D_{\alpha_{i}^{\vee}}(Z)-D_{\alpha_{j}^{\vee}}(Z)\right)
$$

By adjunction, we therefore have a linear equivalence

$$
\left.K_{D_{\alpha_{i}^{\vee}}(Z)_{y}} \sim\left(K_{\tilde{Z}}+D_{\alpha_{i}^{\vee}}(Z)\right)\right|_{D_{\alpha_{i}} \vee}(Z)_{y}=-D_{\alpha_{i}^{\vee}}(Z)_{y} \cap D_{\alpha_{j}^{\vee}}(Z)_{y}=-E .
$$

So $E \subseteq D_{\alpha_{i}^{\vee}}(Z)_{y}$ is an anticanonical divisor, from which it follows that $E \subseteq\left(D_{1}^{\prime}\right)_{y}$ is also an anticanonical divisor in the blow down. So from the explicit identification of the surface $\left(D_{1}^{\prime}\right)_{y}$ given in Proposition 6.6.8, we have

$$
\left(E^{2}\right)_{\left(D_{1}^{\prime}\right)_{y}}=K_{\left(D_{1}^{\prime}\right)_{y}}^{2}= \begin{cases}9, & \text { in types } E \text { and } G, \\ 8, & \text { in type } F .\end{cases}
$$

Substituting the values of $N, n_{0}$ and $d$ into (6.7.3) in each of the different cases gives the desired expressions for $\left(E^{2}\right)_{\left(D_{1}\right)_{y}}$.

We can use similar techniques to study the morphism $Y \times \operatorname{Pic}^{0}(E) \rightarrow Z_{0}$ in types $B, C$ and $D$.

Lemma 6.7.6. Assume that $(G, P, \mu)$ is of type $B, C$ or $D$. Then for any $y$ : $\operatorname{Spec} k \rightarrow$ $Y=0_{\Theta_{Y}^{-1}}$, the morphism $\operatorname{Pic}^{0}(E)=\{y\} \times \operatorname{Pic}^{0}(E) \subseteq \tilde{\chi}_{Z}^{-1}(y) \rightarrow Z_{0}$ has degree 2.

Proof. In these cases, we have again by Theorem 4.6.1 that

$$
K_{\tilde{Z}}=\psi_{Z}^{*} K_{Z} \otimes f^{*} K_{\widetilde{\operatorname{Bun}}_{G, r i g} / \operatorname{Bun}_{G, r i g}}=\psi_{Z}^{*} M \otimes \mathcal{O}\left(-D_{\alpha_{i}^{\vee}}(Z)-D_{\alpha_{j}^{\vee}}(Z)\right)
$$

for some line bundle $M$ on $Z$, where $f: \tilde{Z} \rightarrow \widetilde{\operatorname{Bun}}_{G, \text { rig }}$ is the natural morphism. So by adjunction, we have

$$
\begin{equation*}
K_{D_{\alpha_{i}^{\vee}}(Z)_{y}}=\left.\left(K_{\tilde{Z}} \otimes \mathcal{O}\left(D_{\alpha_{i}^{\vee}}(Z)\right)\right)\right|_{D_{\alpha_{i}^{\vee}}}(Z)_{y}=\left.\psi_{Z}^{*} M\right|_{D_{\alpha_{i}^{\vee}}(Z)_{y}} \otimes \mathcal{O}(-E), \tag{6.7.4}
\end{equation*}
$$

where we write $E=\{y\} \times \operatorname{Pic}^{0}(E) \subseteq D_{\alpha_{i}^{\vee}}(Z)_{y}$. To compute the degree of the finite morphism $E \rightarrow Z_{0}$, choose a $k$-point $z \in Z_{0}$ disjoint from the images of $\theta_{k}^{\prime}(y)$ and the stacky point in type $B$, and let $F_{z} \cong \mathbb{P}_{k}^{1}$ be the fibre of $D_{\alpha_{i}^{\vee}}(Z)_{y} \rightarrow Z_{0}$ over $z$. By (6.7.4) and adjunction, the degree is the intersection product

$$
E \cdot F_{z}=-K_{D_{\alpha_{i}} \vee}(Z)_{y} \cdot F_{z}=-\left(K_{D_{\alpha_{i}} \vee}(Z)_{y}+F_{z}\right) \cdot F_{z}=-\operatorname{deg} K_{F_{z}}=2,
$$

which proves the lemma.
Proof of Theorem 6.7.3. We first prove (3). By construction, $\chi_{Z}^{-1}(0)$ is affine, and the open subset

$$
\chi_{Z}^{-1}(0)^{r e g}=\chi_{Z}^{-1}(0) \times_{\operatorname{Bun}_{G, r i g}} \operatorname{Bun}_{G, r i g}^{r e g}
$$

is big. So choosing any $y$ : Spec $k \rightarrow 0_{\Theta_{Y}^{-1}}$, we have

$$
\chi_{Z}^{-1}(0)=\operatorname{Spec} H^{0}\left(\chi_{Z}^{-1}(0), \mathcal{O}\right)=\operatorname{Spec} H^{0}\left(\chi_{Z}^{-1}(0)^{r e g}, \mathcal{O}\right)=\operatorname{Spec} H^{0}\left(\tilde{\chi}_{Z}^{-1}(y)^{r e g}, \mathcal{O}\right)
$$

where $\tilde{\chi}_{Z}^{-1}(y)^{\text {reg }}=\tilde{\chi}_{Z}^{-1}(y) \cap \psi_{Z}^{-1}\left(\chi_{Z}^{-1}(0)^{\text {reg }}\right) \cong \chi_{Z}^{-1}(y)^{\text {reg }}$. But by Theorem 6.1.9, $\tilde{\chi}_{Z}^{-1}(y)^{\text {reg }}=$ $\left(D_{1}\right)_{y} \backslash E$ is the complement of the zero section in the negative degree line bundle $L^{-1}=$ $\left(D_{1}\right)_{y}$ over $E=\{y\} \times \operatorname{Pic}^{0}(E)$. So

$$
\chi_{Z}^{-1}(0)=\operatorname{Spec} H^{0}\left(\left(D_{1}\right)_{y} \backslash E, \mathcal{O}\right)=\operatorname{Spec} \bigoplus_{n \geq 0} H^{0}\left(E, L^{\otimes n}\right)
$$

Taking completions and applying Lemma 6.7 .5 shows that $\chi_{Z}^{-1}(0)$ has a simply elliptic singularity of the desired degree.

To prove (1) and (2), we argue as follows. First note that by Proposition 5.5.8, we have $\psi_{Z *}^{\prime} \mathcal{O}=\mathcal{O}$, where

$$
\psi_{Z}^{\prime}: \tilde{Z} \longrightarrow Z \times_{\widehat{Y} / / W} \Theta_{Y}^{-1}
$$

is the natural morphism induced by $\psi_{Z}$. For $y: \operatorname{Spec} k \rightarrow 0_{\Theta_{Y}^{-1}}$, we let $\psi_{Z, y}^{\prime}: \tilde{\chi}_{Z}^{-1}(y) \rightarrow$ $\chi_{Z}^{-1}(0)$ denote the restriction of $\psi_{Z}^{\prime}$. We show below that $\mathbb{R}^{i} \psi_{Z, y *}^{\prime} \mathcal{O}=0$ for $i>0$, which implies, since both domain and codomain of $\psi_{Z}^{\prime}$ are flat over $\Theta_{Y}^{-1}$, that $\mathbb{R} \psi_{Z *}^{\prime} \mathcal{O}=\mathcal{O}$, and hence $\mathbb{R} \psi_{Z, y *}^{\prime} \mathcal{O}=\mathcal{O}$ by base change.

Since $\chi_{Z}^{-1}(0) \rightarrow Z_{0}$ is affine by construction, it is enough to show that $\mathbb{R}^{i} \pi_{*} \mathcal{O}=0$ for $i>0$, where $\pi: \tilde{\chi}_{Z}^{-1}(y) \rightarrow Z_{0}$ is the natural morphism. This holds by inspection for the fibre over $y \in Y$ of the reduced normal crossings variety

$$
D=D_{\alpha_{i}^{\vee}}(Z)+D_{\alpha_{j}^{\vee}}(Z)+D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z),
$$

from the explicit descriptions of the components given by Theorems 6.1.9 and 6.6.1, using the fact that $\mathbb{R} f_{*} \mathcal{O}=\mathcal{O}$ whenever $f$ is either a $\mathbb{P}^{1}$-bundle or the blow up of a smooth surface at a point. This proves the claim in types $A, B$ and $D$. In type $C$, we claim that the morphism $\mathbb{R} \pi_{*} \mathcal{O}_{\bar{D}_{y}} \rightarrow \mathbb{R} \pi_{*} \mathcal{O}_{D_{y}}$ is a quasi-isomorphism, where $\bar{D}=\tilde{\chi}_{Z}^{-1}\left(0_{\Theta_{Y}^{-1}}\right)$, from which the desired vanishing follows. To see this, note that we have a short exact sequence

$$
\left.0 \longrightarrow \mathcal{O}(-D)\right|_{D_{\alpha_{i}}(Z)} \longrightarrow \mathcal{O}_{\bar{D}} \longrightarrow \mathcal{O}_{D} \longrightarrow 0
$$

so it is enough to show that $\left.\mathbb{R}^{i} \pi_{*} \mathcal{O}(-D)\right|_{D_{\alpha_{i}^{\vee}}}(Z)_{y}=0$ for all $i$. From the explicit description of $D_{\alpha_{i}^{\vee}}(Z)_{y}$ given in Theorem 6.6.1 and Proposition 6.6.8, it is enough to show that $\left.\mathcal{O}(-D)\right|_{D_{\alpha_{i}}(Z)_{y}}$ has degree 0 on the exceptional curve $\gamma$ of the blowup and degree -1 on every irreducible fibre of $D_{1}^{\prime} \rightarrow Z_{0}=\mathbb{P}^{1}$. But since $\Theta_{Y}$ is trivial on $D_{\alpha_{i}^{\vee}}(Z)_{y}$, we have a linear equivalence

$$
-\left.2 D\right|_{D_{\alpha_{i}^{\vee}}}(Z)_{y} \sim-D_{\alpha_{j}^{\vee}}(Z)_{y} \cap D_{\alpha_{i}^{\vee}}(Z)_{y}-D_{\alpha_{i}^{\vee}+\alpha_{j}^{\vee}}(Z)_{y} \cap D_{\alpha_{i}^{\vee}}(Z)_{y}=-E-\gamma,
$$

from which the claim follows by Lemma 6.7.6.
To complete the proof of (1) and (2), since $\psi_{Z, y_{*}} \mathcal{O}=\mathcal{O}$ in each case and $\chi_{Z}^{-1}(0) \rightarrow Z_{0}$ is affine, we have

$$
\chi_{Z}^{-1}(0)=\operatorname{Spec}_{Z_{0}} \pi_{*} \mathcal{O}_{\bar{D}_{y}}=\operatorname{Spec}_{Z_{0}} \pi_{*} \mathcal{O}_{D_{y}}
$$

for any choice of $y: \operatorname{Spec} k \rightarrow 0_{\Theta_{Y}^{-1}}$. Using Theorems 6.1.9 and 6.6.1, it is easy to see that

$$
\pi_{*} \mathcal{O}_{D_{y}} \cong \pi_{*} \mathcal{O}_{\left(D_{1}\right)_{y}} \times_{\pi_{*} \mathcal{O}_{E}} \pi_{*} \mathcal{O}_{\left(D_{1}^{\prime}\right)_{y}}
$$

where we have identified $\{y\} \times \operatorname{Pic}^{0}(E)=D_{\alpha_{i}^{\vee}}(Z)_{y} \cap D_{\alpha_{j}^{\vee}}(Z)_{y}$ with $E$ and by mild abuse of notation we have also written $\pi$ for the morphisms $\left(D_{1}\right)_{y} \rightarrow Z_{0},\left(D_{1}^{\prime}\right)_{y} \rightarrow Z_{0}$ and $E \rightarrow Z_{0}$. From the explicit descriptions of $\left(D_{1}\right)_{y}$ and $\left(D_{1}^{\prime}\right)_{y}$, it is clear that $\chi_{Z}^{-1}(0)$ is obtained in type $A$ by gluing two line bundles as claimed, and in types $B, C$ and $D$ by contracting the zero section of $\left(D_{1}\right)_{y}$ along $E \rightarrow Z_{0}$.

Let $p: \operatorname{Spec} k \rightarrow Z_{0}$ be any point, and choose a formally smooth morphism Spec $k \llbracket u \rrbracket \rightarrow$ $Z_{0}$ sending the closed point to $p$. In type $A$, the completed pullbacks of $\pi_{*} \mathcal{O}_{\left(D_{1}\right)_{y}}, \pi_{*} \mathcal{O}_{\left(D_{1}^{\prime}\right)_{y}}$ and $\pi_{*} \mathcal{O}_{E}$ are given by $k \llbracket u, v \rrbracket, k \llbracket u, w \rrbracket$ and $k \llbracket u \rrbracket$ respectively, with the maps to $\pi_{*} \mathcal{O}_{E}$ given by setting $v$ and $w$ to 0 . So

$$
R=k \llbracket u, v \rrbracket \times_{k \llbracket u \rrbracket} k \llbracket u, w \rrbracket \cong \frac{k \llbracket x, y, z \rrbracket}{(x y)},
$$

where $x=(v, 0), y=(0, w)$ and $z=u$, has a type $A_{\infty}$ singularity. This proves (1).
In types $B, C$ and $D$, if $E \rightarrow Z_{0}$ is unramified over $p$, then by Lemma 6.7.6, the completed pullbacks of $\pi_{*} \mathcal{O}_{\left(D_{1}\right)_{y}}, \pi_{*} \mathcal{O}_{\left(D_{1}^{\prime}\right)_{y}}$ and $\pi_{*} \mathcal{O}_{E}$ are given by $k \llbracket u_{1}, v_{1} \rrbracket \times k \llbracket u_{2}, v_{2} \rrbracket$, $k \llbracket u \rrbracket$ and $k \llbracket u_{1} \rrbracket \times k \llbracket u_{2} \rrbracket$ respectively, where the maps to $\pi_{*} \mathcal{O}_{E}$ are given by sending $v_{1}$ and $v_{2}$ to 0 and $u$ to ( $u_{1}, u_{2}$ ). So the ring

$$
R=\left(k \llbracket u_{1}, v_{1} \rrbracket \times k \llbracket u_{2}, v_{2} \rrbracket\right) \times_{k \llbracket u_{1} \rrbracket \times k \llbracket u_{2} \rrbracket} k \llbracket u \rrbracket \cong \frac{k \llbracket x, y, z \rrbracket}{(x y)},
$$

where $x=\left(v_{1}, 0,0\right), y=\left(0, v_{2}, 0\right)$ and $z=\left(u_{1}, u_{2}, u\right)$, again has a singularity of type $A_{\infty}$, and hence $\chi_{Z}^{-1}(0)$ has a singularity of type $A_{\infty}$ at $p$. If $p$ is a branch point of $E \rightarrow Z_{0}$, then (since we are assuming $k$ does not have characteristic 2 in these types) the completed pullbacks of $\pi_{*} \mathcal{O}_{\left(D_{1}\right)_{y}}, \pi_{*} \mathcal{O}_{\left(D_{1}^{\prime}\right)_{y}}$ and $\pi_{*} \mathcal{O}_{E}$ are instead given by $k \llbracket v, w \rrbracket, k \llbracket u \rrbracket$ and $k \llbracket w \rrbracket$ respectively, where the maps to $\pi_{*} \mathcal{O}_{E}$ send $v$ to 0 and $u$ to $w^{2}$. So

$$
R=k \llbracket v, w \rrbracket \times_{k \llbracket v \rrbracket} k \llbracket u \rrbracket \cong \frac{k \llbracket x, y, z \rrbracket}{\left(x^{2} y-z^{2}\right)},
$$

where $x=(w, 0), y=\left(v^{2}, u\right)$ and $z=(v w, 0)$, has a singularity of type $D_{\infty}$, and hence $\chi_{Z}^{-1}(0)$ has a singularity of type $D_{\infty}$ at $p$ as claimed.

To complete the proof, it remains to show that $E \rightarrow Z_{0}$ has 3 branch points in type $B$ and 4 in types $C$ and $D$. To see this, note that the composition with the coarse moduli space map $Z_{0} \rightarrow \mathbb{P}^{1}$ is a degree 2 morphism from a smooth elliptic curve over $k$ to $\mathbb{P}^{1}$ and is therefore branched over 4 isolated points since the characteristic of $k$ is not 2 . So in types $C$ and $D, E \rightarrow Z_{0}=\mathbb{P}^{1}$ is branched over these 4 points. In type $B$, on the other hand, one of the branch points must be the branch point of $Z_{0}=\mathbb{P}(1,2) \rightarrow \mathbb{P}^{1}$ (i.e., the stacky point), so $E \rightarrow Z_{0}$ is branched over the remaining 3 points.

## Bibliography

[A] M. F. Atiyah, Vector bundles over an elliptic curve, Proceedings of the London Mathematical Society s3-7 (1957), 414-452.
[ACV] D. Abramovich, A. Corti, and A. Vistoli, Twisted bundles and admissible covers, Communications in Algebra 31 (2003), 3547-3618.
[BB] A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, Comptes Rendus de l'Académie des Sciences 292 (1981), 15-18.
[B1] K. A. Behrend, Semi-stability of reductive group schemes over curves, Mathematische Annalen 301 (1995), 281-305.
[B2] E. Brieskorn, Singular elements of semi-simple algebraic groups, Actes du Congrès International des Mathématiciens (Nice, 1970) 2 (1971), 279-284.
[BG] A. Braverman and D. Gaitsgory, Geometric Eisenstein series, Inventiones Mathematicae 150 (2002), 287-384.
[BM] K. Behrend and Y. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Mathematical Journal 85 (1996), 1-60.
[BS] A. Borel and T. A. Springer, Rationality properties of linear algebraic groups II, Tôhoku Mathematical Journal 20 (1968), 443-497.
[BZN1] D. Ben-Zvi and D. Nadler, Beilinson-Bernstein localization over the Harish-Chandra center, 2012. arXiv:1209.0188.
[BZN2] , Elliptic Springer theory, Compositio Mathematica 151 (2015), 1568-1584.
[C] J. Campbell, A resolution of singularities for Drinfeld's compactification by stable maps, 2016. arXiv:1606.01518.
[DM] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Publications Mathématiques de l'IHÉS 36 (1969), 75-109.
[E] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, SpringerVerlag, New York, 1991.
[FM1] R. Friedman and J. Morgan, Holomorphic principal bundles over elliptic curves, 1998. arXiv:math/9811130.
[FM2] , Holomorphic principal bundles over elliptic curves II: the parabolic construction, Journal of Differential Geometry 56 (2000), no. 2, 301-379.
[GD] A. Grothendieck and J. Dieudonné, Éléments de Géometrie Algébrique I: Le langage des schémas, Publications Mathématiques, Institut des Hautes Études Scientifiques, 1960.
[GR] D. Gaitsgory and N. Rozenblyum, A Study in Derived Algebraic Geometry. Volume I: Correspondences and Duality, Mathematical Surveys and Monographs, American Mathematical Society, Providence, Rhode Island, 2017.
[GSB] I. Grojnowski and N. I. Shepherd-Barron, Del Pezzo surfaces as Springer fibres for exceptional groups, 2018. arXiv:1507.01872.
[HS1] S. Helmke and P. Slodowy, On unstable principal bundles over elliptic curves, Publications of the Research Institue for Mathematical Sciences, Kyoto University 37 (2001), 349-395.
[HS2] , Loop groups, elliptic singularities and principal bundles over elliptic curves, Banach Center Publications 62 (2004), 87-99.
[K] B. Kostant, Lie group representations on polynomial rings, American Journal of Mathematics 85 (1963), 327-404.
[L1] A. Langer, Semistable principal G-bundles in positive characteristic, Duke Mathematical Journal 128 (2005), 511-540.
[L2] E. Looijenga, Root systems and elliptic curves, Inventiones Mathematicae 38 (1976), 17-32.
[L3] G. Lusztig, Character sheaves I, Advances in Mathematics 56 (1985), 193-237.
[LMB] G. Laumon and L. Moret-Bailly, Champs algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge/ A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, Heidelberg, 2000.
[LS] Y. Laszlo and C. Sorger, The line bundles on the moduli of parabolic G-bundles over curves and their sections, Annales Scientifiques de l'ÉNS 30 (1997), 499-525.
[M] D. Mumford, Abelian Varieties, Tata Institute of Fundamental Research Studies in Mathematics, Oxford University Press, Bombay, 1970.
[MN] K. McGerty and T. Nevins, Derived equivalence for quantum symplectic resolutions, Selecta Mathematica 20 (2014), 675-717.
[O1] M. Olsson, Sheaves on Artin stacks, Journal für die reine und angewandte Mathematik 603 (2007), 52-112.
[O2] _ Algebraic Spaces and Stacks, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, Rhode Island, 2016.
[RR] S. Ramanan and A. Ramanathan, Some remarks on the instability flag, Tôhoku Mathematical Journal 36 (1984), 269-291.
[S1] P. Slodowy, Four lectures on simple groups and singularities, Communications of the Mathematical Institute, Rijksuniversiteit, Utrecht, 1980.
[S2] ___ Simple Singularities and Simple Algebraic Groups, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, 1980.
[S3] T. A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Inventiones Mathematicae 36 (1976), 173-207.
[S4] _ A construction of representations of Weyl groups, Inventiones Mathematicae 44 (1978), 279-293.
[S5] R. Steinberg, Regular elements of semi-simple algebraic groups, Publications Mathématiques de l'IHÉS 25 (1965), 49-80.
[S6] _ Conjugacy Classes in Algebraic Groups, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

