Notation

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NOTATION, (in mathematics) the art of adapting arbitrary symbols to the representation of quantities, and the operations to be performed on them. The numerous symbols, which form the language of analysis in the present advanced stage of that science, have been produced by the gradually increasing wants of those who have cultivated it; new and more extensive views continually opening, have compelled them to contract modes of notation already in existence, and the improvements thus introduced into notation, have in their turn directed the attention of the mathematicians to wider and more general views of the science.

The advances towards perfection which have been made in the language of analysis, and the generalization which has been introduced into its principles, alternately acting on each other, as cause and effect, have by their combined influence produced a language of unrivalled power, enabling the mind to carry on processes of deductive reasoning of almost unlimited length, with scarcely the fear of an error, and with a conviction, that, should accident have introduced one, a careful revision will not fail to eradicate it.

Brevity appears to have been the directing principle which guided the early cultivators of the algebraic art; unaware of the immense importance which, in a subsequent state of the science, would be attached to the language whose foundations they were thus unconsciously laying, they contented themselves with avoiding the tedious repetitions of the same words, by employing one or two of the initial, or, in some cases, of the final letters, to denote them. Such was the case with Diophantus, the earliest author on algebra, whose writings have descended to us. The unknown quantity he denominates $\alpha\rho\iota\theta\mu o\varsigma$, and to avoid repeating it uses the final letter ς. He also uses the sign ⋔ or the inverted ψ to denote minus, obviously from the circumstance of its being a prominent letter in the Greek word $\lambda\varepsilon\iota\psi\iota\varsigma$. This author has no sign to denote plus, but uses the word at length. He has represented the various power thus, δ^{ω}, K^{ν}, $\delta\delta^{\nu}$, &c., meaning the square, cube, fourth power, &c., these letters are the same as those which commence the words square, cube, &c.

The earliest algebraical writer after the invention of printing, was Lucas Paciolus, or de Burgo, he uses *p* to signify plus and *m* for minus, and indicates the various power by their two first letters. Such was very nearly the notation employed by his successors, Cardan, Tartaglea, and Ferrari. Stifelius, a German, who published a work, entitled Arithmetica Integra, Norimburg, 1544, added considerably to the use of signs; according to Dr. Hutton he is the first writer who employed the signs $+$ and $-$ and also $\surd$, to designate the root of a quantity. He had no sign to represent equality, a deficiency afterwards supplied by Robert Recorde; we also owe to him the vinculum $\overline{a+b}$ to connect compound quantities, the other mode by means of parentheses $(a+b)$ being afterwards proposed by Girarde. When Stifelius treated of several variables, he denoted them by the letters A, B, C, &c.

Bombelli appears to have made a most valuable innovation in the method of denoting powers, rejecting the plans of attaching their initials to the radix, he marks them thus, $\underset{\smile}{1}$, $\underset{\smile}{2}$, $\underset{\smile}{3}$, $\underset{\smile}{4}$, &c., an improvement, perhaps more valuable than any which has yet been noticed; nearly a similar plan was adopted by Simon Stevin, who denoted the power of the unknown quantity thus, ⓪, ①, ②, ③, and he observed that the power, whose index is zero, is equal to unity. He, however, went a step beyond his predecessors, and denoted roots by fractional indices; thus $\bigcirc\!\!\!\!\!\!\tfrac{1}{2}$ $\bigcirc\!\!\!\!\!\!\tfrac{1}{3}$ with him represent the square and cube roots.

Vieta flourished several years later than Stevin, he added greatly to the science he cultivated, although he did not avail himself of all the improvements in notation, which existed previously. The most important alteration which he introduced was that of denoting *known* as well as *unknown* quantities by letters; for

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the former he employed the consonants, and for the letter the vowels. He appears also to have made the important remark, that negative exponents, perform the same office as positive ones, although his view seems rather to have been restricted to whole numbers.

It is a curious circumstance, that the symbol which now represents equality, was first used to denote subtraction, in which sense it was employed by Albert Girade, and that a word, signifying equality, was always used instead of a sign, until the time of Harriot, who at the same time proposed the signs $>$ and $<$ to denote "greater than" and "less than."

Oughtred, in his "Clavis" appears first to have employed the $\times$ for multiplication, he sometimes also joins the letters without any intervening sign for the same purpose.

Such was the origin of the more common signs, which we now employ. An example of the different modes of expressing the same equation will exhibit this more clearly.

Paciolus	$1\,cu.\ m.\ 6ce.$ P. $11co.$ eguale $6n^i$
Stifelius	$1\,\mathfrak{c} - 6\mathfrak{z} + 11\,\mathfrak{r}$ equantur 6
Bombelli	$1\,\underset{\smile}{3}\ .m.\ 6\underset{\smile}{2}.\ p.\ 11\,\underset{\smile}{1}$ eguale 6
Stevinus	$1 \textcircled{3} - 6 \textcircled{2} + 11 \textcircled{1}$: egale 6
Vieta	$1\,C - 6Q + 11$ N egal 6
Harriot	$1.aaa - 6.aa + 11.a = 6$
Modern	$x^3 - 6x^2 + 11x = 6$

Since the time of Harriot few alternations have been made in the mode of denoting the more simple operations of analysis, although the science itself has received great improvements. The first step which necessitated the invention of new methods of denoting operations was the fluxional or differential calculus. Newton and Leibnitz each gave a notation to this new branch of analysis, and their respective followers entering with more zeal than judgment into the unfortunate controversy respecting the claim of priority of invention, pertinaciously adhered to the notation of their masters without inquiring into their relative merits, and probably without the knowledge requisite to form a correct judgment on the question, had they turned their attention to the inquiry. In fact, the state of the science at the period at which these modes of calculation were given to the world, could not admit of the formation of a correct estimate of their relative value, and the few reasons which could then be adduced, were, on the whole, more favourable to the fluxional notation. It requires, even at the present time, an extensive acquaintance with the subsequent discoveries of the successors of Newton and Leibnitz, and an enlarged view of the present state of mathematical science, duly to appreciate the reasons which concur in rendering the notation of Leibnitz decidedly the best of any which has hitherto been proposed for accomplishing the same object.

As the proper adaptation of notation to the object of inquiry has contributed greatly to the progress of analysis, we propose offering a few general rules for the consideration of those who may have occasion to express new relations, or who may desire to abbreviate those already in use; and we shall compare several of the notations already received with these principles.

The great object of all notation is to convey to the mind, as speedily as possible, a complete idea of operations which are to be, or have been, executed; and since every thing is to be exhibited to the eye, the more compact and condensed the symbols are, the more readily will they be caught, as it were, at a glance.

The first and most obvious rule is, that *all notation should be as simple as the nature of the operations to be indicated will admit*. The reason of this is sufficiently evident, and it scarcely admits of an exception. It must, however, be remarked, that it is, in many cases, absolutely impossible to express the complicated operations required in the highest departments of analysis by formulae that can be called simple. Still, however, they may be simple with reference to the multiplied relations they express.

The next rule we shall propose is, *that we must adhere to one notation for one thing*. It is particularly unphilosophical, and completely contrary to the whole spirits of symbolic reasoning, to employ the same signs for the representation of different operations, yet instances of the infringement of this rule are occasionally met with; and as several examples are to be found in the works of an author of the very highest reputation, it is more particularly necessary to mark this deviation from the propriety of analytical language.

The notation employed by Lagrange to explain the differential calculus will be considered on another occasion. It is curious to observe, that his zeal for introducing the system of accentuation compelled him to abandon a most valuable contrivance which he had formerly employed for designating the variations of quantities which harmonized with the notation of Leibnitz, whilst that which he substituted for it, (the system of dots) besides the inconveniences which it possessed in its original signification, had the additional disadvantage of having been employed in a different sense by all the followers of Newton. In the first edition of the *Théorie des Fonctions*, this notation was not employed, but in the *Leçons sur le Calcul des Fonctions*, published in 1806, the fluxional notation is used for explaining the calculus of variations. The same author has also assumed the quadrant as the unity of angular measure, and represented $\sin\frac{\pi}{2}$: thus, $\sin.$ 1, which, unless previously defined, would indicate the sine of an arc equal to the radius.

Another instance of a breach of this rule is to be observed in the *Théorie des Nombres* of Legendre, who throughout that highly valuable work, employs the sign $=$ in two different senses; first, in its ordinary acceptation; and, secondly, he places it between two quantities, to denote, that when they are divided by the same quantity, they will leave the same remainder. This relation of quantity is of frequent occurrence in the theory of numbers; and the necessity of denoting it with brevity, induced the author of the *Disquisitions Arithmeticae* to invent the symbol $\equiv$; and to adjoin (inclosed in brackets) the quantity which is used as the divisor. A different method has been adopted by Mr. Barlow, who, in his *Treatise on the Theory of Number*, uses the double *f* (*ff*) placed in a horizontal position. These three methods of denoting the same relation stand thus:

L. $a^n = -1$ G. $a^n \equiv -1\,(\text{mod.}p)$ B. $a^n \rightrightarrows pv - 1$. The first of these sins unpardonably against the rule we are endeavouring to enforce; it is much more inconvenient than the use of dots by Lagrange, because, in this instance we are considering, the same symbol $=$ is used in the *same page*, and even in the *same line* in *two different senses*. It has also the disadvantage of requiring the divisor to be expressed in words.

This innovation in the use of a well known sign, probably arose from too strict an adherence to an admitted rule: *Not to multiply the number of mathematical signs without necessity*. We may, however, here be permitted to observe, that the necessity existed; and it is acknowledged by Gauss, whose notation, although much preferable to the one we have been criticising, is decidedly inferior to that of Mr. Barlow. The preference we have given to this latter is also supported by another rule too evident to require much argument. *When it is required to express new relations that are analogous to other for which signs are already contrived, we should employ a notation as nearly allied to those signs as we conveniently can*.

That analogy ought to be our guide in the formation of all new notations, is a truth, which, like many others, has been felt and acted upon, although it may not have been stated in express terms: and it was probably this feeling which induced Stifelius to inquire into the meaning of negative exponents, the consequence of which was the establishment of the connection between the direct and reciprocal power, a deduction which, when enlarged by the consideration of fractional exponents, was no mean addition to the state of algebra at the time it was suggested.

It has long been usual to denote known quantities by the first letters of the alphabet, and to represent unknown ones by the last letters. In the sixteenth and seventeenth centuries, the practice used to be to employ the vowels for the unknown, and the consonants for the known quantities. This is in itself a matter perfectly arbitrary, and, provided it is adhered to, either method of distinguishing known from unknown quantities is equally proper; but whenever one of these, suppose that now in use, is fixed upon for this purpose, if we wish to consider known and unknown functions, and to treat of their relations, it is no longer a matter of indifference how they are to be distinguished. The rule we are now illustrating must be attended to, and if we have used the first letters for the alphabet for known quantities, it compels us to employ the first letters of whatever other alphabet we may fix on to indicate functions for the known ones, and the latter letters of the same alphabet for unknown ones. Thus, if *a, b, c,* &c. are known quantities, and *x, y, z,* &c. unknown ones, α, β, γ, &c. must denote known functions, and ϕ, χ, ψ unknown ones. In another instance, the repetition of a letter *xx* has been marked thus x^2. Analogy, therefore, would direct us when we wish to repeat an operation as $\psi\psi$, to contract it by writing it thus,ψ^2, and similarly for others, as ψ^3, ψ^4.

We have already had occasion to remark that we ought not *to multiply the number of mathematical symbols without necessity*. This is a maxim of so much importance that it deserves a fuller consideration.

The natural tendency of the science is to develop new relations and new combinations of those already known. When these new relations involve complicated combinations of such as are already received, or when they are of frequent occurrence, it becomes necessary, if it were merely for the sake of brevity, that some new symbol should be employed; it is, however, a necessity that ought to be avoided as much as possible; and the rule is one which has suffered many infractions in modern times. Amongst the numberless instances may be remarked the case of a function consisting of the product of a number of terms in arithmetical progression. This species of function is of some importance, and has been frequently considered, yet scarcely any two authors have agreed in the notation they have employed. The first, in point of times, is that of Vandermonde. (*Acad. des Sciences*, 1779). He expressed the quantity, $x.x-a.x-2a\ldots x-\overline{n-1}a$ by the sign $[x,a]^n$, or when $a=1$, more shortly, thus $[x]^n$. Kramp, in his treatise on *Astronomical Refractions**, calls the functions in question *facultées numeriques*, and denote them thus $x^{n1a}=x.x+a\ldots x+\overline{n-1}a$. Arbogast has occasion to treat of the same quantities†, when he represents thus $D^n1^x=x.x-1\ldots x-\overline{n-1}$.

Legendre, in his *Exercises du Calcul Integral*, employs a different notation to express a function very nearly allied to that we are considering. He denotes $x.x-1\ldots 2.1$ by $\gamma(x.)$ Euler also treated of the properties of the co-efficients of a binomial, and denoted them by the characters $\left(\frac{n}{1}\right),\left(\frac{n}{2}\right),\left(\frac{n}{3}\right)$, &c. Of these various methods of denoting the same thing, that of Vandermonde seems to be the most advantageous. The reasons of this preference we shall shortly state. When *x* is repeated *n* times, $x.x\ldots x$, it is represented by x^n; and generally when any letter denoting either operation or quantity is repeated, it has the letter indicating the number of repetitions placed in the right hand corner above it; such is the case in the following instance, x^n, d^n, δ^n, f^n. It is therefore correct, as far as regards analogy, to place the index in that situation; so far then as relates to the position of the letter *n*, both notations of Vandermonde and that of Arbogast are equally admissible; this however, is not the case with the letter *a*. It is used for the difference of two successive factors, and has no regard at all repetition; it is not therefore consistent with analogy to place it in a similar situation.

Another, and a very powerful reason for preferring the system of Vandermonde is, that it admits of unlimited extension, without in the least infringing on typographical facility. When the factors have their second difference zero, they are denoted thus:

$$[x,a]^n = x.x-a\ldots x-\overline{n-1}.a$$

M. Lacroix has assigned to such expressions the appropriate name of power of the second order. If the third difference are zero, they are called powers of the third orders, and are readily denote thus:

$$[x,a,b]^n = \textcircled{x}\overline{x+a}.\overline{x+2a+b}\ldots\overline{x+na+\frac{n.\overline{n-1}.b}{2}}$$

and similarly for the higher orders; this notation therefore admits of a very easy extension to complicated cases without increasing its difficulty of execution. Perhaps it may be urged that the notation of M. Kramp may be extended as follows:

$$x^{nIa,b} = x.x\overline{-a}.x\overline{-2a-b}\ldots x\overline{-na-\frac{n.n-1}{2}b}$$

but the difficulties of the printer will be greatly increased the moment any substitutions are made for *a* or *b*, and the other objections remain the same.

These inconveniences attend the method of Arbogast in at least an equal degree. That which Euler has employed is adapted to the particular purpose he has in view; but if it be allowed that sufficient reasons have been produced to justify the preference we have assigned to that of Vandermonde, it is highly desirable that it should be employed on all future occasions. We have observed that M. Legendre uses $\gamma(x)$ to represent $x.x-1\ldots 2.1$, whether *x* is a whole number or a fraction; this can scarcely be termed a new notation; but

*Strasburg, 1798

†Calcul des Derivations, 1800

it is certainly rather to be regretted that so able an inquirer should not have given his sanction to a notation previously established, and which agrees so well with those rules on which all notation should be founded.

The two signs which denote differentiation and integration, are, in strictness, liable to the objections of this rule; and a similar remark is applicable to the similarly related signs Δ and Σ.

In several of these instances, mathematical signs have been needlessly multiplied by not attending to this principle, that *whenever we wish to denote the inverse of any operation, we must use the same characteristic with the index* — 1. This principle being acknowledged, we shall in future be delivered from one cause of the redundancies of signs. Should, for example, any future inquirer, when considering the calculus of variations, have occasion for the inverse operation of that denoted by δ, he must not, conformably to a hasty view of analogy, represent it by σ; but, directing his attention more deeply to the subject, he will perceive the necessity of indicating it by δ^{-1}.

Again, in the equation $x = \tan y$, the value of *y*, in terms of *x* is usually expressed $y = \arctan x$; this is both long and inelegant. It should, in complying with the rule just laid down, be written thus: $y = \tan^{-1} x$, and similarly with other circular function; this method is likewise attended with the advantage, that all this class of function will then be indicated by three letters.

Another principle, whose importance becomes eminent in proportion as our investigations become general, is, *that every equation ought to be capable of indicating a law*. Signs must not be employed at random, nor must new ones be introduced without grave necessity. When, however, unusual combinations, or the demonstration of new properties, render such a resource indispensable, it becomes particularly desirable that those which are first contrived, should be possessed of such propriety and power as shall effectually preclude future innovators from all temptation to change them. Analogy to those which form the established language of the science, although a precept of great importance, cannot be admitted to supersede the rigid enforcement of that we are now considering: happily however, the two principles will rarely be found at variance, for those symbols, and those inflexions of symbols, (if the term may be allowed,) which long experience has naturalized, generally furnish the most correct models of imitation. A precept, in some measure connected with this principle, although it may perhaps be considered of minor importance, may be introduced.

In case it is required to express a series continued to *n* terms, the *n*th terms should be written, and not the words "to *n* terms" be attached to a few of the first. These different methods would appear thus:

$$y = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} +, \text{\&c. to } n \text{ terms.}$$
$$y = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n}.$$

In this simple case, something is gained in point of brevity; and if deductions are to be made from it, much is gained in perspicuity. The justice of this observation will be more readily acknowledged if it is put into rather different language. Its object may be thus stated: *It is better to make any expression an apparent function of* n, *than to let it consist of operations* n *times repeated*. Thus, if it is required to express the series

$$\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$$

By means of a definite integral, its value would be

$$\int \frac{\mathrm{d}x}{x} \int \frac{\mathrm{d}x}{x} \ldots (n) \int \frac{\mathrm{d}x}{1-x} \qquad \begin{bmatrix} x=0 \\ x=1 \end{bmatrix}$$

Where there are *n* integrations, or as it has been written by Mr. Spence,

$$\int \frac{\mathrm{d}x}{x} \int \frac{\mathrm{d}x}{x} \cdots \int \frac{\mathrm{d}x}{1-x} \qquad \begin{bmatrix} x=0 \\ x=1 \end{bmatrix}$$

Neither of these methods possess that clearness which ought instantly to convey to the mind the system of operations intended to be expressed; for it does not appear quite certain whether the *n* means that there are *n* repetitions of $\int \frac{\mathrm{d}x}{x}$, or that the whole number of integrations is *n*: this latter is the signification in which it is employed by Spence. A better method of expressing it would be,

$$\left(\int \frac{1}{x}\right)^n \frac{x}{1-x}\,\mathrm{d}x^n, \text{ or } \left(\mathrm{d}^{-1}\frac{1}{x}\right)^n \frac{x}{1-x}\,\mathrm{d}x^n$$

To any one acquainted with the integral calculus, this could not convey an incorrect idea, although presented for the first time, and without explanation. The index *n* denoting the repetition of the quantity to which it is attached, gives

$$\int \frac{1}{x} \int \frac{1}{x} \cdots \int \frac{1}{x} \frac{x}{1-x}\,\mathrm{d}x^n.$$

And it is clear that *n dx s* must be assigned to the *n* integrations. This latter expression may indeed be employed as deviating still less from the received one, and as still less liable to misinterpretation.

Another principle, which is chiefly valuable as it conduces to brevity, is, *That all notation should be so contrived as to have its parts capable of being employed separately*. By observing this rule, we avoid being encumbered with any other signs than those actually necessary to express the properties we are considering. This principle has been so universally followed, that it is difficult to produce examples of its neglect. Thus, although it has never been expressly proposed in terms, it has been virtually acted upon by analysts for the last two centuries.

It was in fact to be expected, before any general principles of notation were delivered, that it should be extended in exact proportion to the progress of the inquirer, and this progress proceeding from the simple to the more complicated, his notation would naturally increase by continued additions. Such being its origin, it will necessarily follow, that at any stage it might be used without reference to those additions with which subsequent considerations had obliged him to augment it.

It is a valuable service rendered to any science to embody into language those rules which perhaps insensibly direct or govern the minds of those who improve it; however apparently obvious some of them may be, or however well known amongst the instructed, a considerable time will always elapse before individual penetration will have formed for itself such guides; nor will the advantage be entirely confined to the student. How frequently does it happen, even to the best informed, that they prefer one thing and reject another, from some latent sense of their propriety or impropriety, without being immediately able to state the reasons on which such choice is founded; yet it cannot be doubted, when the selection appears to be the result of correct taste, that it is guided by unwritten rules, themselves the valued offspring of long experience. Any explanation of these is probably rendered unavailing at the instant, for want of having previously fixed them by language.

Another reasons which appears to justify, or rather to oblige us to notice this principle, is, that although it has not hitherto been much infringed, it seems probable that, unless due care be take in the future formation of notation, some of the rules which have been proposed, may themselves lead to its infraction. This observation more particularly applies to that which directs us "to contrive all notation, so that it shall be capable of expressing laws." In attempting to satisfy this condition, we are obliged to take wide and extensive views, and are therefore peculiarly liable to forget that which may appear of minor importance. The two principles are by no means incompatible; and it is very desirable that when we have occasion to employ the latter, the former may always be borne in mind.

Having discussed the general principles on which notation should be formed, it now becomes necessary to explain several rules which are proposed for the proper application of individual signs. Writers on algebra have defined the meaning of the signs they use; but something further than these definitions is requisite in the present state of the science, or, to speak more accurately, some limitations, as well as extensions of the meaning of several of them are required.

There are in analysis two great divisions of symbols, – those which denote quantity, and those which indicate operations. These latter were all of them originally arbitrary marks, such as those employed to express addition, multiplication, &c. At the present time, however, letters are frequently employed to signify operations; and hence arises occasionally a source, if not of error, at least of inconvenience. The use of letters in two senses has been objected to the followers of Leibnitz, who employed the *d* to denote the differential of the quantity to which it is prefixed; and as differentials are much employed in the doctrine of series, it sometimes occurs that the fourth term of the series, $a+bx+cx^2+dx^3+$, enters into calculations in which the *d* also represents an operation preformed on the letter to which it is prefixed. This inconvenience has been avoided by M. Lacroix, who uses the Roman d for differentiation, and the Italic *d* for quantity. Arbogast also, in his work on Derivations, employs the Roman d for the same purpose. This completely removes the inconvenience; but in order to prevents its recurrence in any other shape, and for the purpose of affording a wider choice in the selection of letters, we would propose the following general rule: *All letters that denote quantity should be printed in Italic, but all those which indicate operations, should be printed in a Roman characters*. Thus the letters $x, y, z, \ldots a, b, c, \ldots \alpha, \beta, \ldots \psi, \theta$, would represent quantity; and d, f, A, B, C, would denote operations. The only inconvenience which would attend this method, is of very minor importance: it would become necessary to cast new types for a few of the Greek letters in common use, because the difference between those two modes of printing does not exist in that language.

The class of letters that are termed characteristics, and which represent operations, is divides into two species; those which denote functions, as

$$\mathrm{f}, \mathrm{F}, \phi, \psi, \chi,$$

and those which signify that some alternation is to be made, which depends on the nature of the function on which it is to be executed: thus $f(1+x^2)$ means a function of $1+x^2$. In this case form of the function selected for *f* is quite independent on $1+x^2$; but if we have $\mathrm{d}(1+x^2)$ the effect of the operation denoted by d, depends on the form of the function incolsed within the parentheses: this is the case with symbol

$$\mathrm{d}, \int, \Delta, \Sigma, \mathrm{S}, \delta.$$

Those of the former species are called *functional characteristics*; and as it will be convenient to have some generic name for the latter, we shall appropriate to them that of Derivative Characteristics, since their own form depends on, and is derived from that which follows them.

The first of these species is subject to the following rule: *Every functional characteristics affects all symbols which follow it, just as if they constituted one letter*.

Thus, if $\alpha x = ax + bx^2$, and $fx = 1 - x^2$,

Then, $\alpha fx = a(1-x^2) = a(1-x^2) + b(1-x^2)^2$

And $f\alpha x = 1 - (ax)^2 = 1 - (ax + bx^2)^2$.

Two signs have been made use of to represent multiplication: the cross $(\times)$ and the dot$(\cdot)$; and the simple juxtaposition of two letters, if they are in Italic, means the same thing: This last method is preferable in all simple cases, as

$$abc, \qquad x^2y \qquad \frac{a+x}{y}a^2$$

There exists a difference of meaning between the two former of these methods of medicating multiplication and the latter, which it is necessary to explain, and to which it is convenient to adhere. The dot and the cross imply a kind of disjunctive multiplication, or that, when they are interposed between two quantities before which a derivative characteristic is placed, this symbol only applies to the first of the two:

As, $$\Delta\frac{x+a}{x}. \quad \frac{\varepsilon^x - 1}{x}$$

which represents $$\frac{\varepsilon^x - 1}{x}\Delta\frac{x+a}{x}$$

If the dot or cross were not used in this case, the meaning would be quite different, for $\Delta\frac{x+a}{x}\ \frac{\varepsilon^x-1}{x}$ would signify the complete difference of the quantity $\frac{x+a}{x^2}\ (\varepsilon^x - 1)$.

Another common signification of the dot is, that when it is placed immediately after a derivative characteristic, it implies that the derivation extends until some other dot occurs which may separate it from any succeeding factors: This is sometimes denotes by brackets of various kinds, thus,

$$\mathrm{d}.xv.\ \frac{\mathrm{d}u}{\mathrm{d}x}$$

denote the differential of *u* relative to *x*, multiplied by the complete differential of *xv*. If written thus,

$$\mathrm{d}.xv\ \frac{\mathrm{d}u}{\mathrm{d}x}$$

it would signify the complete differential of $xv\ \frac{\mathrm{d}u}{\mathrm{d}x}$.

In the operations of the differential calculus, it is constantly required to express the second and higher powers of a differential, as $(\mathrm{d}x)^2, (\mathrm{d}x)^3, (\mathrm{d}x)^n$, whilst it has rarely been found necessary to express the differentials of the simple powers $\mathrm{d}(x^2), \mathrm{d}(x^3), \ldots \mathrm{d}(x^4)$. These two series differ in the situation of the parentheses they contain; but as the quantities in the first recur perpetually, it was desirable that they should be contracted by omitting the parentheses. Universal usage has sanctioned this omission, and they are always written thus, $\mathrm{d}x^2, \mathrm{d}x^3, \ldots \mathrm{d}x^n$; and in order to remove the ambiguity which this might occasion in the few cases where the quantities mentioned in the latter series occur, they are distinguished by means of the point thus, $\mathrm{d}.x^2, \mathrm{d}.x^3, \ldots \mathrm{d}.x^n$.

The principle that *parentheses may be omitted, if it can be done without introducing ambiguity*, has been partly adopted in the manner of expressing circular functions; a department of analysis in which the inattention to all general principles of notation has been severely felt. Five methods have been generally employed for expressing the squares and higher powers of the sines and cosines of arcs.

$$(\sin .\theta)^2,\ \sin .\theta^2,\ \sin .^2\theta,\ (\sin\theta)^2,\ \sin\theta^2.$$

The first of these has nothing objectionable in it except the dot, which is not merely useless, but interferes with another principle. The second omits the parentheses, in compliance with the principle we have just stated: like the former, it has the useless appendage of the dot. The third is by far the most objectionable of any, and is completely at variance with strong analogies. An index in the situation in which it there occurs, invariably denotes repletion of the symbol to which it is attached: in this case it would therefore mean $\sin . \sin .\theta$, or the sine of the sine of the arc θ. It is true that this notation may be defined to mean the square of the sine, or any other function of the sine. Although a definition cannot be false, it may be improper; and the impropriety may arise either from its inducing ambiguity, or from its offending against received principles; both which objections occur in the present instance. Besides, if $\sin^2\theta$ were allowed to signify the square of the sine of θ, how should we denote the second sine of θ? It may safely be asserted, that if this notation for the square of the sine were admitted, no convenient method could be devised, which should not also infringe some general principle. Thus one infraction would become the ground of introducing another, and instead of possessing a philosophical language founded on general laws, the science would arrives at one with no regularity to assist the memory, and devoid of those strong analogies which facilitate the acquisition of all others.—The fifth method, $(\sin\theta)^2$, is made use of by Arbogast, in his work on the Calculus of Derivations; an authority which is entitled to the greater weight, because, from the nature of that excellent work, and the powerful use which is made of the various notations employed in it, it is highly improbable that the author should have selected this out of the many received notations, without having well considered its propriety. It differs but little from the last method, the parentheses only being omitted, which gives $\sin\theta^2$ a mode of expression sufficiently clear for all simple cases. If however θ become a compound quantity, the parentheses must again be introduced. This notation has been employed by M. Lacroix, in his great work on the Integral Calculus. M. Gauss and Delambre have also adopted it; and it is likewise used in the latest and most valuable work of language[‡].

[‡]Mecanique Analytique.

In order to connect together various symbols which compose part of an expression, and which are subjected to the same operation, different species of vincula[§] are employed. From the great variety of these which are used by the printer, a considerable latitude of choice is allowed. In general, all symbols collected under a vinculum, are to be considered and operated upon, as if they formed a single symbol. The choice of parentheses although a matter of minor importance, is not altogether arbitrary. An example, in which propriety and impropriety are apparent in different modes of writing the same expression, will be sufficient, without the aid of formal rules.

$$\left[1-\left(\tan\frac{a}{y}\right)^2\right]^2+\left\{1-\left\{\tan\frac{y}{a}\right\}^2\right\}^2$$

$$\left(1-\left[\tan\frac{a}{y}\right]^2\right)^2+\left[1-\left\{\tan\frac{y}{a}\right\}^2\right]^2$$

$$\left\{1-\left(\tan\frac{a}{y}\right)^2\right\}^2+\left\{1-\left(\tan\frac{y}{a}\right)^2\right\}^2$$

Of these three ways of writing the expression, the last is evidently to be preferred: the other two are objectionable, not from any deficiency in clearness, but from a certain want of symmetry, which is very perceptible in both of them.

Modern analysts have frequently had occasions to separate the symbol of operation from the quantities on which they act, an artifice which gives great conciseness to a variety of theorems: this separation has been marked in different ways, either by a dot or the sign of multiplication, or by parentheses: in some instances the double dot (:) has been employed. As it is very desirable to confine each symbol to one meaning, the last plan, or the two dots, is the least objectionable for this purpose, and can create no ambiguity, since it has becomes entirely obsolete in the best writers in any other sense. The theorems of Language would be thus written.

$$\Delta^n u=\left\{\begin{array}{c}\frac{\mathrm{d}}{\mathrm{d}x}\\ \varepsilon-1\end{array}\right\}^n : u, \qquad \Sigma^n u=\left\{\begin{array}{c}\frac{\mathrm{d}}{\mathrm{d}x}\\ \varepsilon-1\end{array}\right\}^{-n} : u.$$

The merits of the two systems of notation employed in the differential calculus have been discussed by Euler, in the preface to his Institutiones Calculi Integralis, and more fully by Mr. Woodhouse, in the preface to the Analytical Calculations. The superiority of the use of the d over the system of dots has become so very apparent, that the course of a few years will, in all probability, render the latter obsolete, a circumstance which induces us to forbear entering into any lengthened detail of their comparative values, and to content ourselves with merely indicating a few of the grounds for rejecting the latter: these are, 1st, The uncertainty which may arise respecting the letters to which the indices refer, and the confusion which arise from having two indices to a letter. 2d, The want of analogy with other established notations, such as those relating to the symbols Δ and δ. 3d, The great difficulty, if not the impossibility, of representing, by their means, theorems relating to the separation of operations from quantities[¶].

[§] Such as braces, parentheses, brackets, bars, &c.

[¶] The Editor has been indebted for this interesting Article to CHARLES BABBAGE, Esq. F. R. S. Lond. and Edin.