The Bass–Quillen conjecture

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OUR GOAL IN THIS TALK is to present Lindel's proof of the geometric case of the Bass–Quillen conjecture.

The statement

Theorem (Serre's conjecture, proved by Quillen and Suslin)**.** *If k is a field, then every finitely generated projective module over $k[t_1, \dots, t_n]$ is free.*[1]

[1] This was discussed last week.

Conjecture (Bass–Quillen)**.** *Suppose A a regular ring, and suppose P a finitely generated projective module over $A[t_1, \dots, t_n]$. Then P is extended from A; that is, one has*

$$P \cong P_0 \otimes_A A[t_1, \dots, t_n],$$

where

$$P_0 \cong P/(t_1, \dots, t_n)P.$$

Theorem (Quillen–Suslin)**.** *The Bass–Quillen conjecture holds for* $\dim A \leq 2$.[2]

[2] This too was discussed last week.

Theorem (Lindel)**.** *The Bass–Quillen conjecture holds for A essentially of finite type*[3] *over a field k.*

[3] A k-algebra is *essentially of finite type* if it is a localization of a finite type k-algebra.

Easy reductions

Lemma. *It suffices to assume that A is a regular* local *ring; in particular, we may assume that*

$$A \cong C_p, \quad \textit{where} \quad C = k[x_1, \dots, x_m]/(f_1, \dots, f_r)$$

and $p \lhd C$ is a prime ideal.

Proof. Quillen patching.[4] □

[4] Also from last week.

Lemma. *It suffices to assume that k is perfect (even prime).*

Proof. Let $k_0 \subset k$ denote the prime field. Since P is projective, it is the image of an idempotent endomorphism α on a free $A[t_1, \dots, t_n]$-module. Now let $k_0 \subset k' \subset k$ denote the subfield generated by the coefficients of $f_1, \dots, f_r$ and the entries of a matrix representing α. Set

$$C' \coloneqq k'[x_1, \dots, x_m]/(f_1, \dots, f_r) \quad \text{and} \quad p' \coloneqq p \cap C' \quad \text{and} \quad A' \coloneqq C'_{p'}.$$

Since α is defined over $A'[t_1, \dots, t_n]$, it follows that

$$P \cong P' \otimes_{A'[t_1,\dots,t_n]} A[t_1, \dots, t_n]$$

for some finitely generated projective $A'[t_1, \dots, t_n]$-module P'.

Now observe that $A \cong A' \otimes_{k'} k$ is a faithfully flat extension of A, whence A' is also regular. Furthermore, k' is finitely generated over k_0, whence A' is essentially of finite type over k_0. □

Induction hypothesis

Write $d \coloneqq \dim A$. The Quillen–Suslin theorem addresses the case $d \leq 2$. So we assume that $d \geq 3$ and that Lindel's theorem holds for k-algebras essentially of finite type of dimension $< d$.

Reducing the dimension via descent

Lemma (Nisnevich descent)**.** *Suppose $\phi\colon S \longrightarrow R$ an étale homomorphism of rings, and suppose $x \in S$. Assume that $\phi(x)$ is not a zerodivisor. If ϕ induces an isomorphism $S/x \xrightarrow{\sim} R/\phi(x)$, then the square*[5]

$$\begin{array}{ccc} S & \xrightarrow{\phi} & R \\ \downarrow & & \downarrow \\ S_x & \longrightarrow & R_{\phi(x)} \end{array}$$

is a pullback of rings, and the corresponding square

$$\begin{array}{ccc} \mathbf{Proj}(S) & \xrightarrow{-\otimes_S R} & \mathbf{Proj}(R) \\ {\scriptstyle (-)_x}\downarrow & & \downarrow{\scriptstyle (-)_{\phi(x)}} \\ \mathbf{Proj}(S_x) & \xrightarrow[-\otimes_{S_x} R_{\phi(x)}]{} & \mathbf{Proj}(R_{\phi(x)}) \end{array}$$

is a pullback square of exact categories.

[5] After applying Spec, this square becomes a *elementary distinguished square*. Consequently, this lemma is the key special case of the assertion that **Proj** is a sheaf of exact categories for the Nisnevich site.

Sketch of proof.[6] The first statement is trivial. For the second statement, we construct a functor

$$\mathbf{Proj}(S_x) \times_{\mathbf{Proj}(R_{\phi(x)})} \mathbf{Proj}(R) \longrightarrow \mathbf{Proj}(S)$$

by the fiber product:

$$(Q, P, \sigma\colon Q \otimes_{S_x} R_{\phi(x)} \cong P_{\phi(x)}) \rightsquigarrow Q \times_{P_{\phi(x)}} P;$$

one notes that the S-module $Q \times_{P_{\phi(x)}} P$ is finitely generated and projective. To complete the proof, one must prove two facts:

- For any (Q, P, σ) as above, one has natural isomorphisms

$$(Q \times_{P_{\phi(x)}} P)_x \cong Q \quad \text{and} \quad (Q \times_{P_{\phi(x)}} P) \otimes_S R \cong P.$$

- For any finitely generated projective S-module E, one has a natural isomorphism

$$E \cong E_x \times_{(E \otimes_S R)_{\phi(x)}} (E \otimes_S R). \qquad \square$$

[6] The proof given by Milnor in Ch. 2 of his text on algebraic K-theory works with only trivial modifications.

Definition. The pair (ϕ, x) together will be called an *elementary Nisnevich cover*.

Corollary. *Suppose $(\phi\colon S \longrightarrow R, x \in S)$ an elementary Nisnevich cover, and suppose P an R-module such that $P_{\phi(x)}$ is free. Then there exists a projective S-module P' such that $P' \otimes_S R \cong P$.*

A key special case

Lemma (Special case). *Suppose* $m = (f(x_1), x_2, \ldots, x_d) \lhd k[x_1, \ldots, x_d]$ *a maximal ideal. Then the Bass–Quillen conjecture holds for the regular local ring*

$$B = k[x_1, \ldots, x_d]_m.$$

Proof. Set

$$B_0 \coloneqq k[x_1, \ldots, x_{d-1}]_{(f(x_1), x_2, \ldots, x_{d-1})},$$

and consider the homomorphism $B_0[x_d] \longrightarrow B$. Note that the pair

$$(B_0[x_d, t_1, \ldots, t_n] \longrightarrow B[t_1, \ldots, t_n], x_d)$$

is an elementary Nisnevich cover.

Suppose M a finitely generated projective B-module. Now $\dim B_{x_d} < d$, so by the induction hypothesis, M_{x_d} is extended from B_{x_d}, hence free. Now by Nisnevich descent, there exists a finitely generated projective $B_0[x_d, t_1, \ldots, t_n]$-module M' such that

$$M \cong M' \otimes_{B_0[x_d, t_1, \ldots, t_n]} B[t_1, \ldots, t_n].$$

Observe that $\dim B_0 < d$ as well. The inductive hypothesis thus implies that M' is free, whence so is M. □

Reduction to the special case

Lemma (Nashier). *There exists an elementary Nisnevich cover* $(\phi\colon B \longrightarrow A, h)$ *such that* $B \cong C_m$ *for*

$$C = k[x_1, \ldots, x_d] \quad \textit{and} \quad m = (f(x_1), x_2, \ldots, x_d) \lhd C$$

a maximal ideal.

$$\begin{array}{ccc} B & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ B_h & \longrightarrow & A_{\phi(h)} \end{array}$$

Proof of Lindel's Theorem from Nashier's Lemma. By the induction hypothesis, since $\dim A_{\phi(h)} < d$, the projective $A_{\phi(h)}$-module $P_{\phi(h)}$ is free. By Nisnevich descent, there exists a finitely generated projective B-module P' such that

$$P' \otimes_B A \cong P.$$

Our special case now applies, so P' is free, whence so is P. □

Proof of Nashier's Lemma. Denote by $m_A \lhd A$ the maximal ideal, and choose a nonzero element $a \in m_A^2$. One may select[7] a regular sequence $a, x_2, \ldots, x_d$ and an element $z \in m_A$ such that $m_A = (z, x_2, \ldots, x_d)$.

In particular, the sub-k-algebra of A generated by $\{a, x_2, \ldots, x_d\}$ is a polynomial ring $k[a, x_2, \ldots, x_d] \subset A$. Let $C' \subset A$ denote the integral closure of $k[a, x_2, \ldots, x_d]$ in A. One notes that[8]

$$m' \coloneqq m_A \cap C'$$

[7] by prime avoidance

[8] since $m_A \cap k[a, x_2, \ldots, x_d] = (a, x_2, \ldots, x_d)$

is maximal in C'.

We now claim that $A = C'_{m'}$. This follows from Zariski's Main Theorem. The important thing to note is that the fraction fields $K(k[a, x_2, \dots, x_d])$ and $K(A)$ have the same transcendence degree over k, whence the extension

$$K(k[a, x_2, \dots, x_d]) \subset K(A)$$

is algebraic. Consequently, $K(C') = K(A)$, and C' is finite over $k[a, x_2, \dots, x_d]$, so Zariski's Main Theorem applies.

For any $s \in C'$, write $\bar{s} \coloneqq s + m' \in C'/m'$. Since k is perfect, there exists $s \in C'$ such that C'/m' is the field $k(\bar{s})$. Let f denote the minimal polynomial of $\bar{s}$.[9] Since $A = C'_{m'}$, the C'/m'-vector space $m'/(m')^2$ is generated by $\{\bar{z}, \bar{x}_1, \dots, \bar{x}_d\}$, whence

$$f(\bar{s}) = b_1\bar{z} + b_2\bar{x}_1 + \cdots + b_d\bar{x}_d$$

with $b_1, \dots, b_d \in C'/m'$. Now replacing s with $s + z$ if $b_1 = 0$, one ensures that

$$m' = (f(s), x_2, \dots, x_d) + (m')^2.$$

Write $m', m'_2, \dots, m'_r \lhd C'$ for the maximal ideals that lie over

$$(a, x_2, \dots, x_d) \lhd k[a, x_2, \dots, x_d].$$

Choose $x_1 \in m'_2 \cap \cdots \cap m'_r$ such that

$$x_1 = s \mod (m')^2.$$

Replacing x_1 by $x_1 + a^t$ if necessary, one ensures that a is integral over $k[x_1, \dots, x_d]$. The only maximal ideal of C' that contains $(a, f(x_1), x_2, \dots, x_d)$ is m', and since $f(x_1) = f(s) \mod (m')^2$, it follows that

$$m' = (f(x_1), x_2, \dots, x_d) + (m')^2,$$

whence $m' = (a, f(x_1), x_2, \dots, x_d)$.

Now set

$$C \coloneqq k[x_1, \dots, x_d], \quad \text{and} \quad m = (f(x_1), x_2, \dots, x_d) \quad \text{and} \quad B \coloneqq C_m.$$

Note that Nakayama's Lemma implies[10]

$$A = C'_{m'} = B[a]_{(m,a)}.$$

Observe also that

$$A = B + aA \quad \text{and} \quad B/(B \cap aA) = A/aA.$$

Let $g(u) = u^n + c_{n-1}u^{n-1} + \cdots + c_0$ be the minimal polynomial of a over C. Set

$$h \coloneqq c_0 = -(c_1 + c_2 a + \cdots + c_{n-1}a^{n-2} + a^{n-1})a.$$

[9] In particular,

$$f(s) \in m' \quad \text{and} \quad f'(s) \notin m',$$

and for any $y \in m'$,

$$f(s + y) = f(s) + f'(s)y \mod (m')^2.$$

[10] since $m_A \cap C = m$ and $m_A = (m, a)$

Our claim is now that the pair $(B \hookrightarrow A, h)$ is the desired elementary Nisnevich cover. To show that $B/hB \longrightarrow A/hA$ is an isomorphism, we must argue that

$$hB = hA \cap B \quad \text{and} \quad A = B + hA.$$

The first claim follows from the fact that $A = B[a]_{(m,a)}$ is flat as a B-module, and hence (since A is local) faithfully flat over B.

To prove the second claim, it suffices to show that

$$c_1 \notin m.$$

For this, since $a \in m_A^2$ and since m_A is generated by m in A, there are elements $\mu_i, \nu_i \in C$ with $\mu_0, \mu_1 \in m$ and $\nu_0 \notin m$ such that

$$(\nu_0 + \cdots + \nu_r a^r)a = \mu_0 + \cdots + \mu_r a^r$$

Consequently, there is $h(u) = \alpha_0 + \cdots + \alpha_r u^r \in C[u]$ such that $h(a) = 0$ but $\alpha_1 \notin m$, and since g divides h, it follows that $c_1 \notin m$. □