

Exodromy for stacks

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Abstract

In this short note we extend the Exodromy Theorem of [3] to a large class of stacks and higher stacks. We accomplish this by extending the Galois category construction to simplicial schemes. We also deduce that the nerve of the Galois category of a simplicial scheme is equivalent to its étale topological type in the sense of Friedlander.

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0 Introduction

In [3], we identified a profinite category $\mathrm{Gal}(X)$ attached to any scheme¹ X [2; 3, Construction 13.5]. The profinite category $\mathrm{Gal}(X)$ classifies nonabelian constructible sheaves on X (our *Exodromy Equivalence* [3, Theorem 11.7]) and the protruncated classifying space of $\mathrm{Gal}(X)$ recovers the étale topological type of X in the sense of Friedlander [9]. A natural question, then, arises: what is the analogue of this construction for a simplicial scheme or stack? For example, what is the correct exodromy representation corresponding to an equivariant constructible sheaf on a scheme with an action of a group scheme?

Here, we answer this question by extending the Galois category construction and the Exodromy Theorem to a large class of stacks and higher stacks. Here is the basic construction.

0.1 Construction. Let Y_* be a simplicial scheme. Denote by $\mathrm{Gal}^\Delta(Y_*)$ the following 1-category. The objects are pairs (m, ν) consisting of an object $m \in \Delta$ and a geometric point $\nu \rightarrow Y_m$. A morphism $(m, \nu) \rightarrow (n, \xi)$ of $\mathrm{Gal}^\Delta(Y_*)$ is a morphism $\sigma : m \rightarrow n$ of Δ and a

¹All our schemes and stacks in this paper will be assumed to be coherent.

specialisation $\nu \leftarrow \sigma^*(\xi)$. This category has an obvious forgetful functor $\text{Gal}^\Delta(Y_*) \rightarrow \Delta$, which is a cartesian fibration. A morphism $(m, \nu) \rightarrow (n, \xi)$ is cartesian over $\sigma: m \rightarrow n$ in Δ if and only if the specialisation $\nu \leftarrow \sigma^*(\xi)$ is an isomorphism.

The fibre over $m \in \Delta$ is the category $\text{Gal}(Y_m)$, which we regard as a profinite category. (See [Definition 1.7](#) for the precise notion of categories fibred in profinite categories.)

Also attached to a simplicial scheme Y_* is the étale topological type of Y_* as constructed by Eric Friedlander [8, §4] and refined by David Cox [7], Ilan Barnea and Tomer Schlank [1], David Carchedi [5], and Chang-Yeon Cho [6]. The étale topological type of Y_* can be identified with the colimit in protruncated spaces of the simplicial object that carries $m \in \Delta$ to the protruncated étale homotopy type of Y_m (see [7, Theorem III.8]). Since the protruncated homotopy type of the fibres of the cartesian fibration $\text{Gal}^\Delta(Y_*) \rightarrow \Delta$ agree with the étale homotopy type of the schemes Y_m , it follows that the protruncated homotopy type of the the total category $\text{Gal}^\Delta(Y_*)$ is the colimit of this simplicial diagram. In other words:

o.2 Theorem. *The classifying protruncated space of $\text{Gal}^\Delta(Y_*)$ recovers the protruncated étale topological type of Y_* .*

This is a consequence of [Proposition 1.15](#) below. We will also show:

o.3 Theorem ([Proposition 2.5](#)). *If Y_* is a presentation of an Artin n -stack X , then the localisation of $\text{Gal}^\Delta(Y_*)$ at the cartesian edges classifies constructible sheaves on X ; in other words, a constructible sheaf on X is tantamount to a functor $\text{Gal}^\Delta(Y_*) \rightarrow \mathbf{S}_\pi$ to π -finite spaces that carries all cartesian edges to equivalences and restricts to a continuous functor $\text{Gal}^\Delta(Y_m) \rightarrow \mathbf{S}_\pi$ for all $m \in \Delta$.*

This theorem speaks only of Artin n -stacks, but it applies just as well to any coherent fpqc stack with a presentation as a simplicial scheme.

Additionally, this theorem speaks only about nonabelian constructible sheaves, but in fact the Galois categories we construct suffice to recover constructible $\overline{\mathbf{Q}}_\ell$ sheaves as well. The proof will appear in a forthcoming note [4].

o.4 Example. Let G be an affine group scheme over a ring k , and let X be a k -scheme with an action of G . Then we have the usual simplicial k -scheme $B_{k,*}(X, G, k)$ whose n -simplices are $X \times_k G^n$; this presents the quotient stack X/G .

Thus the category of G -equivariant (nonabelian) constructible sheaves on X is equivalent to the category of continuous functors

$$\text{Gal}^\Delta(B_{k,*}(X, G, k)) \rightarrow \mathbf{S}_\pi$$

that carry the cartesian edges to equivalences. If Λ is a ring, then the derived category of G -equivariant constructible sheaves of Λ -modules on X is equivalent to the category of continuous functors

$$\text{Gal}^\Delta(B_{k,*}(X, G, k)) \rightarrow \mathbf{Perf}(\Lambda)$$

that carry cartesian edges to equivalences.

The objects of the category $\text{Gal}^\Delta(B_{k,*}(X, G, k))$ can be thought of as tuples

$$(m, \Omega, x_0, g_1, \dots, g_m)$$

in which $m \in \Delta$ is an object, Ω is a separably closed field, and $x_0 : \text{Spec } \Omega \rightarrow X$ and $g_1, \dots, g_m : \text{Spec } \Omega \rightarrow G$ are points with the property that (x_0, g_1, \dots, g_m) is a geometric point of $X \times_k G^m$, so that Ω is the separable closure of the residue field of the image of the (x_0, g_1, \dots, g_m) in the Zariski space of $X \times_k G^m$.

Acknowledgments. The second-named author gratefully acknowledges support from both the MIT Dean of Science Fellowship and NSF Graduate Research Fellowship.

1 Fibred Galois categories

1.1. We use the language and tools of higher category theory, particularly in the model of *quasicategories*, as defined by Michael Boardman and Rainer Vogt and developed by André Joyal and Jacob Lurie. We will generally follow the terminological and notational conventions of Lurie’s trilogy [HTT; HA; SAG], but we will simplify matters by *systematically using words to mean their good homotopical counterparts*. So ‘category’ here means ‘ ∞ -category’, ‘topos’ means ‘ ∞ -topos’, & c.

We write \mathcal{S} for the category of spaces and $\mathcal{S}_\pi \subset \mathcal{S}$ for the full subcategory spanned by the π -finite spaces.

We use [HTT, Corollary 3.2.2.13] systematically to construct cartesian fibrations; we leave the details of this by now standard construction implicit in what follows.

1.2 Notation. If $\mathbf{X} \rightarrow S$ is a topos fibration [HTT, Definition 6.3.1.6], then for any morphism $f : s \rightarrow t$ of S , there is a corresponding geometric morphism $f_* : \mathbf{X}_t \rightarrow \mathbf{X}_s$ of topoi; its left exact left adjoint will be denoted f^* .

1.3 Definition. Let S be a category. A *bounded coherent topos fibration* $\mathbf{X} \rightarrow S$ is a topos fibration in which each fibre \mathbf{X}_s is bounded coherent, and for any morphism $f : t \rightarrow s$ of S , the induced geometric morphism $f_* : \mathbf{X}_s \rightarrow \mathbf{X}_t$ is coherent [SAG, Definitions A.2.0.12 & A.7.1.2; 3, Definition 5.28]. A *spectral topos fibration* $\mathbf{X} \rightarrow S$ is a bounded coherent topos fibration in which each fibre \mathbf{X}_s is a spectral topos (for the canonical profinite stratification [3, Lemma 9.40 & Definition 10.3]).

1.4. The usual straightening/unstraightening equivalence restricts to an equivalence between the category of bounded coherent (respectively, spectral) topos fibrations $\mathbf{X} \rightarrow S$ and the category of functors from S^{op} to the category of bounded coherent (resp., spectral) topoi (cf. [HTT, Proposition 6.3.1.7]).

For a bounded coherent topos fibration $\mathbf{X} \rightarrow S$ we write $\mathbf{X}_{<\infty}^{coh} \subseteq \mathbf{X}$ for the full subcategory spanned by the objects that are truncated and coherent in their fibre [3, Definition 5.18]. Then $\mathbf{X}_{<\infty}^{coh} \rightarrow S$ is a cocartesian fibration that is classified by a functor from S to the category of bounded pretopoi [SAG, Definition A.7.4.1 & Theorem A.7.5.3].

1.5 Example. If X_* is a simplicial (coherent!) scheme, then the fibred topos $X_{*,\acute{e}t} \rightarrow \Delta$ is a spectral topos fibration.

1.6. Hochster duality [3, Theorem 10.10] expresses an equivalence between the category of profinite layered categories² and the category of spectral topoi, which carries a profinite layered category $\Pi = \{\Pi_\alpha\}_{\alpha \in A}$ to the spectral topos $\tilde{\Pi}$ of sheaves in the effective epimorphism topology [SAG, §A.6.2] on the bounded pretopos

$$\text{Fun}^{cts}(\Pi, \mathbf{S}_\pi) := \text{colim}_{\alpha \in A^{op}} \text{Fun}(\Pi_\alpha, \mathbf{S}_\pi)$$

of *continuous functors* $\Pi \rightarrow \mathbf{S}_\pi$. Under Hochster duality, the category of spectral topos fibrations $X \rightarrow S$ is equivalent to the category of functors from S^{op} to the category of profinite layered categories.

A fibred form of Hochster duality is what allows us to construct fibred Galois categories. To define it, we need to make sense categories fibred in profinite stratified spaces.

1.7 **Definition.** Let S be a category. A functor $\Pi \rightarrow S$ will be said to be a *category over S fibred in layered categories* if it is a cartesian fibration whose fibres are layered categories. We write $\text{Lay}_{/S}^{cart}$ for the category of categories over S fibred in layered categories.

1.8 **Construction.** There is a monad T on the category Lay of small layered categories given by sending a layered category Π to the limit over the π -finite layered categories to which it maps.³ The category of T -algebras is equivalent to the category of profinite layered categories. If S is a category, this monad can be applied fibrewise to give a monad T_S on the category $\text{Lay}_{/S}^{cart}$ of categories fibred in layered categories.

Under the straightening/unstraightening identification

$$\text{Lay}_{/S}^{cart} \simeq \text{Fun}(S^{op}, \text{Lay}),$$

the monad T_S corresponds to the monad on $\text{Fun}(S^{op}, \text{Lay})$ given by applying T objectwise. Consequently, the category of T_S -algebras is equivalent to the category of functors from S^{op} to the category of profinite layered categories.

1.9 **Definition.** Let S be a category. A *category over S fibred in profinite layered categories* is a T_S -algebra. If $\Pi \rightarrow S$ is a category fibred in layered categories, then a *fibrewise profinite structure* on $\Pi \rightarrow S$ is a T_S -algebra structure on $\Pi \rightarrow S$. We write $\text{Lay}_{\pi/S}^{cart, \wedge}$ for the category of T_S -algebras.

1.10 **Warning.** One might also contemplate the category $\text{Pro}(\text{Lay}_{\pi/S}^{cart})$ of proobjects in the full subcategory

$$\text{Lay}_{\pi/S}^{cart} \subseteq \text{Lay}_{/S}^{cart}$$

spanned by those cartesian fibrations whose fibres are π -finite layered categories. This is generally *not* equivalent to the category of categories over S fibred in profinite layered categories. Under straightening/unstraightening, the category $\text{Lay}_{\pi/S}^{cart, \wedge}$ is equivalent to the category $\text{Fun}(S^{op}, \text{Lay}_\pi^\wedge)$, whereas $\text{Pro}(\text{Lay}_{\pi/S}^{cart})$ is equivalent to the category $\text{Pro}(\text{Fun}(S^{op}, \text{Lay}_\pi))$. These coincide when S is a finite poset [HTT, Proposition 5.3.5.15], but otherwise typically do not coincide.

²A category C is *layered* if every endomorphism in C is an equivalence.

³That is, T is the right Kan extension of the inclusion $\text{Lay}_\pi \hookrightarrow \text{Lay}$ of π -finite layered categories along itself.

1.11. Let S be a category. Then the category of spectral topos fibrations over S is equivalent to the category $\text{Lay}_{\pi, S}^{\text{cart}, \wedge}$. Let us make the equivalence explicit. If $\mathbf{X} \rightarrow S$ is a spectral topos fibration, then we define a category over S fibred in layered categories

$$\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X}) \rightarrow S$$

as follows. An object of $\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})$ is a pair (s, ν) , where $s \in S$ and $\nu_* : \mathbf{S} \rightarrow \mathbf{X}_s$ is a point. A morphism $(s, \nu) \rightarrow (t, \xi)$ is a morphism $f : s \rightarrow t$ of S and a natural transformation $\nu_* \rightarrow f_* \xi_*$. The category $\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})$ fibred in layered categories admits a canonical fibrewise profinite structure; the fibre $\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})_s$ over an object $s \in S$ is the profinite stratified shape $\Pi_{(\infty, 1)}^{\wedge}(\mathbf{X}_s)$ of [3, Construction 11.1].

In the other direction, if $\Pi \rightarrow S$ is a category over S fibred in profinite layered categories, then let $X_0 \rightarrow S$ denote the cocartesian fibration in which the objects are pairs (s, F) consisting of an object $s \in S$ and a functor $F : \Pi_s \rightarrow \mathbf{S}_\pi$, and a morphism $(f, \phi) : (s, F) \rightarrow (t, G)$ consists of a morphism $f : s \rightarrow t$ of S and a natural transformation $\phi : f_! F \rightarrow G$. Then $(\overline{\Pi})_{< \infty}^{\text{coh}}$ is equivalent to the subcategory of X_0 whose objects are those pairs (s, F) in which F is continuous and whose morphisms are those pairs (f, ϕ) in which ϕ is continuous (1.6).

1.12 Construction. If S is a category and \mathbf{Y} is a bounded coherent topos, then the projection $\mathbf{Y} \times S \rightarrow S$ is a bounded coherent topos fibration. The assignment $\mathbf{Y} \mapsto \mathbf{Y} \times S$ defines a functor from the category of bounded coherent topoi to the category of bounded coherent topos fibrations over S . This functor admits a left adjoint, which we denote by $|\cdot|_S$. At the level of pretopoi, $(|\mathbf{X}|_S)_{< \infty}^{\text{coh}}$ is equivalent to the category of cocartesian sections of $\mathbf{X}_{< \infty}^{\text{coh}} \rightarrow S$, i.e., the limit of the corresponding functor from S to bounded pretopoi.

Now we arrive at the main topos-theoretic result.

1.13 Proposition. *Let S be a category, and let $\mathbf{X} \rightarrow S$ be a spectral topos fibration. Then the pretopos $(|\mathbf{X}|_S)_{< \infty}^{\text{coh}}$ is equivalent to the category of functors $F : \Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X}) \rightarrow \mathbf{S}_\pi$ with the following properties.*

- ▶ F carries any cartesian edge to an equivalence.
- ▶ For any object $s \in S$, the restriction $F|_{\Pi_{(\infty, 1)}^{\wedge}(\mathbf{X}_s)}$ is continuous.
- ▶ F is uniformly truncated in the sense that there exists an $N \in \mathbf{N}$ such that for any object $(s, \nu) \in \Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})$, the space $F(s, \nu)$ is N -truncated.

Proof. The pretopos $(|\mathbf{X}|_S)_{< \infty}^{\text{coh}}$ can be identified with the category of cocartesian sections of $\mathbf{X}_{< \infty}^{\text{coh}} \rightarrow S$. The description of (1.11) completes the proof. \square

Please note that the last condition of **Proposition 1.13** is automatic if S has only finitely many connected components (e.g., $S = \Delta$).

1.14 Example. If X_* is a simplicial scheme, then the category over Δ fibred in profinite layered categories $\Pi_{(\infty, 1)}^{\Delta, \wedge}(X_{*, \text{ét}})$ associated to the spectral topos fibration $X_{*, \text{ét}} \rightarrow \Delta$ is the category $\text{Gal}^{\Delta}(X_*)$ of **Construction 0.1**. In this case, **Proposition 1.13** implies that $(|X_{*, \text{ét}}|_{\Delta})_{< \infty}^{\text{coh}}$ is equivalent to the category of functors $\text{Gal}^{\Delta}(X_*) \rightarrow \mathbf{S}_\pi$ that carry cartesian edges to equivalences and restrict to continuous functors $\text{Gal}^{\Delta}(X_m) \rightarrow \mathbf{S}_\pi$ for all $m \in \Delta$.

Finally, since the profinite stratified shape is a delocalisation of the protruncated shape [9, Theorem 2.5] we deduce the following:

1.15 Proposition. *Let S be a category, and let $X \rightarrow S$ be a spectral topos fibration. Then the protruncated shape of $|X|_S$ is equivalent to the protruncated homotopy type of $\Pi_{(\infty,1)}^{S,\Delta}(X)$.*

1.16 Example. If X_* is a simplicial scheme, then the protruncated homotopy type of the fibrewise profinite category $\text{Gal}^\Delta(X_*)$ is equivalent to the Friedlander étale topological type of X_* [9, Theorem A].

2 Sheaves on stacks

2.1 Construction. Write \mathbf{Aff} for the 1-category of affine schemes. We employ [HTT, Corollary 3.2.2.13] to construct a category $\mathbf{PSh}_{\acute{e}t}$ and a cocartesian fibration

$$\mathbf{PSh}_{\acute{e}t} \rightarrow \mathbf{Aff}^{op}$$

in which the objects of $\mathbf{PSh}_{\acute{e}t}$ are pairs (S, F) consisting of an affine scheme S and a presheaf (of spaces) on the small étale site of S , and a morphism $(S, F) \rightarrow (T, G)$ is a pair (f, ϕ) consisting of a morphism $f: T \rightarrow S$ and a morphism of presheaves $\phi: f^{-1}F \rightarrow G$ on the small étale site of T . Define $\mathbf{Sh}_{\acute{e}t} \subset \mathbf{PSh}_{\acute{e}t}$ to be the full subcategory spanned by those pairs (S, F) in which F is a sheaf; then $\mathbf{Sh}_{\acute{e}t} \rightarrow \mathbf{Aff}^{op}$ is a topos fibration. Define $\mathbf{Constr}_{\acute{e}t} \subset \mathbf{Sh}_{\acute{e}t}$ to be the further full subcategory spanned by those pairs (S, F) in which F is a (nonabelian) constructible sheaf [3, Definition 10.1.1]; then $\mathbf{Constr}_{\acute{e}t} \rightarrow \mathbf{Aff}^{op}$ is a cocartesian fibration.

2.2 Definition. Let $X \rightarrow \mathbf{Aff}$ be a stack, i.e., a right fibration that is classified by an accessible fpqc sheaf $\mathbf{Aff}^{op} \rightarrow \mathbf{S}$. A (nonabelian) constructible sheaf on X is a cocartesian section

$$F: X^{op} \rightarrow \mathbf{Constr}_{\acute{e}t}$$

over \mathbf{Aff}^{op} . We write $\mathbf{Constr}_{\acute{e}t}(X)$ for the category of constructible sheaves on X .

2.3 Warning. This can only be expected to be a reasonable definition for coherent stacks.

2.4. Informally, a constructible sheaf F on X assigns to every affine scheme S over X a constructible sheaf F_S and to every morphism $f: S \rightarrow T$ of affine schemes an equivalence $F_S \simeq f^*F_T$. In other words, the category of constructible sheaves on X is the limit of the diagram $X^{op} \rightarrow \mathbf{Cat}$ given by the assignment $S \mapsto \mathbf{Constr}_{\acute{e}t}(S)$.

Of course, since X is not a small category, it is not obvious that this limit exists in \mathbf{Cat} . However, if X contains a small limit-cofinal full subcategory Y , then the desired limit exists.

Now we conclude:

2.5 Proposition. *If $p: X \rightarrow \mathbf{Aff}$ is a stack, and if X is presented by a simplicial scheme Y_* , then we obtain an equivalence between the category $\mathbf{Constr}_{\acute{e}t}(X)$ and the category of functors*

$$\text{Gal}^\Delta(Y_*) \rightarrow \mathbf{S}_\pi$$

that carry cartesian edges to equivalences and for all $m \in \Delta$ restrict to a continuous functor $\text{Gal}(Y_m) \rightarrow \mathbf{S}_\pi$.

References

- HTT J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009, vol. 170, pp. xviii+925, ISBN: 978-0-691-14049-0; 0-691-14049-9.
- HA _____, *Higher algebra*, Preprint available at math.harvard.edu/~lurie/papers/HA.pdf, Sep. 2017.
- SAG _____, *Spectral algebraic geometry*, Preprint available at math.harvard.edu/~lurie/papers/SAG-rootfile.pdf, Feb. 2018.
1. I. Barnea and T. M. Schlank, *A projective model structure on pro-simplicial sheaves, and the relative étale homotopy type*, Adv. Math., vol. 291, pp. 784–858, 2016. DOI: [10.1016/j.aim.2015.11.014](https://doi.org/10.1016/j.aim.2015.11.014).
 2. C. Barwick, *On Galois categories & perfectly reduced schemes*, Preprint available at [arXiv:1811.06125](https://arxiv.org/abs/1811.06125), Nov. 2018.
 3. C. Barwick, S. Glasman, and P. Haine, *Exodromy*, Preprint available at [arXiv:1807.03281](https://arxiv.org/abs/1807.03281), Nov. 2018.
 4. C. Barwick and P. J. Haine, *Exodromy for ℓ -adic sheaves*, In preparation, Jan. 2019.
 5. D. Carchedi, *On the étale homotopy type of higher stacks*, Preprint available at [arXiv:1511.07830](https://arxiv.org/abs/1511.07830), Jul. 2016.
 6. C.-Y. Cho, *Topological types of Algebraic stacks*. ProQuest LLC, Ann Arbor, MI, 2016, p. 104, Thesis (Ph.D.)–University of California, Berkeley, ISBN: 978-1369-05672-3.
 7. D. A. Cox, *Homotopy theory of simplicial schemes*, Compositio Math., vol. 39, no. 3, pp. 263–296, 1979.
 8. E. M. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982, vol. 104, pp. vii+190, ISBN: 0-691-08288-X; 0-691-08317-7.
 9. P. J. Haine, *Extended étale homotopy groups from profinite Galois categories*, Preprint available at [arXiv:1812.11637](https://arxiv.org/abs/1812.11637), Dec. 2018.