A comment on the vanishing of rational motivic Borel–Moore homology

Clark Barwick Denis Nardin

Abstract

This note concerns a weak form of Parshin's conjecture, which states that the rational motivic Borel–Moore homology of a quasiprojective variety of dimension m over a finite field in bidegree (s,t) vanishes for $s > m+t$. It is shown that this conjecture holds if and only if the cyclic action on the motivic cohomology of an Artin–Schreier field extension in bidegree (i,j) is trivial if $i < j$.

Let k be a finite field of characteristic p; let V be a quasiprojective variety of dimension m over k. The conjecture of Beilinson–Parshin states that if V is smooth and projective, then $K_i(V) \otimes \mathbf{Q} = 0$ for $i > 0$; equivalently, the rational motivic cohomology $H^i(V, \mathbf{Q}(j))$ vanishes unless $i = 2j$. Equivalently, the conjecture states that for V smooth and projective, $H_s^{BM}(V, \mathbf{Q}(t))$ vanishes unless $s = 2t$.

We are interested in the following conjecture for arbitrary (i.e., not necessarily smooth or projective) V, which identifies a more restricted vanishing range:

1 Conjecture. *The rational motivic Borel–Moore homology $H_s^{BM}(V, \mathbf{Q}(t))$ vanishes if $s > m+t$.*

Combined with usual vanishing results in motivic cohomology [3, Th. 3.6 and Th. 19.3], this would imply that when V is smooth (but not necessarily projective), one has (with $i = 2m - s$ and $j = m - t$)

$$H^i(V, \mathbf{Q}(j)) = 0 \ \text{ unless } \ i \in [j, j+m] \cap [j, 2j] \,.$$

Here is a conjecture concerning fields. Let K be a perfect field of characteristic p, and let $L \coloneqq K[y]/(y^p - y - a)$ be an Artin–Schreier extension, on which the cyclic group C_p acts via $y \mapsto y+1$.

2 Conjecture. *The induced action of C_p on $H^i(L, \mathbf{Q}(j))$ is trivial for every $i < j$.*

This would imply that $H^i(L, \mathbf{Q}(j))$ vanishes in this range, so we may regard this as a kind of 'ascent' property for motivic cohomology along Artin–Schreier covers.

The purpose of this note is to prove:

3 Theorem. *Conjecture 2 implies Conjecture 1*

The proof is an induction argument that reduces Conjecture 1 to Conjecture 2. We are grateful to Joseph Ayoub, who kindly informed us that our previous formulation of this result was too strong.

4. If $m = 0$, Conjecture 1 (and indeed the Beilinson–Parshin Conjecture itself) follows from Quillen's computation of the K-theory of finite fields. When $m = 1$, it follows from the celebrated computations of Harder. For the purpose of induction, we now assume this statement for quasiprojective varieties of dimension $< m$.

5. Choose an open immersion $V \hookrightarrow \overline{V}$ into a projective variety of dimension m such that the complement $\overline{V} - V$ (with its reduced scheme structure) is quasiprojective of positive codimension. The localization sequence

$$\cdots \to H_s^{BM}(\overline{V} - V, \mathbf{Q}(t)) \to H_s^{BM}(\overline{V}, \mathbf{Q}(t)) \to H_s^{BM}(V, \mathbf{Q}(t)) \to \cdots$$

now permits us to reduce to the case in which V is projective. It suffices also to assume that V is irreducible.

Now we deploy the following result of Kiran Kedlaya:

6 Theorem (Kedlaya, [2, Theorem 1]). *Suppose X a projective variety, pure of dimension m over our finite field k. Suppose L an ample line bundle on X, D a closed subscheme of dimension less than m, and S a 0-dimensional subscheme of the regular locus not meeting D.*

Then there exists a positive integer r and an $(m+1)$-tuple of linearly independent sections of $L^{\otimes r}$ with no common zero such that the induced finite morphism

$$f : X \to \boldsymbol{P}H^0(X, L^{\otimes r}) \cong \boldsymbol{P}^m$$

of k-schemes enjoys the following conditions.

(6.1) *If $\boldsymbol{P}^{m-1} \cong H \subset \boldsymbol{P}^m$ denotes the hyperplane at infinity, then f is étale away from H.*

(6.2) *The image $f(D)$ is contained in H.*

(6.3) *The image $f(S)$ does not meet H.*

7. We thus obtain a finite morphism $f : V \to \boldsymbol{P}^m$ that is étale over $\boldsymbol{A}^m$. Let's write $Z \coloneqq f^{-1}(H)$ and $U \coloneqq f^{-1}(\boldsymbol{A}^m)$; of course the latter is smooth.

The localization sequence

$$\cdots \to H_s^{BM}(Z, \mathbf{Q}(t)) \to H_s^{BM}(V, \mathbf{Q}(t)) \to H_s^{BM}(U, \mathbf{Q}(t)) \to \cdots,$$

when combined with our induction hypothesis, reduces the problem to showing that the rational motivic cohomology

$$H^i(U, \mathbf{Q}(j)) \cong H_{2m-i}^{BM}(U, \mathbf{Q}(m-j))$$

vanishes whenever $i < j$.

8. At any stage, it will suffice to assume U is connected, and moreover we will be free to pass to a further étale cover of U: indeed, if $g : U' \to U$ is a finite étale map, then the composite $g_* g^* : H^i(U, \mathbf{Z}(j)) \to H^i(U, \mathbf{Z}(j))$ is multiplication by its degree. Hence

$$g^* : H^i(U, \mathbf{Q}(j)) \to H^i(U', \mathbf{Q}(j))$$

is injective, and so it suffices to show that $H^i(U', \mathbf{Q}(j)) = 0$ for $i < j$.

9. As a first application of 8, if $f : U \to \mathbf{A}^m$ is not Galois, we may pass to its Galois closure.

Harbater and van der Put show [1, Example 5.3] that a group is a finite quotient of the étale fundamental group of $\mathbf{A}^m_{\overline{k}}$ (for $\overline{k}$ an algebraic closure of k) just in case it is a quasi-p-group. Hence by a second application of 8, we may pass to a finite extension of k and to connected components if necessary and thereby assume that U is geometrically integral, and the Galois group G of the Galois cover f is a quasi-p-group.

By a third application of 8, we may also pass to a finite extension of k to ensure that the fiber of $f : U \to \mathbf{A}^m$ over 0 contains a rational point.

10. Since rational motivic cohomology satisfies étale descent, we have a convergent spectral sequence

$$E_2^{u,v} \cong H^u(G, H^v(U, \mathbf{Q}(j))) \Rightarrow H^{u+v}(\mathbf{A}^m_k, \mathbf{Q}(j)) \cong \begin{cases} \mathbf{Q} & \text{if } u + v = 0 \text{ and } j = 0; \\ 0 & \text{otherwise,} \end{cases}$$

by homotopy invariance and Quillen. Since $E_2^{u,v}$ vanishes unless $u = 0$, we deduce that $H^i(U, \mathbf{Q}(j))^G = 0$ unless $i = j = 0$.

11. The claim now is that $H^i(U, \mathbf{Q}(j)) = 0$ is trivial when $i < j$; this is clearly true when G is the trivial group. Since G is generated by elements of order a power of p it suffices to show that every such element acts trivially. In particular, the conjecture will follow if for every Galois cover $U \to X$ of order p^n, the action of the Galois group on $H^i(U, \mathbf{Q}(j))$ is trivial. We want to show that it suffices to check the case where $n = 1$. We will prove this by induction on $n \geq 2$.

Suppose we knew the above statement for Galois covers of order p, and let g be a generator of the Galois group of U over X. Suppose $n \geq 2$. Then we can find $0 < e < n$, so that both e and $n - e$ are less than n. In particular, our thesis is true for g^{p^e}, that is the action of g^{p^e} on $H^i(U, \mathbf{Q}(j))$ is trivial. But then

$$H^i(U/g^{p^e}, \mathbf{Q}(j)) = H^i(U, \mathbf{Q}(j))^{g^{p^e}} = H^i(U, \mathbf{Q}(j)) .$$

Moreover, g descends to an automorphism of U/g^{p^e} of order p^e. Hence by our inductive hypothesis g acts trivially on $H^i(U/g^{p^e}, \mathbf{Q}(j)) = H^i(U, \mathbf{Q}(j))$.

Since (as is well-known) Galois extensions of order p are Artin–Schreier extensions, we may now reduce to the following situation.

We suppose A a smooth k-algebra, and we suppose that $A \subset B$ is an Artin–-Schreier extension, so that $B \cong A[y]/(y^p - y - a)$. We assume that $T = \operatorname{Spec} A$ and $U = \operatorname{Spec} B$ are geometrically integral. Hence we may consider the subring $k[a] \subseteq A$; we note that since U and T are assumed geometrically integral, it follows that a is not algebraic over k. Consequently, the function a is a dominant, finite type morphism $a\colon T \to \mathbf{A}^1_k$, and we have a pullback square

$$\begin{array}{ccc} U & \xrightarrow{b} & S \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ T & \xrightarrow[a]{} & \mathbf{A}^1_k, \end{array}$$

in which $S = \operatorname{Spec} k[x,y]/(y^p - y - x)$, and q is the Artin–Schreier cover given by the inclusion $k[x] \subset k[x,y]/(y^p - y - x)$. (Of course $S \cong \boldsymbol{A}^1_k$.)

This, then, is our first reduction of Conjecture 1:

12 Reduction. *The action of C_p on $H^i(U, \mathbf{Q}(j))$ is trivial if $i < j$.*

13. We now reduce the question to one of suitable function fields. That is, we claim that our induction hypothesis implies that if V is smooth and geometrically irreducible, then $H^i(V, \mathbf{Q}(j)) \cong H^i(k(V), \mathbf{Q}(j))$ for $i < j$. Indeed, for any nonempty open subset $W \subsetneq V$, one has the localization sequence

$$\to H^{BM}_{2m-i}(V - W, \mathbf{Q}(m-j)) \to H^i(V, \mathbf{Q}(j)) \to H^i(W, \mathbf{Q}(j)) \to H^{BM}_{2m-i-1}(V - W, \mathbf{Q}(m-j)) \to$$

Let c denote the codimension of W; note that $c \geq 1$, so that if $i < j$ then $2m - i - 1 > m - c + m - j$, whence by the induction hypothesis on the dimension,

$$H^{BM}_{2m-i}(V - W, \mathbf{Q}(m-j)) = H^{BM}_{2m-i-1}(V - W, \mathbf{Q}(m-j)) = 0.$$

Consequently, one has an isomorphism

$$H^i(V, \mathbf{Q}(j)) \cong H^i(W, \mathbf{Q}(j))$$

in this range. Passing to the colimit, one has $H^i(V, \mathbf{Q}(j)) \cong H^i(k(V), \mathbf{Q}(j))$.

14 Reduction. *The action of C_p on $H^i(k(U), \mathbf{Q}(j))$ is trivial if $i < j$.*

15. If B is smooth over a perfect field k, then one may compare rational motivic cohomology of B in the sense of Voevodsky with the Ext groups in the ∞-category $\mathbf{DM}(B; \mathbf{Q})$ of rational motives:

$$H^i(B, \mathbf{Q}(j)) \cong [1_B, 1_B(j)[i]]_{\mathbf{DM}(B;\mathbf{Q})}.$$

In our case, we are interested in the situation in which B is Spec of the function fields $k(T)$ and $k(U)$. We note that these fields are not perfect, but for any field K with perfection K^{perf}, the ∞-category $\mathbf{DM}(K; \mathbf{Q})$ is equivalent to $\mathbf{DM}(K^{perf}; \mathbf{Q})$, so we are free to pass to the context originally contemplated by Voevodsky.

Consequently, we write $K \coloneqq k(T)^{perf}$, and $L \coloneqq K(y)/(y^p - y - a)$.

The task is thus to analyze the Galois action of the cyclic group C_p on the rational motivic cohomology of $L \cong K[y]/(y^p - y - a)$ induced by the action $y \mapsto y + 1$. The final reduction of Conjecture 1 now is

16 Reduction. *The action of C_p on $H^i(L, \mathbf{Q}(j))$ is trivial if $i < j$.*

This is Conjecture 2. Equivalently, if we abuse notation slightly and write L again for the Artin motive of $K \subset L$, then we have shown that Conjecture 1 would follow from the triviality of the action of C_p on the cohomology $H^i(K, L(j))$ of the Artin–Tate motive $L(j)$ for $i < j$.

References

1. D. Harbater and M. van der Put, *Valued fields and covers in characteristic p*, in *Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999)*, Fields Inst. Commun. Vol. 32, With an appendix by Robert Guralnick, Amer. Math. Soc., Providence, RI, 2002, pp. 175–204.
2. K. S. Kedlaya, *More étale covers of affine spaces in positive characteristic*, J. Algebraic Geom., vol. 14, no. 1, pp. 187–192, 2005. DOI: 10.1090/S1056-3911-04-00381-9.
3. C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006, vol. 2, pp. xiv+216, ISBN: 978-0-8218-3847-1; 0-8218-3847-4.