OPERATOR CATEGORIES, MULTICATEGORIES, AND HOMOTOPY COHERENT ALGEBRA

by

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What's all this then?— Here are revised notes for my ill-fated talk in the Norwegian Topology Symposium in Bergen in June, 2007. As a result of poor planning, a tedious narrative structure, and a basic lack of familiarity with computer-based presentations, I did not manage to explore nearly as much material as I had intended. This was, for me, a genuine loss, as I was excited to see what interesting applications "real" topologists might find for my little pet project.

As penance for my failings as a speaker, I've spent the last three months or so revising and expanding the slides from the talk; here is the result. I have included more details and examples throughout, and I have interspersed a fair number of friendly, informal comments and attempts at humor in order to cut down the inevitable turgidity of the prose. The detailed proofs of the results announced here will be compiled elsewhere — mostly as threads developed in [**3**] and [**4**] —, and for the most part, I feel satisfied with sketching the proofs of the more surprising results here. As the completion and revision of the book have taken far longer than expected, this short note can be seen as a kind of progress report on my work so far in these areas.

I never did finish writing up these notes, and they haven't been edited in a year. I'm making them available publicly mostly to express the ideas.

Some introductory remarks. — The aim of the theory of operator categories is to provide a new set of "sharper tools" (to borrow an expression from Hilbert) for discussing the interaction between algebraic structures and homotopy theory. This approach is further enhanced by the introduction of higher-categorical instruments that are carefully tuned to detect the harmonics of the interface between algebra and homotopy theory. Of course this interface has received more than enough attention over the years; indeed, homotopy coherent algebra has experienced a kind of prolonged renaissance over the course of the past half century, with more great minds involved in its unfolding than I dare name here. With this sort of pedigree, it may seem wholly improbable that modern homotopy coherent algebra should retain any mystery whatsoever, particularly none of the sort that might yield to the introduction of a new (rather thin) layer of formalism.

It should be borne in mind, however, that accompanying this marvelous history of successful work has been a bewildering history of errors — many of which are quite difficult to detect — coupled with a technical heft that can make it difficult to determine the status of some problems or their amenability to certain technologies. In particular, I am especially interested in the following question, which I had once thought would prove easy to answer.

Question. — Is the algebraic K-theory of an E_n ring spectrum an E_{n-1} ring spectrum in a canonical fashion?

There have been a number of unsuccessful attempts to approach related questions, but I am so far unable to determine whether this question has found a satisfactory answer before now. The analogous question for topological Hochschild homology has a purported solution by a number of authors, but I have so far not been able to follow their arguments.

Hence our aim here is to establish a useful higher categorical and combinatorial framework in which we can recast some classical theorems, give cleaner, more conceptual proofs of some more modern results, and offer answers to questions akin to the one above (whose answer, incidentally, is *yes*).

Unfortunately, the joy of operator categories, higher categories, multicategories, and all their concomitant alembics may prove difficult to share; the sense in which this subject may be said to be "fun" has always been rather subtle — at times even elusive. So, as a friendly word of advice, may I suggest that any reader who expects to find all this category theory idiotic or painful seriously consider reading these notes *backwards*. Start with the very last section to see what sort of applications I have in mind, and decide whether you care. If (as I hope) you find that you do, then you can look in the penultimate section to see what the technical underpinnings of these applications look like. And so on, until you are either fed up or so convinced that operator categories are a good idea that you are prepared to use them yourself.

Contents

1. Operator Categories

What is an operator category?— Operator categories are to be thought of as categories of indexing sets for multiplication laws; hence operator categories provide an organizational rubric to which a collection of objects must conform in order for them to be multiplied.

Definition 1.1. — An *operator category* is an essentially small category Φ satisfying the following conditions.

(1.1.1) Φ is locally finite; i.e., for any objects I and J of Φ, $\mathrm{Mor}_\Phi(I, J)$ is a finite set.
(1.1.2) There exists a terminal object $\star \in \Phi$.
(1.1.3) For any morphism $f : J \longrightarrow I$ of Φ and any point $i : \star \longrightarrow I$, the fiber of f over i, i.e., the pullback

$$\begin{array}{ccc} J_i & \longrightarrow & J \\ \downarrow & & \downarrow \\ \star & \longrightarrow & I, \end{array}$$

exists.

1.2. — I wrote this definition down a few years ago purely out of sheer laziness. I'd noticed that I was writing the same arguments repeatedly, and I began to suspect that one might be able to extract a short list of axioms for indexing sets for various kinds of monoidal structures, so I could stop using irritating expressions like, "the proof is similar to that of Lemma 1.2.4."

1.3. — Suppose Φ an operator category. An object $I \in \Phi$ is to be thought of as the finite set of points $|I| := \mathrm{Mor}_\Phi(\star, I)$, equipped with some extra structure.

Such an operator category Φ specifies the rubric of multiplication laws in the following manner: in order to multiply a finite set of elements according to some multiplication law $\odot$ under the rubric of Φ, one follows an easy recipe:

(1.3.1) Arrange the elements according to the structure on an object of Φ:

$$(X_I) := (X_i)_{i\in|I|}$$

(an I-*tuple*).
(1.3.2) The multiplication law now permits you to proceed:

$$\odot X_I := \bigodot_{i\in|I|} X_i.$$

1.4. — A multiplication law under the rubric of Φ will be called a Φ-*monoidal structure*. A Φ-*monoid* $(A, \odot)$ in the category of sets is (at least as a first approximation) to be thought of as a functor:

$$\begin{aligned} \Phi &\longrightarrow \mathbf{Set} \\ I &\longmapsto A^{\times|I|} \\ [J \to I] &\longmapsto [(X_J) \mapsto (\odot X_{(J/I)})], \end{aligned}$$

where:

$$(\odot X_{(J/I)}) := (\odot X_{J_i})_{i\in|I|}.$$

This picture will be made much more precise in due course.

Easy examples, in decreasing degree of triviality. — We'll get the ball rolling with some very basic examples, all of which are relatively familiar. Later on we'll have a look at an interesting way to generate new infinite families of examples.

Example 1.5. — The following are all operator categories:

(1.5.1) The linear category $\mathbf{p} := [0 \longrightarrow 1 \longrightarrow \ldots \longrightarrow p]$ (here $p \geq 0$; observe that the empty category $-\mathbf{1}$ is *not* an operator category),

(1.5.2) the full subcategory $\mathbf{O}_{\leq n} \subset \mathbf{Cat}$ comprised of the categories $\mathbf{p}$ for $-1 \leq p \leq n$ — or, equivalently, totally ordered finite sets of cardinality $\leq n+1$,

(1.5.3) the full subcategory $\mathbf{O} \subset \mathbf{Cat}$ comprised of the categories $\mathbf{p}$ for $p \geq -1$ — or, equivalently, totally ordered finite sets,

(1.5.4) the full subcategory $\mathbf{F}_{\leq n} \subset \mathbf{Set}$ comprised of the sets $|\mathbf{p}| := \operatorname{Obj} \mathbf{p}$ for $-1 \leq p \leq n$ — or, equivalently, finite sets of cardinality $\leq n+1$, and

(1.5.5) the full subcategory $\mathbf{F} \subset \mathbf{Set}$ comprised of the sets $|\mathbf{p}|$ for $p \geq -1$ — or, equivalently, finite sets.

Here's a table of these operator categories and their associated rubrics in the category of sets.

Φ	Φ-monoids in **Set**
$\mathbf{0}$	sets
$\mathbf{p}$ $(p > 0)$	pointed sets
$\mathbf{O}_{\leq 1}$	unital magmas
$\mathbf{O}_{\leq n}$ $(n > 1)$	monoids
$\mathbf{O}$	monoids
$\mathbf{F}_{\leq 1}$	commutative, unital magmas
$\mathbf{F}_{\leq n}$ $(n > 1)$	commutative monoids
$\mathbf{F}$	commutative monoids

1.6. — Observe that for any $n > 1$, the categories of Φ-monoids in **Set** for $\Phi = \mathbf{O}_{\leq n}$ and $\Phi = \mathbf{O}$ are all equivalent. One says that the operator categories $\mathbf{O}_{\leq n}$ $(n > 1)$ and $\mathbf{O}$ all *span the same rubric* in **Set**.

1.7. — Note that with the exception of the first, these examples are all *unital* structures. This is no accident: any time Φ contains an object with a plurality of points, the requirement that all fibers of a map exist — even over points not in the image — forces the existence of objects without points; these naturally give rise to units.

There is also a theory of "nonunital" operator categories; morphisms that induce epimorphisms on the sets of points play a particularly significant role in that theory. For now, let us concentrate on the unital theory.

Example 1.8. — Here's a sort of impractical example, just for fun, to demonstrate that the possibility for more exotic structures exists. For any set S, one can also define an operator category $\mathbf{P}_S$ whose set of objects is the set $S_+ := S \sqcup \star$, with exactly one morphism from any $s \in S$ to $\star$, and no other nontrivial morphisms. Then $\mathbf{P}_S$-monoids (in **Set**) are simply objects of the over category $(S/\mathbf{Set})$.

Flat and perfect operator categories. — In algebraic geometry, a property P of schemes usually comes in two flavors: there's a relative version of P and an absolute version of P. The relative version of a property is really a property of a morphism of schemes, whereas the absolute version is simply a property of a single scheme. Frequently, as one might expect, it is simpler to verify that a given scheme is "absolutely P" than it is to verify that a given morphism is "relatively P."

This is where flatness enters; in many cases of interest, a flat morphism whose fibers are absolutely P is a relatively P morphism. In other words, flatness is precisely the condition needed to guarantee that information can be "spread out" from the fibers of a morphism to the morphism itself. This idea can be fruitfully translated into the world of operator categories as well.

1.9. — The existence of fibers in an operator category Φ allows one to define functors

$$\begin{aligned} \sigma_I : \quad (\Phi/I) &\longrightarrow \Phi^{\times |I|} \\ [J \to I] &\longmapsto (J_I) \end{aligned}$$

for any $I \in \Phi$.

Definition 1.10. — An operator category Φ is *flat* if for any $I \in \Phi$ the functor σ_I is fully faithful; Φ is *faithfully flat* if for any $I \in \Phi$ the functor σ_I is an equivalence.

1.11. — As promised, flatness in an operator category is precisely what is needed to "spread out" information from fibers: if $[J \longrightarrow I] \in \Phi$ and $[K \longrightarrow I] \in \Phi$, then any collection of morphisms $(J_i \longrightarrow K_i)_{i\in|I|}$ is the I-tuple of fibers of a morphism $J \longrightarrow K$ over I.

Example 1.12. — The *only* easy examples of operator categories we have seen that are *not* flat are the categories $\mathbf{p}$ for $p > 1$.

On the other hand, only the operator categories $\mathbf{0}$, $\mathbf{1}$, $\mathbf{O}$, and $\mathbf{F}$ (as well as our impractical example, $\mathbf{P}_S$) are faithfully flat; the others are merely flat.

1.13. — Although flat operator categories include the more familiar examples, the technical criterion that will be most relevant for our work here is that of *perfection*, to which we now turn.

Definition 1.14. — (1.14.1) A *point classifier* for an operator category Φ is a pointed object $(T, t) \in (\star/\Phi)$ with the following universal property: for any pointed object $(V, v) \in (\star/\Phi)$, there exists a unique morphism $[\chi_v : V \longrightarrow T] \in \Phi$ such that

$$\begin{array}{ccc} \star & \xrightarrow{v} & V \\ \| & & \downarrow{\scriptstyle \chi_v} \\ \star & \xrightarrow[t]{} & T \end{array}$$

is a pullback square.

(1.14.2) If (T, t) is a point classifier for Φ, then the point $t \in |T|$ will be called the *special point* of T, and any other point of T will be called a *generic point*; the set of generic points will be denoted $|T|_\gamma := |T| \setminus t$.

(1.14.3) An operator category Φ is *perfect* if:

(1.14.3.1) Φ contains a *point classifier* (T, t), and

(1.14.3.2) the functor

$$(-)_t : (\Phi/T) \longrightarrow \Phi$$

has a right adjoint, which we also denote T.

(1.14.4) The *complexity* of a perfect operator category Φ is defined to be the number of generic points of the point classifier:

$$C(\Phi) := \#|T|_\gamma$$

1.15. — The concept of a point classifier may remind you a little of the notion of a subobject classifier from topos theory. When Φ is faithfully flat, that's almost exactly what it is.

In a perfect operator category Φ, the right adjoint $T : \Phi \longrightarrow (\Phi/T)$ effectively adds points in as many directions as possible. The number of those directions is the complexity of Φ. Another way of looking at the complexity is that it is an estimate of the minimum number of "components" left over after you remove a general point of an object of Φ. This idea can actually be made precise, but I will not do so here.

Example 1.16. — Let us investigate the perfection of our easy examples.

(1.16.1) The initial operator category $\mathbf{0}$ is perfect, and the point classifier is the only object; $C(\mathbf{0}) = 0$.

(1.16.2) The operator category $\mathbf{1}$ is perfect, and the terminal object is the point classifier; $C(\mathbf{1}) = 0$. More generally, precisely the same is true for $\mathbf{p}$ and for $\mathbf{P}_S$.

(1.16.3) The "partial" operator categories $\mathbf{O}_{\leq n}$ and $\mathbf{F}_{\leq n}$ are not perfect.

(1.16.4) The category $\mathbf{O}$ is perfect; the point classifier is $\mathbf{2}$ with special point $1 \in |\mathbf{2}|$; $C(\mathbf{O}) = 2$.

(1.16.5) The category $\mathbf{F}$ is perfect as well; the point classifier here is $|\mathbf{1}|$ with special point $1 \in |\mathbf{1}|$; $C(\mathbf{F}) = 1$.

1.17. — One may be led by the examples I have given to wonder whether the intersection of these two classes of operator categories — perfect operator categories that are also flat — is significant. It appears, however, that the relationship between the conditions of flatness and perfection are relatively subtle, and their domains of usefulness are in some ways orthogonal. It seems that operations that tend to preserve flatness tend to destroy perfection, and vice versa. Related to these ideas are the following easy propositions.

Proposition 1.18. — *If Φ is flat, then any object I with no points receives no nontrivial morphisms; in other words, (Φ/I) is a contractible groupoid.*

Proposition 1.19. — *A faithfully flat operator category containing a point classifier is perfect.*

Operator morphisms. — What makes the theory of operator categories vaguely promising is the observation that there is a natural notion of *morphisms* of operator categories. This makes it possible to investigate the functoriality of various standard constructions.

Definition 1.20. — Suppose Φ and Ψ operator categories. An *operator morphism* $f : \Psi \longrightarrow \Phi$ is a functor satisfying the following properties.

(1.20.1) The functor f respects terminal objects — that is, $f(\star) \longrightarrow \star$ is an isomorphism.

(1.20.2) The functor f respects fibers — that is, $f(J_i) \longrightarrow (fJ)_{f(i)}$ is an isomorphism for any $[J \longrightarrow I] \in \Psi$ and any $i \in |I|$.

(1.20.3) For any object I of Ψ, the induced morphism $|I| \longrightarrow |fI|$ is a *surjection.*

This gives a $(2,1)$-category **Op** of operator categories, and full sub-$(2,1)$-categories **Flop**, **Fflop**, and **Plop** of flat, faithfully flat, and perfect[(1)] operator categories.

Lemma 1.21. — *The operator category* **0** *is homotopy initial in* **Op***:*

$$\natural : \mathbf{0} \longrightarrow \Phi$$
$$\star \longmapsto \star .$$

Lemma 1.22. — *The operator category* **F** *is homotopy terminal in* **Op***:*

$$U : \Phi \longrightarrow \mathbf{F}$$
$$I \longmapsto |I|.$$

1.23. — The $(2,1)$-category **Op** can thus be seen as a kind of *axis of structure* varying between the operator category **0** — representing no structure at all — to the operator category **F** — representing maximal structure:

$$\mathbf{0} \longrightarrow \Phi \longrightarrow \mathbf{F}$$

Example 1.24. — (1.24.1) Suppose Φ an operator category, $\Psi \subset \Phi$ a full subcategory containing $\star$ and closed under subobjects. Then Ψ is also an operator category, and the inclusion $\Psi \longrightarrow \Phi$ is an operator morphism.

(1.24.2) Any functor $\phi : \mathbf{p} \longrightarrow \mathbf{q}$ with the property that $\phi(i) = q$ iff $i = p$ is an *operator morphism.*

(1.24.3) For any $n \geq m \geq 0$, the obvious diagram of functors

$$\begin{array}{ccccc} \mathbf{O}_{\leq m} & \longrightarrow & \mathbf{O}_{\leq n} & \longrightarrow & \mathbf{O} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{F}_{\leq m} & \longrightarrow & \mathbf{F}_{\leq n} & \longrightarrow & \mathbf{F} \end{array}$$

is a diagram of *operator morphisms.*

1.25. — I found it surprising that the $(2,1)$-category **Op** has quite a few homotopy limits and homotopy colimits. It is surely neither homotopy complete nor homotopy cocomplete; however, there are enough homotopy limits and colimits that some useful constructions can be made. I won't discuss them all, but here's a handy little result.

Lemma 1.26. — *The* $(2,1)$*-category* **Op** *has finite homotopy limits and arbitrary homotopy coproducts.*

(1) You're going to complain that there is no "l" in the word "perfect," and that makes **Plop** a bit of an odd choice. I appreciate that objection, but I find rhyme and prosody much more influential factors than orthographical faithfulness in the naming of $(2,1)$-categories.

Sketch of proof. — If $\Phi \longrightarrow \mathrm{X}$ and $\Psi \longrightarrow \mathrm{X}$ are operator morphisms, then the homotopy fiber product $\Phi \times^h_{\mathrm{X}} \Psi$ in **Cat** is also an operator category, and is the homotopy fiber product in **Op** as well. If Φ_i are operator categories, then the "homotopically pointed sum" $\coprod^{h,\star} \Phi_i$ in **Cat** is an operator category, and is the homotopy coproduct in **Op**. □

2. Multicategories

Φ-multicategories. — Here I give the definitions and basic properties of Φ-multicategories. The definitions are, of course, naïve analogues of the definitions from the classical theory of multicategories.

The most important aspect of the theory as I describe it here is its functoriality in every conceivable direction. This functoriality is both conceptually satisfying and technically powerful.

Definition 2.1. — Suppose Φ an operator category. A Φ-*multicategory* $\mathcal{C}$ consists of:

(2.1.A) a set of *objects* $\operatorname{Obj}\mathcal{C}$,

(2.1.B) an *I-polymorphism set* $I\operatorname{Mor}_{\mathcal{C}}((M_I), N)$ for any $I \in \Phi$, any I-tuple $(M_I) \in (\operatorname{Obj}\mathcal{C})^{\times|I|}$, and any $N \in \operatorname{Obj}\mathcal{C}$,

(2.1.C) an *identity element* $\mathrm{id}_M \in \operatorname{Mor}_{\mathcal{C}}(M, M)$ for every $M \in \operatorname{Obj}\mathcal{C}$, and

(2.1.D) a *polycomposition map*

$$I\operatorname{Mor}_{\mathcal{C}}((M_I), N) \times (J/I)\operatorname{Mor}_{\mathcal{C}}((L_J), (M_I)) \longrightarrow J\operatorname{Mor}_{\mathcal{C}}((L_J), N)$$

for any morphism $[J \longrightarrow I] \in \Phi$, any J-tuple $(L_J) \in (\operatorname{Obj}\mathcal{C})^{\times|J|}$, any I-tuple $(M_I) \in (\operatorname{Obj}\mathcal{C})^{\times|I|}$, and any object N, where we use the shorthand

$$(J/I)\operatorname{Mor}_{\mathcal{C}}((L_J), (M_I)) := \prod_{i \in |I|} J_i \operatorname{Mor}_{\mathcal{C}}((L_{J_i}), M_i)$$

These data are subject to the following axioms.

(2.1.1) *Associativity*: For any $[K \twoheadrightarrow J \twoheadrightarrow I] \in \Phi$, these diagrams commute:

$$
\begin{array}{ccc}
 & (I\operatorname{Mor}_{\mathcal{C}}((N_I),P)\times(J/I)\operatorname{Mor}_{\mathcal{C}}((M_J),(N_I)))\times(K/J)\operatorname{Mor}_{\mathcal{C}}((L_K),(M_J)) & \\
\swarrow\sim & & \searrow \\
I\operatorname{Mor}_{\mathcal{C}}((N_I),P)\times\prod_{i\in|I|}\left(J_i\operatorname{Mor}_{\mathcal{C}}((M_{J_i}),N_i)\times(K_i/J_i)\operatorname{Mor}_{\mathcal{C}}((L_{K_i}),(M_{J_i}))\right) & & \\
\downarrow & & J\operatorname{Mor}_{\mathcal{C}}((M_J),P)\times(K/J)\operatorname{Mor}_{\mathcal{C}}((L_K),(M_J)) \\
I\operatorname{Mor}_{\mathcal{C}}((N_I),P)\times\prod_{i\in|I|}K_i\operatorname{Mor}_{\mathcal{C}}((L_{K_i}),N_i) & & \\
\| & & \swarrow \\
I\operatorname{Mor}_{\mathcal{C}}((N_I),P)\times(K/I)\operatorname{Mor}_{\mathcal{C}}((L_K),(N_I)) & & \\
\searrow & & \\
 & K\operatorname{Mor}_{\mathcal{C}}((L_K),P). &
\end{array}
$$

(2.1.2) *Identity*: For any $I \in \Phi$, the identity maps of I-polymorphism sets $I\operatorname{Mor}_{\mathcal{C}}((M_I), N)$ correspond both to

(2.1.2.a) $(\mathrm{id}_{M_I}) \in (I/I)\operatorname{Mor}_{\mathcal{C}}(M_I, M_I)$ under the polycomposition map induced by $[I = I] \in \Phi$, and

(2.1.2.b) $\mathrm{id}_N \in \operatorname{Mor}_{\mathcal{C}}(N, N)$ under the polycomposition map induced by $[I \longrightarrow \star] \in \Phi$.

A Φ-multicategory with exactly one object is called a Φ-*operad*.

2.2. — It's perhaps amusing to think of Φ-multicategories as the ghosts of Φ-monoidal structures (to be defined a little later). A Φ-monoidal structure allows one to multiply objects. This you cannot necessarily do in a Φ-multicategory, but you can say what maps out of that tensor product would look like, if only it existed.

This thought might lead one to suspect that any full subcategory of a Φ-monoidal category inherits a Φ-multicategory structure, and as we shall see, this is in fact the case.

Example 2.3. — Despite the abstract definition, the concept is familiar for our "easy" examples of perfect operator categories.

(2.3.1) A **0**-multicategory is simply a category. A **0**-operad is a monoid.
(2.3.2) A **1**-multicategory is a category C equipped with a functor $C \longrightarrow \mathbf{Set}$. A **1**-operad is a pair (M, X) consisting of a monoid M and an M-set X.
(2.3.3) An **O**-multicategory is a (nonsymmetric) multicategory. An **O**-operad is a(n) (nonsymmetric) operad.
(2.3.4) A **F**-multicategory is a symmetric multicategory. A **F**-operad is a symmetric operad.

Φ-multifunctors. — It is relatively important to stress that the theory of Φ-multicategories is *not* the same as that of colored Φ-operads. Though the objects in question are obviously equivalent, the categorical structure is entirely different: whereas morphisms of colored Φ-operads preserve the colors, Φ-multifunctors may mix the colors. When $\Phi = \mathbf{0}$, this is the difference between categories and categories with a fixed object set. In this subsection we investigate color-mixing morphisms, called *multifunctors*.

Definition 2.4. — Suppose Φ an operator category, and suppose $\mathcal{C}$ and $\mathcal{D}$ Φ-multicategories. A Φ-*multifunctor* $G : \mathcal{C} \longrightarrow \mathcal{D}$ is a map $\operatorname{Obj} G : \operatorname{Obj} \mathcal{C} \longrightarrow \operatorname{Obj} \mathcal{D}$ and, for any $I \in \Phi$, a morphism

$$\gamma_I : I \operatorname{Mor}_{\mathcal{C}}(M_I, N) \longrightarrow I \operatorname{Mor}_{\mathcal{D}}(GM_I, GN)$$

that are compatible with composition in the sense that the following diagrams commute for any morphism $J \longrightarrow I$ of Φ:

$$\begin{array}{ccc} I \operatorname{Mor}_{\mathcal{C}}((M_I), N) \times (J/I) \operatorname{Mor}_{\mathcal{C}}((L_J), (M_I)) & \longrightarrow & J \operatorname{Mor}_{\mathcal{C}}((L_J), N) \\ \downarrow & & \downarrow \\ I \operatorname{Mor}_{\mathcal{D}}((G^I M_I), GN) \times (J/I) \operatorname{Mor}_{\mathcal{D}}((G^J L_J), (G^I M_I)) & \longrightarrow & J \operatorname{Mor}_{\mathcal{D}}((G^J L_J), GN). \end{array}$$

2.5. — Multifunctors (G, γ) and (H, η) compose to yield the functor $H \circ G$, equipped with the composition $(\eta G) \circ \gamma$. I leave it to the interested reader (if there are any) to formulate the notion of a morphism of Φ-multifunctors.

Denote by $\mu^{\Phi}\mathbf{Cat}$ the resulting 2-category of Φ-multicategories,[2] and denote by $\mathbf{Operad}^{\Phi}$ the subcategory of Φ-operads.

Lemma 2.6. — *We now have* $(2,1)$-functors

$$\begin{array}{rcl} \mu\mathbf{Cat} : \mathbf{Op}^{\mathrm{op}} & \longrightarrow & \mathbf{2Cat} \\ \Phi & \longmapsto & \mu^{\Phi}\mathbf{Cat} \\ f & \longmapsto & (-)^f \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbf{Operad} : \mathbf{Op}^{\mathrm{op}} & \longrightarrow & \mathbf{2Cat} \\ \Phi & \longmapsto & \mathbf{Operad}^{\Phi} \\ f & \longmapsto & (-)^f. \end{array}$$

Corollary 2.7. — *Suppose* Φ *an* operator category.

(2.7.1) A Φ*-multicategory* $\mathcal{C}$ *is in particular a category* $\mathcal{C}^{\natural}$.
(2.7.2) A symmetric multicategory $\mathcal{C}$ *is in particular a* Φ*-multicategory* $\mathcal{C}^{U}$.

2.8. — Suppose now D a category. Denote by $\mu^{\Phi}\mathbf{Cat}(D)$ the homotopy fibre of the forgetful 2-functor

$$(-)^{\natural} : \mu^{\Phi}\mathbf{Cat} \longrightarrow \mathbf{Cat}$$

over D — i.e., the 2-category of pairs $(\mathcal{C}, u)$ consisting of a Φ-multicategory $\mathcal{C}$ and an equivalence of categories $u : \mathcal{C}^{\natural} \longrightarrow D$.

[2] I am, of course, being sloppy about set-theory concerns here. There is an obvious way to choose universes (in the sense of Grothendieck) so this all makes sense.

Lemma 2.9. — *For any category D, the* $(2,1)$*-functor* $\mu\mathbf{Cat}$ *restricts to a* $(2,1)$*-functor*

$$\begin{aligned}\mu\mathbf{Cat}(D) : \mathbf{Op}^{\mathrm{op}} &\longrightarrow 2\mathbf{Cat}\\ \Phi &\longmapsto \mu^{\Phi}\mathbf{Cat}(D)\\ f &\longmapsto (-)^{f}.\end{aligned}$$

Colored Φ-operads. — The theory of colored Φ-operads plays a role in the sequel.

Definition 2.10. — Suppose S a set, Φ an operator category. Then the category $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}$ of *S-colored Φ-operads* is the homotopy fiber of the functor $\mathrm{Obj} : \mu^{\Phi}\mathbf{Cat} \longrightarrow \mathbf{Set}$ over the set S. Hence an object of $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}$ is a pair $(\mathcal{C}, e)$, where $e : S \longrightarrow \mathrm{Obj}\,\mathcal{E}$ is a bijection.

2.11. — Whereas $\mu^{\Phi}\mathbf{Cat}$ is most naturally a 2-category, $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}$ is most naturally a 1-category. Of course when $S = \star$, the category of S-colored Φ-operads and that of Φ-operads coincide. Moreover, under certain circumstances, Φ-operads give rise to S-colored Φ-operads.

Definition 2.12. — Suppose S a set, Φ an operator category.

(2.12.1) Consider the functor

$$\begin{aligned}\epsilon_S : \Phi^{\mathrm{op}} &\longrightarrow \mathbf{Set}\\ I &\longrightarrow S^{|I|} \times S\\ \phi &\longrightarrow (\phi^{\star}, \mathrm{id}_S).\end{aligned}$$

A *color sieve for S under Φ* is a full subcategory R of the category $\mathrm{Tot}\,\epsilon_S$ of elements of ϵ_S satisfying the following conditions.

(2.12.1.1) For any element $x \in S$, the element $(\star, x, x) \in R$.

(2.12.1.2) For any morphism $J \longrightarrow I$ of Φ, any I-tuple $(y_I) \in S^{|I|}$, any J-tuple $(x_J) \in S^{|J|}$, and any element $z \in S$, if $(I, (y_I), z) \in R$, and if for any $i \in |I|$, $(J_i, (x_{J_i}), y_i) \in R$, then $(J, (x_J), z) \in R$ as well.

(2.12.2) If R is a color sieve for S under Φ, and $\mathcal{P}$ is any Φ-operad, then the *$(\mathcal{P}, R)$-decorated S-colored Φ-operad* $\mathcal{D}^{\Phi}_R(\mathcal{P})$ is the S-colored operad in which

$$I\,\mathrm{Mor}_{\mathcal{D}^{\Phi}_R(\mathcal{P})}((x_I), y) := \begin{cases}\mathcal{P}(I) & \text{if } (I, (x_I), y) \in R,\\ \varnothing & \text{else,}\end{cases}$$

where the polycomposition law is inherited from $\mathcal{P}$.

2.13. — This defines a functor

$$\mathcal{D}^{\Phi}_R : \mathbf{Operad}^{\Phi} \longrightarrow \mathbf{Col}(S)\mathbf{Operad}^{\Phi}.$$

Example 2.14. — Here is a very special color sieve that will come up later in our study of modules. Suppose Φ perfect. Then the *pure perfection sieve* is the following color sieve R for $|T|$ (the set of points of the point classifier of Φ) under Φ. A triple $(I, (x_I), y)$ is an element of R if and only if one of the following holds.

(2.14.1) The point y is the special point, and there exists a point $i \in |I|$ such that the I-tuple $(x_I) = (|\chi_i|_I)$ is given by the classifying morphism $\chi_i : I \longrightarrow T$.

(2.14.2) The point y is a generic point, and for every $i \in |I|$, $x_i = y$.

If $\mathcal{P}$ is a Φ-operad, then $\mathcal{D}^{\Phi}_R(\mathcal{P})$ for this color sieve will simply be denoted $\mathcal{H}_{\mathcal{P}}$.

Corepresentable Φ-multicategories. — Corepresentable Φ-multicategories are the ones that are essentially uniquely specified by a kind of lax multiplication law on the category under the rubric of Φ.

Suppose here that Φ is an operator category.

Definition 2.15. — Suppose $\mathcal{C}$ a Φ-multicategory. One says that $\mathcal{C}$ is *corepresentable* if the functors

$$I\,\mathrm{Mor}_{\mathcal{C}}((L_I), -) : \mathcal{C}^{\natural} \longrightarrow \mathbf{Set}$$

are all corepresentable for any $I \in \Phi$.

2.16. — Denote by $\mu^{\Phi,c}\mathbf{Cat}$ the full sub-2-category of $\mu^{\Phi}\mathbf{Cat}$ consisting of corepresentable Φ-multicategories.

2.17. — If Φ-multicategories are the ghosts of Φ-monoidal categories, then in this macabre parable we are led to think of corepresentable Φ-multicategories as the undead — deranged zombies who wander the mathematical landscape in search of third-rate actors in campy horror flicks to terrorize.[(3)]

Lemma 2.18. — *The* $(2,1)$*-functor* $\mu\mathbf{Cat}$ *restricts to a* $(2,1)$*-functor*

$$\begin{aligned} \mu^c\mathbf{Cat} : \mathbf{Op}^{\mathrm{op}} &\longrightarrow 2\mathbf{Cat} \\ \Phi &\longmapsto \mu^{\Phi,c}\mathbf{Cat} \\ f &\longmapsto (-)^f. \end{aligned}$$

2.19. — Suppose $\mathcal{C}$ a corepresentable Φ-multicategory. Let us denote a *corepresenting object* for $I\operatorname{Mor}_{\mathcal{C}}((L_I),-)$ by $\otimes L_I$. Such an object is, of course, essentially unique. If $[J \longrightarrow I] \in \Phi$, and $(M_J) \in (\operatorname{Obj}\mathcal{C})^{\times|J|}$, then let us also write $(\otimes M_{(J/I)})$ for the *I-tuple* $(\otimes M_{J_i})_{i\in|I|}$.

Lemma 2.20. — *Suppose* $\mathcal{C}$ *a corepresentable* Φ*-multicategory. To any morphism* $[\phi : J \longrightarrow I] \in \Phi$ *is associated a natural* comparison morphism

$$\alpha_\phi : \otimes M_J \longrightarrow \otimes\left(\otimes M_{(J/I)}\right),$$

natural in $(M_J) \in (\operatorname{Obj}\mathcal{C})^{\times|J|}$.

Proposition 2.21. — *A corepresentable* Φ*-multicategory structure on a category* C *is essentially uniquely specified by the following data:*

(2.21.A) *a* tensor product functor *for any* $I \in \Phi$*:*

$$\begin{aligned} \otimes_I : C^{\times|I|} &\longrightarrow C, \\ (L_I) &\longmapsto \otimes L_I, \end{aligned}$$

and

(2.21.B) *a natural* comparison morphism $\alpha_\phi : \otimes M_J \longrightarrow \otimes\left(\otimes M_{(J/I)}\right)$ *for any* $[\phi : J \longrightarrow I] \in \Phi$,

subject to the following axioms.

(2.21.1) *The functor* $C \longrightarrow C$ *corresponding to* $\star \in \Phi$ *is the identity.*

(2.21.2) *For any* $[K \to J \to I] \in \Phi$*, these diagrams commute:*

$$\begin{array}{ccc} \otimes L_K & \longrightarrow & \otimes(\otimes L_{(K/J)}) \\ \downarrow & & \downarrow \\ \otimes(\otimes(L_{(K/I)}) & \longrightarrow & \otimes(\otimes(\otimes L_{((K/J)/I)})). \end{array}$$

Furthermore, if $\mathcal{C}$ *and* $\mathcal{D}$ *are corepresentable* Φ*-multicategories, then a* Φ*-multifunctor* $\mathcal{C} \longrightarrow \mathcal{D}$ *is essentially uniquely specified by the data of a functor* $F : \mathcal{C}^\natural \longrightarrow \mathcal{D}^\natural$ *and a morphism* $\otimes(FL_I) \longrightarrow F(\otimes L_I)$ *satisfying the by now obvious compatibilities.*

Φ-monoidal categories. — Loosely speaking, Φ-monoidal categories are corepresentable Φ-multicategories whose opposites are also corepresentable Φ-multicategories.

Again, suppose Φ an operator category.

Definition 2.22. — A corepresentable Φ-multicategory $\mathcal{C}$ is a Φ*-monoidal category* if for any morphism $[\phi : J \longrightarrow I] \in \Phi$, the comparison morphisms

$$\alpha_\phi : \otimes(M_J) \longrightarrow \otimes\left(\otimes M_{(J/I)}\right)$$

are isomorphisms.

2.23. — In our scary little story, Φ-monoidal categories are, of course, to be thought of as the living. They come equipped with a kind of multiplication law under the rubric of Φ.

(3) In all seriousness, this analogy does have some content; see the subsection "Raising the dead" in the following section for evidence.

Lemma 2.24. — *A category C* admits a Φ-monoidal structure *if and only if C^{op} does also, in the obvious fashion.*

Definition 2.25. — Suppose $\mathcal{C}$ and $\mathcal{D}$ Φ-monoidal categories.

(2.25.1) A *lax Φ-monoidal functor* $\mathcal{C}\longrightarrow\mathcal{D}$ is a morphism $\mathcal{C}\longrightarrow\mathcal{D}$ of $\mu^{\Phi}\mathbf{Cat}$.

(2.25.2) A *colax Φ-monoidal functor* $\mathcal{C}\longrightarrow\mathcal{D}$ is a morphism $\mathcal{C}^{\mathrm{op}}\longrightarrow\mathcal{D}^{\mathrm{op}}$ of $\mu^{\Phi}\mathbf{Cat}$.

(2.25.3) A lax Φ-monoidal functor $F : \mathcal{C}\longrightarrow\mathcal{D}$ is *pseudo-Φ-monoidal* if the comparison morphisms $\otimes(FL_I)\longrightarrow F(\otimes L_I)$ are all isomorphisms.

2.26. — (2.26.1) Denote by $\mu^{\Phi,\otimes}\mathbf{Cat}^{\mathbf{lax}}$ the 2-category of Φ-monoidal categories with *lax Φ-monoidal functors.*

(2.26.2) Denote by $\mu^{\Phi,\otimes}\mathbf{Cat}^{\mathbf{colax}}$ the 2-category of Φ-monoidal categories with *colax Φ-monoidal functors.*

(2.26.3) Denote by $\mu^{\Phi,\otimes}\mathbf{Cat}$ the 2-category of Φ-monoidal categories with *pseudo-Φ-monoidal functors.*

Lemma 2.27. — *The $(2,1)$-functor $\mu^{c}\mathbf{Cat}$* restricts *to $(2,1)$-functors*

$$\begin{aligned}\mu^{\otimes}\mathbf{Cat}^{\alpha} : \mathbf{Op}^{\mathrm{op}} &\longrightarrow 2\mathbf{Cat}\\ \Phi &\longmapsto \mu^{\Phi,\otimes}\mathbf{Cat}^{\alpha}\\ f &\longmapsto (-)^{f}.\end{aligned}$$

for $\alpha \in \{\mathbf{lax},\mathbf{colax},\varnothing\}$.

Proposition 2.28. — *The forgetful functor $U^{\Phi,\mu} : \mu^{\Phi,\otimes}\mathbf{Cat}\longrightarrow\mu^{\Phi}\mathbf{Cat}$ has a left adjoint*

$$\mathrm{Free}^{\Phi,\otimes} : \mu^{\Phi}\mathbf{Cat}\longrightarrow\mu^{\Phi,\otimes}\mathbf{Cat}$$

Enriched Φ-multicategories. — Now is perhaps a good time to observe that in any $\mathbf{F}$-multicategory $\mathcal{E}$, the notions of Φ-multi-$\mathcal{E}$-category, Φ-operad in $\mathcal{E}$, corepresentable Φ-multi-$\mathcal{E}$-category, Φ-monoidal $\mathcal{E}$-category, and their morphisms, are all perfectly sensible. I leave it to the obsessive-compulsive reader to formulate these notions precisely.

This yields 2-categories $\mu^{\Phi}(\mathcal{E})\mathbf{Cat}$, $\mathbf{Operad}^{\Phi}(\mathcal{E})$, $\mu^{\Phi,c}(\mathcal{E})\mathbf{Cat}$, and $\mu^{\Phi,\otimes}(\mathcal{E})\mathbf{Cat}$, and $(2,1)$-functors $\mu(\mathcal{E})\mathbf{Cat}$, $\mathbf{Operad}(\mathcal{E})$, $\mu^{c}(\mathcal{E})\mathbf{Cat}$, and $\mu^{\otimes}(\mathcal{E})\mathbf{Cat}$, to which I shall unabashedly appeal in the sequel.

3. Avatars of structure

Raising the dead. — It is clear that any full subcategory of a Φ-multicategory $\mathcal{D}$ inherits a Φ-multicategory structure from $\mathcal{D}$. In particular, if $\mathcal{D}$ is a corepresentable Φ-multicategory, then any full subcategory $\mathcal{C}\subset\mathcal{D}$ inherits a Φ-multicategory structure in which

$$I\,\mathrm{Mor}_{\mathcal{C}}((L_I),M) := \mathrm{Mor}_{\mathcal{D}}(\otimes L_I, M).$$

The purpose of this subsection is to observe that all Φ-multicategories arise in this manner. That is, Φ-multicategory structures on a category are always corepresentable Φ-multicategory structures on a larger category. This is some kind of justification for our “macabre parable” of the previous section.

Suppose Φ an operator category, and suppose C a category.

3.1. — The first point to be made here is the observation that a corepresentable Φ-multicategory structure on the copresheaf category $\mathbf{Set}^{C}$ induces a Φ-monoidal structure on C itself. The Yoneda embedding $y : C^{\mathrm{op}}\longrightarrow\mathbf{Set}^{C}$ induces functors

$$y^{I,\star} : (\mathbf{Set}^{C})^{(\mathbf{Set}^{C})^{\times|I|}}\longrightarrow(\mathbf{Set}^{C})^{(C^{\mathrm{op}})^{\times|I|}}$$

for any $I\in\Phi$.

Proposition 3.2. — *Suppose $\otimes$ a corepresentable Φ-multicategory structure on the functor category $\mathbf{Set}^{C}$. By adjunction, the functors*

$$y^{I,\star}\otimes_I : (C^{\mathrm{op}})^{\times|I|}\longrightarrow\mathbf{Set}^{C}$$

correspond to functors

$$I\,\mathrm{Mor}_{\mathcal{C}} : (C^{\mathrm{op}})^{\times|I|}\times C\longrightarrow\mathbf{Set},$$

and these define a Φ-multicategory structure $\mathcal{C}$ on C.

Proposition 3.3. — *This construction defines a 2-functor*

$$y^{\star} : \mu^{\otimes}\mathbf{Cat}(\mathbf{Set}^{C}) \longrightarrow \mu\mathbf{Cat}(C).$$

3.4. — On the other hand, notice that a Φ-multicategory structure $\mathcal{C}$ on C indicates how to form the tensor product of any I-tuple of corepresentable functors $C \longrightarrow \mathbf{Set}$. The idea now is to use the fact that the corepresentables generate the category $\mathbf{Set}^{C}$ to extend this tensor product. Observe that the left Kan extensions

$$y_!^{I} : (\mathbf{Set}^{C})^{(C^{\mathrm{op}})^{\times|I|}} \longrightarrow (\mathbf{Set}^{C})^{(\mathbf{Set}^{C})^{\times|I|}}$$

exist and are fully faithful for $I \in \Phi$.

Proposition 3.5. — *Suppose $\mathcal{C}$ a Φ-multicategory structure on C. Then the functors $y_!^{I}\, I\,\mathrm{Mor}_{\mathcal{C}}$ define a corepresented Φ-multicategory structure on $\mathbf{Set}^{C}$.*

Lemma 3.6. — *This yields a fully faithful 2-functor*

$$y_! : \mu^{\Phi}\mathbf{Cat}(C) \longrightarrow \mu^{\Phi,c}\mathbf{Cat}(\mathbf{Set}^{C}),$$

left adjoint (in the $\mathbf{Cat}$-enriched sense) to $y^{\star}$.

Definition 3.7. — Suppose $\otimes$ a corepresentable Φ-multicategory structure $\otimes$ on $\mathbf{Set}^{C}$. Consider, for any $I \in \Phi$ and any I-tuple F_I of copresheaves $C \longrightarrow \mathbf{Set}$, the colimit $P(F_I)$ of the diagram

$$\begin{array}{ccc} \prod_{i\in|I|}(C^{\mathrm{op}}/F_i) & \longrightarrow & \mathbf{Set}^{C} \\ (X_I) & \longmapsto & \otimes y(X)_I; \end{array}$$

then one says that $\otimes$ is *generated by C* if for any $I \in \Phi$ and any I-tuple F_I of copresheaves $C \longrightarrow \mathbf{Set}$, the canonical morphism

$$\otimes F_I \longrightarrow P(F_I)$$

is an isomorphism.

***Proposition 3.8* (Raising the dead).** — *The* adjunction

$$y_! : \mu^{\Phi}\mathbf{Cat}(C) \rightleftarrows \mu^{\Phi,c}\mathbf{Cat}(\mathbf{Set}^{C}) : y^{\star}$$

induces an equivalence of 2-categories between $\mu^{\Phi}\mathbf{Cat}(C)$ and the full sub-2-category

$$\mu^{\Phi,c}\mathbf{Cat}(\mathbf{Set}^{C})\langle C\rangle \subset \mu^{\Phi,c}\mathbf{Cat}(\mathbf{Set}^{C})$$

comprised of corepresented Φ-multicategory structures generated by C.

Φ as a Φ-multicategory. — Using the same ideas as the "raising the dead" results of the previous subsection, we can observe a Φ-multicategory structure on Φ itself.

Theorem 3.9. — *Suppose Φ a flat operator category. Then Φ is a Φ-multicategory, with*

$$I\,\mathrm{Mor}_{\Phi}(J_I, -) := (\sigma_{I,!}F_I^{\star}y)(J_I),$$

where F_I is the forgetful functor $(\Phi/I)^{\mathrm{op}} \longrightarrow \Phi^{\mathrm{op}}$, so that the following diagram commutes:

$$\begin{array}{ccc} (\Phi/I)^{\mathrm{op}} & \xrightarrow{\sigma_I} & (\Phi^{\times|I|})^{\mathrm{op}} \\ {\scriptstyle F_I}\downarrow & & \downarrow{\scriptstyle I\,\mathrm{Mor}_{\Phi}} \\ \Phi^{\mathrm{op}} & \xrightarrow[y]{} & \mathbf{Set}^{\Phi}. \end{array}$$

3.10. — For Φ a flat operator category, we obtain the formula

$$I\,\mathrm{Mor}_{\Phi}(J_I, K) := \operatorname{colim}_{J' \in (\Phi/I)^{\mathrm{op}}, (J_i \to J'_i)_{i\in|I|}} \mathrm{Mor}_{\Phi}(J', K).$$

Theorem 3.11. — *The following are equivalent for a flat operator category* Φ.

1. *The* Φ*-multicategory structure on* Φ *is corepresentable.*
2. *The functor* $I\,\mathrm{Mor}_\Phi$ lifts*:*

$$\begin{array}{ccc} (\Phi/I)^{\mathrm{op}} & \xrightarrow{\sigma_I} & (\Phi^{\times|I|})^{\mathrm{op}} \\ {\scriptstyle F_I}\downarrow & \swarrow{\scriptstyle \otimes_I} & \downarrow{\scriptstyle I\,\mathrm{Mor}_\Phi} \\ \Phi^{\mathrm{op}} & \xrightarrow[y]{} & \mathbf{Set}^\Phi. \end{array}$$

3. Φ *is faithfully flat.*
4. *The* Φ*-multicategory structure on* Φ *is* Φ*-monoidal.*

Example 3.12. — At first blush, this may look a little surprising. But in fact it's just a generalization of a few well-known examples. The category $\mathbf{0}$ is a category; the category $\mathbf{1}$ is pointed (at 0); the category $\mathbf{O}$ is symmetric monoidal with the "concatenation" product; and the category $\mathbf{F}$ is symmetric monoidal with the coproduct.

3.13. — In effect, these results establish Φ as a kind of universal Φ-multicategory; when Φ is faithfully flat, Φ is by the same token the universal Φ-monoidal category.

Φ-monoids and the Leinster category $\mathcal{L}_\Phi$. — Now we come to the first nontrivial results. The notion of Φ-monoid is a fairly predictable one, but the real insight is that Φ-monoids in categories E with finite limits can be described entirely as functors to E.

3.14. — As far as I understand, this observation was first made by T. Leinster for $\Phi = \mathrm{O}$ and $\Phi = \mathrm{F}$; hence the construction below bears his name.

Definition 3.15. — Supppose Φ a flat operator category; $\mathcal{E}$ an $\mathbf{F}$-monoidal category. Then there is a Φ-multi-$\mathcal{E}$-category $\mathbf{1}$ with a unique object $\star$ and

$$I\,\underline{\mathrm{Mor}}_{\mathbf{1}}((\star_I),\star) := \mathbf{1}_{\mathcal{E}}$$

A Φ*-monoid* in a Φ-multi-$\mathcal{E}$-category $\mathcal{C}$ is a Φ-multifunctor $\mathbf{1}\longrightarrow\mathcal{C}$. Write

$$\mathbf{Mon}^\Phi_{\mathcal{E}}(\mathcal{C}) := \underline{\mathrm{Mor}}_{\mu^\Phi(\mathcal{E})\mathbf{Cat}}(\mathbf{1},\mathcal{C})$$

Lemma 3.16. — *Suppose* Φ *perfect, and suppose* $\mathcal{C}$ *a* Φ*-monoidal* $\mathcal{E}$*-category; then there is a* natural equivalence

$$\mathbf{Mon}^\Phi_{\mathcal{E}}(\mathcal{C}) \simeq \underline{\mathrm{Mor}}_{\mu^{\Phi,\otimes}(\mathcal{E})\mathbf{Cat}}(\mathrm{Free}^{\Phi,\otimes}_{\mathcal{E}}(\mathbf{1}),\mathcal{C}).$$

Lemma 3.17. — *If* Φ *is faithfully flat and* $\mathcal{E} = \mathbf{Set}$*, then* $\mathrm{Free}^{\Phi,\otimes}(\star) \simeq \Phi$*, so that*

$$\mathbf{Mon}^\Phi_{\mathbf{Set}}(\mathcal{C}) \simeq \underline{\mathrm{Mor}}_{\mu^{\Phi,\otimes}\mathbf{Cat}}(\Phi,\mathcal{C}).$$

Example 3.18. — It is now an easy exercise to verify that the Φ-monoids in $\mathbf{Set}$ for our "easy examples" of operator categories Φ are exactly as we described them:

Φ	$\mathbf{Mon}^\Phi(\mathbf{Set})$
$\mathbf{0}$	$\mathbf{Set}$
$\mathbf{p}$ $(p > 0)$	$(\star/\mathbf{Set})$
$\mathbf{O}_{\leq 1}$	$\mathbf{Mag}^1$
$\mathbf{O}_{\leq n}$ $(n > 1)$	$\mathbf{Mon}$
$\mathbf{O}$	$\mathbf{Mon}$
$\mathbf{F}_{\leq 1}$	$\mathbf{Commag}^1$
$\mathbf{F}_{\leq n}$ $(n > 1)$	$\mathbf{Common}$
$\mathbf{F}$	$\mathbf{Common}$

Theorem 3.19. — *Suppose* Φ *a perfect operator category. Then there exist:*

(3.19.A) *a category* $\mathcal{L}_\Phi$ *— called the* Leinster category *of* Φ *—,*

(3.19.B) *an* adjunction

$$T : \Phi \rightleftarrows \mathcal{L}_\Phi : R$$

(3.19.C) *a* functorial *assignment to any $I \in \Phi$ the following data:*

(3.19.C.1) a finite category P_I,

(3.19.C.2) a functor $D_I : P_I \longrightarrow \mathcal{L}_\Phi$,

(3.19.C.3) a morphism of functors $[\mathbf{const}_{TI} \longrightarrow D_I] \in (\mathcal{L}_\Phi)^{P_I}$,

that satisfy the following conditions.

(3.19.1) For any category E with finite products*, there is a natural equivalence*

$$\underline{\mathrm{Mor}}_{\mu^\Phi \mathbf{Cat}^{\mathbf{colax}}}(\mathrm{Free}^{\Phi,\otimes}(\star), E) \simeq E^{\mathcal{L}_\Phi}.$$

(3.19.2) For any category E with all finite limits*, this equivalence induces an equivalence between $\mathbf{Mon}^\Phi_{\mathbf{Set}}(E)$ and the full subcategory $\mathbf{Seg}^\Phi(E) \subset E^{\mathcal{L}_\Phi}$ comprised of $F : \mathcal{L}_\Phi \longrightarrow E$ such that every element of*

$$S_\Phi := \{\, F(TI) \longrightarrow \lim(F \circ D_I)\,\}$$

is an isomorphism of E.

Sketch of proof. — Here the perfection of Φ is of critical import: since Φ is perfect, there is a right adjoint $T : \Phi \longrightarrow (\Phi/T)$ to the "special fiber" functor. Composing this with the forgetful functor $(\Phi/T) \longrightarrow \Phi$ gives an endofunctor of Φ that is in fact a monad on Φ. We define $\mathcal{L}_\Phi$ to be the Kleisli category of this monad.

For the diagram $D_I : P_I \longrightarrow \mathcal{L}_\Phi$, one can simply take P_I to be the full subcategory of (I/Φ) comprised of those morphisms $I \longrightarrow J$ that (1) induce an epimorphism $|I| \longrightarrow |J|$, and that (2) factor through one of the morphisms $\chi_i : I \longrightarrow T$ $(i \in |I|)$ guaranteed by the universal property of T. Then D_I is just the restriction of the functor $\Phi \longrightarrow \mathcal{L}_\Phi$ to P_I, and the morphism $\mathbf{const}_{TI} \longrightarrow D_I$ is obvious. □

Example 3.20. — For our "easy examples" of perfect operator categories Φ, we can write explicitly what $\mathcal{L}_\Phi$ and S_Φ are:

Φ	$\mathcal{L}_\Phi$	S_Φ
$\mathbf{0}$	$\mathbf{0}$	$\varnothing$
$\mathbf{1}$	$\mathbf{1}$	$\{A_0 \to \star\}$
$\mathbf{O}$	Δ^{op}	$\{A_p \to A_1 \times_{A_0} \cdots \times_{A_0} A_1 \mid p \geq 0\}$
$\mathbf{F}$	Γ^{op}	$\{A_p \to A_1 \times_{A_0} \cdots \times_{A_0} A_1 \mid p \geq 0\}$

That's a perfectly clear explanation of what the categories Δ^{op} and Γ^{op} and the corresponding Segal maps have to do with monoids and commutative monoids in the conventional sense.(4)

Φ-operads and the category $\mathcal{M}_\Phi$. — In the last subsection, we saw that the Leinster category allowed us to treat Φ-monoids in categories E with finite limits as certain kinds of diagrams in E. A good question to ask now is: can we do the same thing with operads? That is, can we view operads in E as certain diagrams in E? The answer is yes.

Theorem 3.21. — *Suppose Φ an operator category. Then there exist:*

(3.21.A) a category $\mathcal{M}_\Phi$,

(3.21.B) a functor $\mathcal{M}_\Phi \longrightarrow \mathcal{L}_\Phi$, if Φ is perfect,

(3.21.C) a functorial assignment to any $X \in \mathcal{M}_\Phi$ the following data:

(3.21.C.a) a finite category Q_X,

(3.21.C.b) a functor $D_X : Q_X \longrightarrow \mathcal{M}_\Phi$,

(3.21.C.c) a morphism of functors $[\mathbf{const}_X \longrightarrow D_X] \in (\mathcal{M}_\Phi)^{Q_X}$,

that satisfies the following condition.

(4) While it's very clear that pointed finite sets are the Kleisli category of the monad T for $\mathbf{F}$, it's a little less obvious that Δ^{op} is the Kleisli category of the monad T for $\mathbf{O}$. To see this, one need only observe that Δ^{op} is equivalent to the category of ordered finite sets with a distinct top and bottom. This is one of the peculiarities of $\mathbf{O}$: it contains a copy of its own opposite. As I understand it, this observation is due to Joyal.

(3.21.0) For any category E with all finite limits, there is a natural fully faithful functor

$$\mathbf{Operad}^{\Phi}(E) \longrightarrow E^{\mathcal{M}_{\Phi}},$$

where the essential image is comprised of functors $G : \mathcal{M}_{\Phi} \longrightarrow E$ such that every element of

$$T_{\Phi} := \{ G(X) \longrightarrow \lim(G \circ D_X) \}$$

is an isomorphism of E.

Sketch of proof. — Consider first the pseudofunctor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ whose value on $\mathbf{p}$ is the category of Φ-valued presheaves on $\mathbf{p}$ — i.e., the functor category $\Phi^{(\mathbf{p}^{\mathrm{op}})}$. Now $\mathcal{M}_{\Phi}$ is the lluf subcategory of the total category $\operatorname{Tot} A$ whose morphisms $(\phi, f) : (\mathbf{q}, J) \longrightarrow (\mathbf{p}, I)$ are cartesian in the sense that for any $0 \leq r < q$, the induced squares

$$\begin{array}{ccc} J(r) & \longrightarrow & J(0) \\ \downarrow & & \downarrow \\ I(\phi(r)) & \longrightarrow & I(\phi(0)) \end{array}$$

have the property that for any $i \in |I(\phi(r))|$, the morphism $J(r)_i \longrightarrow J(0)_i$ is an isomorphism.

The analogues of the Segal maps are pretty complicated, but the idea is quite clear from the following. From a Φ-operad P in a category E with all finite limits, one can define an associated functor

$$\begin{aligned} N^{\Phi}P : \quad \mathcal{M}_{\Phi} &\longrightarrow E \\ (\mathbf{p}, J) &\longmapsto \textstyle\prod_{0 \leq i < p} \prod_{a \in |J(i)|} P(J(i+1)_a), \end{aligned}$$

where $J(i+1)_a$ denotes the fiber of the map $J(i+1) \longrightarrow J(i)$ over the point $a \in |J(i)|$. This defines the functor

$$N^{\Phi} : \mathbf{Operad}^{\Phi}(E) \longrightarrow E^{\mathcal{M}_{\Phi}},$$

which is fully faithful, with a left adjoint.

The functor $\mathcal{M}_{\Phi} \longrightarrow \mathcal{L}_{\Phi}$ is easy: it's simply the assignment $(\mathbf{p}, I) \longmapsto TI(0)$. □

Example 3.22. — If $\Phi = \star$, then $\mathcal{M}_{\Phi} \simeq \Delta^{\mathrm{op}}$, and the analogue of the Segal maps are simply the usual Segal maps $A_p \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ for $p > 0$. It is now an elementary exercise to verify that these are the same things as ordinary monoids in E.

3.23. — Some people seem to be quite fond of trees. I find this affinity a little odd, but *de gustibus non est disputandum*. In any case, dendrophiles may be gratified to learn that $\mathcal{M}_{\Phi}$ can be described as a category of Φ-*trees*. The objects of this category are trees (with a root and tails, as in Kontsevich-Soibelman), equipped with an isomorphism between the set of points of a given object $I(r)$ of Φ and the vertices of height r, so that the edge maps are induced by specified morphisms of Φ. In other words, an object $I(p) \to I(p-1) \to \ldots \to I(0)$ of $\mathcal{M}_{\Phi}$ can be conceived as a "Φ-numbered" tree in which the tails are the points of $I(p)$, the internal vertices are the elements of $\bigcup_{0 \leq r < p} |I(r)|$, and in which an edge connects a point j of $I(r+1)$ to a point i of $I(r)$ if and only if j is a point of the fiber $I(r+1)_i$.

The *valency* $\operatorname{val} v$ of a given vertex v of height r in the tree corresponding to $I(p) \to I(p-1) \to \ldots \to I(0)$ is then to be thought of as the fiber of the map $I(r+1) \longrightarrow I(r)$ over the point $i \in |I(r)|$ corresponding to v. So for an operad $\mathcal{P}$, one sees that the value of $N^{\Phi}\mathcal{P}$ on a Φ-tree T is simply the product of $\mathcal{P}(\operatorname{val} v)$ over all internal vertices v of T.

In particular, when $\Phi = \mathbf{F}$, the objects of $\mathcal{M}_{\Phi}$ simply are trees. The morphisms are a little special, however: a morphism $\sigma : \mathcal{T} \longrightarrow \mathcal{S}$ of $\mathcal{M}_{\mathbf{F}}$ is in particular required to have the property that for any internal vertex v of $\mathcal{S}$, we have

$$\operatorname{val} v = \sum_{w \in \sigma^{-1}(v)} \operatorname{val} w.$$

Φ-operads and the F-multicategory $\mathcal{A}^{\Phi}$. — The upshot of the previous subsection was the existence of a category $\mathcal{M}^{\Phi}$ with the property that operads in a category E with all finite limits are precisely functors $\mathcal{M}^{\Phi}\longrightarrow E$ preserving certain finite limits. Similarly, the upshot of this subsection is the existence of an $\mathbf{F}$-multicategory $\mathcal{A}^{\Phi}$ with the property that operads in an $\mathbf{F}$-multicategory $\mathcal{E}$ are precisely $\mathbf{F}$-multifunctors $\mathcal{A}^{\Phi}\longrightarrow\mathcal{E}$.

Theorem 3.24. — *Suppose Φ a perfect operator category. Then the 2-functor*

$$\begin{aligned}\mathbf{Operad}^{\Phi}:\ \mu^{\mathrm{F}}\mathbf{Cat} &\longrightarrow \mathbf{Cat}\\ \mathcal{E} &\longmapsto \mathbf{Operad}^{\Phi}(\mathcal{E})\end{aligned}$$

is pseudocorepresentable by an $\mathbf{F}$-multicategory $\mathcal{A}^{\Phi}$.

Sketch of proof. — This follows from the usual representability theorems, once one observes that $\mathbf{Operad}^{\Phi}$ preserves all homotopy limits. ☐

Corollary 3.25. — *There is a* universal Φ-operad $\mathcal{U}^{\Phi}$ *in $\mathcal{A}^{\Phi}$ with the property that any Φ-operad $\mathcal{P}$ in any symmetric multicategory $\mathcal{E}$ is isomorphic to the image of $\mathcal{U}^{\Phi}$ under the functor $\mathbf{Operad}^{\Phi}(\mathcal{A}^{\Phi})\longrightarrow\mathbf{Operad}^{\Phi}(\mathcal{E})$ induced by the multifunctor $\mathcal{A}^{\Phi}\longrightarrow\mathcal{E}$ corresponding to $\mathcal{P}$.*

Example 3.26. — When $\Phi=\mathbf{0}$, one verifies easily that $\mathcal{A}^{\mathbf{0}}$ is the associative operad, i.e., the image of the terminal object $\star$ under the left adjoint to the forgetful functor $\mathbf{Operad}^{\mathbf{F}}(\mathcal{E})\longrightarrow\mathbf{Operad}^{\mathbf{O}}(\mathcal{E})$. The universal $\mathbf{0}$-operad $\mathcal{U}^{\mathbf{0}}$ is now clear.

3.27. — In general, $\mathcal{A}^{\Phi}$ is a symmetric multicategory with object set $\operatorname{Obj}\mathcal{A}^{\Phi}=\operatorname{Obj}\Phi$; to make the distinction clear, I will denote the object of $\mathcal{A}^{\Phi}$ corresponding to $I\in\Phi$ by AI. The morphisms of $\mathcal{A}^{\Phi}$ are generated by elements

$$Af\in(T|I|)\operatorname{Mor}_{\mathcal{A}^{\Phi}}((AI,(AJ_I)),AJ)$$

associated to every morphism $[f:J\longrightarrow I]\in\Phi$, subject to the relations in $T(|J|\sqcup|K|)\operatorname{Mor}_{\mathcal{A}^{\Phi}}((AI,(AJ_I),(AK_J)),AK)$ for every composable pair of morphisms $[K\to J\to I]\in\Phi$ arising from the associativity axiom for operads. The universal Φ-operad $\mathcal{U}^{\Phi}$ is again clear.

3.28. — It is an interesting problem to find a clean, explicit description of $\mathcal{A}^{\Phi}$. One can construct $\mathcal{A}^{\Phi}$ as above or, alternatively, by forming a suitable quotient of the free symmetric multicategory generated by $\mathcal{M}_{\Phi}$. However, I have found it difficult to describe the combinatorics of $\mathcal{A}^{\Phi}$ and the universal Φ-operad $\mathcal{U}^{\Phi}$ in a concrete, intrinsic manner. I suspect that there are satisfactory characterizations using a suitable category of Φ-trees (no doubt pleasing the tree-huggers), but a precise description along these lines has eluded me thus far.[5]

Algebras, chiralities, and modules. — In this subsection I define algebras, chiralities, and modules over algebras with given chiralities. Although the notion of a chirality appears to be completely nonstandard, it is a crucial ingredient in our work here.

3.29. — Suppose here Φ an operator category and $\mathcal{E}$ a cocomplete symmetric monoidal category, in which the tensor product $\otimes$ commutes with colimits in each variable.

Definition 3.30. — Suppose $\mathcal{P}$ and $\mathcal{Q}$ two Φ-multi-$\mathcal{E}$-multicategories. Then the category of $\mathcal{P}$-algebras in $\mathcal{Q}$ is the category

$$\mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}}(\mathcal{Q}):=\underline{\operatorname{Mor}}_{\mu^{\Phi}(\mathcal{E})\mathbf{Cat}}(\mathcal{P},\mathcal{Q}).$$

This specifies a 2-functor

$$\mathbf{Alg}^{\Phi}_{\mathcal{E}}:\mu^{\Phi}(\mathcal{E})\mathbf{Cat}^{\mathrm{op}}\times\mu^{\Phi}(\mathcal{E})\mathbf{Cat}\longrightarrow\mathbf{Cat}.$$

[5] Kontsevich and Soibelman purport to give a description of a related object when $\Phi=\mathbf{F}$, but unfortunately I do not understand it.

3.31. — If $S = \operatorname{Obj}\mathcal{P}$, then there is an obvious forgetful functor $\mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}}(\mathcal{Q}) \longrightarrow \mathcal{Q}^{\times S}$, and if $\mathcal{Q}$ has enough colimits, then this functor has a left adjoint. In particular, if $\mathcal{P}$ is a Φ-operad then there is the forgetful functor $\mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}}(\mathcal{Q}) \longrightarrow \mathcal{Q}$, and if $\mathcal{Q}$ has enough $\mathcal{E}$-colimits, there is a free $\mathcal{P}$-algebra functor $\mathcal{Q} \longrightarrow \mathbf{Alg}^{\Phi}_{\mathcal{P}}(\mathcal{Q})$.

Example 3.32. — Examples of algebras over **F**-operads are ubiquitous and very well documented elsewhere. Let us here turn our attention to Φ-operads and Φ-multicategories for other operator categories Φ.

(3.32.1) A **0**-operad is a monoid M in $\mathcal{E}$, and an M-algebra in an $\mathcal{E}$-category (=**0**-multi-$\mathcal{E}$-category) C is an object X of C an homomorphism of monoids $M \longrightarrow \operatorname{End} X$.

(3.32.2) A **1**-operad in $\mathcal{E}$ is in fact a pair (M, X) consisting of a **0**-operad M and an M-algebra X; hence a

(3.32.3) By definition, $\mathbf{Mon}^{\Phi}_{\mathcal{E}}(\mathcal{Q}) = \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathbf{1}}(\mathcal{Q})$.

(3.32.4) There is a natural equivalence of categories $\mathbf{Operad}^{\Phi}(\mathcal{E}) \simeq \mathbf{Alg}^{\mathbf{F}}_{\mathbf{Set},\mathcal{A}^{\Phi}}(\mathcal{E})$.

3.33. — In a moment I will have to give two rather long definitions, which I believe are new. Before I do, however, let me try to give a word of explanation for the auxiliary notion of *chirality*. When dealing with algebras over a Φ-operad, it is not necessarily clear what it means to have an algebra *act* on an object. We are quite used to thinking of associative algebras acting either on the left or on the right; this is one sort of chirality, but for more general sort of algebras, it is not immediately clear what this means, and it may be that a more "exotic" kind of action is more appropriate in a given setting.

Moreover, it is not always entirely clear how to formalize notions of modules in which more than one algebra is permitted to act (e.g., (R, S)-bimodules). The number of algebras that can act on a single object is in fact precisely equal to the complexity of Φ. It is necessary also to allow the possibility that these actions do not commute, but act on one another as well, i.e., to allow "twists" of various kinds. A *chirality*, then, is meant to provide a formalization of the idea of possibly several algebras, possibly over different Φ-operads, acting in various, possibly noncommuting, ways.

3.34. — Suppose Φ a perfect operator category with point classifier (T, t), suppose $(\mathcal{P}_{|T|_\gamma}) = (\mathcal{P}_\eta)_{\eta \in |T|_\gamma}$ a $|T|_\gamma$-tuple of Φ-operads in $\mathcal{E}$.[6]

Definition 3.35. — (3.35.1) A Φ-*chirality in* $\mathcal{E}$ is a Φ-multi-$\mathcal{E}$-category $\mathcal{H}$ with $\operatorname{Obj}\mathcal{H} = |T|$ satisfying the following conditions.

(3.35.1.1) For any I-tuple (k_I) of objects of $\mathcal{H}$, the polymorphism object $I\operatorname{Mor}_{\mathcal{H}}((k_I), t) = \varnothing$ unless there exists $i \in |I|$ such that the map $k : |I| \longrightarrow \operatorname{Obj}\mathcal{H} = |T|$ is induced by the classifying morphism $\chi_i : I \longrightarrow T$ in Φ.

(3.35.1.2) For any I-tuple (k_I) of elements of T, and any generic point $\eta \in |T|_\gamma$, the polymorphism object $I\operatorname{Mor}_{\mathcal{H}}((k_I), \eta) = \varnothing$ unless $k : |I| \longrightarrow \operatorname{Obj}\mathcal{H} = |T|$ factors through a map $|I| \longrightarrow |T|_\gamma$.

(3.35.2) A Φ-chirality $\mathcal{H}$ in $\mathcal{E}$ is said to be *pure* if for any I-tuple (k_I) of elements of T, and any generic point $\eta \in |T|_\gamma$, the polymorphism object $I\operatorname{Mor}_{\mathcal{H}}((k_I), \eta) = \varnothing$ unless $k_i = \eta$ for every $i \in |I|$.

(3.35.3) A *morphism of* Φ-*chiralities* $\mathcal{H} \longrightarrow \mathcal{K}$ is a morphism of $|T|$-colored Φ-operads in $\mathcal{E}$ — i.e., Φ-multi-$\mathcal{E}$-functor that induces the identity on $|T|$. Denote by $\mathbf{Chr}^{\Phi}(\mathcal{E})$ the category of Φ-chiralities in $\mathcal{E}$ and their morphisms.

(3.35.4) Suppose $\mathcal{H}$ a Φ-chirality in $\mathcal{E}$. Then for any generic point $\eta \in |T|_\gamma$, the Φ-operad $\mathcal{H}\langle\eta\rangle$ generated by η in $\mathcal{H}$ — i.e., the full sub-Φ-multi-$\mathcal{E}$-category consisting of the object η alone[7] — is the *structural* Φ-*operad of* $\mathcal{H}$ *at* η; this defines functors

$$\operatorname{Str}_\eta : \mathbf{Chr}^{\Phi}(\mathcal{E}) \longrightarrow \mathbf{Operad}^{\Phi}(\mathcal{E}) \quad \text{and} \quad \operatorname{Str} : \mathbf{Chr}^{\Phi}(\mathcal{E}) \longrightarrow \mathbf{Operad}^{\Phi}(\mathcal{E})^{\times |T|_\gamma}.$$

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[6] One can of course define *colored chiralities*, in which the $\mathcal{P}_\eta$ are permitted to be colored Φ-operads, but this is more generality than I know how to use here.

[7] also known as the *endomorphism* Φ-*operad* of η

(3.35.5) A $(\mathcal{P}_{|T|_\gamma})$-*chirality in* $\mathcal{E}$ is a Φ-chirality $\mathcal{H}$ equipped with an isomorphism $\mathcal{P}_\eta \longrightarrow \mathcal{H}\langle\eta\rangle$ for each generic point $\eta \in |T|_\gamma$. The category $\mathbf{Chr}^{\Phi}_{(\mathcal{P}_{|T|_\gamma})}(\mathcal{E})$ of $(\mathcal{P}_{|T|_\gamma})$-chiralities in $\mathcal{E}$ is thus the homotopy fibre of Str over the $|T|_\gamma$-tuple $(\mathcal{P}_{|T|_\gamma})$.

(3.35.6) If the Φ operads $\mathcal{P}_\eta$ are all equal to a Φ-operad $\mathcal{P}$, then a $(\mathcal{P}_{|T|_\gamma})$-chirality in $\mathcal{E}$ is called a $\mathcal{P}$-*chirality in* $\mathcal{E}$, and the category of such is denoted $\mathbf{Chr}^{\Phi}_{\mathcal{P}}(\mathcal{E})$.

3.36. — As usual, one can think of $\mathbf{Chr}^{\Phi}(\mathcal{E})$ as a category fibred over the category $\mathbf{Operad}^{\Phi}(\mathcal{E})^{\times|T|_\gamma}$ or as a pseudofunctor

$$\begin{aligned}\mathbf{Chr}^{\Phi}(\mathcal{E}) :\ \mathbf{Operad}^{\Phi}(\mathcal{E})^{\times|T|_\gamma} &\longrightarrow (\mathcal{E})\mathbf{Cat} \\ (\mathcal{P}_{|T|_\gamma}) &\longmapsto \mathbf{Chr}^{\Phi}_{(\mathcal{P}_{|T|_\gamma})}(\mathcal{E})\end{aligned}\quad .$$

3.37. — Just as we are required to say whether a module over an associative algebra is a left or right module, so are we required to give a chirality for a module over an algebra, or, more generally, over a $|T|_\gamma$-tuple of algebras, each possibly over a different Φ-operad.

A module with a given chirality is in fact nothing more than an algebra over that chirality. Such an algebra is to be considered a module over the induced algebras over the structural operads.

Definition 3.38. — (3.38.1) Suppose $\mathcal{H}$ a $(\mathcal{P}_{|T|_\gamma})$-chirality in $\mathcal{E}$, and suppose $\mathcal{Q}$ a Φ-multi-$\mathcal{E}$-category. Then for any generic point $\eta \in |T|_\gamma$, the *structural* $\mathcal{P}_\eta$-*algebra* $X\langle\eta\rangle$ of an $\mathcal{H}$-algebra X is the induced Φ-multi-$\mathcal{E}$-functor

$$\mathrm{Str}_\eta X : \mathcal{P}_\eta \cong \mathcal{H}\langle\eta\rangle \longrightarrow \mathcal{Q}.$$

This defines functors

$$\mathrm{Str}_\eta : \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{H}}(\mathcal{Q}) \longrightarrow \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}_\eta}(\mathcal{Q}) \quad \text{and} \quad \mathrm{Str} : \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{H}}(\mathcal{Q}) \longrightarrow \textstyle\prod_{\eta\in|T|_\gamma} \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}_\eta}(\mathcal{Q}).$$

(3.38.2) Suppose $\mathcal{H}$ a $(\mathcal{P}_{|T|_\gamma})$-chirality in $\mathcal{E}$, suppose $\mathcal{Q}$ a Φ-multi-$\mathcal{E}$-category, and suppose, for any generic point $\eta \in T$, A_η a $\mathcal{P}_\eta$-algebra in $\mathcal{Q}$. Then *an* $(A_{|T|_\gamma})$-*module with chirality* $\mathcal{H}$ is an $\mathcal{H}$-algebra X in $\mathcal{Q}$, equipped with an isomorphism $A_\eta \longrightarrow X\langle\eta\rangle$ of $\mathcal{P}_\eta$-algebras for every generic point $\eta \in |T|_\gamma$. The category $\mathbf{Mod}^{\Phi}_{\mathcal{E}}((A_{|T|_\gamma});\mathcal{H})$ of $(A_{|T|_\gamma})$-modules with chirality $\mathcal{H}$ is the homotopy fibre of Str over $(A_{|T|_\gamma})$.

(3.38.3) If the Φ-operads $\mathcal{P}_\eta$ are all equal to a single Φ-operad $\mathcal{P}$, and if the $\mathcal{P}$-algebras A_η are all equal to a single $\mathcal{P}$-algebra A, then an $(A_{|T|_\gamma})$-module with chirality $\mathcal{H}$ is called an A-*module with chirality* $\mathcal{H}$, and the category of such is denoted $\mathbf{Mod}^{\Phi}_{\mathcal{E}}(A;\mathcal{H})$.

3.39. — Again, one can think of $\mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{H}}(\mathcal{Q})$ as a category fibred over the category $\prod_{\eta\in|T|_\gamma} \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}_\eta}(\mathcal{E})$ or as a pseudofunctor

$$\begin{aligned}\mathbf{Mod}^{\Phi}_{\mathcal{E}}(-;\mathcal{H}) :\ \textstyle\prod_{\eta\in|T|_\gamma} \mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}_\eta}(\mathcal{E}) &\longrightarrow \mathbf{Cat} \\ (A_{|T|_\gamma}) &\longmapsto \mathbf{Mod}^{\Phi}_{\mathcal{E}}((A_{|T|_\gamma});\mathcal{H})\end{aligned}\quad .$$

Example 3.40. — Observe that for any Φ-operad $\mathcal{P}$, there is a unique pure $\mathcal{P}$-chirality $\mathcal{H}_{\mathcal{P}}$ satisfying the following conditions.

(3.40.1) For any object $I \in \Phi$ and any point $i \in |I|$, one has

$$I\,\mathrm{Mor}_{\mathcal{H}_{\mathcal{P}}}((|\chi_i|_I), t) = \mathcal{P}(I),$$

where $(|\chi_i|_I)$ is the I-tuple of elements of $|T|$ given by the classifying morphism $\chi_i : I \longrightarrow T$.

(3.40.2) For any generic point $\eta \in |T|_\gamma$, one has

$$\mathcal{H}_{\mathcal{P}}\langle\eta\rangle = \mathcal{P}.$$

(3.40.3) The polycompositions of $\mathcal{H}_{\mathcal{P}}$ are all inherited from the polycomposition laws for the operad $\mathcal{P}$.

The chirality $\mathcal{H}_{\mathcal{P}}$ is the $(\mathcal{P}, R)$-decorated $|T|$-colored Φ-operad for the pure perfection color sieve R (2.14).

If $(A_{|T|_\gamma})$ is a $|T|_\gamma$-tuple of $\mathcal{P}$-algebras, then an $(A_{|T|_\gamma})$-module with chirality $\mathcal{H}_{\mathcal{P}}$ will be called a $(A_{|T|_\gamma})$-*omnimodule*. Here the same operad that controls all the algebras A_η also controls the actions of these algebras on a module with this chirality; furthermore, all these action commute, because the chirality is pure.

Hence when $\Phi = \mathbf{F}$, there is only a single $\mathcal{P}$-algebra A, and an A-omnimodule is precisely what is called by Getzler-Kapranov and Kriz-May an A-module.

When $\Phi = \mathbf{O}$, there are two $\mathcal{P}$-algebras A and B. If $\mathcal{P} = \mathbf{1}$, then A and B are of course monoids, and an (A, B)-omnimodule is precisely the same as an (A, B)-bimodule in the conventional sense.

4. Wreath products of operator categories

The definition. — In this section I define a weak monoidal structure on **Op**, which I have called a *wreath product*. Loosely speaking, a $(\Phi \wr \Psi)$-monoid is a Ψ-monoid in Φ-monoids. One way to look at this monoidal structure (a perspective that grew out of a conversation with J. Rognes) is as a recontextualization of the Boardman-Vogt tensor product, whose homotopical properties as a tensor product of symmetric operads is — shall we say — subtle.

In any case, the wreath product will provide us with numerous new and interesting examples of operator categories.

Definition 4.1. — Suppose Φ and Ψ two operator categories. Then we have a pseudofunctor:

$$\begin{aligned}\Sigma^{\Psi}_{\Phi} : \ \Psi^{\mathrm{op}} &\longrightarrow \mathbf{Cat}\\ I &\longmapsto \Phi^{\times |I|}.\end{aligned}$$

Now define the *wreath product operator category* as

$$\Phi \wr \Psi := \operatorname{Tot} \Sigma^{\Psi}_{\Phi}.$$

Hence the objects of $\Phi \wr \Psi$ are pairs $((K_I), I)$ consisting of an object $I \in \Psi$ and an I-tuple of objects $(K_I) \in \Phi^{\times |I|}$.[8] A morphism $((L_J), J) \longrightarrow ((K_I), I)$ is a morphism $\psi : J \longrightarrow I$ of Ψ, and a J-tuple of morphisms $(L_j \longrightarrow K_{\psi(j)})_{j \in |J|}$ of Φ.

4.2. — Suppose $J \in \mathbf{O}$, and suppose (Φ_J) an J-tuple of operator categories. Then one can define the *iterated wreath product* $\mathrm{Wr}_J \, \Phi_J$ as the category whose objects are J-tuples $(K^J) = (K^j)_{j \in |J|}$, wherein each K^j is itself a $\sqcup |K^{j+1}|$-tuple of objects of Φ_j.

Lemma 4.3. — *The iterated wreath product makes* **Op** *into a weak* **O***-monoidal* $(2, 1)$*-category, wherein* **0** *is the unit.*

Sketch of proof. — This is just a direct verification, but it must be borne in mind that this is a *weak* **O**-monoidal category, so that the associativity isomorphisms are suitably coherent equivalences of operator categories.[9] □

Corollary 4.4. — *The wreath product comes equipped with operator morphisms*

$$\begin{aligned}w : \Phi \cong \ \Phi \wr \mathbf{0} &\longrightarrow \Phi \wr \Psi \qquad \text{and} \qquad & v : \Psi \cong \ \mathbf{0} \wr \Psi &\longrightarrow \Phi \wr \Psi\\ K &\longmapsto ((K), \star) & I &\longmapsto ((\star_I), I).\end{aligned}$$

4.5. — Observe that the wreath product is highly noncommutative; in particular the wreath product does not give **Op** a weakly braided monoidal structure.

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(8) Though it seems, I freely admit, a tad perverse, it's actually handy to write the pairs in this "backwards" order.

(9) A precise definition of this notion will appear in the subsection of weakly monoidal weakly enriched categories below.

Lemma 4.6. — *In* $\Phi \wr \Psi$,

$$|((K_I), I)| \cong \coprod_{i \in |I|} |K_i|.$$

More, generally, suppose $J \in \mathbf{O}$, *suppose* $0 \in |J|$ *the initial object of* J, *and suppose* J^+ *is the full subcategory of* J *comprised of every object apart from* 0. *Then in* $\mathrm{Wr}_J \Phi_J$,

$$|(K^J)| \cong \coprod_{i \in |(K^{J^+})|} |K_i^0|,$$

where (K^{J^+}) *is the obvious* J^+*-tuple consisting of all but the first tuple* (K^0).

Proposition 4.7. — *If* Φ *and* Ψ *are perfect, then so is* $\Phi \wr \Psi$.

Sketch of proof. — The point classifier of $\Phi \wr \Psi$ is the pair $((T^{\Psi}_{\Phi,T_\Psi}), T_\Psi)$, where (T_Ψ, t) is point classifier of Ψ, and (T^{Ψ}_{Φ,T_Ψ}) is the T_Ψ-tuple of objects of Φ wherein $T^{\Psi}_{\Phi,t} = T_\Phi$ is the point classifier for T, and, for any generic point $\eta \in |T|_\gamma$, $T^{\Psi}_{\Phi,\eta} = \star$. ☐

Corollary 4.8. — *If* $J \in \mathbf{O}$, *and* (Φ_J) *is an* J*-tuple of perfect operator categories, then the wreath product* $\mathrm{Wr}_J \Phi_J$ *is perfect of complexity*

$$C(\mathrm{Wr}_J \Phi_J) = \sum_{j \in |J|} C(\Phi_j).$$

Example 4.9. — This provides us with our very first example of a perfect operator category that is not fully faithful: $\mathbf{O} \wr \mathbf{O}$, which has complexity 4.

4.10. — The following little confusion seems to be easy to make. Suppose Φ is faithfully flat, and suppose $f : \Psi \longrightarrow \Phi$ an operator morphism. Since Φ is faithfully flat, it carries a Φ-monoidal, and hence a Ψ-monoidal, structure, and it is tempting (or at least it was briefly for me) to try to define a left adjoint $p_f : \Phi \wr \Psi \longrightarrow \Phi$ to w by the formula

$$p_f((K_I), I) := \otimes_I K_I.$$

It is critical to note, however, that this formula does *not* in general define an operator morphism (or even a functor). Indeed, consider the following morphism $((L_J), J) \longrightarrow ((K_I), I)$ of $\mathbf{O} \wr \mathbf{O}$: the object $((K_I), I)$ consists of $I = \mathbf{0}$ and the $\mathbf{0}$-tuple $K_0 = \mathbf{2}$; the object $((L_J), J)$ consists of $J = \mathbf{1}$ and the $\mathbf{1}$-tuple $(L_0, L_1) = (\mathbf{2}, \mathbf{2})$; and the map $((L_J), J) \longrightarrow ((K_I), I)$ is given by the unique map $J \longrightarrow I$ and the isomorphisms $L_0 \longrightarrow K_0$ and $L_1 \longrightarrow K_0$. Note that there is a unique morphism $L_0 \otimes L_1 \simeq \mathbf{5} \longrightarrow \mathbf{1}$ whose fibers are all isomorphic to $\mathbf{1}$, but it is not induced by the isomorphisms $L_0 \longrightarrow K_0$ and $L_1 \longrightarrow K_0$.

On the other hand, if $\Phi = \mathbf{F}$, then the above definition of p_f works just fine for $f = U : \Psi \longrightarrow \mathbf{F}$. This leads one to the following proposition.

Proposition 4.11. — *For any operator category* Ψ, *the wreath product* $\mathbf{F} \wr \Psi$ *is canonically equivalent to the category of* Ψ*-partitions of finite sets — i.e., triples consisting of a finite set* K, *an object* $I \in \Phi$, *and a map* $K \longrightarrow |I|$. *The operator category* $\mathbf{F}$ *is a localization of* $\mathbf{F} \wr \Psi$.

4.12. — This does not quite say that $\mathbf{F}$ is a zero object for the wreath product; $\mathbf{F}$ is not necessarily equivalent to $\mathbf{F} \wr \Psi$. However, it does follow that $\mathbf{F}$ and $\mathbf{F} \wr \Psi$ are *rubric-equivalent.*

Corollary 4.13. — *Suppose* $\mathcal{E}$ *a symmetric monoidal category. Then the functor*

$$\mu^{\mathbf{F}}(\mathcal{E})\mathbf{Cat} \longrightarrow \mu^{(\mathbf{F}\wr\Psi)}(\mathcal{E})\mathbf{Cat}$$

is an equivalence of $\mathcal{E}$*-categories.*

4.14. — The aim of the wreath product of operator categories was to formalize the notion of a Ψ-monoid in Φ-monoids. Let us now turn to a series of results intended to make this intuition precise.

Suppose now $\mathcal{E}$ a symmetric monoidal category. Observe that a $(\Phi \wr \Psi)$-monoid A in $\mathcal{E}$ consists of an object A, equipped with a product

$$A^{\otimes |((K_I), I)|} \longrightarrow A,$$

for any object $((K_I), I)$ of $\Phi \wr \Psi$, satisfying various conditions.

Proposition 4.15. — *Any $(\Phi \wr \Psi)$-monoid A in $\mathcal{E}$ is in particular both a Φ-monoid and a Ψ-monoid, and, for any object $((K_I), I)$ of $\Phi \wr \Psi$, the following diagram commutes:*

$$\begin{array}{ccc} \bigotimes_{i\in|I|} A^{\otimes|K_i|} & \xrightarrow{\sim} & A^{\otimes|((K_I),I)|} \\ \downarrow & & \downarrow \\ A^{\otimes|I|} & \longrightarrow & A. \end{array}$$

4.16. — Suppose Φ and Ψ perfect operator categories; consider the pseudofunctor

$$\begin{aligned} \mathcal{L}\Sigma_\Phi^\Psi : \qquad \mathcal{L}_\Psi^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \\ I &\longmapsto \mathcal{L}_\Phi^{\times|I|} \\ [\psi : J \to I] &\longmapsto [\ell_\psi : \mathcal{L}_\Phi^{\times|I|} \to \mathcal{L}_\Phi^{\times|J|}], \end{aligned}$$

where the functor $\ell_\psi : \mathcal{L}_\Phi^{\times|I|} \longrightarrow \mathcal{L}_\Phi^{\times|J|}$ is defined in the following manner. Recall that the morphism $\psi : J \longrightarrow I$ of $\mathcal{L}_\Psi$ is a morphism $\psi : J \longrightarrow TI$ of Ψ; now for any I-tuple (K_I) of objects of $\mathcal{L}_\Phi$, write

$$\ell_\psi(K_I) := \begin{cases} K_{\psi(j)} & \text{if } j \in |J| \times_{|TI|} |I| \\ \star & \text{else.} \end{cases}$$

Proposition 4.17. — *If Φ and Ψ are perfect operator categories, the Leinster category $\mathcal{L}_{(\Phi\wr\Psi)}$ is the total category of $\mathcal{L}\Sigma_\Phi^\Psi$.*

Corollary 4.18. — *It thus follows that $\mathcal{L}_{(\Phi\wr\Psi)}$ is endowed with a fibration $\mathcal{L}_{(\Phi\wr\Psi)} \longrightarrow \mathcal{L}_\Psi$, and the fiber over $I \in \mathcal{L}_\Psi$ is equivalent to $\mathcal{L}_\Phi^{\times|I|}$.*

Pairings of operads and the tensor product of Boardman-Vogt. — The wreath product of operator categories is closely related to the notion of a *pairing* of operads, as studied by J. P. May. I turn now to a generalization of May's notion, and its relationship to a generalization of the tensor product of J. M. Boardman and R. M. Vogt. To this end, suppose $\mathcal{E}$ a complete and cocomplete symmetric monoidal category, in which the tensor product $\otimes$ commutes with colimits in each variable.

4.19. — If Φ and Ψ are operator categories, then for any $K \in \Phi$ and $I \in \Psi$, write $K \wr I$ for the pair $((K_I), I) \in \Phi \wr \Psi$ in which $K_i = K$ for every $i \in |I|$.

Definition 4.20. — Suppose Φ and Ψ operator categories, $\mathcal{A}$ a Φ-multi-$\mathcal{E}$-category, $\mathcal{B}$ a Ψ-multi-$\mathcal{E}$-category, and $\mathcal{C}$ a $(\Phi \wr \Psi)$-multi-$\mathcal{E}$-category. Then a (Φ, Ψ)-*pairing*

$$\pi : (\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{C}$$

consists of

(4.20.A) a map

$$\pi : \operatorname{Obj} \mathcal{A} \times \operatorname{Obj} \mathcal{B} \longrightarrow \operatorname{Obj} \mathcal{C}$$

and

(4.20.B) a morphism

$$K\, \underline{\mathrm{Mor}}_{\mathcal{A}}((W_K), Y) \otimes I\, \underline{\mathrm{Mor}}_{\mathcal{B}}((X_I), Z) \longrightarrow (K \wr I)\, \underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(W_K, X_I), \pi(Y, Z))$$

for every $K \in \Phi$, every $I \in \Psi$, every K-tuple $(W_K) \in (\operatorname{Obj} \mathcal{A})^{\times|K|}$, every I-tuple $(X_I) \in (\operatorname{Obj} \mathcal{B})^{\times|I|}$, every $Y \in \operatorname{Obj} \mathcal{A}$, and $Z \in \operatorname{Obj} \mathcal{B}$.

These data are subject to the following constraints.

(4.20.1) *Associativity*: For every $[L \longrightarrow K] \in \Phi$ and every $[J \longrightarrow I] \in \Psi$, the diagrams

$$K\,\underline{\mathrm{Mor}}_{\mathcal{A}}((W_K),Y)\otimes(L/K)\,\underline{\mathrm{Mor}}_{\mathcal{A}}((U_L),(W_K))\otimes I\,\underline{\mathrm{Mor}}_{\mathcal{B}}((X_I),Z)\otimes(J/I)\,\underline{\mathrm{Mor}}_{\mathcal{B}}((V_J),(X_I))$$

$$\xrightarrow{\ \sim\ } K\,\underline{\mathrm{Mor}}_{\mathcal{A}}((W_K),Y)\otimes I\,\underline{\mathrm{Mor}}_{\mathcal{B}}((X_I),Z)\otimes \bigotimes_{\substack{k\in|K|\\ i\in|I|}}(L_k\,\underline{\mathrm{Mor}}_{\mathcal{A}}((U_{L_k}),W_k)\otimes J_i\,\underline{\mathrm{Mor}}_{\mathcal{B}}((V_{J_i}),X_i))$$

$$\longrightarrow (K\wr I)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(W_K,X_I),\pi(Y,Z))\otimes \bigotimes_{\substack{k\in|K|\\ i\in|I|}}(L_k\wr J_i)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(U_{L_k},V_{J_i}),\pi(W_k,X_i))$$

$$\xrightarrow{\ \sim\ } (K\wr I)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(W_K,X_I),\pi(Y,Z))\otimes(L\wr J/K\wr I)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(U_L,V_J),\pi(W_K,X_I))$$

$$\longrightarrow (L\wr J)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(U_L,V_J),\pi(Y,Z)),$$

and

$$K\,\underline{\mathrm{Mor}}_{\mathcal{A}}((W_K),Y)\otimes(L/K)\,\underline{\mathrm{Mor}}_{\mathcal{A}}((U_L),(W_K))\otimes I\,\underline{\mathrm{Mor}}_{\mathcal{B}}((X_I),Z)\otimes(J/I)\,\underline{\mathrm{Mor}}_{\mathcal{B}}((V_J),(X_I)) \longrightarrow L\,\underline{\mathrm{Mor}}_{\mathcal{A}}((U_L),Y)\otimes J\,\underline{\mathrm{Mor}}_{\mathcal{B}}((V_J),Z) \longrightarrow (L\wr J)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(U_L,V_J),\pi(Y,Z)).$$

commute.

(4.20.2) *Identity*: For any objects $W \in \operatorname{Obj}\mathcal{A}$ and $Z \in \operatorname{Obj}\mathcal{B}$, the diagram

$$\begin{array}{ccc} \mathbf{1}_{\mathcal{E}}\otimes\mathbf{1}_{\mathcal{E}} & \xrightarrow{\ \sim\ } & \mathbf{1}_{\mathcal{E}} \\ {\scriptstyle \mathrm{id}_W\otimes\mathrm{id}_Z}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\pi(W,Z)}} \\ \underline{\mathrm{Mor}}_{\mathcal{A}}(W,W)\otimes\underline{\mathrm{Mor}}_{\mathcal{B}}(Z,Z) & \longrightarrow & (K\wr I)\,\underline{\mathrm{Mor}}_{\mathcal{C}}(\pi(W,Z),\pi(W,Z)) \end{array}$$

commutes.

4.21. — That associativity diagram looks pretty intimidating, but I find it much easier to follow than that description using "elements" one often finds in the literature (especially before 1990). Writing out the diagram when $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are operads of their respective rubrics should do a nice job of putting your mind at ease.

Observe that the role of the operator categories here is very significant. In particular, since the wreath product is noncommutative, a (Φ,Ψ)-pairing is very different from a (Ψ,Φ)-pairing, unless $\Phi = \Psi$, in which case the notions coincide. If we eliminated the role of the operator categories, we would have a many-object version of May's original notion.

The notion of pairing is obviously related to the Boardman-Vogt interchange condition for two morphisms of operads, but the interchange condition seems to make sense only when there are diagonals in $\mathcal{E}$.

Example 4.22. — When $\Phi = \Psi = \mathbf{0}$, the notion of a pairing reduces to that of an $\mathcal{E}$-bifunctor.

4.23. — Given operator categories Φ, Ψ, we have a functor

$$\begin{array}{rcl} \mathbf{Pair}^{(\Phi,\Psi)}:\ \mu^{\Phi}(\mathcal{E})\mathbf{Cat}^{\mathrm{op}}\otimes\mu^{\Psi}(\mathcal{E})\mathbf{Cat}^{\mathrm{op}}\otimes\mu^{(\Phi\wr\Psi)}(\mathcal{E})\mathbf{Cat} & \longrightarrow & \mathbf{Set} \\ (\mathcal{A},\mathcal{B},\mathcal{C}) & \longrightarrow & \mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},\mathcal{B}),\mathcal{C}) \end{array}$$

where $\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},\mathcal{B}),\mathcal{C})$ denotes the set of (Φ,Ψ)-pairings $(\mathcal{A},\mathcal{B})\longrightarrow\mathcal{C}$.

This can in fact be lifted to a functor

$$\mathbf{Pair}^{(\Phi,\Psi)}:\mu^{\Phi}(\mathcal{E})\mathbf{Cat}^{\mathrm{op}}\otimes\mu^{\Psi}(\mathcal{E})\mathbf{Cat}^{\mathrm{op}}\otimes\mu^{(\Phi\wr\Psi)}(\mathcal{E})\mathbf{Cat}\longrightarrow(\mathcal{E})\mathbf{Cat},$$

but it is not necessary to do so directly, in light of the following result.

Proposition 4.24. — *Suppose Φ and Ψ operator categories, $\mathcal{A}$ a Φ-multi-$\mathcal{E}$-category, $\mathcal{B}$ a Ψ-multi-$\mathcal{E}$-category, and $\mathcal{C}$ a $(\Phi\wr\Psi)$-multi-$\mathcal{E}$-category. Then the covariant functor $\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},\mathcal{B}),-)$ is corepresentable, and the contravariant functors $\mathbf{Pair}^{(\Phi,\Psi)}((-,\mathcal{B}),\mathcal{C})$ and $\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},-),\mathcal{C})$ are representable.*

Definition 4.25. — Suppose Φ, Ψ, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as above.

(4.25.1) The *generalized Boardman-Vogt tensor product* ${\mathcal{A}}_{\Phi}\!\otimes_{\Psi}\!\mathcal{B}$ is the corepresenting object of the functor $\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},\mathcal{B}),-)$.
(4.25.2) The *generalized Boardman-Vogt enrichment* $\underline{\mathrm{Mor}}^{(\Phi,\Psi)}(\mathcal{B},\mathcal{C})$ is the representing object of the functor $\mathbf{Pair}^{(\Phi,\Psi)}((-,\mathcal{B}),\mathcal{C})$.
(4.25.3) The *generalized Boardman-Vogt cotensor* $\mathrm{mor}^{(\Phi,\Psi)}(\mathcal{A},\mathcal{C})$ is the representing object of the functor $\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},-),\mathcal{C})$.

Proposition 4.26. — *The object-sets of these multicategories are given by*

$$\begin{aligned} \mathrm{Obj}({\mathcal{A}}_{\Phi}\!\otimes_{\Psi}\!\mathcal{B}) &\cong (\mathrm{Obj}\,\mathcal{A})\times(\mathrm{Obj}\,\mathcal{B}),\\ \mathrm{Obj}\,\underline{\mathrm{Mor}}^{(\Phi,\Psi)}(\mathcal{B},\mathcal{C}) &\cong \mathrm{Obj}\,\mathrm{Mor}_{\mu^{(\Phi\wr\Psi)}(\mathcal{E})\mathbf{Cat}}(\mathcal{B},\mathcal{C}),\\ \mathrm{Obj}\,\mathrm{mor}^{(\Phi,\Psi)}(\mathcal{A},\mathcal{C}) &\cong \mathrm{Obj}\,\mathrm{Mor}_{\mu^{(\Phi\wr\Psi)}(\mathcal{E})\mathbf{Cat}}(\mathcal{A},\mathcal{C}). \end{aligned}$$

Proposition 4.27. — *Suppose now $f:\Phi\longrightarrow\Phi'$ and $g:\Psi\longrightarrow\Psi'$ two operator morphisms, and suppose $\mathcal{A}$ a Φ-multicategory in $\mathcal{E}$, $\mathcal{B}$ a Ψ-multicategory in $\mathcal{E}$, and $\mathcal{C}$ a $(\Phi'\wr\Psi')$-multicategory in $\mathcal{E}$. Then there is a canonical bijection*

$$\mathbf{Pair}^{(\Phi,\Psi)}((\mathcal{A},\mathcal{B}),(f\wr g)^{\star}\mathcal{C})\cong\mathbf{Pair}^{(\Phi',\Psi')}((f_!\mathcal{A},g_!\mathcal{B}),\mathcal{C}).$$

Corollary 4.28. — *In the situation of the proposition above, we have the following three formulæ:*

$$(f\wr g)_!({\mathcal{A}}_{\Phi}\!\otimes_{\Psi}\!\mathcal{B}) \cong (f_!\mathcal{A})_{\Phi'}\!\otimes_{\Psi'}\!(g_!\mathcal{B}), \tag{4.28.1}$$

$$\underline{\mathrm{Mor}}^{(\Phi,\Psi)}(\mathcal{B},(f\wr g)^{\star}\mathcal{C}) \cong f^{\star}\,\underline{\mathrm{Mor}}^{(\Phi',\Psi')}(g_!\mathcal{B},\mathcal{C}), \tag{4.28.2}$$

$$\mathrm{mor}^{(\Phi,\Psi)}(\mathcal{A},(f\wr g)^{\star}\mathcal{C}) \cong g^{\star}\,\mathrm{mor}^{(\Phi',\Psi')}(f_!\mathcal{A},\mathcal{C}). \tag{4.28.3}$$

4.29. — This variant of the tensor product comes equipped with more information than the classical Boardman-Vogt tensor product. For example, if $\mathcal{P}$ and $\mathcal{Q}$ are each Φ-operads, the structure on ${\mathcal{P}}_{\Phi}\!\otimes_{\Phi}\!\mathcal{Q}$ is that of a $(\Phi\wr\Phi)$-operad. If $\otimes_{\mathrm{BV}}$ denotes the classical Boardman-Vogt tensor product on $\mathbf{F}$-operads, then one can verify that $U_!({\mathcal{P}}_{\Phi}\!\otimes_{\Phi}\!\mathcal{Q})$ is naturally isomorphic to $U_!\mathcal{P}\otimes_{\mathrm{BV}}U_!\mathcal{Q}$. So ${\mathcal{P}}_{\Phi}\!\otimes_{\Phi}\!\mathcal{Q}$ is a more finely-structured tensor product.

4.30. — Suppose Φ and Ψ operator categories. Then one defines a functor

$$\begin{aligned} \omega:\quad \mathcal{M}_{\Phi}\times\mathcal{M}_{\Psi} &\longrightarrow \mathcal{M}_{(\Phi\wr\Psi)}\\ ((\mathbf{r},K),(\mathbf{p},I)) &\longmapsto (\min(\mathbf{r},\mathbf{p}),K\wr I), \end{aligned}$$

wherein $(K\wr I)(j):=K(j)\wr I(j)$ for any $0\leq j\leq\min(r,p)$. Now the Day convolution product gives a product $E^{\mathcal{M}_{\Phi}}\times E^{\mathcal{M}_{\Phi}}\longrightarrow E^{\mathcal{M}_{(\Phi\wr\Psi)}}$, which, as the following result demonstrates, models the generalized Boardman-Vogt tensor product.

Proposition 4.31. — *Suppose E a cocomplete category with all finite limits in which the cartesian product commutes with colimits in each variable. Then the following diagram commutes for any flat operator categories Φ and Ψ.*

$$\begin{array}{ccccc} E^{\mathcal{M}_{\Phi}}\times E^{\mathcal{M}_{\Phi}} & \xrightarrow{\ \boxtimes\ } & E^{(\mathcal{M}_{\Phi}\times\mathcal{M}_{\Psi})} & \xrightarrow{\ \omega_!\ } & E^{\mathcal{M}_{(\Phi\wr\Psi)}}\\ \downarrow & & & & \downarrow\\ \mathbf{Operad}^{\Phi}(E)\times\mathbf{Operad}^{\Psi}(E) & & \xrightarrow[\ {}_{\Phi}\otimes_{\Phi}\ ]{} & & \mathbf{Operad}^{\Phi\wr\Psi}(E), \end{array}$$

where $\boxtimes$ denotes the external product:

$$(A\boxtimes B)((\mathbf{r},K),(\mathbf{p},I)):=A(\mathbf{r},K)\times B(\mathbf{p},I).$$

5. Homotopical structure

Remark 5.1. — Now at last we're ready to introduce homotopy theory and higher categorical instruments into the picture. So Φ, Ψ, ... will here denote flat operator categories in this entire section. Sometimes I will need them to be perfect as well.

Homotopy theory of strict algebraic structures. — The following theorems follow from arguments very closely modeled on M. Spitzweck's original arguments; however, a key difference is that, thanks to J. Smith's combinatorial model categories, it is no longer necessary to construct sets both of generating cofibrations and of generating trivial cofibrations. Now it is enough to concoct a sufficiently nice generating set of cofibrations. This key technical insight gives a pretty foolproof version of Kan's lemma for lifting model structures, and, as E. Getzler reminded me, it means that it's no longer necessary to talk about left semimodel categories.(10)

Definition 5.2. — A model category is said to be *tractable* if it is combinatorial and a set I of generating cofibrations can be chosen so that the domains of I are all cofibrant. Denote by **ModCat** the $(2,1)$-category of tractable model categories and left Quillen functors.

Definition 5.3. — (5.3.1) A *tractable, Φ-monoidal model category* is a tractable model category $\mathbf{M}$, equipped with a Φ-monoidal structure such that the following axioms are satisfied.

(5.3.1.1) For any $I \in \Phi$, the functor

$$\otimes_I : \mathbf{M}^{\times |I|} \longrightarrow \mathbf{M}$$

preserves all colimits.

(5.3.1.2) For any $I \in \Phi$, consider the subposet T_I of $2^{|I|}$ (ordered by inclusion) comprised of all proper subsets of $|I|$. For any I-tuple $f_I : L_I^0 \longrightarrow L_I^1$ of cofibrations of $\mathbf{M}$, write $\otimes L^f$ for the colimit of the diagram $T_I \longrightarrow \mathbf{M}$ that assigns to any subset $F \subset |I|$ the object $\otimes(L_I^F)$, where for any $i \in |I|$,

$$L_i^F := \begin{cases} L_I^0 & \text{if } i \in F; \\ L_I^1 & \text{if } i \notin F. \end{cases}$$

The condition then is: the morphism $\otimes L^f \longrightarrow \otimes L_I^1$ is a cofibration that is trivial if any of the f_i are.

(5.3.2) Suppose $\mathbf{M}$ a tractable, Φ-monoidal model category. Denote by R the class of morphisms

$$\otimes_I f_I : X_I \longrightarrow Y_I,$$

in which there exists an $i \in |I|$ such that f_i is a trivial cofibration, and for any $j \in |I| \setminus i$, the morphism f_j is an isomorphism. Then $\mathbf{M}$ is said to *satisfy the monoid axiom* if any tranfinite composition of pushouts of elements of R is a weak equivalence.

(5.3.3) If $\mathbf{V}$ is a tractable Φ-monoidal model category, and $f : \Psi \longrightarrow \Phi$ is an operator morphism, then a *tractable, Ψ-monoidal model $\mathbf{V}$-category* is a tractable, Ψ-monoidal model category $\mathbf{M}$ and a Φ-monoidal left Quillen functor $\mathbf{V}^f \longrightarrow \mathbf{M}$. The operator morphism f will always be clear from the context.

5.4. — Notice that the generalized pushout-product axiom above forces unit objects to be cofibrant. In some situations it is obviously not desirable to require this; I leave it to the reader to adjust the definition and the results that follow in order to accommodate such situations.

Proposition 5.5. — *This defines $(2,1)$-functors*

$$\mathbf{MonModCat} : \mathbf{Op}^{\mathrm{op}} \longrightarrow (2,1)\mathbf{Cat}$$
$$\Phi \longmapsto \mathbf{Mon}^{\Phi}\mathbf{ModCat},$$

and, for any operator category Φ,

$$\mathbf{MonMod}(-)\mathbf{Cat} : (\mathbf{Op}/\Phi)^{\mathrm{op}} \times \mathbf{Mon}^{\Phi}\mathbf{ModCat}^{\mathrm{op}} \longrightarrow (2,1)\mathbf{Cat}$$
$$(\Psi, \mathbf{V}) \longmapsto \mathbf{Mon}^{\Psi}\mathbf{Mod}(\mathbf{V})\mathbf{Cat},$$

where $\mathbf{Mon}^{\Phi}\mathbf{ModCat}$ is the $(2,1)$-category of tractable Φ-monoidal model categories satisfying the monoid axiom and pseudo-Φ-monoidal left Quillen functors, and $\mathbf{Mon}^{\Psi}\mathbf{Mod}(\mathbf{V})\mathbf{Cat}$ is the $(2,1)$-category of

(10) In addition, I make no use of path-object arguments or the Hopf intervals of Berger-Moerdijk here.

tractable Ψ-monoidal model $\mathbf{V}$*-categories satisfying the monoid axiom and pseudo-*Ψ*-monoidal left Quillen* $\mathbf{V}$*-functors.*

Theorem 5.6. — *Suppose* $\mathbf{V}$ *a tractable,* $\mathbf{F}$*-monoidal model category satisfying the monoid axiom. Then the category* $\mathbf{Operad}^{\Phi}(\mathbf{V})$ *is a tractable model category with a projective model structure.*

Proposition 5.7. — *This defines a* $(2,1)$*-functor*

$$\mathbf{Operad}: \mathbf{Op} \times \mathbf{Mon^F ModCat} \longrightarrow \mathbf{ModCat}$$
$$(\Phi, \mathbf{V}) \longmapsto \mathbf{Operad}^{\Phi}(\mathbf{V})$$

with the property that for any pseudo-$\mathbf{F}$*-monoidal left Quillen equivalence* $\mathbf{V} \longrightarrow \mathbf{W}$ *of tractable* $\mathbf{F}$*-monoidal model categories, the induced left Quillen functor*

$$\mathbf{Operad}^{\Phi}(\mathbf{V}) \longrightarrow \mathbf{Operad}^{\Phi}(\mathbf{W})$$

is a Quillen equivalence.

5.8. — This result can be generalized to produce a functorial projective model category of colored operads in the following manner.

Theorem 5.9. — *For any set* S*, the category* $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{V})$ *of* Φ*-multicategories enriched in* $\mathbf{V}$ *with object set* S *with morphisms* preserving the colors — *i.e., inducing the identity on* S — *is a tractable model category with a projective model structure.*

Proposition 5.10. — *This defines a* $(2,1)$*-functor*

$$\mathbf{Col}(S)\mathbf{Operad}: \mathbf{Op} \times \mathbf{Mon^F ModCat} \longrightarrow \mathbf{ModCat}$$
$$(\Phi, \mathbf{V}) \longmapsto \mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{V})$$

with the property that for any pseudo-$\mathbf{F}$*-monoidal left Quillen equivalence* $\mathbf{V} \longrightarrow \mathbf{W}$ *of tractable* $\mathbf{F}$*-monoidal model categories, the induced left Quillen functor*

$$\mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{V}) \longrightarrow \mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{W})$$

is a Quillen equivalence.

5.11. — Note, however, that these results do *not* provide a model category of Φ-multi-$\mathbf{V}$-categories. The weak equivalences of such a model category structure on $\mu^{\Phi}(\mathbf{V})\mathbf{Cat}$ are essentially surjective multifunctors that induce weak equivalences on all polymorphism objects, and there is Quillen pair relating $(S/\mu^{\Phi}(\mathbf{V})\mathbf{Cat})$ and $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{V})$.

Thanks to terribly useful conversations with J. Lurie and J. Bergner, I have at last worked out how to produce such a model structure; I'll write about this elsewhere.

Definition 5.12. — One can assemble the total categories of the $(2,1)$-functors above into a single $(2,1)$-category of *contexts*:

$$\mathbf{Context} := \mathrm{Tot}_c\, \mathbf{Operad} \times^h_{\mathbf{Flop} \times \mathbf{Mon^F ModCat}} \mathrm{Tot}^{\mathrm{op}}\, \mathbf{MonModCat}$$

whose objects are octuples $(\Phi, \mathbf{V}, \mathcal{P}, \Phi', \mathbf{V}', \mathbf{M}, f, F)$, where in Φ and Φ' are operator categories, $f : \Phi \longrightarrow \Phi'$ is an equivalence thereof, $\mathbf{V}$ and $\mathbf{V}'$ are tractable symmetric monoidal model categories satisfying the monoid axiom, $F : \mathbf{V} \longrightarrow \mathbf{V}'$ is an equivalence thereof,(11) $\mathcal{P}$ is a cofibrant Φ-operad in $\mathbf{V}$, and $\mathbf{M}$ is a tractable, Φ'-monoidal model $\mathbf{V}'$-category satisfying the monoid axiom.

The context $(\Phi, \mathbf{V}, \mathcal{P}, \Phi', \mathbf{V}', \mathbf{M}, f, F)$ is said to be *perfect*, *flat*, or *faithfully flat* if Φ (equivalently, Φ') is so.

(11) not a Quillen equivalence, an equivalence of categories respecting every bit of the model structure

Definition 5.13. — Likewise, $\mathbf{Col}(S)\mathbf{Context}$ denotes the $(2,1)$-category of *S-colored contexts*, which is just like the usual $(2,1)$-category of contexts, save only that $\mathcal{P}$ is a Φ-multi-$\mathbf{V}$-category with object set S, cofibrant in $\mathbf{Col}(S)\mathbf{Operad}^{\Phi}(\mathbf{V})$:

$$\mathbf{Col}(S)\mathbf{Context} := \mathrm{Tot}_c\,\mathbf{Col}(S)\mathbf{Operad} \times^h_{\mathbf{Flop}\times\mathbf{Mon^F ModCat}} \mathrm{Tot}^{\mathrm{op}}\,\mathbf{MonModCat}.$$

5.14. — By an altogether inoffensive abuse, I'll treat (S-colored) contexts as though they were quadruples $(\Phi,\mathbf{V},\mathcal{P},\mathbf{M})$, but in order to get all the functorialities right, it's better to have the redundant information.

Theorem 5.15. — *Suppose* $(\Phi,\mathbf{V},\mathcal{P},\mathbf{M})$ *an S-colored context. Then the category*

$$\mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{P}}(\mathbf{M}) := \underline{\mathrm{Mor}}_{\mu^{\Phi}(\mathbf{V})\mathbf{Cat}}(\mathcal{P},\mathbf{M})$$

is a tractable model category with the projective model structure.

Proposition 5.16. — *This defines a* $(2,1)$*-functor*

$$\begin{aligned}\mathbf{Alg}:\ \mathbf{Col}(S)\mathbf{Context} &\longrightarrow \mathbf{ModCat}\\ (\Phi,\mathbf{V},\mathcal{P},\mathbf{M}) &\longmapsto \mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{P}}(\mathbf{M})\end{aligned}$$

with the following properties.

(5.16.1) For any S-colored context $(\Phi,\mathbf{V},\mathcal{P},\mathbf{M})$ *and any pseudo-*Φ*-monoidal left Quillen* $\mathbf{V}$*-equivalence* $\mathbf{M}\longrightarrow\mathbf{N}$ *of tractable* Φ*-monoidal model* $\mathbf{V}$*-categories satisfying the monoid axiom, the induced left Quillen functor*

$$\mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{P}}(\mathbf{M})\longrightarrow\mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{P}}(\mathbf{N})$$

is a Quillen equivalence.

(5.16.2) For S-colored context $(\Phi,\mathbf{V},\mathcal{P},\mathbf{M})$ *and any weak equivalence* $\mathcal{P}\longrightarrow\mathcal{Q}$ *of cofibrant S-colored operads, the induced left Quillen functor*

$$\mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{P}}(\mathbf{M})\longrightarrow\mathbf{Alg}^{\Phi}_{\mathbf{V},\mathcal{Q}}(\mathbf{M})$$

is a Quillen equivalence.

Definition 5.17. — A *chirality context* $(\Phi,\mathbf{V},\mathcal{H},\mathbf{M})$ is a colored context in which $\mathcal{H}$ is a Φ-chirality in $\mathbf{V}$.

Theorem 5.18. — *Suppose* $(\Phi,\mathbf{V},\mathcal{H},\mathbf{M})$ *a chirality context, and suppose, for any generic point* $\eta\in|T|_{\gamma}$*,* A_{η} *a* $\mathcal{H}\langle\eta\rangle$*-algebra. Then the category* $\mathbf{Mod}^{\Phi}_{\mathbf{V}}((A_{|T|_{\gamma}});\mathcal{H})$ *is a tractable model category with a projective model structure.*

5.19. — The assignment $(\Phi,\mathbf{V},\mathcal{H},\mathbf{M},(A_{|T|_{\gamma}}))\longmapsto\mathbf{Mod}^{\Phi}_{\mathbf{V}}((A_{|T|_{\gamma}});\mathcal{H})$ is $(2,1)$-functorial as well, of course, but I leave this to the reader to formulate. You get the idea.

Weakly enriched categories. — In this section I will sketch a theory of weakly enriched categories, which is a generalization of work of C. Rezk. This particular theory has very special properties, which seemingly do not appear in other (often equivalent) theories with similar aims.

One major challenge for this subsection is to develop a model category of categories weakly enriched over an arbitrary (nice) symmetric monoidal model category $(\mathbf{V},\otimes)$. I expect to resolve this issue soon. Fortunately, $\mathcal{L}_{\Phi}$ and $\mathcal{M}_{\Phi}$ make life easy when $\mathbf{V}$ is *internal*, and this is the case we will be most interested in.

Definition 5.20. — An *enrichment model category* $\mathbf{V}$ is an internal, simplicial, left proper, tractable model category in which the terminal object $\star$ is cofibrant.

Denote by **Enr** the $(2,1)$-category of *enrichment model categories* and *product-preserving, left Quillen functors*.

5.21. — Among the Segal-style conditions for the theory of monoids produced by the Leinster category, there was a unitality condition. This condition forced any functor $F : \mathcal{L}_\Phi \longrightarrow E$ corresponding to a monoid in E to take the value $\star \in E$ on any object of $\mathcal{L}_\Phi$ corresponding to an object of Φ with no points.

In order to get a multi-object version of the theory for $\Phi = \mathbf{O}$, one needs to modify this condition. One option would be to force $F(0)$ to be discrete; this leads to the theory of Segal categories developed by Dwyer, Kan, Smith, Dunn, Tamsamani, Simpson, and Hirschowitz. Unfortunately, this leads to some technical complications in understanding the homotopy theory of these gadgets. An alternative, whose idea is due I think to C. Rezk, is to place a condition on $F(0)$ that will effectively force it to be a kind of "interior" for F. This is the *completeness condition*, and it requires a little paradigm shift about what enrichment in a model category really means.

Theorem 5.22. — *There is an endo-$(2,1)$-functor, which I like to call* Rezk categorification

$$\mathbf{Wk}(-)\mathbf{Cat} : \mathbf{Enr} \longrightarrow \mathbf{Enr}$$

equipped with a morphism of endo-$(2,1)$-functors

$$\mathrm{id} \longrightarrow \mathbf{Wk}(-)\mathbf{Cat}$$

satisfying the following conditions for any enrichment model category $\mathbf{V}$.

(5.22.1) The underlying category *of $\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ is the category $s\mathbf{V}$ of* simplicial objects *of $\mathbf{V}$.*
(5.22.2) The left Quillen functor *$\mathbf{V} \longrightarrow \mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ is the* diagonal *functor.*
(5.22.3) The cofibrations *of $\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ are the* Reedy *cofibrations.*
(5.22.4) An object $A \in \mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ is fibrant — as I call it, a weak $\mathbf{V}$-category *— if and only if it satisfies the following conditions.*
  (5.22.4.1) $A \in s\mathbf{V}$ is Reedy fibrant.
  (5.22.4.2) The Segal morphism

$$A_p \longrightarrow A_1 \times^h_{A_0} \cdots \times^h_{A_0} A_1$$

  is an isomorphism of $\mathrm{Ho}\,\mathbf{V}$.
  (5.22.4.3) The Rezk morphism

$$A_0 \longrightarrow \operatorname{holim}_{p \in (\Delta/\overline{1})^{\mathrm{op}}} A_p$$

  is an isomorphism of $\mathrm{Ho}\,\mathbf{V}$, where $\overline{1}$ is the unique contractible groupoid with two objects, and $(\Delta/\overline{1})$ is the category of functors $\mathbf{p} \longrightarrow \overline{1}$, or equivalently the category of simplices of the nerve $\nu_\bullet(\overline{1})$.
(5.22.5) Weak equivalences between fibrant objects are objectwise.

Sketch of proof. — This is a model structure on $s\mathbf{V}$ constructed on the model provided by C. Rezk. In effect, one forms an *enriched* left Bousfield localization of the Reedy model structure with respect to morphisms representing the Segal morphisms and the Rezk morphism. □

Example 5.23. — Define

$$\begin{aligned}
\mathbf{Wk}(\infty,0)\mathbf{Cat} &:= s\mathbf{Set} \\
\mathbf{Wk}(n,0)\mathbf{Cat} &:= L_{\{S^k \to \star \ \mid\ k>n\}}\mathbf{Wk}(\infty,0)\mathbf{Cat} \\
\mathbf{Wk}(n,m)\mathbf{Cat} &:= \mathbf{Wk}(\mathbf{Wk}(n,m-1)\mathbf{Cat})\mathbf{Cat}.
\end{aligned}$$

These are *fantastic* models for *weak (n,m)-categories* — i.e., weak n-categories such that the i-morphisms for $i > m$ are weakly invertible.

More generally, write

$$\begin{aligned}
\mathbf{Wk}(\mathbf{V},0)\mathbf{Cat} &:= \mathbf{V} \\
\mathbf{Wk}(\mathbf{V},m)\mathbf{Cat} &:= \mathbf{Wk}(\mathbf{Wk}(\mathbf{V},m-1)\mathbf{Cat})\mathbf{Cat}.
\end{aligned}$$

We have a diagram of right Quillen functors:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbf{Wk}(\mathbf{V},n)\mathbf{Cat} & \longrightarrow \cdots \longrightarrow & \mathbf{Wk}(\mathbf{V},1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(\mathbf{V},0)\mathbf{Cat} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathbf{Wk}(\infty,n)\mathbf{Cat} & \longrightarrow \cdots \longrightarrow & \mathbf{Wk}(\infty,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(\infty,0)\mathbf{Cat} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \xrightarrow[\sim]{} & \mathbf{Wk}(n,n)\mathbf{Cat} & \longrightarrow \cdots \longrightarrow & \mathbf{Wk}(n,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(n,0)\mathbf{Cat} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \mathbf{Wk}(1,n)\mathbf{Cat} & \longrightarrow \cdots \xrightarrow[\sim]{} & \mathbf{Wk}(1,1)\mathbf{Cat} & \longrightarrow & \mathbf{Wk}(1,0)\mathbf{Cat} \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \mathbf{Wk}(0,n)\mathbf{Cat} & \longrightarrow \cdots \longrightarrow & \mathbf{Wk}(0,1)\mathbf{Cat} & \xrightarrow[\sim]{} & \mathbf{Wk}(0,0)\mathbf{Cat}
\end{array}
$$

The right Quillen functors should be viewed as giving an "interior," i.e., the maximum subobject with the prescribed structure. In the case of the upward pointing maps, this reduces to a mere forgetful functor.

I should emphasize that this appears to be unique in all of higher category theory, special to the Rezk categorification I have described here. As far as I know, no other theory of higher categories comes with a diagram of enrichment model categories like the one above.[12]

Example 5.24. — If $\mathcal{C}$ is a category with weak equivalences, we have the *Rezk nerve*:

$$
\begin{aligned}
N\mathcal{C} :\ \Delta^{\mathrm{op}} &\longrightarrow \mathbf{Wk}(\infty,0)\mathbf{Cat} \\
p &\longrightarrow \nu_{\bullet} w(\mathcal{C}^p)
\end{aligned}
$$

which is "close" to being fibrant in $\mathbf{Wk}(\infty,1)\mathbf{Cat}$. If $\mathcal{C}$ is a model category, for example, then an objectwise fibrant replacement of $N\mathcal{C}$ is fibrant.

If $\mathcal{C}$ is a model $\mathbf{V}$-category, then there is a weak $\mathbf{V}$-category $N_{\mathbf{V}}\mathcal{C}$ such that $N\mathcal{C}$ is the image under the right adjoint

$$\mathrm{Ho}(\mathbf{Wk}(\mathbf{V})\mathbf{Cat}) \longrightarrow \mathrm{Ho}(\mathbf{Wk}(\infty,1)\mathbf{Cat}).$$

Moreover this association $\mathcal{C} \longmapsto N_{\mathbf{V}}\mathcal{C}$ is functorial.

Weakly Φ-monoidal objects. — Given an enrichment category $\mathbf{V}$, one can use the Leinster category to give a definition of a model category of weak Φ-monoids in $\mathbf{V}$ when Φ is perfect. When $\Phi = \mathbf{O}$, this is Quillen equivalent to the category of weakly $\mathbf{V}$-enriched categories with a single object.

Again, it would be very gratifying to see similar results for other symmetric monoidal model categories.

Suppose Φ a perfect operator category.

Theorem 5.25. — *Suppose* $\mathbf{V} \in \mathbf{Enr}$*; then there exists a left-proper, simplicial, tractable model* $\mathbf{V}$*-category* $\mathbf{WkMon}^{\Phi}(\mathbf{V})$ *with the following properties.*

(5.25.1) The underlying category of $\mathbf{WkMon}^{\Phi}(\mathbf{V})$ *is the functor category* $\mathbf{V}^{\mathcal{L}_{\Phi}}$.

(5.25.2) The cofibrations of $\mathbf{WkMon}^{\Phi}(\mathbf{V})$ *are precisely the projective cofibrations.*

(5.25.3) An object $A \in \mathbf{WkMon}^{\Phi}(\mathbf{V})$ *is fibrant if and only if the following conditions are satisfied.*

(5.25.3.1) A is objectwise fibrant.

[12] In fact, using the strictification theorem below, it is even possible to "fill in" the far left line in this diagram, with theories of weak (∞,∞)-categories, AKA ω-categories.

(5.25.3.2) *Every element of*

$$\{A(TI) \longrightarrow \operatorname{holim}(A \circ D_I) \mid I \in \Phi\}$$

is an isomorphism of $\operatorname{Ho}\mathbf{V}$.

(5.25.4) *The weak equivalences between fibrant objects are objectwise.*

This defines a $(2,1)$*-functor*

$$\begin{aligned}\mathbf{WkMon} : \mathbf{Plop} \times \mathbf{Enr} &\longrightarrow \mathbf{ModCat} \\ (\Phi, \mathbf{V}) &\longrightarrow \mathbf{WkMon}^{\Phi}(\mathbf{V}).\end{aligned}$$

Example 5.26. — The categories $\mathbf{WkMon}^{\Phi}(\mathbf{Wk}(n,m)\mathbf{Cat})$ are well-behaved model categories of Φ-monoidal weak (n,m)-categories.

When $\Phi = \mathbf{F}$, $n = \infty$, $m = 0$, you know this as a model category of connective spectra. That is, symmetric monoidal ∞-groupoids are connective spectra.

Example 5.27. — If $\mathcal{C}$ is a Φ-monoidal model $\mathbf{V}$ category, then $N_{\mathbf{V}}\mathcal{C}$ is naturally a Φ-monoidal weak $\mathbf{V}$-category.

5.28. — If you've read about the so-called "periodic table" of n-categories, it may amuse you to see a little piece of it here. The rest of it will be extracted below, in the section of the Eckmann-Hilton tower.

Proposition 5.29. — *(5.29.1) There is a Quillen adjunction*

$$B : \mathbf{WkMon}^{\mathbf{O}}(\mathbf{V}) \rightleftarrows (\star/\mathbf{Wk}(\mathbf{V})\mathbf{Cat}) : \Omega$$

in which

$$\begin{aligned}(BA)_p &:= A_p/A_0, \\ (\Omega_a E)_p &:= \star \times_{(E_0^{\times(p+1)})} E_p,\end{aligned}$$

for any $A \in \mathbf{WkMon}^{\mathbf{O}}(\mathbf{V})$ *and any* $E \in (\star/\mathbf{Wk}(\mathbf{V})\mathbf{Cat})$.

(5.29.2) *Furthermore,* $\mathbf{WkMon}^{\mathbf{O}}(\mathbf{V})$ *is a colocalization of* $(\star/\mathbf{Wk}(\mathbf{V})\mathbf{Cat})$*, in the sense that the natural morphism* $\mathrm{id} \longrightarrow \mathbf{R}\Omega\mathbf{L}B$ *of endofunctors of* $\operatorname{Ho}\mathbf{WkMon}^{\mathbf{O}}(\mathbf{V})$ *is an isomorphism.*

(5.29.3) *The essential image of* $\mathbf{L}B$ *is comprised of those pointed weak* $\mathbf{V}$*-categories* (A, a) *such that any object of* A *is equivalent to* a.

5.30. — The left Quillen functor here is the reinterpretation of a weak $\mathbf{O}$-monoid in $\mathbf{V}$ as a weak $\mathbf{V}$-category with a distinguished object, and the right adjoint simply discards the connected components of the weak $\mathbf{V}$-category that do not contain the special point.

Weak multi-V-categories. — Having used the Leinster category $\mathcal{L}_\Phi$ to provide a good theory of weak Φ-monoids in an enrichment model category, and having used simplicial objects and the Rezk categorification to produce a good theory of weakly enriched $\mathbf{V}$-categories, one can ask whether one can weave these two sorts of structure together to define a Φ-multicategory enriched in an enrichment model category, using our $\mathcal{M}_\Phi$. Indeed we can! But there are two points that have to be addressed somewhat carefully first.

Suppose Φ a perfect operator category.

5.31. — First and foremost, since the model category of weakly enriched Φ-multi categories is going to be given as a left Bousfield localization of an objectwise model structure on a diagram category, it would be good to say a word or two about what that objectwise model structure is. In order to guarantee maximum functoriality, it is convenient to use a model structure that sits between the injective and projective structures. This will be the *blended model structure* on the diagram category $\mathbf{V}^{\mathcal{M}_\Phi}$.

In particular, recall that $\mathcal{M}_\Phi$ was constructed as a lluf subcategory of the total category of the simplicial category $\mathbf{p} \longmapsto \Phi^{(\mathbf{p}^{\mathrm{op}})}$. Broadly speaking, we ask that the structure be a Reedy structure in the simplicial direction, and a projective structure in the Φ direction. This is an important point, but it's also a technical point, so I will not go into much more detail here.

Proposition 5.32. — *For any tractable model category* $\mathbf{V}$*, there exists a model structure* $\mathbf{V}^{\mathcal{M}_\Phi}_{\mathbf{blended}}$ *with the property that the functor* $\mathbf{V}^{\Delta^{\mathrm{op}}}_{\mathbf{Reedy}} \longrightarrow \mathbf{V}^{\mathcal{M}_\Phi}_{\mathbf{blended}}$ *is left Quillen and the functor* $\mathbf{V}^{\mathcal{L}_\Phi}_{\mathbf{proj}} \longrightarrow \mathbf{V}^{\mathcal{M}_\Phi}_{\mathbf{blended}}$ *is right Quillen.*

5.33. — Recall that in our original theorem asserting the existence of $\mathcal{M}_\Phi$, we had to include a condition asserting that a given value of the functor $G : \mathcal{M}_\Phi \longrightarrow E$ corresponding to a Φ-E-operad was $\star \in E$. For the multi-object version, we need to replace that using a condition analogous to the completeness condition we discussed in the theory of weakly enriched categories. So we have to doctor some of the diagrams $D_X : Q_X \longrightarrow \mathcal{M}_\Phi$, to get diagrams $D'_X : Q_X \longrightarrow \mathcal{M}_\Phi$. Armed with these, we have the following theorem.

Theorem 5.34. — *Suppose* $\mathbf{V} \in \mathbf{Enr}$*; then there exists a left-proper, simplicial, tractable model* $(\mathbf{WkVCat})$*-category* $\mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ *with the following properties.*

(5.34.1) The underlying category of $\mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ *is the functor category* $\mathbf{V}^{\mathcal{M}_\Phi}$*.*
(5.34.2) The cofibrations of $\mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ *are precisely the blended cofibrations.*
(5.34.3) An object $B \in \mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ *is fibrant if and only if the following conditions are satisfied.*
  (5.34.3.1) B *is blended fibrant.*
  (5.34.3.2) Every element of

$$\{B(X) \longrightarrow \operatorname{holim}(B \circ D'_X) \mid X \in \mathcal{M}_\Phi\}$$

  is an isomorphism of $\operatorname{Ho}\mathbf{V}$*.*
(5.34.4) The weak equivalences between fibrant objects are objectwise.

This defines a $(2,1)$*-functor*

$$\begin{aligned} \mathbf{Mon} : \ \mathbf{Plop} \times \mathbf{Enr} &\longrightarrow \mathbf{Modcat} \\ (\Phi, \mathbf{V}) &\longrightarrow \mu^\Phi(\mathbf{V})\mathbf{Cat}. \end{aligned}$$

Example 5.35. — Of course for $\Phi = \star$, we simply recover the old theory of weakly enriched categories we've already investigated:

$$\mu^\star\mathbf{Wk}(\mathbf{V})\mathbf{Cat} = \mathbf{Wk}(\mathbf{V})\mathbf{Cat}.$$

More generally, $\mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$ is a good model category of weak Φ-multi-$\mathbf{V}$-categories, and by the functoriality, there is a right Quillen functor

$$(-)^\natural : \mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat} \longrightarrow \mathbf{Wk}(\mathbf{V})\mathbf{Cat}$$

Proposition 5.36. — *The functor* $\mathcal{M}_\Phi \longrightarrow \Delta^{\mathrm{op}} \times \mathcal{L}_\Phi$ *induces a* $\mathbf{Wk}(\mathbf{V})\mathbf{Cat}$*-enriched Quillen adjunction*

$$\mathbf{Free}^{\Phi,\otimes} : \mu^\Phi(\mathbf{V})\mathbf{Cat} \rightleftarrows \mathbf{Mon}^\Phi(\mathbf{Wk}(\mathbf{V}) : U^{\Phi,\mu}$$

which is functorial in Φ *and* $\mathbf{V}$*.*

Corollary 5.37. — *In particular, the right Quillen forgetful functor*

$$\mathbf{WkMon}^\Phi(\mathbf{Wk}(\mathbf{V})\mathbf{Cat}) \longrightarrow \mathbf{Wk}(\mathbf{V})\mathbf{Cat}$$

factors as a composable pair of right Quillen functors:

$$\mathbf{WkMon}^\Phi(\mathbf{Wk}(\mathbf{V})\mathbf{Cat}) \longrightarrow \mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat} \longrightarrow \mathbf{Wk}(\mathbf{V})\mathbf{Cat}\ .$$

Weak Φ-operads. — Using the same style of thinking as for Φ-monoids, one can develop a theory of weak Φ-operads.

Suppose Φ a perfect operator category.

Theorem 5.38. — *Suppose* $\mathbf{V} \in \mathbf{Enr}$*; then there exists a left-proper, simplicial, tractable model* $\mathbf{V}$*-category* $\mathbf{WkOperad}^\Phi(\mathbf{V})$ *with the following properties.*

(5.38.1) The underlying category of $\mathbf{WkOperad}^\Phi(\mathbf{V})$ *is the functor category* $\mathbf{V}^{\mathcal{M}_\Phi}$*.*
(5.38.2) The cofibrations of $\mathbf{WkOperad}^\Phi(\mathbf{V})$ *are precisely the projective cofibrations.*
(5.38.3) An object $P \in \mathbf{WkOperad}^\Phi(\mathbf{V})$ *is fibrant if and only if the following conditions are satisfied.*
  (5.38.3.1) P *is objectwise fibrant.*

(5.38.3.2) Every element of

$$\{ P(X) \longrightarrow \operatorname{holim}(A \circ D_X) \mid X \in \mathcal{M}_\Phi \}$$

is an isomorphism of $\operatorname{Ho}\mathbf{V}$.

(5.38.4) The weak equivalences between fibrant objects are objectwise.

This defines a $(2,1)$*-functor*

$$\begin{aligned} \mathbf{WkOperad} :\ \mathbf{Plop} \times \mathbf{Enr} &\longrightarrow \mathbf{ModCat} \\ (\Phi, \mathbf{V}) &\longrightarrow \mathbf{WkOperad}^\Phi(\mathbf{V}). \end{aligned}$$

5.39. — We can prove an analogue of our proto-periodic table result above that applies to weak Φ-operads and weak Φ-multicategories. To show that this result is a genuine generalization of our previous result, we will require a comparison result from the next section.

Proposition 5.40. — *(5.40.1) There is a Quillen adjunction*

$$\mathrm{B}^\Phi : \mathbf{WkOperad}^\Phi(\mathbf{V}) \rightleftarrows (\star/\mu^\Phi \mathbf{Wk}(\mathbf{V})\mathbf{Cat}) : \Omega^\Phi$$

(5.40.2) Furthermore, $\mathbf{WkOperad}^\Phi(\mathbf{V})$ *is a colocalization of* $(\star/\mu^\Phi\mathbf{Wk}(\mathbf{V})\mathbf{Cat})$*, in the sense that the natural morphism* $\mathrm{id} \longrightarrow \mathbf{R}\Omega^\Phi\mathbf{L}\mathrm{B}^\Phi$ *of endofunctors of* $\operatorname{Ho}\mathbf{WkOperad}^\Phi(\mathbf{V})$ *is an isomorphism.*

(5.40.3) The essential image of $\mathbf{L}\mathrm{B}^\Phi$ *is comprised of those pointed weak* Φ*-multi-*$\mathbf{V}$*-categories* (A, a) *such that any object of* A *is equivalent to* a.

6. Strictifications and comparisons

6.1. — Here is the technical core of this work. In this section are the key results that compare various different homotopy theories of structure. After reviewing the Categorical Strictification (CS) Theorem, I proceed to formulate Operadic Strictification, Algebraic Strictification, and Modular Strictification (OS, AS, and MS) Theorems. Over enrichment model categories, the auxiliary categories $\mathcal{L}_\Phi$ and $\mathcal{M}_\Phi$ provide direct proofs of these results, but in fact all of these results follow from the General Algebraic Strictification (GAS) Theorem, which I formulate in the final subsection.

Categorical strictification. — I begin with a quick review of the classical statements of strictification. These are beautiful, highly nontrivial, results, and they will provide us with mercifully brief proofs of some special cases of the "algebraic" strictification results and conjectures that we are going to discuss in the next section.

***Theorem 6.2* (Categorical Strictification)**. — *Suppose* $\mathbf{M}$ *a tractable left (respectively, right) Quillen presheaf on* D*. Then we have the following.*

(6.2.1) There exists a tractable injective (resp., projective) model structure on the on the category $\mathbf{Sect}^L(\mathbf{M})$ *of left sections (resp., on the on the category* $\mathbf{Sect}^L(\mathbf{M})$ *of left sections) in which the weak equivalence and cofibrations (resp, the weak equivalences and fibrations) are defined objectwise.*

(6.2.2) There is an equivalence of $\mathbf{Wk}(\infty,1)\mathbf{Cat}$*:*

$$N\mathbf{Sect}^L_{\mathrm{inj}}(\mathbf{M}) \longrightarrow \operatorname{holim}^{\mathrm{lax}}_{d\in D^{\mathrm{op}}} \mathbf{M}_d \quad \textit{(resp.,} \quad N\mathbf{Sect}^R_{\mathrm{proj}}(\mathbf{M}) \longrightarrow \operatorname{holim}^{\mathrm{lax}}_{d\in D^{\mathrm{op}}} N\mathbf{M}_d \textit{)}.$$

(6.2.3) There exists a tractable right Bousfield localization $\mathbf{Sect}^L_{\mathrm{holim}}(\mathbf{M})$ *of* $\mathbf{Sect}^L_{\mathrm{inj}}(\mathbf{M})$ *(resp., a left Bousfield localization* $\mathbf{Sect}^R_{\mathrm{holim}}(\mathbf{M})$ *of* $\mathbf{Sect}^R_{\mathrm{proj}}(\mathbf{M})$*) in which the cofibrant (resp., fibrant) objects are the cofibrant (resp., fibrant), homotopy cartesian objects of* $\mathbf{Sect}(\mathbf{M})$.

(6.2.4) The equivalences of (6.2.2) equivalences of $\mathbf{Wk}(\infty,1)\mathbf{Cat}$

$$N\mathbf{Sect}^R_{\mathrm{holim}}(\mathbf{M}) \longrightarrow \operatorname{holim}_{d\in D^{\mathrm{op}}} N\mathbf{M}_d \quad \textit{(resp.,} \quad N\mathbf{Sect}^L_{\mathrm{holim}}(\mathbf{M}) \longrightarrow \operatorname{holim}_{d\in D^{\mathrm{op}}} N\mathbf{M}_d \textit{)}.$$

About the Proof. — The first and third assertions have appeared in [**1**, 1.30, 1.32, 2.44] and [**2**, 2.23]. Somewhat more specific versions of the second and fourth assertions have been proved by Hirschowitz-Simpson, Toën-Vezzosi, and Lurie. In volume 1 of my book will appear a complete proof of the general statement. □

Corollary 6.3. — *If* $\mathbf{M}$ *is a tractable model category, and* D *a category, then there is an equivalence of* $\mathbf{Wk}(\infty,1)\mathbf{Cat}$*:*

$$\mathbf{R}\,\underline{\mathrm{Mor}}(ND, N\mathbf{M}) \simeq N(\mathbf{M}^D)_{\mathbf{proj}}.$$

6.4. — Using entirely abstract techniques, one can show that the lax homotopy limit and the homotopy limit of the theorem are Rezk nerves of *some* tractable model categories. But the advantage of this result is that it gives one an explicit — and very pleasant — model for the resulting $(\infty,1)$-category. The price one has to pay for this seems to be that sometimes one has to think about right Bousfield localizations of model categories that are not right proper. C'est la vie.

Example 6.5. — Recall that we had, for any enrichment model category $\mathbf{V}$, a kind of tower of enrichment categories

$$\cdots \longrightarrow \mathbf{Wk}(\mathbf{V},n)\mathbf{Cat} \longrightarrow \cdots \longrightarrow \mathbf{Wk}(\mathbf{V},1)\mathbf{Cat} \longrightarrow \mathbf{Wk}(\mathbf{V},0)\mathbf{Cat}$$

Using the Strictification Theorem, one can now compute the homotopy limit of this tower as a model structure on the category of sequences (A_n,ϕ_n), in which $A_n \in \mathbf{Wk}(\mathbf{V},n)\mathbf{Cat}$ and $\phi_n : A_{(n-1)} \longrightarrow A_n$ is a morphism of $\mathbf{Wk}(\mathbf{V},n-1)\mathbf{Cat}$.

This is a great model of weak $(\mathbf{V},\infty)$-categories. When $\mathbf{V} = s\mathbf{Set}$, this is a beautiful model of ω-categories, fully compatible with every piece of structure around. It would be interesting to know precisely how this compares to Dominic Verity's model category of ω-categories, especially since these seem to play a big role in Mike Hopkins's recent work.

Operadic Strictification. — Here we compare Φ-operads and weak Φ-operads. This takes the form of a rigidification theorem for Φ-operads.

Theorem 6.6. — *Suppose* Φ *a perfect operator category, and suppose* $\mathbf{V}$ *and enrichment model category. Then the functor* $\mathbf{N}^\Phi : \mathbf{Operad}^\Phi(\mathbf{V}) \longrightarrow \mathbf{V}^{\mathcal{M}_\Phi}$ *is part of a Quillen equivalence*

$$\mathbf{P}^\Phi : \mathbf{WkOperad}^\Phi(\mathbf{V}) \rightleftarrows \mathbf{Operad}^\Phi(\mathbf{V}) : \mathbf{N}^\Phi$$

Sketch of proof. — Once it has been established that $\mathbf{N}^\Phi$ is right Quillen, it is clear that

$$\mathbf{RN}^\Phi : \mathrm{Ho}\,\mathbf{Operad}^\Phi(\mathbf{V}) \longrightarrow \mathrm{Ho}\,\mathbf{WkOperad}^\Phi(\mathbf{V})$$

is fully faithful; it thus suffices to show that $\mathbf{P}^\Phi$ reflects weak equivalences between cofibrant objects. This is a relatively straightforward computation. □

6.7. — From one point of view, the previous result is not surprising: it is, after all, a result of Dugger's that any tractable model category has a presentation, i.e., a Quillen equivalence with a left Bousfield localization of the category of simplicial presheaves on some category; one might regard the previous result as the mere selection of a particular presentation.

On the other hand, the previous result also says something quite interesting; namely, any weak Φ-operad Q in $\mathbf{V}$ has a "strictification," i.e., an isomorphism $Q \longrightarrow \mathbf{RN}^\Phi\mathbf{LP}^\Phi Q$ in $\mathrm{Ho}\,\mathbf{WkOperad}^\Phi(\mathbf{V})$ wherein the target is a strict Φ-operad in $\mathbf{V}$.

Theorem 6.8. — *For any perfect operator category* Φ *and for any enrichment model category* $\mathbf{V}$*, there is an equivalence of weak* $\mathbf{V}$*-categories*

$$\mathbf{N_V}(\mathbf{WkOperad}^\Phi(\mathbf{V})) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\mathbf{F}}\mathbf{Wk}(\mathbf{V})\mathbf{Cat}}(\mathbf{N_V}\mathcal{A}^\Phi, \mathbf{N_V}\mathbf{V}).$$

Sketch of proof. — It's enough to show this for $\mathbf{V} = \mathbf{Wk}(\infty,0)\mathbf{Cat}$. By the classical Strictification Theorem, the $(\infty,1)$-category $\mathbf{N}(\mathbf{WkOperad}^\Phi(\mathbf{Wk}(\infty,0)\mathbf{Cat}))$ is a reflexive full sub-$(\infty,1)$-category of

$$\mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Wk}(\infty,1)\mathbf{Cat}}(\mathbf{N}\mathcal{M}_\Phi, \mathbf{NWk}(\infty,0)\mathbf{Cat}) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\mathbf{F}}\mathbf{Wk}(\infty,1)\mathbf{Cat}}(U_!\mathbf{N}\mathcal{M}_\Phi, \mathbf{NWk}(\infty,0)\mathbf{Cat}).$$

A straightforward computation now implies that the essential image is precisely $\mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\mathbf{F}}\mathbf{Wk}(\infty,1)\mathbf{Cat}}(\mathbf{N}\mathcal{A}^\Phi, \mathbf{NWk}(\infty,0)\mathbf{Cat})$. □

6.9. — Using $\mathcal{A}^\Phi$, one can give a quite general definition of the notion of weak Φ-operad in any weak $\mathbf{F}$-multi-$\mathbf{V}$-category.

Definition 6.10. — Suppose Φ perfect, B a weak $\mathbf{F}$-multi-$\mathbf{V}$-category. Then define the weak $\mathbf{V}$-category of *weak operads in* A by the formula

$$\mathbf{WkOperad}^{\Phi}(B) := \mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\mathbf{F}}\mathbf{Wk(V)Cat}}(\mathbf{N_V}\mathcal{A}^{\Phi}, B).$$

***Theorem 6.11* (Operadic Strictification).** — *For any perfect operator category Φ and any symmetric monoidal model $\mathbf{V}$-category $\mathbf{M}$, there is a functorial equivalence of weak $\mathbf{V}$-categories*

$$\mathbf{N_V Operad}^{\Phi}(\mathbf{M}) \simeq \mathbf{WkOperad}^{\Phi}(U^{\Phi,\mu} N_{\mathbf{V}}\mathbf{M}).$$

6.12. — I have thus far not seen any version of this result formulated elsewhere, though presumably some "dendroidal" formulation is possible. As with the other results of this section, this result follows from the GAS Theorem, which I will discuss shortly.

Algebraic Strictification. — In this subsection we compare the homotopy theory of weak algebras with the homotopy theory of strict algebras. The former is a higher categorical concept; the latter, which uses model categories, was introduced at the beginning of the previous section.

Definition 6.13. — Suppose Φ perfect. Then for any weak Φ-multi-$\mathbf{V}$-category A, and any Φ-multi-$\mathbf{V}$-category P, define the weak $\mathbf{V}$-category of *weak P-algebras* in A as:

$$\mathbf{WkAlg}^{\Phi}_{P}(A) := \mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\Phi}\mathbf{Wk(V)Cat}}(\mathbf{B}^{\Phi}P, A)$$

Corollary 6.14. — *The Quillen adjunction $(\mathbf{Free}^{\Phi,\otimes}, U^{\Phi,\mu})$ yields the following formula for any weak Φ-monoidal weak $\mathbf{V}$-category A any Φ-multi-$\mathbf{V}$-category P:*

$$\mathbf{WkAlg}^{\Phi}_{P}(U^{\Phi,\mu}A) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Mon}^{\Phi}(\mathbf{Wk(V)Cat})}(\mathbf{LFree}^{\Phi,\otimes}(P), A)$$

Corollary 6.15. — *If Φ is faithfully flat, then $\mathbf{LFree}^{\Phi,\otimes}(\star) \simeq N\Phi$, whence*

$$\mathbf{WkAlg}^{\Phi}_{\star}(U^{\Phi,\mu}A) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Mon}^{\Phi}(\mathbf{Wk(V)Cat})}(N\Phi, A).$$

Example 6.16. — It follows from this corollary that if A is a symmetric monoidal $(\infty,1)$-category, then the weak $(\infty,1)$-category of weak commutative algebras in A is canonically equivalent to the weak $(\infty,1)$-category of weakly symmetric monoidal weak $(\infty,1)$-functors $\mathbf{F} \longrightarrow A$.

6.17. — When E is an $(\infty,1)$-category with all finite limits, the Leinster category plays a very much similar role to the one it played in the theory of "traditional" Φ-monoids.

Theorem 6.18. — *For any weak $(\infty,1)$-category E with* all finite limits *— viewed as an $\mathbf{F}$-multicategory and thus a Φ-multicategory via homotopy products —, there is a fully faithful morphism of $(\infty,1)$-categories*

$$\mathbf{WkAlg}^{\Phi}_{\star}(E) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mu^{\Phi}\mathbf{Wk(V)Cat}}(\star, E) \longrightarrow \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Wk(V)Cat}}(\mathcal{L}_{\Phi}, E)$$

whose essential image is comprised of $F : \mathcal{L}_{\Phi} \longrightarrow E$ such that every element of

$$S_{\Phi} := \{\, F(TI) \longrightarrow \mathrm{holim}(F \circ D_I)\,\}$$

is a weak equivalence of E.

Corollary 6.19. — *For any enrichment category $\mathbf{V}$, we have*

$$\begin{array}{ccc}\mathbf{WkAlg}^{\Phi}_{\star}(U^{\Phi,\mu}N\mathbf{V}) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Mon}^{\Phi}(\mathbf{Wk}(\infty,1)\mathbf{Cat})}(\mathbf{LFree}^{\Phi,\otimes}(\star), N\mathbf{V}) & \xrightarrow{\sim} & N\mathbf{WkMon}^{\Phi}(\mathbf{V}) \\ \downarrow & & \downarrow \\ \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Wk(V)Cat}}(N\mathcal{L}_{\Phi}, N\mathbf{V}) & \xrightarrow[\sim]{} & N(\mathbf{V}^{\mathcal{L}_{\Phi}})_{\mathbf{proj}}\end{array}$$

a commutative diagram in $\mathrm{Ho}\,\mathbf{Wk}(\infty,1)\mathbf{Cat}$ *in which the horizontal morphisms are isomorphisms and the vertical morphisms are fully faithful.*

Theorem 6.20. — *Suppose Φ a perfect operator category, $\mathbf{V}$ an enrichment model category $\mathbf{V}$, and Q a cofibrant replacement for the terminal Φ-operad in $\mathbf{V}$. Then there is an equivalence of weak $\mathbf{V}$-categories*

$$N_{\mathbf{V}}\mathbf{WkMon}^{\Phi}(\mathbf{V}) \simeq N_{\mathbf{V}}\mathbf{Alg}^{\Phi}_{Q}(\mathbf{V}).$$

Corollary 6.21. — *Putting everything together, we have, for any* $\mathbf{V} \in \mathbf{Enr}$ *and a homotopically terminal* Φ*-operad* $\star$ *in* $\mathbf{V}$*,*

$$\mathbf{WkAlg}^{\Phi}_{\star}(U^{\Phi,\mu}N\mathbf{V}) \simeq N_{\mathbf{V}}\mathbf{WkMon}^{\Phi}(\mathbf{V}) \simeq N_{\mathbf{V}}\mathbf{Alg}^{\Phi}_{Q}(\mathbf{V}).$$

Theorem 6.22 **(Algebraic Strictification).** — *For any tractable* Φ*-monoidal model category* $\mathbf{M}$*, and for any cofibrant* Φ*-operad* O *in* $\mathbf{M}$*, there is an equivalence of* $(\infty,1)$*-categories*

$$\mathbf{WkAlg}^{\Phi}_{O}(U^{\Phi,\mu}N\mathbf{M}) \simeq N\mathbf{Alg}^{\Phi}_{O}(\mathbf{M})$$

6.23. — This result is a common generalization of some conjectures of Toën from about seven years ago. More recently, Lurie has addressed related questions. The result here follows from the GAS Theorem.

Modular Strictification. — Having strictified algebras, it seems natural to attempt to strictify modules over these algebras. The GAS Theorem also implies strictification corollaries that permit one to strictify any kind of module, but the precise formulation would take us too far afield. I will instead be satisfied with stating the following corollary, which I suspect is widely believed but till now not satisfactorily proved.

Corollary 6.24. — *Suppose* $\mathcal{M}$ *as above. Suppose* A *an* A_∞*-algebra in* $\mathcal{M}$*. Write* $\mathbf{Mod}^r(A)$ *for the category of right* A*-modules, and write* $\mathbf{Mod}(A)$ *for the category of* A*-bimodules. Then the* $(\infty,1)$*-category* $N\mathbf{Mod}(A)$ *inherits the structure of a weak* $\mathbf{O}$*-monoidal* $(\infty,1)$*-category. There are equivalences among the spaces of weak* $\mathrm{Wr}^{(k+1)}(\mathbf{O})$*-monoidal structures on* $N\mathbf{Mod}^r(A)$ *recovering the weak* $\mathbf{O}$*-monoidal structure on* A*, of weak* $\mathrm{Wr}^{(k)}(\mathbf{O})$*-monoidal structures on* $N\mathbf{Mod}(A)$ *recovering the weak* $\mathbf{O}$*-monoidal structure on* A*, and of* E_k*-algebra structures on* A *recovering the* A_∞*-algebra structures on* A*:*

$$(\mathbf{WkMonStr}^{\mathrm{Wr}^{(k+1)}(\mathbf{O})}(N\mathbf{Mod}^r(A))/A) \simeq (\mathbf{WkMonStr}^{\mathrm{Wr}^{(k)}(\mathbf{O})}(N\mathbf{Mod}(A))/A) \simeq (\mathbf{WkAlgStr}_{E_k}(A)/A).$$

The following subcorollary answers a question of Miller. In effect, it supplies an E_{k+1} algebra structure on the "E_k topological Hochschild cohomology" of an E_k ring spectrum A.

Corollary 6.25. — *Suppose* A *an* E_k *ring spectrum for* $1 \leq k \leq \infty$*. Then the endomorphism spectrum* $\mathbf{End}(A)$ *of* A *in* $\mathbf{Mod}(A)$ *inherits a natural structure as an* E_{k+1} *ring spectrum.*

General Algebraic Strictification. — In this subsection, I discuss some notes on a general conjecture that implies all the other conjectures of this section. I consider the following theorem my deepest result in the realm of higher categories and homotopy coherent algebra. It provides an incredibly powerful sort of strictification procedure whereby one recovers strict models of weak algebras over a colored Φ-operad.

Theorem 6.26 **(General Algebraic Strictification).** — *Suppose* $(\Phi,\mathcal{E},S,\mathcal{P},\mathcal{M})$ *a tuple in which* Φ *is an operator category,* $\mathcal{E}$ *an enrichment model category,* S *is a set,* $\mathcal{P}$ *a cofibrant* S*-colored* Φ*-operad in* $\mathcal{E}$*, and* $\mathcal{M}$ *a tractable* Φ*-monoidal model* $\mathcal{E}$*-category satisfying the monoid axiom. Then there is a canonical equivalence of weak* $\mathcal{E}$*-categories:*

$$N_{\mathcal{E}}\mathbf{Alg}^{\Phi}_{\mathcal{E},\mathcal{P}}(\mathcal{M}) := N_{\mathcal{E}}\,\underline{\mathrm{Mor}}_{\mu^{\Phi}(\mathcal{E})\mathbf{Cat}}(\mathcal{P},\mathcal{M}) \xrightarrow[\sim]{} \mathbf{R}\,\underline{\mathrm{Mor}}^{\mathbf{Wk}(\mathcal{E})\mathbf{Cat}}_{\mathbf{Wk}\mu^{\Phi}(\mathcal{E})\mathbf{Cat}}(\mathcal{P},\mathcal{M}) =: \mathbf{WkAlg}^{\Phi}_{\mathcal{E},\mathcal{P}}(N_{\mathcal{E}}\mathcal{M}).$$

A special case of this result was (at least partially) demonstrated some time ago by Spitzweck and me, but the method of proof required for the result above is different from the technique used in our previous work.

7. Filtering algebraic structures

Associative structures. — Let us begin by examining the section of the $(2,1)$-category $\mathbf{Op}$ between $\mathbf{0}$ and $\mathbf{O}$. It is well-known among topologists that the operads A_n are meant to provide signposts along the way from no structure to associative or A_∞ structure. This is now easy to reformulate using the operator categories $\mathbf{O}_{\leq n}$. Just for fun, I will give some indications of how one might go about this. (This is an outgrowth of a conversation with Sarah Whitehouse, who pointed out to me that this would be help legitimize the theory.)

Lemma 7.1. — *An* A_∞ *operad is a homotopically terminal* $\mathbf{O}$*-operad.*

Proposition 7.2. — *For any* $n \geq 0$*, consider the inclusion* $j_n : \mathbf{O}_{\leq n} \longrightarrow \mathbf{O}$ *of operator categories; then the image of the homotopically terminal* $\mathbf{O}_{\leq n}$*-operad under the derived left adjoint* $\mathbf{L}j_{n,!}$ *is an* A_{n+1} *operad.*

Sketch of proof. — Suppose K a homotopically terminal $\mathbf{O}$-operad. Then $\mathbf{R}j_n^{\star}K$ is a homotopically terminal $\mathbf{O}_{\leq n}$-operad, and $\mathbf{L}J_{n,!}\mathbf{R}j_n^{\star}K$ is the $\mathbf{O}$-operad generated by the first $(n+1)$ spaces of K. ☐

7.3. — Thus an A_{n-1}-algebra (or space) is precisely the same thing as an algebra for the homotopically terminal $\mathbf{O}_{\leq n}$-operad.

The Eckman-Hilton tower and the Freudenthal-Breen-Baez-Dolan Stabilization Hypothesis. —

Definition. — The *Eckman-Hilton sequence* is the sequence of operator categories

$$\star \longrightarrow \mathbf{O} \longrightarrow \mathbf{O} \wr \mathbf{O} \longrightarrow \mathbf{O} \wr \mathbf{O} \wr \mathbf{O} \longrightarrow \cdots \longrightarrow \mathbf{Wr}^{(n)}(\mathbf{O}) \longrightarrow \cdots.$$

Suppose $\mathbf{V} \in \mathbf{Enr}$; then there is the associated the *Eckman-Hilton tower*

$$\mathbf{EH_V} := [\ \mathbf{Wk(V)Cat} \longleftarrow \mu^{\mathbf{O}}\mathbf{Wk(V)Cat} \longleftarrow \mu^{(\mathbf{O}\wr\mathbf{O})}\mathbf{Wk(V)Cat} \longleftarrow \mu^{(\mathbf{O}\wr\mathbf{O}\wr\mathbf{O})}\mathbf{Wk(V)Cat} \longleftarrow \cdots \longleftarrow \mu^{\mathbf{Wr}^{(n)}(\mathbf{O})}\mathbf{Wk(V}$$

a tower of right Quillen functors.

Lemma. — *The Leinster category*

$$\mathcal{L}_{\mathbf{Wr}^{(n)}(\mathbf{O})} \simeq (\Delta^{\mathrm{op}})^{\times n},$$

and

$$\mathbf{Mon}^{\mathbf{Wr}^{(n)}(\mathbf{O})}(\mathbf{V}) \simeq \mathbf{Mon}^{\mathbf{O}}(\mathbf{Mon}^{\mathbf{O}}(\ldots\mathbf{Mon}^{\mathbf{O}}(\mathbf{V})\ldots))$$

Remark. — For any $n \geq 0$, we have, by functoriality, a Quillen adjunction

$$U_! : \ \mu^{\mathbf{Wr}^{(n)}(\mathbf{O})}\mathbf{Wk(V)Cat} \rightleftarrows \mu^{\mathbf{F}}\mathbf{Wk(V)Cat} \ : U^{\star}.$$

***Theorem* (Also see Batanin, Fiedorowicz-Vogt).** — *Suppose* $\mathbf{V} = s\mathbf{Set}$*. Then for any* $n > 0$*,* $\mathbf{L}U_!(\star)$ *is an* E_n *operad.*

Corollary. — *For any* $n > 0$*, the* $(\infty,1)$*-category of* E_n*-algebras in* $\mathbf{V}$ *is modeled by the tractable model category*

$$\mathbf{Mon}^{\mathbf{Wr}^{(n)}(\mathbf{O})}(\mathbf{V}).$$

Theorem. — $\operatorname{holim} N\mathbf{EH_V} \simeq \mu^{\mathbf{F}}\mathbf{Wk(V)Cat}$.

Remark. — Our results show that an E_n-algebra A in $\mathbf{V}$ can be viewed as an $(\mathbf{V},n)$-category.

Theorem. — *There is an equivalence of* $(\mathbf{V},n)$*-categories*

$$N_{\mathbf{V}}\mathbf{Mod}_{\star,\mathbf{V}}^{\mathbf{Wr}^{(n)}(\mathbf{O})}(A) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Wk(V},n)\mathbf{Cat}}^{\mathbf{Wk(V)Cat}}(A, N_{\mathbf{Wk(V},n-1)\mathbf{Cat}}\mathbf{Wk(V},n-1)\mathbf{Cat})$$

Theorem. — *An* E_1*-algebra* A *in* $\mathbf{V}$ *is an* E_n*-algebra if and only if the weak* $\mathbf{V}$*-category*

$$N_{\mathbf{V}}\mathbf{Mod}_{\star,\mathbf{V}}^{\mathbf{O}}(A) \simeq \mathbf{R}\,\underline{\mathrm{Mor}}_{\mathbf{Wk(V)Cat}}(A, N_{\mathbf{V}}\mathbf{V})$$

carries a $\mathbf{Wr}^{(n-1)}(\mathbf{O})$*-monoidal structure.*

The F-filtration. — The filtration

$$\star \longrightarrow \mathbf{F}_{\leq 1} \longrightarrow \mathbf{F}_{\leq n} \longrightarrow \mathbf{F}_{\leq 3} \longrightarrow \cdots \longrightarrow \mathbf{F}_{\leq n} \longrightarrow \cdots$$

is essentially the same as the Robinson filtration on the E_∞ operad.

Applications. — Here are a couple results one can prove easily using the ideas sketched here. There are lots of results like these.

***Theorem 7.4* (Deligne Conjecture).** — *For any E_n ring spectrum A, the spectra* $\mathrm{THH}(A)$ *and* $K(A)$ *are E_{n-1} ring spectra in a canonical fashion.*

***Theorem 7.5*.** — *For any E_n ring spectrum A, one may define the* topological Hochschild cohomology $\mathrm{THC}(A)$ *as the endomorphism ring in the category of A-omnimodules. This has a canonical E_{n+1} ring spectrum structure.*

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Norwegian Topology Symposium: Bergen, 2007

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