

# Exodromy

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## Abstract

Let  $X$  be a scheme. Let  $\text{Gal}(X)$  be the topological category whose objects are geometric points of  $X$  and whose morphisms are specialisations thereof. If  $X$  is a scheme of finite type over a finitely generated field  $k$  of characteristic zero, then the category  $\text{Gal}(X)$  acquires a continuous action of the absolute Galois group  $G_k$  of  $k$ . Our main result is that the resulting functor from reduced normal schemes of finite type over  $k$  to topological categories with an action of  $G_k$  is fully faithful.

The category  $\text{Gal}(X)$  is a form of MacPherson's exit-path category for the étale topology. *Exodromy* refers to the equivalence between representations of  $\text{Gal}(X)$  and constructible sheaves on  $X$ . Together with a higher categorical form of Hochster Duality, this equivalence ensures that the entire étale topos of a coherent scheme can be reconstructed from  $\text{Gal}(X)$ . Voevodsky's proof of a conjecture of Grothendieck then implies our main theorem.

## Contents

<b>Introduction</b>	<b>5</b>
Monodromy for topological spaces . . . . .	6
Monodromy for schemes . . . . .	6
Exodromy for topological spaces . . . . .	7
Exodromy for schemes . . . . .	9
Hochster Duality for higher topoi . . . . .	10
Stratified Riemann Existence . . . . .	11
Technical overview . . . . .	11
Open problems . . . . .	12
Acknowledgements . . . . .	13
<b>o Terminology &amp; notations</b>	<b>13</b>
Set theoretic conventions . . . . .	13
Higher categories . . . . .	14
Proobjects in higher categories . . . . .	15
Recollements . . . . .	16
Relative adjunctions . . . . .	17
<b>I Stratified spaces</b>	<b>19</b>

<b>1</b>	<b>Aide-mémoire on the topology of posets &amp; profinite posets</b>	<b>19</b>
	Alexandroff Duality . . . . .	19
	Stratifications of topological spaces . . . . .	20
	Hochster duality . . . . .	20
	Materialisation . . . . .	22
	Profinite stratifications . . . . .	22
<b>2</b>	<b>Stratified spaces</b>	<b>22</b>
	Stratified spaces as conservative functors . . . . .	22
	Strata & links . . . . .	23
	Repairing functors that are not conservative . . . . .	23
	The stratified Postnikov tower . . . . .	24
	Finite stratified spaces . . . . .	26
<b>3</b>	<b>Profinite stratified spaces</b>	<b>27</b>
	Stratified prospaces over proposets . . . . .	27
	The horizontal materialisation . . . . .	28
	Profinite stratified spaces . . . . .	30
<b>4</b>	<b>Spatial décollages</b>	<b>31</b>
	Complete Segal spaces & spatial décollages . . . . .	31
	The nerve of a stratified space . . . . .	32
	Profinite spatial décollages . . . . .	33
<b>II</b>	<b>Elements of higher topos theory</b>	<b>35</b>
<b>5</b>	<b>Aide-mémoire on higher topoi</b>	<b>35</b>
	Higher topoi . . . . .	35
	Boundedness . . . . .	36
	Coherence . . . . .	37
	Classification of bounded coherent $\infty$ -topoi via $\infty$ -pretopoi . . . . .	39
	Coherence of inverse limits . . . . .	41
	Coherence & preservation of filtered colimits . . . . .	41
	Points, Conceptual Completeness, & Deligne Completeness . . . . .	43
	Shape theory . . . . .	45
	Profinite spaces & Stone $\infty$ -topoi . . . . .	46
<b>6</b>	<b>Oriented pushouts &amp; oriented fibre products</b>	<b>48</b>
	Recollections of higher topoi . . . . .	48
	Oriented squares . . . . .	51
	Oriented pushouts . . . . .	52
	Internal homs & path $\infty$ -topoi . . . . .	53
	Oriented fibre products . . . . .	55
	Generating $\infty$ -sites for oriented fibre products . . . . .	57
	Compatibility of oriented fibre products & étale geometric morphisms . . . . .	61

<b>7</b>	<b>Local <math>\infty</math>-topoi &amp; localisations</b>	<b>64</b>
	Quasi-equivalences . . . . .	64
	Local $\infty$ -topoi . . . . .	65
	Nearby cycles & localisations . . . . .	66
	Localisation à la Grothendieck–Verdier . . . . .	68
	Coherence of localisations . . . . .	70
	Geometric examples of localisations . . . . .	70
<b>8</b>	<b>Beck–Chevalley conditions &amp; gluing squares</b>	<b>71</b>
	The Beck–Chevalley transformation & Beck–Chevalley conditions . . . . .	71
	Localisations & the bounded Beck–Chevalley condition . . . . .	73
	Functoriality of oriented fibre products in oriented diagrams . . . . .	76
	Proof of the Beck–Chevalley condition for oriented fibre products . . . . .	79
	Applications of Beck–Chevalley . . . . .	81
	Gluing squares . . . . .	82
<b>III</b>	<b>Stratified higher topos theory</b>	<b>84</b>
<b>9</b>	<b>Stratified higher topoi</b>	<b>84</b>
	Higher topoi attached to posets & proposets . . . . .	84
	Stratifications over posets . . . . .	85
	Toposic décollages . . . . .	87
	The nerve of a stratified $\infty$ -topos . . . . .	89
	Stratifications over spectral topological spaces . . . . .	91
	The natural stratification of a coherent $\infty$ -topos . . . . .	92
	Stratified spaces & profinite stratified spaces as stratified $\infty$ -topoi . . . . .	94
<b>10</b>	<b>Spectral higher topoi</b>	<b>96</b>
	Stone $\infty$ -topoi & oriented fibre products . . . . .	97
	Spectral $\infty$ -topoi & toposic décollages . . . . .	97
	Hochster duality for higher topoi . . . . .	99
	Constructible sheaves . . . . .	99
<b>11</b>	<b>Profinite stratified shape</b>	<b>104</b>
	The definition of the profinite stratified shape . . . . .	104
	Points & materialisation . . . . .	105
	Stratified homotopy types via décollages . . . . .	107
	Van Kampen theorem . . . . .	109
<b>IV</b>	<b>Stratified étale homotopy theory</b>	<b>110</b>
<b>12</b>	<b>Aide-mémoire on the étale homotopy type</b>	<b>110</b>
	Definition . . . . .	110
	Examples . . . . .	111

<b>13 Stratified étale homotopy types</b>	<b>111</b>
Definition . . . . .	111
Stratified Riemann Existence Theorem . . . . .	114
<b>14 Topologically rigid schemes &amp; reconstruction of absolute schemes</b>	<b>116</b>
Topological rigidity & seminormality . . . . .	117
Topological rigidification . . . . .	118
Grothendieck's conjecture . . . . .	119
 <b>References</b>	 <b>121</b>

## Introduction

Let  $X$  be a scheme with underlying Zariski topological space  $X^{zar}$ . Consider the following category  $\text{Gal}(X)$ .

- ▶ An object is a *geometric point*  $x \rightarrow X$ , by which we mean a point whose residue field  $\kappa(x)$  is a separable closure of the residue field  $\kappa(x_0)$  of the image  $x_0 \in X^{zar}$  of  $x$ .
- ▶ For two geometric points  $x \rightarrow X$  and  $y \rightarrow X$ , a morphism  $x \rightarrow y$  is a *specialisation*  $x \leftarrow y$  – that is, a geometric point  $y \rightarrow X_{(x)}$  of the strict localisation  $X_{(x)}$  lying over  $y \rightarrow X$ .

Specialisations  $x \leftarrow y$  and  $y \leftarrow z$  compose to give a specialisation  $x \leftarrow z$ .

The category  $\text{Gal}(X)$  is a kind of categorification of the absolute Galois group. The assignment  $x \mapsto x_0$  is a conservative functor to the *specialisation poset* of  $X^{zar}$  – that is, the poset of points in which  $x_0 \leq y_0$  if and only if  $x_0$  lies in the closure of  $y_0$ . The fibre over a point  $x_0$  is  $BG_{\kappa(x_0)}$ , where  $G_{\kappa(x_0)}$  is the absolute Galois group of  $\kappa(x_0)$ . If  $X$  is normal, then the space of sections over a map  $x_0 \leq y_0$  is  $BD_{x_0 \leq y_0}$ , where  $D_{x_0 \leq y_0}$  is the decomposition group of  $x_0$  in the closure of  $y_0$ .

As with absolute Galois groups, there is a natural topology on the set of morphisms of  $\text{Gal}(X)$ , which is generated as follows. For any point  $u \rightarrow X$  that is finite over its image  $u_0 \in X^{zar}$ , we form the unramified extension  $A$  of the henselisation  $O_{X,u_0}^h$  with residue field the separable closure of  $\kappa(u_0)$  in  $\kappa(u)$ , and we write  $X_{(u)} := \text{Spec } A$ . Now if  $v \rightarrow X$  is finite over its image  $v_0 \in X^{zar}$ , then a *specialisation*  $u \leftarrow v$  is a point  $v \rightarrow X_{(u)}$  of  $X_{(u)}$  lying over  $v \rightarrow X$ . For any such specialisation  $u \leftarrow v$ , we define the subset  $U(u \leftarrow v)$  of the set of morphisms of  $\text{Gal}(X)$  consisting of those specialisations  $x \leftarrow y$  that lie over  $u \leftarrow v$ . We endow the morphisms of  $\text{Gal}(X)$  with the topology generated by the sets  $U(u \leftarrow v)$ . With this topology,  $\text{Gal}(X)$  becomes a *topological category*.

**A Theorem** (see [Theorem 14.16](#)). *Let  $k$  be a finitely generated field of characteristic zero. Then the assignment  $X \mapsto \text{Gal}(X)$  is fully faithful as a functor from reduced, normal  $k$ -schemes of finite type to topological categories with a continuous action of the absolute Galois group  $G_k$ . That is, one obtains bijections*

$$\text{Mor}_k(X, Y) \simeq \text{Fun}_{cts}^{G_k}(\text{Gal}(X), \text{Gal}(Y))$$

for any reduced, normal  $k$ -schemes  $X$  and  $Y$  of finite type.

As a formal consequence, we deduce that a morphism  $f: X \rightarrow Y$  of reduced normal  $k$ -schemes of finite type is an isomorphism if and only if  $f$  induces an equivalence  $\text{Gal}(X) \rightarrow \text{Gal}(Y)$  of ordinary categories (*without* a topology). This theorem is a categorical version of the *Anabelian Conjecture* of Alexander Grothendieck: in effect, it states that purely Galois-theoretic information can be assembled to give a complete invariant of normal varieties over  $k$ .

The category  $\text{Gal}(X)$  is in effect an *étale exit-path category*. Bob MacPherson introduced the exit-path categories of stratified topological spaces to classify constructible sheaves in what we call the *exodromy equivalence*. Accordingly, our proof of [Theorem A](#) involves the development of a *stratification* of the *étale homotopy type* and the new theory of *exodromy* in the étale context.

## Monodromy for topological spaces

It is a truth universally acknowledged, that a local system of  $\mathbf{C}$ -vector spaces on a connected topological manifold  $X$  is completely determined by its attached *monodromy representation*, so that the choice of a point  $x \in X$  specifies an equivalence of categories

$$M_x : \mathbf{LS}(X, \mathbf{Vect}(\mathbf{C})) \simeq \mathbf{Rep}_{\mathbf{C}}(\pi_1(X, x)) .$$

If one wants to avoid selecting a point, or if one wants to drop the connectivity hypothesis on  $X$ , then one may combine the set of connected components and the various fundamental groups of  $X$  to form the *fundamental groupoid*  $\Pi_1(X)$ . Then the monodromy equivalence becomes

$$M : \mathbf{LS}(X, \mathbf{Vect}(\mathbf{C})) \simeq \mathbf{Fun}(\Pi_1(X), \mathbf{Vect}(\mathbf{C})) .$$

An early insight of Dan Kan was that in a similar fashion, *all* the homotopy groups and all the  $k$  invariants of  $X$  could, in effect, be combined to form a single combinatorial gadget – a simplicial set  $\Pi_{\infty}(X)$  called the *singular simplicial set* or, in contemporary parlance, the *fundamental  $\infty$ -groupoid* of  $X$  – which knows everything about the homotopy type of  $X$ .

Perhaps the clearest formulation of this insight was that of Dan Quillen, who showed that the category  $\mathbf{TSpc}$  of topological spaces and the category  $\mathbf{sSet}$  of simplicial sets each admit model structures – each with the conventional choice of weak equivalence – relative to which the functor

$$\Pi_{\infty} : \mathbf{TSpc} \rightarrow \mathbf{sSet}$$

is a right Quillen equivalence. Nowadays we go a step farther and think of  $\Pi_{\infty}$  as an equivalence  $\mathbf{S} \simeq \mathbf{Gpd}_{\infty}$  between the underlying  $\infty$ -category of spaces and that of  $\infty$ -groupoids.

This fundamental  $\infty$ -groupoid of  $X$  appears in derived versions of the monodromy equivalence: for instance, the monodromy of a local system of *complexes* of  $\mathbf{C}$ -vector spaces is a functor from  $\Pi_{\infty}(X)$  to complexes, and this induces an equivalence of  $\infty$ -categories

$$M : \mathbf{LS}(X, \mathbf{Cplx}(\mathbf{C})) \simeq \mathbf{Fun}(\Pi_{\infty}(X), \mathbf{Cplx}(\mathbf{C})) .$$

All of these equivalences follow from the ur-example of local systems of *spaces* on  $X$ , which are known as *parametrised homotopy types* in the homotopy theory literature [35]. These form an  $\infty$ -category  $\mathbf{LS}(X)$ , and there is a natural monodromy equivalence of  $\infty$ -categories

$$M : \mathbf{LS}(X) \simeq \tilde{\Pi}_{\infty}(X) := \mathbf{Fun}(\Pi_{\infty}(X), \mathbf{S}) .$$

## Monodromy for schemes

To replace the manifold in this story with a scheme, Grothendieck identified étale local systems on a suitable connected scheme  $X$  with representations of its étale fundamental group. Here it is not the Zariski topological space of  $X$  that is germane but its *étale topos*, and one obtains not a group but a progroup: the *extended étale fundamental group*  $\pi_1^{\text{ét}, \text{ext}}(X)$  – or, if preferred, its profinite reflection: the usual *étale fundamental group*  $\pi_1^{\text{ét}}(X)$ .

The étale fundamental group is an information-dense invariant, and Grothendieck's *Anabelian Conjectures* are roughly an investigation of the extent to which it is a *complete* invariant for certain classes of schemes. In dimension 0, the classical theorem of Neukirch and Uchida [39; 40; 51] ensures that two number fields are isomorphic if and only if their absolute Galois groups are. In dimension 1, Akio Tamagawa [48] and Shinichi Mochizuki [36] show that dominant morphisms between smooth hyperbolic curves over suitable fields of characteristic zero can be detected at the level of fundamental groups. Florian Pop [42, Theorem 1] shows that an isomorphism between two function fields over finitely generated fields can be detected at the level of Galois groups.

Eduardo Dubuc [13, §§5–6] generalised the étale fundamental group by extracting from a topos  $\mathbf{X}$  a *fundamental progroupoid*  $\Pi_1(\mathbf{X})$  and a monodromy equivalence

$$\mathbf{X}^{\text{locsys}} \simeq \text{Fun}(\Pi_1(\mathbf{X}), \mathbf{Set})$$

between the local systems of sets on  $\mathbf{X}$  and  $\mathbf{Set}$ -valued functors on the  $\Pi_1(\mathbf{X})$  (in the ‘pro’ sense). Following this, from an  $\infty$ -topos  $\mathbf{X}$ , Jacob Lurie extracted a fundamental  $\infty$ -groupoid  $\Pi_\infty(\mathbf{X})$  whose representations are monodromy representations. The caveat is again that one is forced to contend with proobjects:  $\Pi_\infty(\mathbf{X})$  is most naturally a prospace, called the *shape* of  $\mathbf{X}$ , and its profinite completion is the *homotopy type*  $\Pi_\infty^\wedge(\mathbf{X})$  of  $\mathbf{X}$ . Lurie shows [SAG, Theorem E.2.3.2] that for any  $\infty$ -topos  $\mathbf{X}$ , one has a natural monodromy equivalence of  $\infty$ -categories

$$\mathbf{X}^{\text{lisse}} \simeq \text{Fun}(\Pi_\infty^\wedge(\mathbf{X}), \mathcal{S}_\pi)$$

between the *lisse* sheaves on  $\mathbf{X}$  – i.e., locally constant sheaves of  $\pi$ -finite spaces on  $\mathbf{X}$  – and functors on  $\Pi_\infty^\wedge(\mathbf{X})$  valued in the  $\infty$ -category  $\mathcal{S}_\pi$  of  $\pi$ -finite spaces. This monodromy equivalence is a form of *galoisian duality*. At the most abstract level, this duality arises from the fully faithful inclusion  $\mathcal{S}_\pi \hookrightarrow \mathbf{Top}_\infty$  given by  $\Pi \mapsto \widetilde{\Pi} := \text{Fun}(\Pi, \mathbf{S})$  and its proëxistent left adjoint. Marc Hoyois showed that if  $X_{\text{ét}}$  is the (1-localic) étale  $\infty$ -topos of a locally noetherian scheme  $X$ , then the profinite space  $\Pi_\infty^\wedge(X_{\text{ét}})$  coincides with the *étale homotopy type*  $\Pi_\infty^{\text{ét},\wedge}(X)$  of Mike Artin and Barry Mazur [25, Corollary 5.6].

If the étale fundamental group  $\pi_1^{\text{ét}}$  is information-dense, then the étale homotopy type  $\Pi_\infty^{\text{ét},\wedge}$  must be even more so. Indeed, Alexander Schmidt and Jacob Stix [46, Theorem 1.2] show that over a finitely generated field  $k$  of characteristic 0, if  $X$  and  $Y$  are smooth, geometrically connected varieties that can be embedded as locally closed subschemes of a product of hyperbolic curves, then the map

$$\text{Isom}_k(X, Y) \rightarrow \text{Isom}_{BG_k}(\Pi_\infty^{\text{ét},\wedge}(X), \Pi_\infty^{\text{ét},\wedge}(Y))$$

is a split injection with a natural retraction, where  $\text{Isom}_{BG_k}$  denotes the set of homotopy classes of equivalences of profinite spaces over  $BG_k$ .

## Exodromy for topological spaces

A string of results has suggested the possibility that *stratified spaces* and *constructible sheaves* might be modelled in a similarly combinatorial fashion. Bob MacPherson proved that constructible sheaves of sets on a (suitably nice) stratified topological space  $X$  over a

poset  $P$  determine and are determined by a functor from the *exit-path category*  $\Pi_{(1,1)}(X; P)$  of  $X$ , whose objects are points of  $X$  and whose morphisms are stratified homotopy equivalence classes of *exit paths* – paths from a stratum  $X_p$  to a stratum  $X_q$  for  $q \geq p$ . We call this equivalence

$$E^P : \mathbf{Sh}_{\leq 0}(X)^{P\text{-constr}} \simeq \text{Fun}(\Pi_{(1,1)}(X; P), \mathbf{Set})$$

between  $P$ -constructible sheaves of sets on  $X$  and functors  $\Pi_{(1,1)}(X; P) \rightarrow \mathbf{Set}$  the *exodromy* equivalence.<sup>1</sup> One notes that  $\Pi_{(1,1)}(X; P)$  is a category with a conservative functor to  $P$  itself. Over each point  $p \in P$ , the fibre of this functor over  $p$  is the fundamental groupoid  $\Pi_1(X_p)$  of the stratum  $X_p$ .

David Treumann [50] then extended MacPherson’s result to give an exodromy equivalence between constructible *stacks* with functors from an *exit-path 2-category* of  $X$  valued in groupoids. Lurie [HA, Appendix A] extended this further to give an exodromy equivalence

$$E^P : \mathbf{Sh}(X)^{P\text{-constr}} \simeq \widetilde{\Pi}_{(\infty,1)}(X; P) := \text{Fun}(\Pi_{(\infty,1)}(X; P), \mathbf{S})$$

between  $P$ -constructible sheaves of spaces on  $X$  and functors from an *exit-path  $\infty$ -category*  $\Pi_{(\infty,1)}(X; P)$ . The objects are points of  $X$ , the morphisms are exit-paths, the 2-morphisms are stratified homotopies, the 3-morphisms are stratified homotopies of homotopies, etc., etc., *ad infinitum*. One notes that  $\Pi_{(\infty,1)}(X; P)$  is an  $\infty$ -category with a conservative functor to  $P$  itself. Over each point  $p \in P$ , the fibre of this functor is the fundamental  $\infty$ -groupoid  $\Pi_{\infty}(X_p)$  of the stratum  $X_p$ .

One is led to seek an analogue of the Kan–Quillen theorem that states that the formation of the exit-path category is an equivalence of suitable homotopy theories between stratified spaces and suitable  $\infty$ -categories. A geometric form of this result was proved by David Ayala, John Francis, and Nick Rozenblyum [4], who showed that the exit-path  $\infty$ -category construction is fully faithful from a homotopy theory of *conically smooth* stratified spaces to  $\infty$ -categories.

A still closer stratified analogue of the Kan–Quillen equivalence has now been provided by the simultaneous, work of three authors: Sylvain Douteau [12], Stephen Nand-Lal and Jon Woolf [38], and the third-named author [20]. These papers each take a slightly different point of view, but for our purposes here, the salient point is this: the functor  $\Pi_{(\infty,1)}(-; P)$  is an equivalence between the following homotopy theories:

- ▶ topological spaces with a stratification over  $P$  – in which a weak equivalence of such is a weak equivalence on strata and (homotopy) links – and
- ▶  $\infty$ -categories with a conservative functor to  $P$ .

We are thus entitled to refer to  $\infty$ -categories with a conservative functor to a poset  $P$  as  *$P$ -stratified spaces*. This makes it possible to port some of the ideas of stratified homotopy theory to the study of schemes. Importantly, if  $S$  is a spectral topological space (i.e., the underlying Zariski topological space  $X^{\text{zar}}$  of a coherent scheme  $X$ , or equivalently a profinite poset), then we are able to extend this description to define the homotopy theory of  *$S$ -stratified spaces*.

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<sup>1</sup>Ξξω: outer; δρόμος: avenue



## Exodromy for schemes

In the present paper, we define *P-stratified*  $\infty$ -topoi and more generally *S-stratified*  $\infty$ -topoi, and we study the *constructible sheaves* therein. For any S-stratified space  $\Pi$ , the  $\infty$ -topos  $\widetilde{\Pi} := \text{Fun}(\Pi, \mathbf{S})$  admits a natural S-stratification. This defines a functor  $\mathbf{Str}_S \rightarrow \mathbf{StrTop}_{\infty, S}^\wedge$ . Restricting to profinite stratified spaces, we obtain a fully faithful functor  $\mathbf{Str}_{\pi, S}^\wedge \hookrightarrow \mathbf{StrTop}_{\infty, S}^\wedge$  and its left adjoint  $\Pi_{(\infty, 1)}^{S, \wedge}$ .

**B Theorem (Theorem 11.7).** *For any S-stratified  $\infty$ -topos  $\mathbf{X}$ , the counit  $\widetilde{\Pi}_{(\infty, 1)}^{S, \wedge}(\mathbf{X}) \rightarrow \mathbf{X}$  of the adjunction to profinite stratified spaces restricts to an equivalence*

$$\text{Fun}(\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X}), \mathbf{S}_\pi) \simeq \mathbf{X}^{S\text{-constr}}$$

between the  $\infty$ -category of functors valued in  $\pi$ -finite spaces and S-constructible sheaves  $\mathbf{X}$ . We call this identification the *exodromy equivalence for stratified  $\infty$ -topoi*.

We call the profinite  $\infty$ -category  $\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})$  the *S-stratified homotopy type of  $\mathbf{X}$* . This is a *refinement* of the usual homotopy type of  $\mathbf{X}$ : the classifying profinite space of  $\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X})$  is precisely  $\Pi_\infty^\wedge(\mathbf{X})$ .

Profinite stratified spaces admit Postnikov towers  $\Pi \rightarrow \dots \rightarrow h_2\Pi \rightarrow h_1\Pi \rightarrow h_0\Pi$ ; thus an S-stratified  $\infty$ -topos  $\mathbf{X}$  has attached fundamental profinite  $(n, 1)$ -categories

$$\Pi_{(n, 1)}^{S, \wedge}(\mathbf{X}) := h_n\Pi_{(\infty, 1)}^{S, \wedge}(\mathbf{X}).$$

Our interest in these refinements arose primarily due to the following example.

**C Example.** If  $X$  is a coherent scheme, then we have the 1-localic  $\infty$ -topos  $X_{\acute{e}t}$ , which admits a natural  $X^{\text{zar}}$ -stratification, and so we obtain the profinite  $\infty$ -category

$$\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X) := \Pi_{(\infty, 1)}^{X^{\text{zar}}, \wedge}(X_{\acute{e}t}),$$

which we call the *stratified étale homotopy type of  $X$* .

For a finite ring  $\Lambda$ , the exodromy equivalence yields in particular

$$\text{Fun}(\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X), \mathbf{Perf}(\Lambda)) \simeq \mathbf{D}_{\text{constr}}^b(X; \Lambda).$$

Passing to suitable limits, we find that  $\ell$ -adic constructible sheaves on  $X$  ‘are’  $\ell$ -adic representations of the stratified étale homotopy type of  $X$ , in just the same way as  $\ell$ -adic local systems on  $X$  ‘are’  $\ell$ -adic representations of the étale homotopy type of  $X$ .

An important point is that the stratified étale homotopy type turns out to be 1-truncated, so that  $\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X) = \Pi_{(1, 1)}^{\acute{e}t, \wedge}(X)$ . For stratified 1-types, we are able to identify them with 1-categories equipped with a suitable topology. Under this correspondence, the stratified étale homotopy type agrees with the topological category  $\text{Gal}(X)$  of points of  $X$  that we introduced just before the statement of **Theorem A**.

## Hochster Duality for higher topoi

The main novel step in our proof of [Theorem A](#) is that the whole étale  $\infty$ -topos of any coherent scheme can be completely recovered from the stratified étale homotopy type. This is a generalisation of what we call *Hochster Duality*.

Melvin Hochster's thesis [22; 23] identifies the category of profinite posets with the category of *spectral topological spaces* – those topological spaces that underlie coherent schemes. This functions as a simultaneous generalisation of Alexandroff Duality (which identifies finite posets with finite topological spaces) and Stone Duality (which identifies profinite sets with quasicompact and totally separated topological spaces).

Lurie has already extended Stone Duality to the context of higher topoi: he proves that the functor that carries a profinite space  $\mathbb{I}$  to the  $\infty$ -topos  $\widetilde{\mathbb{I}}$  is fully faithful, and its essential image consists of bounded coherent  $\infty$ -topoi in which the truncated coherent objects coincide with the *lisse* sheaves. We call these  $\infty$ -topoi *Stone  $\infty$ -topoi*. (Lurie calls them *profinite  $\infty$ -topoi*.)

In this paper, we prove the following:

**D Theorem** ( $\infty$ -Categorical Hochster Duality; [Theorem 10.10](#)). *The assignment that carries a profinite stratified space  $\mathbb{I}$  to the  $\infty$ -topos  $\widetilde{\mathbb{I}}$  is fully faithful, and its essential image consists of bounded coherent  $\infty$ -topoi in which the truncated coherent objects coincide with the constructible sheaves.*

We call these  $\infty$ -topoi *spectral  $\infty$ -topoi* ([Definition 10.3](#)). This is partially justified by the fact that they are the natural higher categorical extension of Hochster's spectral topological spaces. Better still, we have the following.

**E Example.** Let  $X$  be a coherent scheme. Then the étale  $\infty$ -topos  $X_{\acute{e}t}$  is spectral.

Thus the étale  $\infty$ -topos of a coherent scheme is of the form  $\widetilde{\mathbb{I}}$  for some profinite  $\infty$ -category  $\mathbb{I}$ , which turns out in this case to be a 1-category.

Since one may identify the constructible sheaves on  $X$  with the truncated and coherent objects of  $X_{\acute{e}t}$ , we deduce that in fact  $X_{\acute{e}t}$  is equivalent to the  $\infty$ -topos  $\widetilde{\mathbb{I}}_{(\infty,1)}^{\acute{e}t,\wedge}(X)$ . In other words, the stratified étale homotopy type of  $X$  recovers the entire étale  $\infty$ -topos attached to  $X$ .

Armed with this, [Theorem A](#) follows as soon as we know that our schemes can be recovered from their étale  $\infty$ -topoi. On this score, in his letter to Gerd Faltings, Grothendieck conjectured – and Vladimir Voevodsky proved [52] – that the assignment  $X \mapsto X_{\acute{e}t}$  is a fully faithful functor from reduced, normal schemes of finite type over a finitely generated field  $k$  of characteristic 0 to  $\infty$ -topoi with an action of the absolute Galois group  $G_k$ . Combined with our results on the profinite stratified shape, we obtain our [Theorem A](#).

In effect, whereas the étale homotopy type of a scheme can only be hoped to be a complete invariant only for certain varieties constructed iteratively from hyperbolic curves, the addition of the natural stratification on the étale homotopy type turns it into a complete invariant for *all* varieties. The stratified étale homotopy type identifies reduced normal schemes over  $k$  with a full subcategory of the  $\infty$ -category of profinite  $\infty$ -categories with an action of  $G_k$ .

In characteristic  $p$  and for more general arithmetic schemes, the presence of inseparable extensions forces us to give a more careful formulation of Grothendieck's conjecture ([Conjecture 14.13](#)), and both it and the analogue of [Theorem A](#) remain open.

## Stratified Riemann Existence

If  $X$  is a  $\mathbf{C}$ -scheme of finite type, then the Riemann Existence Theorem amounts to an equivalence between the étale homotopy type  $\Pi_{\infty}^{\text{ét},\wedge}(X)$  and the profinite completion  $\Pi_{\infty}^{\wedge}(X^{\text{an}})$  of the homotopy type of the topological space  $X^{\text{an}}$  of complex points of  $X$  with its analytic topology [[3](#), [Theorem 12.9](#); [8](#), [Proposition 4.12](#)]. In the same vein, the *stratified* Riemann Existence Theorem provides the following.

**F Theorem** (Stratified Riemann Existence; [Proposition 13.15](#)). *Let  $X$  be a  $\mathbf{C}$ -scheme of finite type, and  $X \rightarrow P$  a finite constructible stratification. Then there is a natural equivalence*

$$\Pi_{\infty}^{\text{ét},\wedge}(X; P) \simeq \Pi_{\infty}^{\wedge}(X^{\text{an}}; P).$$

## Technical overview

The first three parts of this paper reflect the three ingredients necessary to construct the stratified étale homotopy type and to prove the central Hochster Duality Theorem for higher categories ([Theorem D=Theorem 10.10](#)). The last part is then focused applying this machinery to the étale  $\infty$ -topoi of schemes.

The first ingredient is a small (and quite elementary) piece of abstract homotopy theory in the study of stratified spaces and profinite stratified spaces. Most of this work is relatively formal, but one important notion is that of a *spatial décollage*, which is a presheaf on the subdivision of a poset satisfying a Segal condition. We prove that the  $\infty$ -category of stratified spaces is equivalent to that of spatial décollages via a nerve construction. The upshot is that a stratified space can be recovered from its ‘unglued’ form<sup>2</sup> – a collection of strata and links, suitably organised.

On the toposic side, one wants to be able to perform the same ungluing procedure, so that one can recover an  $\infty$ -topos  $\mathbf{X}$  from the data of a closed subtopos  $\mathbf{Z}$ , its open complement  $\mathbf{U}$ , and the gluing information in the form of the *deleted tubular neighbourhood*  $\mathbf{W}$  of  $\mathbf{Z}$  in  $\mathbf{U}$ . This is the second major ingredient – *gluing squares* of  $\infty$ -topoi, which are certain squares

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{q_*} & \mathbf{U} \\ p_* \downarrow & \swarrow \sigma & \downarrow j_* \\ \mathbf{Z} & \xrightarrow{i_*} & \mathbf{X} \end{array}$$

of geometric morphisms with a noninvertible natural transformation  $\sigma$ . In order to make sense of this, there are three nontrivial tasks:

- We must work – systematically and *ab initio* – with *bounded coherent*  $\infty$ -topoi. This involves some care, particularly as these conditions are not stable under the formation of recollements.

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<sup>2</sup>Whence the term ‘décollage’.

- ▶ We must develop the higher categorical analogue of Pierre Deligne’s *oriented fibre product* [27; 32; 41]. The *tubular neighbourhood* of  $Z$  in  $X$  is the *evanescent*  $\infty$ -topos  $Z \tilde{\times}_X X$ , and the *deleted tubular neighbourhood*  $W$  is then the open subtopos  $Z \tilde{\times}_X U \subseteq Z \tilde{\times}_X X$ .
- ▶ Finally, and most crucially, we must prove a rather delicate *Beck–Chevalley Theorem*, which ensures that the two gluing functors  $i^* j_*$  and  $p_* q^*$  agree, at least on truncated objects.

We define *stratified*  $\infty$ -topoi in a manner completely analogous to our definition of stratified topological spaces, but our study of gluing squares now permits us to prove that the  $\infty$ -category of *bounded coherent* stratified  $\infty$ -topoi are equivalent to an  $\infty$ -category of *toposic décollages* – i.e., presheaves of  $\infty$ -topoi on the subdivision of a poset that satisfy a kind of *oriented Segal condition*. This condition ensures that a string  $\{p_0 \leq \dots \leq p_n\}$  is carried to an iterated oriented fibre product  $X_{p_0} \tilde{\times}_X \dots \tilde{\times}_X X_{p_n}$  of the strata. We may also pass to profinite objects in the base, which permits us to contemplate stratified  $\infty$ -topoi over spectral topological spaces.

Among the bounded coherent stratified  $\infty$ -topoi are those in which the strata are Stone  $\infty$ -topoi. These are the *spectral*  $\infty$ -topoi. They turn out to agree with those bounded coherent stratified  $\infty$ -topoi in which the truncated coherent objects are exactly the *constructible sheaves* – i.e., those sheaves that restrict to a lisse sheaf on any stratum. If  $\mathbf{II}$  is a profinite stratified space, then the stratified  $\infty$ -topos  $\tilde{\mathbf{II}}$  is spectral in this sense. As in Lurie’s  $\infty$ -Categorical Stone Duality, there is a left adjoint to the functor  $\mathbf{II} \mapsto \tilde{\mathbf{II}}$ , which carries a stratified  $\infty$ -topos to its *stratified homotopy type*.

Now the  $\infty$ -Categorical Hochster Duality Theorem – which provides an equivalence between spectral  $\infty$ -topoi with profinite stratified spaces – follows from a sequence of three moves:

- ▶ We reduce to the case of a finite poset  $P$ . This is formal.
- ▶ We then show that the stratified homotopy type of a spectral  $\infty$ -topos can be computed by ungluing to the toposic décollage, forming the homotopy type objectwise to get a spatial décollage, and then regluing to a profinite stratified space.
- ▶ We then appeal to Lurie’s  $\infty$ -Categorical Stone Duality Theorem.

## Open problems

There are a number of questions we have not answered in this paper. Here are just a few.

**Question.** Our work here leaves [Conjecture 14.13](#) frustratingly open. In effect, it predicts that a large class of *absolute schemes*  $X$  (see [Definition 14.12](#)) can be reconstructed from  $\text{Gal}(X)$ .

**Question.** We may ask whether one can recover an absolute scheme  $X$  from the profinite stratified space at a finite stage. That is, is there a finite constructible stratification  $X \rightarrow P$  such that for any absolute scheme  $Y$ , the map

$$\text{Mor}_k(X, Y) \simeq \text{Map}_{BG_k}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(Y), \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X)) \rightarrow \pi_0 \text{Map}_{BG_k}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(Y), \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; P))$$

is a bijection? (One might expect that it suffices to choose stratification in which the strata in  $X$  are strongly hyperbolic Artin neighbourhoods; at this point, we do not know.)

**Question.** If  $X$  is a topologically noetherian scheme and  $E$  is a local field or  $\overline{\mathbf{Q}}_l$ , then our result is insufficient to recover the full derived  $\infty$ -category of constructible sheaves of  $E$ -complexes on  $X$ . To do this, one should instead pass to the *proétale*  $\infty$ -topos of Bhargav Bhatt and Peter Scholze [6], and the profinite completion of the stratification will not suffice – one will need infinitary information in the *tame stratified shape*. What is this tame shape?

## Acknowledgements

The Université Montpellier has recently released a collection of notes of Alexander Grothendieck, including ‘Cote n° 151: Espaces stratifiés’, in which he develops some elements of stratified topos theory and some elements of an attached shape theory, to which he referred in his *Esquisse d’un Programme* [19, p. 36]. It is not clear to us how much of the work here he anticipated.

We have used the framework and results in Jacob Lurie’s three big books [HTT], [HA], and [SAG] everywhere here. The impact of his ideas here is obvious and extensive. We are also grateful to him for his very helpful answers to a number of technical questions we pelted him with over the course of this project.

Much of our understanding of stratified spaces is directly the result of our conversations with David Ayala, who independently developed the ‘spatial décollage’ perspective on stratified spaces. We are exceedingly grateful to him for sharing with us a portion of these ideas. Ayala’s thinking has had a tremendous influence on us, and conversations with him in the first phase of this project have been vital to our work. Additionally, he has read some early editions of this work, spotted some mistakes, and helped us improve our writing.

Dustin Clausen and Jacob Lurie have each developed a portion of the topos-theoretic material here. In particular, it seems that they were each already aware of the  $\infty$ -Categorical Hochster Duality Theorem. We are grateful to them both for sharing their insights on this matter.

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## o Terminology & notations

### Set theoretic conventions

o.1. Recall that if  $\kappa$  is a strongly inaccessible cardinal (which we always assume to be uncountable), then the set  $V_\kappa$  of all sets of rank strictly less than  $\kappa$  is a Grothendieck

universe [SGA<sub>4</sub> I, Exposé I, Appendix] of rank and cardinality  $\kappa$ . Conversely, if  $V$  is a Grothendieck universe that contains an infinite cardinal, then  $V = V_\kappa$  for some inaccessible cardinal  $\kappa$ . A set, group, simplicial set, category, etc., will be said to be  $\kappa$ -small if it lies in  $V_\kappa$ . We say that a set, group, simplicial set, category, etc., is *essentially  $\kappa$ -small* if it is equivalent (in the appropriate sense) to a  $\kappa$ -small one.

In order to deal honestly with set-theoretic problems arising from the consideration of ‘large’ collections (like that of all  $\infty$ -topoi), we append to ZFC the Axiom of Universes (AU). This asserts that any cardinal is dominated by a strongly inaccessible cardinal. We won’t use the full strength of this axiom, however; in particular, we shall fix two strongly inaccessible cardinals  $\kappa_0 < \kappa_1$  in this paper, whose existence would suffice for our work here.

o.2. We write  $N$  for the poset of *nonnegative integers*. We write  $N^* := N \setminus \{0\}$ , and  $N^\infty := N \cup \{\infty\}$ .

## Higher categories

o.3. We use the language and tools of higher category theory, particularly in the model of *quasicategories*, as defined by Michael Boardman and Rainer Vogt and developed by André Joyal and Lurie. We will generally follow the terminological and notational conventions of Lurie’s trilogy [HTT; HA; SAG]:

- ▶ An  $\infty$ -category here will always mean *quasicategory*.
- ▶ A *subcategory*  $C'$  of an  $\infty$ -category  $C$  is a simplicial subset that is stable under composition in the strong sense, so that if  $\sigma : \Delta^n \rightarrow C$  is an  $n$ -simplex of  $C$ , then  $\sigma$  factors through  $C' \subseteq C$  if and only if each of the edges  $\sigma(\Delta^{\{i,i+1\}})$  does so.
- ▶ An  $n$ -category here means a quasicategory with unique inner horn fillers in dimensions strictly greater than  $n$ .
- ▶ Accessibility of  $\infty$ -categories and functors and presentability of  $\infty$ -categories will always refer to accessibility and presentability with respect to some  $\kappa_0$ -small cardinal. Please observe that an accessible  $\infty$ -category is always essentially  $\kappa_1$ -small and locally  $\kappa_0$ -small.

We also employ some slight variations from Lurie’s conventions:

- ▶ If  $C$  is a 1-category, we shall abuse notation and write  $C$  also for its nerve, which is an  $\infty$ -category.
- ▶ We will use the terms  *$\infty$ -groupoid* or *space* interchangeably for an  $\infty$ -category in which every morphism is invertible. If  $C$  is an  $\infty$ -category, the largest  $\infty$ -groupoid  $\iota C \subseteq C$  contained in  $C$  will be called the *interior* of  $C$ .
- ▶ For a strongly inaccessible cardinal  $\kappa$ , we shall write  $S_\kappa$  for the  $\infty$ -category of  $\kappa$ -small  $\infty$ -groupoids or spaces and  $\text{Cat}_{\infty,\kappa}$  for the  $\infty$ -category of  $\kappa$ -small  $\infty$ -categories. Relative to our fixed choices of inaccessible cardinals, we shall write  $S := S_{\kappa_0}$  and  $\text{Cat}_\infty := \text{Cat}_{\infty,\kappa_0}$ .

- We use the term *isofibration* in place of the term *categorical fibration*.
- Let  $C$  be an  $\infty$ -category and  $W \subseteq C_1$  a set of morphisms of  $C$ . Then we write  $W^{-1}C$  for the result of inverting the morphisms of  $W$ . If  $\kappa$  is an inaccessible cardinal for which  $C$  is  $\kappa$ -small, then  $W^{-1}C$  is  $\kappa$ -small as well. This  $\infty$ -category comes equipped with a functor  $C \rightarrow W^{-1}C$  that, for any  $\infty$ -category  $D$ , induces a fully faithful functor

$$\mathrm{Fun}(W^{-1}C, D) \hookrightarrow \mathrm{Fun}(C, D)$$

that identifies  $\mathrm{Fun}(W^{-1}C, D)$  with the full subcategory spanned by those functors  $C \rightarrow D$  that carry the morphisms of  $W$  to equivalences in  $D$ . One can (rather inexplicitly) describe  $W^{-1}C$  by forming the model category of ( $\kappa$ -small) marked simplicial sets (over  $\Delta^0$ ), and forming a fibrant replacement of the marked simplicial set  $(C, W)$ .

o.4. For any  $n \in \mathbb{N}^+$ , write  $\mathrm{Cat}_n \subseteq \mathrm{Cat}_\infty$  for the full subcategory spanned by those  $\infty$ -categories that are equivalent to  $n$ -categories; that is, an  $\infty$ -category  $C$  lies in  $\mathrm{Cat}_n$  if and only if for any  $x, y \in C$ , the  $\infty$ -groupoid  $\mathrm{Map}_C(x, y)$  is equivalent to an  $(n-1)$ -groupoid. In particular,  $\mathrm{Cat}_0 \simeq \mathrm{poSet}$ , the 1-category of partially ordered sets.

The inclusion  $\mathrm{Cat}_n \subseteq \mathrm{Cat}_\infty$  admits a left adjoint  $h_n$ . If  $C$  is an  $\infty$ -category, then the unit  $C \rightarrow h_n C$  exhibits  $h_n C$  as the  $n$ -categorical truncation, so that the objects of  $h_n C$  are exactly those of  $C$  and whose mapping spaces are defined by the condition that the map

$$\mathrm{Map}_C(x, y) \rightarrow \mathrm{Map}_{h_n C}(x, y)$$

exhibits  $\mathrm{Map}_{h_n C}(x, y)$  as the  $(n-1)$ -truncation of  $\mathrm{Map}_C(x, y)$ . The 1-categorical truncation  $h_1 C$  is also known as the *homotopy category* of  $C$ . The 0-categorical truncation is equivalent to the poset whose elements are the equivalence classes of objects of  $C$  in which  $x \leq y$  if and only if there exists a morphism  $x \rightarrow y$ .

## Proobjects in higher categories

o.5. We say that a  $\kappa_0$ -small  $\infty$ -category  $\Lambda$  is *inverse* if and only if its opposite  $\Lambda^{op}$  is filtered. Hence an *inverse system* in an  $\infty$ -category  $C$  is a functor  $\Lambda \rightarrow C$  from an inverse  $\infty$ -category  $\Lambda$ , and an *inverse limit* is a limit of an inverse system.

For any accessible  $\infty$ -category  $C$  that admits all finite limits, a *proobject* of  $C$  is an accessible left exact functor  $C \rightarrow \mathcal{S}$ . We define  $\mathrm{Pro}(C) \subseteq \mathrm{Fun}(C, \mathcal{S})^{op}$  for the full subcategory spanned by the proobjects. We have a Yoneda embedding

$$j: C \hookrightarrow \mathrm{Pro}(C),$$

composition along which defines an equivalence

$$\mathrm{Fun}^{inv}(\mathrm{Pro}(C), D) \simeq \mathrm{Fun}(C, D)$$

for any  $\infty$ -category  $D$  with all  $\kappa_0$ -small inverse limits, where  $\mathrm{Fun}^{inv}$  denotes the  $\infty$ -category of functors that preserve  $\kappa_0$ -small inverse limits.

Recall that an essentially  $\kappa_0$ -small  $\infty$ -category  $C$  is idempotent complete if and only if  $C$  is accessible, and every functor from  $C$  is accessible. Hence in this case, the formation of proobjects is dual to the formation of indobjects in the sense that  $\text{Pro}(C)^{op} \simeq \text{Ind}(C^{op})$ .

If  $X: \Lambda \rightarrow C$  is an inverse system, then its limit in  $\text{Pro}(C)$  is the functor

$$Y \mapsto \text{colim}_{\alpha \in \Lambda^{op}} \text{Map}_C(X_\alpha, Y),$$

we will abuse notation and denote this proobject by  $X = \{X_\alpha\}_{\alpha \in \Lambda}$ . Any proobject of  $C$  can be exhibited in this manner, and for proobjects  $X = \{X_\alpha\}_{\alpha \in \Lambda}$  and  $Y = \{Y_\beta\}_{\beta \in M}$  we obtain the familiar formula

$$\text{Map}_{\text{Pro}(C)}(X, Y) \simeq \lim_{\beta \in M} \text{colim}_{\alpha \in \Lambda^{op}} \text{Map}_C(X_\alpha, Y_\beta).$$

We will thus often speak of objects of  $\text{Pro}(C)$  as if they *were* inverse systems. In particular, a proobject  $X$  is said to be *constant* if and only if it lies in the essential image of  $j$ ; equivalently,  $X$  is constant if and only if, as a functor  $C \rightarrow \mathcal{S}$ , it preserves inverse limits.

**o.6.** Let  $\kappa \geq \kappa_0$  be an inaccessible cardinal,  $C$  a locally  $\kappa$ -small  $\infty$ -category that admits all  $\kappa_0$ -small limits,  $D$  an accessible  $\infty$ -category that admits finite limits, and  $u: D \rightarrow C$  a left exact functor. The functor  $u$  will not in general admit a left adjoint, but passage to proobjects often repairs this. Indeed, one may extend  $u$  to a (unique) functor  $U: \text{Pro}(D) \rightarrow C$  that preserves inverse limits, and in the other direction, one may consider the composite

$$F := u^* \circ j: C \rightarrow \text{Fun}(C, \mathcal{S}_\kappa)^{op} \rightarrow \text{Fun}(D, \mathcal{S}_\kappa)^{op}$$

of the Yoneda embedding  $j$  with the restriction along  $u$ . The functor  $F$  carries an object  $x \in C$  to the assignment  $y \mapsto \text{Map}_C(x, u(y))$ . We have to make two set-theoretic assumptions:

- ▶ Assume that for any object  $x \in C$  and any object  $y \in D$ , the space  $\text{Map}_C(x, u(y))$  is  $\kappa_0$ -small.
- ▶ Assume that for any object  $x \in C$ , there exists a regular cardinal  $\tau < \kappa_0$  such that for any  $\tau$ -filtered diagram  $y: A \rightarrow D$ , the natural map

$$\text{colim}_{a \in A} \text{Map}_C(x, u(y_a)) \rightarrow \text{Map}_C(x, \text{colim}_{a \in A} u(y_a))$$

is an equivalence.

In this case, the functor  $F$  lands in  $\text{Pro}(D)$ , and  $F$  is left adjoint to  $U$ . We shall call  $F$  the *proexistent left adjoint to  $u$* . If  $u$  already admits a left adjoint  $f$ , then  $F$  lands in  $D$  and coincides with  $f$ .

## Recollements

**o.7.** Given functors  $F: X \rightarrow Z$  and  $G: Y \rightarrow Z$  between  $\infty$ -categories, we write

$$X \downarrow_Z Y := X \times_{\text{Fun}(\{0\}, Z)} \text{Fun}(\Delta^1, Z) \times_{\text{Fun}(\{1\}, Z)} Y.$$

This is the *oriented fibre product* of  $\infty$ -categories.



o.8. Let  $X$  and  $Y$  be essentially  $\kappa_0$ -small  $\infty$ -categories, let  $Z$  be a locally  $\kappa_0$ -small  $\infty$ -category, and let  $F: X \rightarrow Z$  and  $G: Y \rightarrow Z$  be functors. Write  $Z' \subset Z$  for the full subcategory spanned by those objects in the image of  $F$  or the image of  $G$ . Then  $Z'$  is essentially  $\kappa_0$ -small and the oriented fibre product  $X \downarrow_Z Y$  is equivalent to  $X \downarrow_{Z'} Y$ , whence  $X \downarrow_Z Y$  is essentially  $\kappa_0$ -small.

o.9 (see [HA, §A.8]). Let  $C$  be an  $\infty$ -category that admits finite limits. Then two functors  $i_*: C_Z \rightarrow C$  and  $j_*: C_U \rightarrow C$  exhibit  $C$  as a *recollement* of  $C_Z$  and  $C_U$  if and only if the following conditions are satisfied.

- ▶ Both  $i_*$  and  $j_*$  are fully faithful.
- ▶ There are left exact left adjoints  $i^*$  and  $j^*$  to the functors  $i_*$  and  $j_*$ .
- ▶ The functor  $j^*i_*$  is constant at the terminal object of  $U$ .
- ▶ The functor  $(i^*, j^*): C \rightarrow C_Z \times C_U$  is conservative.

We refer to the  $\infty$ -category  $C_Z$  as the *closed subcategory*, the  $\infty$ -category  $C_U$  as the *open subcategory*, and the functor  $i^*j_*: C_U \rightarrow C_Z$  as the *gluing functor*.

If  $C$  is the recollement of  $\infty$ -categories  $C_Z$  and  $C_U$ , then  $C_Z$  is canonically equivalent to the *kernel* of  $j^*$  (i.e., the full subcategory spanned by those objects  $x$  such that  $j^*(x) \simeq 1_{C_U}$ ).

If  $C_Z$  and  $C_U$  are any  $\infty$ -categories with finite limits, and if  $\phi: C_U \rightarrow C_Z$  is left exact, then we write

$$C_Z \tilde{\cup}^\phi C_U := C_Z \downarrow_{C_Z} C_U.$$

The projections  $i^*: C_Z \tilde{\cup}^\phi C_U \rightarrow C_Z$  and  $j^*: C_Z \tilde{\cup}^\phi C_U \rightarrow C_U$  admit right adjoints  $i_*: C_Z \rightarrow C_Z \tilde{\cup}^\phi C_U$  and  $j_*: C_U \rightarrow C_Z \tilde{\cup}^\phi C_U$  that together exhibit  $C_Z \tilde{\cup}^\phi C_U$  as a recollement of  $C_Z$  and  $C_U$ . Furthermore, any recollement is of this form, where  $\phi$  is the gluing functor.

If  $C_Z$  contains an initial object, then  $j^*$  admits a further left adjoint  $j_!$ , so in this case we may also write  $j^! := j^*$ . If, moreover,  $C$  contains a zero object (whence so do  $C_Z$  and  $C_U$ ), then  $i_*$  admits a further right adjoint  $i^!$ , so in this case we may also write  $i_! := i_*$ .

o.10. Let  $C$  be an  $\infty$ -category with finite limits and let  $i_*: C_Z \hookrightarrow C$  and  $j_*: C_U \hookrightarrow C$  be two functors which exhibit  $C$  as a recollement of  $C_Z$  and  $C_U$ . Then for any integer  $n \geq -2$ , since the left exact functor  $(i^*, j^*): C \rightarrow C_Z \times C_U$  is conservative, a morphism  $f$  of  $C$  is  $n$ -truncated if and only if  $i^*(f)$  and  $j^*(f)$  are both  $n$ -truncated.

## Relative adjunctions

o.11. Given a commutative triangle of  $\infty$ -categories

$$\begin{array}{ccc} C & \xleftarrow{G} & D \\ & \searrow p & \swarrow q \\ & & E \end{array}$$

where  $p$  and  $q$  are isofibrations, we say that  $G$  admits a *left adjoint relative to  $E$*  if the following condition holds:

- There exists a functor  $F: C \rightarrow D$  and a natural transformation  $\eta: \text{id}_C \rightarrow GF$  which exhibits  $F$  as a left adjoint to  $G$  such that  $p\eta: p \rightarrow pGF \simeq qF$  is an equivalence in  $\text{Fun}(C, E)$ .

In this situation, given a functor  $E' \rightarrow E$ , define  $C_{E'} := C \times_E E'$ ,  $D_{E'} := D \times_E E'$ , and write  $G_{E'}: D_{E'} \rightarrow C_{E'}$  and  $F_{E'}: C_{E'} \rightarrow D_{E'}$  for the induced functors on pullbacks. Then the induced natural transformation  $\text{id}_{C_{E'}} \rightarrow G_{E'}F_{E'}$  exhibits  $F_{E'}$  as a left adjoint to  $G_{E'}$  relative to  $E'$ .

If  $p$  and  $q$  are cartesian fibrations,  $G$  admits a left adjoint relative to  $E$  if and only if the following conditions obtain:

- For every object  $e \in E$ , the induced functor  $G_e: D_e \rightarrow C_e$  admits a left adjoint.
- The functor  $G$  carries  $p$ -cartesian morphisms in  $D$  to  $q$ -cartesian morphisms in  $C$ .

In this case, if  $f: a \rightarrow b$  is a morphism of  $E$ , then one has a natural equivalence

$$f^*G_b \simeq G_a f^* .$$

Dually, if  $p$  and  $q$  are cocartesian fibrations,  $G$  admits a left adjoint relative to  $E$  if and only if the following (somewhat more complicated) conditions obtain:

- For every object  $e \in E$ , the induced functor  $G_e: D_e \rightarrow C_e$  admits a left adjoint  $F_e$ .
- Let  $c \in C$  and  $\alpha: e \rightarrow e'$  be a morphism of  $e$  where  $e \simeq p(c)$ . Let  $\tilde{\alpha}: F_e(c) \rightarrow d$  be a  $q$ -cocartesian morphism in  $D$  lying over  $\alpha$ , and let  $\beta: c \rightarrow G(d)$  be the composite  $\beta := G(\tilde{\alpha}) \circ \eta(c)$ . Choose a factorisation of  $\beta$  as

$$\beta: c \xrightarrow{\beta'} c' \xrightarrow{\beta''} G(d) ,$$

where  $\beta'$  is a  $p$ -cocartesian morphism lifting  $\alpha$  and  $\beta''$  is a morphism in  $C_{e'}$ . Then  $\beta''$  induces an equivalence  $F_{e'}(c') \rightarrow d$  in the  $\infty$ -category  $D_{e'}$ .

In this case, if  $f: a \rightarrow b$  is a morphism of  $E$ , then one has a natural equivalence

$$G_b f_! \simeq f_! G_a .$$

## Part I

# Stratified spaces

## 1 Aide-mémoire on the topology of posets & profinite posets

### Alexandroff Duality

**1.1 Definition.** If  $P$  is a preorder (which we shall always assume to be  $\kappa_0$ -small), then we endow  $P$  with the *Alexandroff topology*, in which a subset  $U \subseteq P$  is open if and only if  $U$  is a *cosieve* (i.e., if and only if, for any points  $p, q \in P$  with  $p \leq q$ , if  $p \in U$  then  $q \in U$ ), and a subset  $Z \subseteq P$  is closed if and only if  $Z$  is a *sieve* (i.e., if and only if, for any points  $p, q \in P$  with  $p \leq q$ , if  $q \in Z$  then  $p \in Z$ ). A subset  $A \subseteq P$  is locally closed if and only if  $A$  is an interval (i.e., if and only if, for any points  $p, q, r \in P$  with  $p \leq q \leq r$ , if  $p, r \in A$  then  $q \in A$ ).

In the other direction, if  $X$  is a topological space, then the preorder on  $X$  in which  $x \leq y$  if and only if  $x \in \overline{\{y\}}$  is called the *specialisation preorder*.

Alexandroff topologies admit a well-known characterisation.

**1.2 Proposition.** *The following are equivalent for a topological space  $X$ .*

- ▶ *The space  $X$  is finitely generated; that is, a subset  $U \subseteq X$  is open if, for any finite topological space  $A$  and any continuous map  $f : A \rightarrow X$ , the inverse image  $f^{-1}(U)$  is open.*
- ▶ *Any union of closed subsets of  $X$  is again closed.*
- ▶ *The topology on  $X$  coincides with the Alexandroff topology attached to the specialisation preorder on  $X$ .*

**1.3 (Alexandroff Duality).** The formation  $A$  of the Alexandroff topology and the formation  $S$  of the specialisation preorder are therefore inverse equivalences between the category of preorders and that of finitely generated topological spaces. In particular,  $A$  and  $S$  restrict to an equivalence between the category of finite preorders and that of finite topological spaces.

The functors  $A$  and  $S$  also restrict to an equivalence between:

- ▶ the category of posets and that of Kolmogoroff finitely generated topological spaces,
- ▶ the category of *noetherian* preorders (i.e., those for which every nonempty subset contains a maximal element) and that of quasi-sober finitely generated topological spaces, and thus
- ▶ the category of noetherian posets and that of sober finitely generated topological spaces.

**1.4 Notation.** Let  $P$  be a preorder. For any subset  $W \subseteq P$ , we write  $P_{\geq W}$  for the cosieve generated by  $W$ , which is the smallest open neighbourhood of  $W$ . Dually, we write  $P_{\leq W}$  for the sieve generated by  $W$ , which is the closure of  $W$ .

We call the sets of the form  $P_{\geq p}$  for  $p \in P$  the *principal open sets*, and we call the sets of the form  $P_{\leq p}$  the *principal ideals*.

Similarly, we write  $P_{> p} := P_{\geq p} \setminus \{p\}$  and  $P_{< p} := P_{\leq p} \setminus \{p\}$ .

**1.5.** A poset is quasicompact if and only if its set of minimal elements is finite and limit-cofinal. A monotone map  $f: Q \rightarrow P$  is quasicompact if and only if, for any  $p \in P$ , the poset  $f^{-1}(P_{\geq p})$  is quasicompact.

**1.6 Notation.** For a poset  $P$ , recall that  $\text{sd}(P)$  denotes the nerve of the poset of *strings* in  $P$  – i.e., finite, nonempty, totally ordered subsets  $\Sigma \subseteq P$  – ordered by inclusion. One has the natural forgetful functor  $\text{sd}(P) \rightarrow \Delta$ .

If  $\Sigma \subseteq P$  is a string, then a closed subset  $Z \subseteq \Sigma$  is again a string, and the inclusion is denoted  $i_{Z \subseteq \Sigma}$  (or simply  $i$  if  $Z$  and  $\Sigma$  are clear from the context). Dually, an open subset  $U \subseteq \Sigma$  is also a string, and the inclusion is denoted  $j_{U \subseteq \Sigma}$  (or again simply  $j$  if  $U$  and  $\Sigma$  are clear from the context).

In more general situations, we will generally write  $e_{W \subseteq \Sigma}: W \hookrightarrow \Sigma$  for an inclusion  $W \subseteq \Sigma$  that is not known to be either closed or open.

## Stratifications of topological spaces

**1.7 Definition.** A *stratification* of a topological space  $X$  is poset  $P$  and a continuous map  $f: X \rightarrow P$ . For any point  $p \in P$ , we write

$$\begin{aligned} X_{\geq p} &:= f^{-1}(P_{\geq p}), \\ X_{> p} &:= f^{-1}(P_{> p}), \\ X_{\leq p} &:= f^{-1}(P_{\leq p}), \\ X_{< p} &:= f^{-1}(P_{< p}), \\ X_p &:= X_{\geq p} \cap X_{\leq p}. \end{aligned}$$

The subspaces  $X_{\geq p}$  and  $X_{> p}$  are open in  $X$ , and  $X_{\leq p}$  and  $X_{< p}$  are closed in  $X$ . The subspace  $X_p \subseteq X$ , which is locally closed, is called the  *$p$ -th stratum*.

We say that the stratification  $f: X \rightarrow P$  is *nondegenerate* if each stratum  $X_p$  is nonempty, and for any  $p, q \in P$ , if  $p \leq q$ , then  $X_p \subseteq \overline{X_q}$ . We say that it is *connective* if it is nondegenerate, and each stratum  $X_p$  is connected.

We say that a stratification is *finite* or *noetherian* if and only if its base poset is so. We say that the stratification  $f: X \rightarrow P$  is *constructible* if and only if, for any  $p \in P$ , the open subset  $X_{\geq p} \subseteq X$  is retrocompact – i.e., its intersection with any quasicompact open  $V \subseteq X$  is again quasicompact.

## Hochster duality

The functor  $A$  can also be extended to *profinite posets* – i.e., proobjects in the category of finite posets. In order to study stratifications on schemes, this turns out to be convenient.

**1.8 Notation.** We write the *poSet* for 1-category of posets, and  $\text{poSet}^{\text{fin}}$  for the 1-category of finite posets. Passing to proobjects, we obtain the 1-category  $\text{Pro}(\text{poSet})$  of proposets

and the full subcategory  $\text{Pro}(\text{poSet}^{\text{fin}})$  of proobjects in the category of finite posets – which we call *profinite posets*.

**1.9 Definition.** For any topological space  $X$ , we write  $\text{FC}(X)$  for the 1-category of finite, nondegenerate, constructible stratifications  $X \rightarrow P$ . Please observe that  $\text{FC}(X)$  is an inverse 1-category that is (equivalent to) a poset.

A topological space  $S$  is said to be *spectral*<sup>3</sup> if and only if  $S$  is the limit of its finite, nondegenerate, constructible stratifications; that is, if and only if

$$S \simeq \lim_{P \in \text{FC}(S)} P$$

in the 1-category of topological spaces.

**1.10.** The formation of the Alexandroff topology extends to an equivalence of 1-categories  $A: \text{Pro}(\text{poSet}^{\text{fin}}) \simeq \text{TSpc}^{\text{spec}}$ , where  $\text{TSpc}^{\text{spec}}$  is the 1-category of spectral topological spaces and quasicompact continuous maps. We will therefore fail to distinguish between a spectral topological space and its corresponding profinite poset.

**1.11 Theorem** (Hochster duality [22; 23]). *The following are equivalent for a topological space  $S$ .*

- ▶ *The space  $S$  is spectral.*
- ▶ *The space  $S$  is sober, quasicompact, and quasiseparated; additionally, the set of quasicompact open subsets forms a base for the topology of  $S$ .*
- ▶ *The space  $S$  is homeomorphic to  $\text{Spec } R$  for some ring  $R$ .*
- ▶ *The space  $S$  is homeomorphic to the underlying Zariski topological space  $Y^{\text{zar}}$  of some coherent scheme  $Y$ .*

**1.12.** On one hand, Alexandroff Duality characterises posets as finitely generated topological spaces; on the other, *Stone Duality* characterises profinite sets as *Stone spaces* – totally separated quasicompact topological spaces. Hochster duality provides a common extension of each of these forms of duality. The situation is summarised in the cube

$$\begin{array}{ccc}
 & \text{poSet}^{\text{fin}} & \xrightarrow{\sim} & \text{TSpc}^{\text{fin, disc}} \\
 \swarrow & \uparrow & & \swarrow \\
 \text{Pro}(\text{poSet}^{\text{fin}}) & \xrightarrow{\sim} & \text{TSpc}^{\text{Stone}} & \\
 \downarrow & \downarrow & \downarrow & \downarrow \\
 & \text{poSet}^{\text{fin}} & \xrightarrow{\sim} & \text{TSpc}^{\text{fin}} \\
 \swarrow & \uparrow & & \swarrow \\
 \text{Pro}(\text{poSet}^{\text{fin}}) & \xrightarrow{\sim} & \text{TSpc}^{\text{spec}} & ,
 \end{array}$$

where the horizontal functors marked ‘ $\sim$ ’ are equivalences of 1-categories.

<sup>3</sup>Others call such topological spaces *coherent*; see for example [SAG, A.1; 28, Chapter III §3.4 & p. 78]. We use Hochster’s algebro-geometric terminology [22; 23].

One of our main technical results here – the  $\infty$ -Categorical Hochster Duality Theorem ([Theorem D=Theorem 10.10](#)) – will be an extension of this square of dualities to one in which the 1-category of finite sets is replaced with the  $\infty$ -category of finite spaces. Part of this extension is already established in the literature: Lurie proves [[SAG, §E.3](#)] an  $\infty$ -categorical form of Stone Duality, which identifies the  $\infty$ -category  $\mathbf{S}_\pi^\wedge$  of profinite spaces with the  $\infty$ -category of what we call *Stone  $\infty$ -topoi*.<sup>4</sup>

## Materialisation

**1.13 Definition.** The *materialisation* of posets is the essentially unique functor

$$\text{mat} : \text{Pro}(\text{poSet}) \rightarrow \text{poSet}$$

that preserves inverse limits and extends the identity functor  $\text{poSet} \rightarrow \text{poSet}$ . If  $\mathbf{P}$  is exhibited as an inverse system  $\{\mathbf{P}_\alpha\}_{\alpha \in \Lambda}$  of posets, then  $\text{mat}(\mathbf{P})$  is the limit  $\lim_{\alpha \in \Lambda} \mathbf{P}_\alpha$  computed in  $\text{poSet}$ .

**1.14 Example.** If  $S$  is a spectral topological space, then  $\text{mat}(S)$  is the specialisation poset on the set of points of  $S$ .

## Profinite stratifications

The the theory of stratifications also works well for profinite stratifications.

**1.15 Definition.** A *profinite stratification* of a topological space  $X$  is a spectral topological space  $S$  and a continuous map  $f : X \rightarrow S$ . We say that it is *constructible* if and only if, for any quasicompact open subset  $U \subseteq S$ , the inverse image  $f^{-1}(U) \subseteq X$  is retrocompact.

**1.16.** A profinite stratification with base  $S$  is the same as a compatible family of stratifications with base  $P$  for each nondegenerate, finite, constructible stratification  $S \rightarrow P$ .

## 2 Stratified spaces

### Stratified spaces as conservative functors

The equivalence between the homotopy theory of topological spaces and that of simplicial sets justifies (at least partially) the treatment of the  $\infty$ -category of Kan complexes as ‘the’ homotopy theory of spaces. Analogously, the results of Nand-Lal and Woolf [[38](#)] and the third-named author [[20](#)] furnish an equivalence between the homotopy theory of stratified topological spaces and that of  $\infty$ -categories with a conservative functor to a poset. We therefore feel entitled to give the following definition.

**2.1 Definition.** We define the  $\infty$ -category  $\mathbf{Str}$  as the full subcategory of  $\text{Fun}(\Delta^1, \text{Cat}_\infty)$  spanned by those functors  $f : \Pi \rightarrow P$  in which  $P$  is a poset and  $f$  is a conservative functor. We regard the 1-category  $\text{poSet}$  of posets (always  $\kappa_0$ -small) and monotonic maps as a full subcategory of  $\text{Cat}_\infty$ ; indeed one has  $\text{poSet} \simeq \text{Cat}_0$ .

<sup>4</sup>Lurie calls these *profinite  $\infty$ -topoi*.

The fibre  $\mathbf{Str}_P$  of the target functor  $t : \mathbf{Str} \rightarrow \mathit{poSet}$  over a poset  $P$  is the underlying  $\infty$ -category of the third-named author's Joyal–Kan model category  $s\mathbf{Set}_P$ , whose underlying  $\infty$ -category is equivalent to the  $\infty$ -category of  $P$ -stratified topological spaces [20]. Consequently, we shall call an object of  $\mathbf{Str}$  a *stratified space* and more particularly an object of  $\mathbf{Str}_P$  a  *$P$ -stratified space*.

2.2. Please observe that if  $\Pi$  and  $\Pi'$  are two  $P$ -stratified spaces, then the  $\infty$ -category  $\mathbf{Fun}_P(\Pi, \Pi')$  of functors  $\Pi \rightarrow \Pi'$  over  $P$  is an  $\infty$ -groupoid. We regard this space of functors as the *stratified mapping space*.

## Strata & links

2.3 **Definition.** If  $f : \Pi \rightarrow P$  is a stratified space, then for every point  $p \in P$ , the space

$$\Pi_p = \mathbf{Map}_P(\{p\}, X)$$

will be called the  *$p$ -th stratum* of  $\Pi$ , and for every pair of points  $p, q \in P$  with  $p \leq q$ , the space

$$N_p(\Pi)\{p, q\} := \mathbf{Map}_P(\{p, q\}, \Pi)$$

will be called the *link*<sup>5</sup> from the  $p$ -th stratum to the  $q$ -th stratum.

Please observe that the link comes equipped with source and target maps

$$(s, t) : N_p(\Pi)\{p, q\} \rightarrow \Pi_p \times \Pi_q,$$

the fibres of which over a point  $(x, y)$  is precisely the space  $\mathbf{Map}_\Pi(x, y)$ . When  $p = q$ , each of  $s$  and  $t$  is an equivalence, whence  $(s, t)$  is equivalent to the diagonal  $\Pi_p \rightarrow \Pi_p \times \Pi_p$ .

2.4. A morphism  $\Pi' \rightarrow \Pi$  of  $\mathbf{Str}_P$  is an equivalence if and only if, for every pair of points  $p, q \in P$  with  $p \leq q$ , the map on links  $\mathbf{Map}_P(\{p, q\}, \Pi') \rightarrow \mathbf{Map}_P(\{p, q\}, \Pi)$  is an equivalence (whence in particular, when  $p = q$ , the map on strata  $\Pi'_p \rightarrow \Pi_p$  is an equivalence).

## Repairing functors that are not conservative

2.5 **Construction.** Let  $P$  be a poset. The forgetful functor  $\mathbf{Str}_P \rightarrow \mathbf{Cat}_{\infty, P}$  admits a left adjoint. Indeed, if  $\Pi$  is an  $\infty$ -category, and  $f : \Pi \rightarrow P$  is any functor (not necessarily conservative), we may formally invert those morphisms of  $\Pi$  that are sent to identities in  $P$  as follows. We form

$$\mathbf{Ex}_P(\Pi) := \mathbf{Ex}(\Pi) \times_{\mathbf{Ex}(P)} P,$$

so that an  $n$ -simplex of  $\mathbf{Ex}_P(\Pi)$  is a commutative square

$$\begin{array}{ccc} \mathbf{sd}(\Delta^n) & \longrightarrow & \Pi \\ \lambda \downarrow & & \downarrow f \\ \Delta^n & \longrightarrow & P \end{array}$$

<sup>5</sup>Our link corresponds to what Frank Quinn and others called the *homotopy link* or *holink*. The significance of our chosen notation will become clear in [Construction 4.4](#).

where  $\lambda$  is the last vertex map. Now  $\lambda$  induces a functor  $\Pi \rightarrow \text{Ex}_p(\Pi)$ , and so we are entitled to form the colimit

$$\text{Ex}_P^\infty(\Pi) := \text{colim}_{n \in \mathbb{N}} \text{Ex}_P^n(\Pi) \cong \text{Ex}^\infty(\Pi) \times_{\text{Ex}^\infty(P)} P .$$

Since  $f$  is an inner fibration (as its target is the nerve of an ordinary category), so is  $\text{Ex}^\infty(f)$ , whence so is the projection  $\text{Ex}_P^\infty(\Pi) \rightarrow P$ ; it is also conservative, since the fibre over a point  $p \in P$  is the  $\infty$ -groupoid  $\text{Ex}^\infty(\Pi_p)$ . The functor  $\Pi \rightarrow \text{Ex}_P^\infty(\Pi)$ , natural in  $\Pi$ , is the unit of the desired adjunction.

**2.6 Proposition.** *The forgetful functor  $t : \mathbf{Str} \rightarrow \text{poSet}$  is a bicartesian fibration.*

*Proof.* Let  $P$  and  $Q$  be posets, and let  $f : P \rightarrow Q$  be a monotonic map; if  $q : \Xi \rightarrow Q$  is a  $Q$ -stratified space, then one obtains a  $P$ -stratified space  $f^*(q) : \Xi \times_P Q \rightarrow Q$ . The resulting square

$$\begin{array}{ccc} \Xi \times_P Q & \longrightarrow & \Xi \\ f^*(q) \downarrow & & \downarrow q \\ P & \xrightarrow{f} & Q \end{array}$$

is a cartesian edge lying over  $f$ . In the other direction, let  $p : \Pi \rightarrow P$  be a  $P$ -stratified space. Then the composite  $f \circ p$  is not in general conservative if  $f$  is not a monomorphism, but one may formally invert those morphisms of  $\Pi$  that are sent to identities by  $f \circ p$ . The square

$$\begin{array}{ccc} \Pi & \longrightarrow & \text{Ex}_Q^\infty(\Pi) \\ p \downarrow & & \downarrow f_!(p) \\ P & \xrightarrow{f} & Q \end{array}$$

is a cocartesian morphism of  $\mathbf{Str}$  over  $f$ . □

**2.7.** The limit of a diagram  $\alpha \mapsto [\Pi_\alpha \rightarrow P_\alpha]$  of stratified spaces can be computed as follows. We first form the limit  $P := \lim P_\alpha$ ; then pulling back along the various projections  $p_\alpha : P \rightarrow P_\alpha$ , we obtain the diagram  $\alpha \mapsto p_\alpha^* \Pi_\alpha$  of  $P$ -stratified spaces. One then forms the limit  $\Pi := \lim \Pi_\alpha$  in  $\mathbf{Cat}_\infty$ ; the functor  $\Pi \rightarrow P$  is then a stratified space, and it is the limit of our diagram.

## The stratified Postnikov tower

**2.8 Definition.** Let  $P$  be a poset and  $\Pi$  a  $P$ -stratified space. Then we obtain a tower of  $P$ -stratified spaces

$$\Pi \rightarrow \cdots \rightarrow h_3 \Pi \rightarrow h_2 \Pi \rightarrow h_1 \Pi \rightarrow h_0 \Pi \rightarrow P ,$$

called the *stratified Postnikov tower*.

In particular, please observe that  $h_0 \Pi \rightarrow P$  is a monotonic map of posets.



2.9. If  $P = \{0\}$ , then the stratified Postnikov tower coincides with the usual Postnikov tower of spaces.

2.10. The following are equivalent for a poset  $P$ , a  $P$ -stratified space  $f: \Pi \rightarrow P$ , and a nonnegative integer  $n \in \mathbf{N}$ :

- ▶ the  $\infty$ -category  $\Pi$  is equivalent to an  $n$ -category;
- ▶ the natural functor  $\Pi \rightarrow h_n \Pi$  is an equivalence;
- ▶ for any objects  $x, y \in \Pi$ , the space  $\text{Map}_\Pi(x, y)$  is  $(n - 1)$ -truncated;
- ▶ for any pair of points  $p, q \in P$  with  $p \leq q$ , the map

$$(s, t): N_p(\Pi)\{p, q\} \rightarrow \Pi_p \times \Pi_q$$

is  $(n - 1)$ -truncated (whence in particular, when  $p = q$ , the stratum  $\Pi_p$  is  $n$ -truncated).

2.11 **Definition.** Let  $P$  be a poset and  $n \in \mathbf{N}$ . We say that a  $P$ -stratified space  $\Pi$  is *n-truncated* if  $\Pi$  satisfies the equivalent conditions of (2.10). We write  $\mathbf{Str}_{P, \leq n} \subset \mathbf{Str}_P$  for the full subcategory spanned by the  $n$ -truncated  $P$ -stratified spaces.

We caution that an  $n$ -truncated  $P$ -stratified space is generally *not* the same thing as an  $n$ -truncated object of the  $\infty$ -category  $\mathbf{Str}_P$  in the sense of Lurie [HTT, Definition 5.5.6.1]. Nor is it the same thing as a  $P$ -stratified space whose strata are  $n$ -truncated; truncatedness in our sense involves a condition on the links as well.

2.12. Dually, the following are equivalent for a poset  $P$ , a  $P$ -stratified space  $f: \Pi \rightarrow P$ , and a nonnegative integer  $n \in \mathbf{N}$ :

- ▶ the natural functor  $h_n \Pi \rightarrow P$  is an equivalence;
- ▶ for any objects  $x, y \in \Pi$  such that  $f(x) \leq f(y)$ , the space  $\text{Map}_\Pi(x, y)$  is  $n$ -connective;
- ▶ for any pair of points  $p, q \in P$  with  $p \leq q$ , the map

$$(s, t): N_p(\Pi)\{p, q\} \rightarrow \Pi_p \times \Pi_q$$

is  $n$ -connective (whence in particular, when  $p = q$ , the stratum  $\Pi_p$  is  $(n + 1)$ -connective).

2.13 **Definition.** Let  $P$  be a poset and  $n \in \mathbf{N}$ . We say that a  $P$ -stratified space  $\Pi$  is *n-connective* if  $\Pi$  satisfies the equivalent conditions of (2.12). We write  $\mathbf{Str}_{P, \geq n} \subset \mathbf{Str}_P$  for the full subcategory spanned by the  $n$ -connective  $P$ -stratified spaces.

2.14 **Definition.** We say that a 1-category is *layered*<sup>6</sup> if and only if every endomorphism is an isomorphism. We say that an  $\infty$ -category  $\Pi$  is *layered* if and only if its homotopy category  $h_1 \Pi$  is a layered category. This holds if and only if the natural functor  $\Pi \rightarrow h_0(\Pi)$  is conservative. Thus a layered  $\infty$ -category  $\Pi$  is naturally an  $h_0(\Pi)$ -stratified space.

We write  $\mathbf{Lay}_\infty$  for the full subcategory of  $\mathbf{Cat}_\infty$  spanned by the layered  $\infty$ -categories.

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<sup>6</sup>or *EL*, as they are more usually called

2.15. The assignment  $[II \rightarrow P] \mapsto II$  defines a functor  $\mathbf{Str} \rightarrow \mathbf{Lay}_\infty$  with a fully faithful left adjoint that carries  $II$  to the  $h_0(II)$ -stratified space  $II$ . Consequently, we obtain an identification

$$\mathbf{Lay}_\infty \simeq \mathbf{Str}_{\geq 0},$$

where  $\mathbf{Str}_{\geq 0} \subset \mathbf{Str}$  is the full subcategory spanned by the 0-connective stratified spaces.

## Finite stratified spaces

We conclude this section by identifying a good finiteness property on stratified spaces.

2.16 **Recollection** ([SAG, Definition E.o.7.8]). An  $\infty$ -groupoid  $K$  is said to be  $\pi$ -finite if and only if the following conditions are satisfied.

- The set  $\pi_0(K)$  is finite.
- For any point  $x \in K$  and any  $i \geq 1$ , the group  $\pi_i(K, x)$  is finite.
- The  $\infty$ -groupoid  $K$  is equivalent to an  $n$ -groupoid for some  $n \in \mathbf{N}$ .

We write  $\mathbf{S}_\pi \subset \mathbf{S}$  for the full subcategory spanned by the  $\pi$ -finite  $\infty$ -groupoids.

We caution that a  $\pi$ -finite  $\infty$ -groupoid is not the same thing as what is normally called a *finite space* – one obtained via finite colimits from  $\Delta^0$ . In fact, the overlap between these two classes of spaces is essentially trivial. In this paper, we shall never refer to finite spaces in this latter sense.

2.17 **Definition**. We say that a stratified space  $II \rightarrow P$  is  $\pi$ -finite if and only if the following conditions are satisfied.

- The poset  $P$  is finite.
- For any point  $p \in P$ , the set  $\pi_0(II_p)$  is finite.
- For any morphism  $\phi: x \rightarrow y$  of  $II$ , and every  $i \geq 1$ , the group  $\pi_i(\mathrm{Map}_{II}(x, y), \phi)$  is finite.
- The  $\infty$ -category  $II$  is equivalent to an  $n$ -category for some  $n \in \mathbf{N}$ .

In particular, a nondegenerate stratified space  $II \rightarrow P$  is  $\pi$ -finite if and only if  $II$  has finitely many objects up to equivalence and is *locally  $\pi$ -finite* in the sense that each mapping space  $\mathrm{Map}_{II}(x, y)$  is  $\pi$ -finite.

We write  $\mathbf{Str}_\pi \subset \mathbf{Str}$  for the full subcategory spanned by the  $\pi$ -finite stratified spaces, and for any finite poset  $P$ , we write  $\mathbf{Str}_{\pi, P} \subset \mathbf{Str}_P$  for the full subcategory spanned by the  $\pi$ -finite  $P$ -stratified spaces.

2.18. The forgetful functor  $t: \mathbf{Str}_\pi \rightarrow \mathbf{poSet}^{fn}$  is a cartesian fibration, but is not a co-cartesian fibration because pullback doesn't admit a left adjoint in the finite realm. However, the pullback does preserve finite limits, and there is a proëxistent left adjoint, which we will discuss in the next section.

2.19 **Lemma**. *The full subcategory  $\mathbf{Str}_\pi \subset \mathbf{Str}$  is an accessible subcategory that is closed under finite limits.*

*Proof.* Finite limits of finite posets are finite, pullbacks of finite stratified spaces along maps of finite posets are finite, and limits of locally  $\pi$ -finite  $\infty$ -categories are locally  $\pi$ -finite. Finally,  $\mathbf{Str}_\pi$  is essentially  $\kappa_0$ -small and idempotent complete.  $\square$

In light of (0.6), this entitles us to speak of profinite stratified spaces, to which we now turn.

### 3 Profinite stratified spaces

#### Stratified propspaces over proposets

**3.1 Definition.** We call objects of the  $\infty$ -category  $\mathbf{Pro}(\mathbf{Str})$  *stratified propspaces*; the forgetful functor  $t: \mathbf{Str} \rightarrow \mathit{poSet}$  from stratified spaces to posets extends to a forgetful functor

$$t: \mathbf{Pro}(\mathbf{Str}) \rightarrow \mathbf{Pro}(\mathit{poSet}) .$$

The fibre  $\mathbf{Pro}(\mathbf{Str})_P$  over a poset  $P$ , regarded as a constant propset, can be identified with the  $\infty$ -category  $\mathbf{Pro}(\mathbf{Str}_P)$  of *P-stratified propspaces* – i.e., of proobjects in  $\mathbf{Str}_P$ .

Similarly, if  $\mathbf{P}$  is a propset, then the fibre  $\mathbf{Pro}(\mathbf{Str})_{\mathbf{P}}$  of  $t$  over  $\mathbf{P}$  will be called the  $\infty$ -category of *P-stratified propspaces*.

**3.2.** A stratified propspace can be exhibited as an inverse system  $\{\Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  of stratified spaces. The functor  $t$  carries this stratified propspace to the propset  $\{P_\alpha\}_{\alpha \in \Lambda}$ .

**3.3 Example.** Our primary interest is when the base is a spectral topological space  $S$  regarded as a profinite poset, whence we obtain the  $\infty$ -category  $\mathbf{Pro}(\mathbf{Str})_S$  of *S-stratified propspaces*. Still, in order to reason effectively with these, it is occasionally necessary to deal with more general stratified propspaces.

**3.4.** Please observe that the forgetful functor  $t: \mathbf{Pro}(\mathbf{Str}) \rightarrow \mathbf{Pro}(\mathit{poSet})$  is a cartesian fibration. Indeed, if  $\{P'_\alpha\}_{\alpha \in \Lambda} \rightarrow \{P_\alpha\}_{\alpha \in \Lambda}$  is a morphism of propsets, and if  $\{\Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  is a stratified propspace, then one may form  $\{\Pi_\alpha \times_{P_\alpha} P'_\alpha\}_{\alpha \in \Lambda}$ .

**3.5 Construction.** Let  $\eta: \mathbf{P} \rightarrow Q$  a morphism of propsets where  $Q$  is constant, so that  $\eta \in \mathbf{P}(Q)$ . For a  $\mathbf{P}$ -stratified propspace  $\mathbf{\Pi}$ , there exists a  $t$ -cocartesian edge  $\mathbf{\Pi} \rightarrow \eta_! \mathbf{\Pi}$  covering  $\eta$ ; indeed, for any  $Q$ -stratified space  $X$ , one has

$$(\eta_! \mathbf{\Pi})(X) \simeq \mathbf{\Pi}(X) \times_{P(Q)} \{\eta\} .$$

Equivalently, if we exhibit  $\mathbf{\Pi}$  as an inverse system  $\{\Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  in  $\mathbf{Str}$ , then the  $Q$ -stratified propspace  $\eta_! \mathbf{\Pi}$  can be exhibited as the inverse system  $\Lambda \times_{\mathit{poSet}} \mathit{poSet}/Q \rightarrow \mathbf{Str}_Q$  given by

$$(\alpha, P_\alpha \rightarrow Q) \mapsto \mathrm{Ex}_Q^\infty(\Pi_\alpha) .$$

Note in particular that if  $\mathbf{P}$  and  $\mathbf{\Pi}$  are constant, then so is  $\eta_! \mathbf{\Pi}$ .

In the  $\infty$ -category  $\mathbf{Pro}(\mathbf{Str})$ , the inverse system  $\mathit{poSet}_{P'} \rightarrow \mathbf{Str}$  given by  $\eta \mapsto \eta_! \mathbf{\Pi}$  is identified with  $\mathbf{\Pi}$  itself.

Now if  $\theta: \mathbf{P}' \rightarrow \mathbf{P}$  is any morphism of propsets and if  $\mathbf{\Pi}'$  is a  $\mathbf{P}'$ -stratified propspace, then we may form the inverse system  $\mathit{poSet}_{P'} \rightarrow \mathbf{Str}$  given by  $\eta \mapsto (\eta \circ \theta)_! \mathbf{\Pi}'$ , which

defines a propset  $\theta_! \mathbf{\Pi}'$ , and as this notation suggests, the morphism  $\mathbf{\Pi}' \rightarrow \theta_! \mathbf{\Pi}'$  is a  $t$ -cocartesian edge over  $\theta$ . Thus  $t: \text{Pro}(\mathbf{Str}) \rightarrow \text{Pro}(\text{poSet})$  is a cocartesian fibration.

We thus combine the previous two points:

**3.6 Proposition.** *The forgetful functor  $t: \text{Pro}(\mathbf{Str}) \rightarrow \text{Pro}(\text{poSet})$  is a bicartesian fibration.*

## The horizontal materialisation

**3.7 Definition.** Let  $\mathbf{P}$  be a propset. A  $\mathbf{P}$ -stratified prospace is said to be a  $\mathbf{P}$ -stratified space if it can be exhibited as an inverse system  $\{\mathbf{\Pi}_\alpha \rightarrow \mathbf{P}_\alpha\}_{\alpha \in \Lambda}$  of  $\mathbf{Str}$  that carries any morphism  $\alpha \rightarrow \beta$  of  $\Lambda$  to a  $t$ -cocartesian edge  $\mathbf{\Pi}_\alpha \rightarrow \mathbf{\Pi}_\beta$  of  $\mathbf{Str}$ . We write  $\mathbf{Str}_{\mathbf{P}} \subset \text{Pro}(\mathbf{Str})_{\mathbf{P}}$  for the full subcategory spanned by the  $\mathbf{P}$ -stratified spaces.

**3.8 Lemma.** *Let  $\mathbf{P}$  be a propset; then the following are equivalent for a  $\mathbf{P}$ -stratified prospace  $\mathbf{\Pi}$ .*

- ▶ For every map  $\eta: \mathbf{P} \rightarrow \mathbf{Q}$  in which  $\mathbf{Q}$  is constant, the  $\mathbf{Q}$ -stratified prospace  $\eta_! \mathbf{\Pi}$  is constant.
- ▶ The  $\mathbf{P}$ -stratified prospace  $\mathbf{\Pi}$  is a  $\mathbf{P}$ -stratified space.

*Proof.* If  $\eta_! \mathbf{\Pi}$  is constant for each  $[\eta: \mathbf{P} \rightarrow \mathbf{Q}] \in \text{poSet}_{\mathbf{P}}$ , then the inverse system  $\{\eta_! \mathbf{\Pi}\}_{\eta \in \text{poSet}_{\mathbf{P}}}$  exhibits  $\mathbf{\Pi}$  as a  $\mathbf{P}$ -stratified space.

Conversely, if  $\mathbf{\Pi}$  is a  $\mathbf{P}$ -stratified space, then exhibit  $\mathbf{\Pi}$  as an inverse system  $\{\mathbf{\Pi}_\alpha \rightarrow \mathbf{P}_\alpha\}_{\alpha \in \Lambda}$  such that for any  $\alpha' \rightarrow \alpha$  of  $\Lambda$ , the edge  $\mathbf{\Pi}_{\alpha'} \rightarrow \mathbf{\Pi}_\alpha$  is  $t$ -cocartesian. It follows that the morphisms  $\mathbf{\Pi} \rightarrow \mathbf{\Pi}_\alpha$  are all  $t$ -cocartesian, and for any  $[\eta: \mathbf{P} \rightarrow \mathbf{Q}] \in \text{poSet}_{\mathbf{P}}$ , there exists an  $\alpha \in \Lambda$  and a poset map  $\phi: \mathbf{P}_\alpha \rightarrow \mathbf{Q}$ , and one can identify the  $t$ -cocartesian edge  $\mathbf{\Pi} \rightarrow \eta_! \mathbf{\Pi}$  with the composite

$$\mathbf{\Pi} \rightarrow \mathbf{\Pi}_\alpha \rightarrow \phi_! \mathbf{\Pi}_\alpha,$$

whose target is constant. □

**3.9.** In other words, the cocartesian fibration  $\mathbf{Str} \rightarrow \text{poSet}$  is classified by a functor  $\text{poSet} \rightarrow \text{Cat}_\infty$  that carries a poset  $\mathbf{P}$  to the  $\infty$ -category  $\mathbf{Str}_{\mathbf{P}}$ . This functor extends to a unique functor  $\text{Pro}(\text{poSet}) \rightarrow \text{Cat}_\infty$  that preserves inverse limits, and  $\mathbf{Str}_{\mathbf{P}}$  is its value on the propset  $\mathbf{P}$ . In this manner, the cocartesian fibration  $\mathbf{Str} \rightarrow \text{poSet}$  extends (via right Kan extension) to a cocartesian fibration  $\mathbf{Str}^{\text{pro}} \rightarrow \text{Pro}(\text{poSet})$  whose fibre over  $\mathbf{P}$  is the  $\infty$ -category  $\mathbf{Str}_{\mathbf{P}}$ .

Now if one extends this cocartesian fibration to an inverse-limit preserving functor  $\text{Pro}(\mathbf{Str}^{\text{pro}}) \rightarrow \text{Pro}(\text{poSet})$ , then one simply recovers the cocartesian fibration  $\text{Pro}(\mathbf{Str}) \rightarrow \text{Pro}(\text{poSet})$ . In particular, please observe that for any propset  $\mathbf{P}$ , there is no ambiguity in the phrase  $\mathbf{P}$ -stratified prospace: we have a canonical equivalence of  $\infty$ -categories

$$\text{Pro}(\mathbf{Str}_{\mathbf{P}}) \simeq \text{Pro}(\mathbf{Str})_{\mathbf{P}}.$$

**3.10 Definition.** The *materialisation* of stratified prospace is the unique functor

$$\text{mat} : \text{Pro}(\mathbf{Str}) \rightarrow \mathbf{Str}$$

that preserves inverse limits and extends the identity functor  $\mathbf{Str} \rightarrow \mathbf{Str}$ . If we exhibit a stratified prospace as an inverse system  $\{\mathbf{II}_\alpha \rightarrow \mathbf{P}_\alpha\}_{\alpha \in \Lambda}$ , then its materialisation is the limit

$$\lim_{\alpha \in \Lambda} \mathbf{II}_\alpha \rightarrow \lim_{\alpha \in \Lambda} \mathbf{P}_\alpha = \text{mat}(\mathbf{P})$$

formed in  $\mathbf{Str}$ . Thus  $\text{mat}$  defines a functor  $\text{Pro}(\mathbf{Str})_{\mathbf{P}} \rightarrow \mathbf{Str}_{\text{mat}(\mathbf{P})}$ .

When restricted to the subcategory of  $\mathbf{P}$ -stratified spaces, it turns out that materialisation is actually an *equivalence*. That is, we can identify the  $\infty$ -category of spaces stratified by  $\mathbf{P}$  with that of spaces stratified by the materialisation  $\text{mat}(\mathbf{P})$ . One should exercise caution on this point, however, because this equivalence does not respect the embedding into the  $\infty$ -category  $\text{Pro}(\mathbf{Str})$ . In  $\text{Pro}(\mathbf{Str})$ , the objects  $\text{id}_{\mathbf{P}} : \mathbf{P} \rightarrow \mathbf{P}$  and  $\text{id}_{\text{mat}(\mathbf{P})} : \text{mat}(\mathbf{P}) \rightarrow \text{mat}(\mathbf{P})$  are very different. When  $\mathbf{P}$  is a spectral topological space  $S$ , this is the difference between  $S$  and the Alexandroff topology attached to its specialisation poset.

**3.11 Proposition.** For any proposit  $\mathbf{P}$ , the materialisation defines an equivalence of  $\infty$ -categories

$$\text{mat} : \mathbf{Str}_{\mathbf{P}} \simeq \mathbf{Str}_{\text{mat}(\mathbf{P})} .$$

*Proof.* The functor  $\text{mat}$  admits a left adjoint that carries a  $\text{mat}(\mathbf{P})$ -stratified space  $\mathbf{II}$  to the  $\mathbf{P}$ -stratified space given by the inverse system  $\{\text{Ex}_Q^\infty(\mathbf{II})\}_{Q \in \text{poSet}_{\mathbf{P}}}$ . Since  $\mathbf{II} \rightarrow \text{mat}(\mathbf{P})$  is conservative, the functor  $\mathbf{II} \rightarrow \lim_{Q \in \text{poSet}_{\mathbf{P}}} \text{Ex}_Q^\infty(\mathbf{II})$  is an equivalence. On the other hand, if  $\mathbf{II}$  is a  $\mathbf{P}$ -stratified space,  $\mathbf{II}$  can be exhibited as the inverse system  $\eta \mapsto \eta_{\mathbf{I}} \mathbf{II}$ , and one observes that for any  $[\eta : \mathbf{P} \rightarrow Q] \in \text{poSet}_{\mathbf{P}}$ , the  $Q$ -stratified space  $\eta_{\mathbf{I}} \mathbf{II}$  is equivalent to  $\text{Ex}_Q^\infty(\text{mat}(\mathbf{II}))$ .  $\square$

The upshot of this result is that it is relatively easy to specify a  $\mathbf{P}$ -stratified space: it is determined by its materialisation. Furthermore, we can extend this equivalence to proobjects:

**3.12 Corollary.** For any proposit  $\mathbf{P}$ , the materialisation extends to an equivalence of  $\infty$ -categories

$$\text{mat}^h : \text{Pro}(\mathbf{Str}_{\mathbf{P}}) \simeq \text{Pro}(\mathbf{Str})_{\mathbf{P}} \simeq \text{Pro}(\mathbf{Str})_{\text{mat}(\mathbf{P})} .$$

**3.13 Definition.** We call the equivalence  $\text{mat}^h$  above the *horizontal materialisation*. If  $\mathbf{P}$  is exhibited as an inverse system  $\{\mathbf{P}_\alpha\}$  of posets, and if  $\mathbf{II} \rightarrow \mathbf{P}$  is exhibited as an inverse system  $\{\mathbf{II}_\alpha \rightarrow \mathbf{P}_\alpha\}_{\alpha \in \Lambda}$  of stratified spaces, then the  $\text{mat}(\mathbf{P})$ -stratified prospace  $\text{mat}^h(\mathbf{II})$  is given by the inverse system  $\{\mathbf{II}_\alpha \times_{\mathbf{P}_\alpha} \text{mat}(\mathbf{P})\}_{\alpha \in \Lambda}$ .

In effect, there are two directions of proobjects we must contemplate: vertically, leading us to stratified prospace, and horizontally, leading us to stratifications over proposit. The horizontal materialisation effectively permits us – *over a fixed base proposit* – to work with stratifications of prospace over ordinary posets.

**3.14 Example.** Let  $S$  be a spectral topological space. Then an  $S$ -stratified space is determined by an  $\infty$ -category  $\mathbf{I}$  along with a conservative functor to the specialisation poset  $\text{mat}(S)$  of  $S$ . One extracts from this the  $S$ -stratified space as the inverse system  $\{\text{Exp}_P^\infty(\mathbf{I})\}_{P \in \text{FC}(S)}$ .

### Profinite stratified spaces

**3.15 Definition.** A *profinite stratified space* is a proobject of the  $\infty$ -category  $\mathbf{Str}_\pi$ . We write  $\mathbf{Str}_\pi^\wedge := \text{Pro}(\mathbf{Str}_\pi)$ . The forgetful functor  $[\mathbf{I} \rightarrow S] \mapsto S$  is a cartesian fibration

$$t: \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{TSp}^{\text{spec}} \simeq \text{Pro}(\text{poSet}^{\text{fin}}),$$

and for any spectral topological space  $S$ , we denote by  $\mathbf{Str}_{\pi,S}^\wedge$  the fibre over  $S$ . This is the  $\infty$ -category of *profinite  $S$ -stratified spaces*.

The inclusion  $\mathbf{Str}_\pi \hookrightarrow \mathbf{Str}$  extends to a fully faithful functor  $\mathbf{Str}_\pi^\wedge \hookrightarrow \text{Pro}(\mathbf{Str})$ , which admits a left adjoint  $\mathbf{I} \mapsto \mathbf{I}_\pi^\wedge$  given by restriction. We call the profinite stratified space  $\mathbf{I}_\pi^\wedge$  the *profinite completion* of  $\mathbf{I}$ .

**3.16.** The profinite completion functor  $\mathbf{I} \mapsto \mathbf{I}_\pi^\wedge$  is not itself a relative left adjunction over  $\text{Pro}(\text{poSet})$ ; however, the inclusion  $\mathbf{Str}_\pi \hookrightarrow \mathbf{Str}$  induces a fully faithful functor

$$\mathbf{Str}_\pi^\wedge \hookrightarrow \text{Pro}(\mathbf{Str}) \times_{\text{Pro}(\text{poSet})} \mathbf{TSp}^{\text{spec}},$$

and profinite completion *does* define a relative left adjoint over  $\mathbf{TSp}^{\text{spec}}$ . In particular, if  $S$  is a spectral topological space and  $\mathbf{I}$  is an  $S$ -stratified prospace, then  $\mathbf{I}_\pi^\wedge$  is a profinite  $S$ -stratified space, and the morphism  $\mathbf{I} \rightarrow \mathbf{I}_\pi^\wedge$  lies over  $S$ .

**3.17 Construction.** Let  $\theta: S' \rightarrow S$  be a quasicompact continuous map of spectral topological spaces, and let  $\mathbf{I}' \rightarrow S'$  be a profinite  $S'$ -stratified space. Then following [Construction 3.5](#), we obtain an  $S$ -stratified prospace  $\theta_! \mathbf{I}' \rightarrow S$ , and so we may form its profinite completion  $(\theta_! \mathbf{I}')_\pi^\wedge \rightarrow S$ . The map  $\mathbf{I}' \rightarrow (\theta_! \mathbf{I}')_\pi^\wedge$  is thus a cocartesian edge over  $\theta$  for the forgetful functor  $t: \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{TSp}^{\text{spec}}$ .

We thus obtain:

**3.18 Proposition.** *The forgetful functor  $t: \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{TSp}^{\text{spec}}$  is a bicartesian fibration.*

**3.19.** Let  $S$  be a spectral topological space. Using the horizontal materialisation, we obtain an equivalence of  $\infty$ -categories between  $\mathbf{Str}_{\pi,S}^\wedge$  and the full subcategory of the  $\infty$ -category  $\text{Pro}(\mathbf{Str})_{\text{mat}(S)}$  spanned by those  $\text{mat}(S)$ -stratified prospace  $\mathbf{I}$  such that for any points  $x, y \in S$  with  $x \leq y$ , the link prospace

$$\text{Map}_{\text{mat}(S)}(\{x, y\}, \mathbf{I})$$

is profinite.

**3.20 Proposition.** *Let  $S$  be a spectral topological space. Then the natural functor*

$$\mathbf{Str}_{\pi,S}^\wedge \rightarrow \lim_{P \in \text{FC}(S)} \mathbf{Str}_{\pi,P}^\wedge$$

*is an equivalence.*

*Proof.* The formation of the limit in  $\mathbf{Str}_\pi^\wedge$  is an inverse. □

## 4 Spatial décollages

### Complete Segal spaces & spatial décollages

**4.1 Recollection.** An  $\infty$ -category can be modelled as a simplicial *space*. In effect, if  $C$  is an  $\infty$ -category, then one may extract a functor  $N(C): \Delta^{op} \rightarrow \mathbf{S}$  in which  $N(C)_m$  is the  $\infty$ -groupoid of functors  $\Delta^m \rightarrow C$  (the ‘moduli space of sequences of arrows in  $C$ ’). The simplicial space  $N(C)$  is what Charles Rezk [43] called a *complete Segal space* – i.e., a functor  $D: \Delta^{op} \rightarrow \mathbf{S}$  such that the following conditions obtain.

- For any  $m \in \mathbf{N}^*$ , the natural map

$$D_m \rightarrow D\{0, 1\} \times_{D\{1\}} D\{1, 2\} \times_{D\{2\}} \cdots \times_{D\{m-1\}} D\{m-1, m\}$$

is an equivalence.

- If  $E$  denotes the unique contractible 1-groupoid with two objects, then the natural map

$$D_0 \rightarrow \text{Map}(E, D)$$

is an equivalence.

Joyal and Tierney [31] showed that the assignment  $C \mapsto N(C)$  is an equivalence between the  $\infty$ -category  $\text{Cat}_\infty$  of  $\infty$ -categories and the  $\infty$ -category  $\text{CSS}$  of complete Segal spaces.

We can isolate the  $\infty$ -groupoids in  $\text{CSS}$ : an  $\infty$ -category  $C$  is an  $\infty$ -groupoid if and only if  $N(C): \Delta^{op} \rightarrow \mathbf{S}$  is left Kan extended from  $\{0\} \subset \Delta^{op}$ .

We shall demonstrate that the homotopy theory of stratified spaces admits an analogous description.

**4.2 Definition.** Let  $P$  be a poset. A functor  $D: \text{sd}^{op}(P) \rightarrow \mathbf{S}$  is said to be a *spatial décollage (over  $P$ )* if and only if, for any string  $\{p_0, \dots, p_m\} \subseteq P$ , the map

$$D\{p_0, \dots, p_m\} \rightarrow D\{p_0, p_1\} \times_{D\{p_1\}} D\{p_1, p_2\} \times_{D\{p_2\}} \cdots \times_{D\{p_{m-1}\}} D\{p_{m-1}, p_m\}$$

is an equivalence. We write

$$\text{Déc}_P(\mathbf{S}) \subseteq \text{Fun}(\text{sd}^{op}(P), \mathbf{S})$$

for the full subcategory spanned by the spatial décollages.

**4.3 Construction.** Write  $J$  for the following 1-category. The objects are pairs  $(P, \Sigma)$  consisting of a poset  $P$  and a string  $\Sigma \subseteq P$ . A morphism  $(P, \Sigma) \rightarrow (Q, T)$  is a monotonic map  $f: P \rightarrow Q$  such that  $T \subseteq f(\Sigma)$ . The assignment  $(P, \Sigma) \mapsto P$  is a cocartesian fibration  $J \rightarrow \text{poSet}$  whose fibre over a poset  $P$  is the poset  $\text{sd}^{op}(P)$ .

We write

$$\text{Pair}_{\text{poSet}}(J, \mathbf{S})$$

for the simplicial set over  $\text{poSet}$  defined by the following universal property: for any simplicial set  $K$  over  $\text{poSet}$ , one demands a bijection

$$\text{Mor}_{\text{sSet}/\text{poSet}}(K, \text{Pair}_{\text{poSet}}(J, \mathbf{S})) \cong \text{Mor}_{\text{sSet}}(K \times_{\text{poSet}} J, \mathbf{S}),$$

natural in  $K$ . By [HTT, Corollary 3.2.2.13], the functor

$$\text{Pair}_{\text{poSet}}(J, \mathbf{S}) \rightarrow \text{poSet}$$

is a cartesian fibration whose fibre over a poset  $P$  is the  $\infty$ -category  $\text{Fun}(\text{sd}^{op}(P), \mathbf{S})$ . Now let

$$\mathbf{D}\acute{\text{e}}\text{c}(\mathbf{S}) \subset \text{Pair}_{\text{poSet}}(J, \mathbf{S})$$

denote the full subcategory spanned by the pairs  $(P, D)$  in which  $D$  is a spatial d collage. Since  $\mathbf{D}\acute{\text{e}}\text{c}(\mathbf{S})$  contains all the cartesian edges, the functor  $\mathbf{D}\acute{\text{e}}\text{c}(\mathbf{S}) \rightarrow \text{poSet}$  is a cartesian fibration.

### The nerve of a stratified space

We shall now show that the  $\infty$ -category  $\mathbf{Str}$  of stratified spaces and the  $\infty$ -category  $\mathbf{D}\acute{\text{e}}\text{c}(\mathbf{S})$  of d collages are equivalent over  $\text{poSet}$ .

**4.4 Construction.** Let  $P$  be a poset. Any string contained in  $P$  can be regarded as a  $P$ -stratified space via the inclusion map. This assignment is a functor  $\text{sd}(P) \rightarrow \mathbf{Str}_P$ . Now for any  $P$ -stratified space  $\Pi$ , let us define  $N_P(\Pi) : \text{sd}^{op}(P) \rightarrow \mathbf{S}$  to be the functor given by the assignment  $\Sigma \mapsto \text{Map}_P(\Sigma, \Pi)$ . (This is the moduli space of sections over  $\Sigma$ .) An equivalence of  $P$ -stratified spaces is carried to an objectwise equivalence of functors; hence this defines a functor

$$N_P : \mathbf{Str}_P \rightarrow \text{Fun}(\text{sd}^{op}(P), \mathbf{S}).$$

Furthermore, the assignment  $[\Pi \rightarrow P] \mapsto (P, N_P(\Pi))$  defines a functor

$$N : \mathbf{Str} \rightarrow \text{Pair}_{\text{poSet}}(J, \mathbf{S}).$$

**4.5 Example.** For any poset  $P$ , any  $P$ -stratified space  $\Pi$ , and any points  $p, q \in P$  such that  $p \leq q$ , the space

$$N_P(\Pi)\{p, q\} \simeq \text{Map}_P(\{p, q\}, \Pi)$$

is the link between the  $p$ -th and  $q$ -th strata of  $\Pi$ .

Let us demonstrate that the functor  $N$  lands in the full subcategory

$$\mathbf{D}\acute{\text{e}}\text{c}(\mathbf{S}) \subset \text{Pair}_{\text{poSet}}(J, \mathbf{S}).$$

**4.6 Lemma.** For any poset  $P$  and any  $P$ -stratified space  $\Pi$ , the functor  $N_P(\Pi)$  is a spatial d collage.

*Proof.* In  $\text{Cat}_{\infty, P}$ , for any string  $\{p_0, \dots, p_n\} \subseteq P$ , one has an equivalence

$$\{p_0, p_1\} \cup^{\{p_1\}} \dots \cup^{\{p_{n-1}\}} \{p_{n-1}, p_n\} \simeq \{p_0, \dots, p_n\},$$

which induces an equivalence

$$\text{Map}_P(\{p_0, \dots, p_n\}, \Pi) \simeq \text{Map}_P(\{p_0, p_1\}, \Pi) \times_{\Pi_{p_1}} \dots \times_{\Pi_{p_{n-1}}} \text{Map}_P(\{p_{n-1}, p_n\}, \Pi),$$

as desired.  $\square$



**4.7 Theorem.** *The functor  $N: \mathbf{Str} \rightarrow \mathbf{D}\acute{e}c(\mathbf{S})$  is an equivalence of  $\infty$ -categories over  $poSet$ .*

*Proof.* Let  $P$  be a poset. The Joyal–Tierney theorem [31] implies that the functor

$$N: \mathbf{Cat}_{\infty, /P} \rightarrow \mathbf{Fun}(\Delta_{/P}^{op}, \mathbf{S})_{/NP} \simeq \mathbf{Fun}(\Delta_{/P}^{op}, \mathbf{S})$$

is fully faithful, and the essential image  $\mathbf{CSS}_{/NP}$  consists of those functors  $\Delta_{/P}^{op} \rightarrow \mathbf{S}$  that satisfy both the Segal condition and the completeness condition. At the same time, the fully faithful functor  $i: sd(P) \hookrightarrow \Delta_{/P}$  induces, via right Kan extension, a fully faithful functor  $\mathbf{D}\acute{e}c_P(\mathbf{S}) \hookrightarrow \mathbf{CSS}_{/NP}$  whose essential image consists of those complete Segal spaces  $C \rightarrow NP$  such that for any  $p \in P$ , the complete Segal space  $C_p$  is an  $\infty$ -groupoid.  $\square$

**4.8.** The nerve  $N$  restricts to an equivalence of  $\infty$ -categories  $\mathbf{Str}_\pi \simeq \mathbf{D}\acute{e}c(\mathbf{S}_\pi)$ , where  $\mathbf{D}\acute{e}c(\mathbf{S}_\pi)$  denotes the full subcategory of  $\mathbf{D}\acute{e}c(\mathbf{S})$  spanned by those pairs  $(P, D)$  where  $P$  is a finite poset and  $D$  is a spatial décollage on  $P$  whose values are all  $\pi$ -finite.

## Profinite spatial décollages

**4.9.** We extend  $N$  to proobjects to obtain an equivalence of  $\infty$ -categories

$$N: \mathbf{Pro}(\mathbf{Str}) \simeq \mathbf{Pro}(\mathbf{D}\acute{e}c(\mathbf{S}))$$

over  $\mathbf{Pro}(poSet)$ .

**4.10 Recollection.** We regard  $\mathbf{S}_\pi^\wedge := \mathbf{Pro}(\mathbf{S}_\pi)$  as a full subcategory of the  $\infty$ -category  $\mathbf{Pro}(\mathbf{S})$ . Precomposition with the inclusion  $\mathbf{S}_\pi \hookrightarrow \mathbf{S}$  is *profinite completion*  $X \mapsto X_\pi^\wedge$ , which exhibits  $\mathbf{S}_\pi^\wedge$  as a localisation of  $\mathbf{Pro}(\mathbf{S})$ .

There are two monoidal structures on  $\mathbf{Pro}(\mathbf{S})$  one may contemplate. On one hand, one has the cartesian symmetric monoidal structure. On the other, the composition of two prospaces is again a prospace, whence we obtain a monoidal structure  $(X, Y) \mapsto X \circ Y$ . The identity functor, which is the unit for  $\circ$ , is terminal in  $\mathbf{Pro}(\mathbf{S})$ , and there certainly is a morphism  $X \circ Y \rightarrow X \times Y$  that is natural in  $X$  and  $Y$ , but it is not an equivalence in general.

However, on the  $\infty$ -category  $\mathbf{S}_\pi^\wedge$  of profinite  $\infty$ -groupoids, we can consider the profinite completion  $(X, Y) \mapsto (X \circ Y)_\pi^\wedge$ , and we claim that the morphism  $(X \circ Y)_\pi^\wedge \rightarrow X \times Y$  is an equivalence. Indeed, we claim that the value of the natural transformation  $X \times Y \rightarrow X \circ Y$  on any truncated space  $K$  is an equivalence.<sup>7</sup> Exhibit  $X$  and  $Y$ , respectively, as inverse systems  $\{X_\alpha\}_{\alpha \in \Lambda}$  and  $\{Y_\beta\}_{\beta \in M}$  of  $\pi$ -finite  $\infty$ -groupoids. For each  $\alpha \in \Lambda$ , the  $\infty$ -groupoid  $X_\alpha$  can be exhibited as a simplicial set with only finitely many nondegenerate simplices of each dimension, whence the functor corepresented by  $X_\alpha$  preserves filtered colimits of uniformly truncated spaces. Since  $K$  is truncated, the filtered diagram  $\beta \mapsto \text{Map}(Y_\beta, K)$  is uniformly truncated. Hence

$$\begin{aligned} (X \times Y)(K) &\simeq \text{colim}_{\alpha \in \Lambda^{op}} \text{Map}(X_\alpha, \text{colim}_{\beta \in M^{op}} \text{Map}(Y_\beta, K)) \\ &\simeq \text{colim}_{(\alpha, \beta) \in \Lambda^{op} \times M^{op}} \text{Map}(X_\alpha \times Y_\beta, K) \simeq (X \circ Y)(K), \end{aligned}$$

<sup>7</sup>We are grateful to Jacob Lurie for this observation.

as desired.

This is helpful for describing fibre products in  $\mathcal{S}_\pi^\wedge$  as well: if  $p: X \rightarrow Z$  and  $q: Y \rightarrow Z$  are two morphisms of profinite  $\infty$ -groupoids, then one may identify the pullback  $X \times_Z Y$  of  $p$  along  $q$  with a cobar construction:

$$X \times_Z Y \simeq \lim_{m \in \Delta} (X \circ Z^{\circ m} \circ Y)_\pi^\wedge .$$

**4.11 Construction.** For any spectral topological space  $S$ , write  $\mathbf{D}\acute{e}c_S(\mathcal{S}_\pi^\wedge)$  for the full subcategory of  $\text{Fun}(\text{sd}^{op}(\text{mat}(S)), \mathcal{S}_\pi^\wedge)$  spanned by those functors

$$D: \text{sd}^{op}(\text{mat}(S)) \rightarrow \mathcal{S}_\pi^\wedge$$

such that for any string  $\{p_0, \dots, p_n\} \subseteq P$ , the natural map

$$D\{p_0, \dots, p_n\} \rightarrow \lim_{m \in \Delta} (D\{p_0, p_1\} \circ D\{p_1\}^{\circ m} \circ D\{p_1, p_2\} \circ D\{p_2\}^{\circ m} \circ \dots \circ D\{p_{n-1}, p_n\})_\pi^\wedge$$

is an equivalence of profinite spaces. We call objects of  $\mathbf{D}\acute{e}c_S(\mathcal{S}_\pi^\wedge)$  *profinite d\acute{e}collages over S*. Now we can compose the nerve with the horizontal materialisation to obtain an equivalence of  $\infty$ -categories

$$\mathbf{Str}_{\pi, S}^\wedge \simeq \mathbf{D}\acute{e}c_S(\mathcal{S}_\pi^\wedge) .$$

## Part II

# Elements of higher topos theory

## 5 Aide-mémoire on higher topoi

In this section we recall a number of important results from higher topos theory (mostly from Jacob Lurie’s [SAG, Appendices A & E]), and we develop some basic results that we’ll use throughout the rest of the paper. This section is here mostly for ease of reference, and we make no pretence to originality.

### Higher topoi

We begin by setting our basic notational conventions for higher topoi.

**5.1 Notation.** We use here the theory of  $n$ -topoi for  $n \in \mathbb{N}^*$ ; see [HTT, Chapter 6]. We write  $\mathbf{Top}_n \subset \mathbf{Cat}_{\infty, \kappa_1}$  for the subcategory of  $\kappa_1$ -small  $n$ -topoi and geometric morphisms. All of the examples in this paper will have  $n \in \{0, 1, \infty\}$ .

For any  $\kappa_0$ -small  $\infty$ -category  $C$ , we write  $\mathcal{P}(C) := \mathbf{Fun}(C^{op}, \mathbf{S})$  for the  $\infty$ -topos of presheaves of spaces on  $C$ .

**5.2 Example.** Recall that 0-topoi are locales (which are essentially  $\kappa_0$ -small) [HTT, Proposition 6.4.2.5], and 1-topoi are topoi in the classical sense of Grothendieck [HTT, Remark 6.4.1.3].

**5.3 Example.** Let  $m, n \in \mathbb{N}^*$  with  $m \leq n$ . By an  $m$ -site, we mean a  $\kappa_0$ -small  $m$ -category  $X$  equipped with a Grothendieck topology  $\tau$ . Attached to this  $m$ -site is the  $n$ -topos  $\mathbf{Sh}_{\tau, \leq (n-1)}(X)$  of sheaves of  $\kappa_0$ -small  $(n-1)$ -groupoids on  $X$ .

Not all  $\infty$ -topoi are of the form  $\mathbf{Sh}_\tau(X)$  for some  $\infty$ -site  $X$ ; however, if  $n \in \mathbb{N}$ , then every  $n$ -topos is of the form  $\mathbf{Sh}_{\tau, \leq (n-1)}(X)$  for some  $n$ -site  $(X, \tau)$  [HTT, Theorem 6.4.1.5(1)].

**5.4 Example.** For any topological space  $W$ , denote by  $\widetilde{W}$  the 0-localic  $\infty$ -topos of sheaves of ( $\kappa_0$ -small) spaces on  $W$ .

**5.5 Notation.** The  $\infty$ -topos  $\mathbf{S}$  is terminal in  $\mathbf{Top}_{\infty}$ . For any  $\infty$ -topos  $X$ , we write  $\Gamma_{X,*}$  or  $\Gamma_*$  for the essentially unique geometric morphism  $X \rightarrow \mathbf{S}$ ; the functor  $\Gamma_*$  is corepresented by the terminal object  $1_X \in X$ . A *point* of  $X$  is a geometric morphism  $x_* : \mathbf{S} \rightarrow X$ ; we may also write  $\tilde{x}$  for this copy of  $\mathbf{S}$ , regarded as lying over  $X$  via  $x_*$ .

**5.6 Recollection.** Let  $X$  and  $Y$  be  $\infty$ -topoi. A geometric morphism  $j_* : X \rightarrow Y$  is *étale* if  $j^*$  admits a further left adjoint  $j_! : X \rightarrow Y$  that exhibits  $X$  as the slice  $\infty$ -topos  $Y_{/j_!(1_X)}$ . By [HTT, Corollary 6.3.5.6], the functor

$$\mathbf{Fun}_*(Z, X) \rightarrow \mathbf{Fun}_*(Z, Y)$$

is a right fibration whose fibre over a geometric morphism  $f_* : Z \rightarrow Y$  is the (essentially  $\kappa_0$ -small) Kan complex  $\mathbf{Map}_X(1_X, f^* j_!(1_X))$ .

**5.7 Notation.** Let  $X$  and  $Y$  be two  $n$ -topoi for some  $n \in \mathbb{N}^\flat$ . We write  $\text{Fun}_*(X, Y) \subseteq \text{Fun}(X, Y)$  for the full subcategory spanned by the geometric morphisms. We note that  $\text{Fun}_*(X, Y)$  is accessible [HTT, Proposition 6.3.1.13]. We write  $\text{Fun}^*(Y, X) \subseteq \text{Fun}(Y, X)$  for the full subcategory spanned by those functors that are left exact left adjoints, so that  $\text{Fun}^*(Y, X) \simeq \text{Fun}_*(X, Y)^{op}$ .

**5.8.** If  $X$  and  $Y$  are  $\infty$ -topoi, the product  $X \times Y$  in  $\text{Top}_\infty$  is *not* the product of  $\infty$ -categories; rather, it can be identified with the tensor product of presentable  $\infty$ -categories.<sup>8</sup>

Similarly, if  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  are geometric morphisms, then the pullback  $X \times_Z Y$  in  $\text{Top}_\infty$  exists [HTT, Proposition 6.3.4.6], but it is not the pullback of  $\infty$ -categories.

Finally, there is an *oriented fibre product* of  $\infty$ -topoi – which we will study in detail in Section 6 – which also does not coincide with the oriented fibre product of  $\infty$ -categories. We will therefore endeavour to indicate clearly when a product, pullback, or oriented fibre product is meant to be formed in  $\text{Top}_\infty$  or some  $\text{Cat}_{\infty, \kappa}$ .

We repeatedly make use of the fact that inverse limits in  $\text{Top}_\infty$  are computed in  $\text{Cat}_{\infty, \kappa_1}$ .

**5.9 Theorem** ([HTT, Theorem 6.3.3.1]). *The forgetful functor  $\text{Top}_\infty \rightarrow \text{Cat}_{\infty, \kappa_1}$  preserves inverse limits.*

## Boundedness

We now turn to the first of two finiteness conditions that we impose on almost all of the  $\infty$ -topoi we consider in this paper.

**5.10 Notation.** If  $m, n \in \mathbb{N}^\flat$  with  $m < n$ , then passage to  $(m - 1)$ -truncated objects is a functor

$$\tau_{\leq m-1} : \text{Top}_n \rightarrow \text{Top}_m .$$

In particular, when  $m = 0$ , we write  $\text{Open}$  for  $\tau_{\leq -1}$ , and we call a  $(-1)$ -truncated object of an  $n$ -topos  $X$  an *open in  $X$* .

For any  $\infty$ -topos  $X$ , write

$$X_{< \infty} := \text{colim}_{n \in \mathbb{N}} \tau_{\leq n} X \subseteq X$$

for the full subcategory spanned by the truncated objects.

**5.11 Definition.** If  $m, n \in \mathbb{N}^\flat$  with  $m < n$ , then the functor  $\tau_{\leq m-1} : \text{Top}_n \rightarrow \text{Top}_m$  admits a fully faithful right adjoint. Write  $\text{Top}_n^m \subseteq \text{Top}_n$  for the essential image of this functor; this consists of those  $n$ -topoi  $X$  such that, for every  $n$ -topos  $Y$ , the functor

$$\text{Fun}_*(Y, X) \rightarrow \text{Fun}_*(\tau_{\leq m-1} Y, \tau_{\leq m-1} X)$$

is an equivalence. We call such  $n$ -topoi  *$m$ -localic* [HTT, §6.4.5].

<sup>8</sup>For this reason, Lurie writes  $X \otimes Y$  for the product in  $\text{Top}_\infty$ .

5.12. If  $n \in \mathbf{N}$ , then the proof of [HTT, Proposition 6.4.5.9] demonstrates that an  $\infty$ -topos  $\mathbf{X}$  is  $n$ -localic if and only if  $\mathbf{X} \simeq \mathbf{Sh}_\tau(X)$ , where  $(X, \tau)$  is a  $\kappa_0$ -small  $n$ -site. One can always arrange that  $X$  admit all finite limits.

5.13 **Example.** If  $W$  is a topological space, then  $\widetilde{W}$  is 0-localic.

5.14 **Example.** If  $X$  is a scheme, then the  $\infty$ -topos  $X_{\acute{e}t}$  of étale sheaves on the 1-site of étale  $X$ -schemes is 1-localic.

5.15 **Definition.** Denote by  $\mathbf{Top}_\infty^\wedge$  the inverse limit of  $\infty$ -categories

$$\mathbf{Top}_\infty^\wedge := \lim_{n \in \mathbf{N}^{op}} \mathbf{Top}_n$$

along the various truncation functors  $\tau_{\leq m-1}$ . This is the  $\infty$ -category of sequences  $\{\mathbf{X}_n\}_{n \in \mathbf{N}}$  in which each  $\mathbf{X}_n$  is an  $n$ -topos, along with identifications  $\mathbf{X}_m \simeq \tau_{\leq m-1} \mathbf{X}_n$  whenever  $m \leq n$ . The truncation functors provide a functor

$$\tau : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty^\wedge,$$

which carries an  $\infty$ -topos  $\mathbf{X}$  to the sequence  $\{\tau_{\leq n-1} \mathbf{X}\}_{n \in \mathbf{N}}$ .

5.16 **Construction.** The functor  $\tau : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty^\wedge$  admits a fully faithful right adjoint, which identifies  $\mathbf{Top}_\infty^\wedge$  with the full subcategory of  $\mathbf{Top}_\infty$  spanned by the *bounded  $\infty$ -topoi*. These are the  $\infty$ -topoi that can be exhibited as inverse limits in  $\mathbf{Top}_\infty$  of a diagram of localic  $\infty$ -topoi.

On the other hand, the functor  $\tau : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty^\wedge$  also admits a left adjoint, which is necessarily fully faithful. This identifies  $\mathbf{Top}_\infty^\wedge$  with the full subcategory of  $\mathbf{Top}_\infty$  spanned by the *Postnikov complete  $\infty$ -topoi*. These are the  $\infty$ -topoi that can be exhibited in  $\mathbf{Cat}_{\infty, \kappa_1}$  as the inverse limit of their truncations.

5.17. The relationship between bounded  $\infty$ -topoi and Postnikov complete  $\infty$ -topoi is formally analogous to the relationship between  $p$ -nilpotent and  $p$ -complete abelian groups. Of course  $p$ -nilpotent and  $p$ -complete abelian groups form equivalent categories, but their embeddings into the category of all abelian groups differ.

## Coherence

The second finiteness conditions that we impose on almost all of the  $\infty$ -topoi we consider is *coherence*.

5.18 **Definition.** Let  $\mathbf{X}$  be an  $\infty$ -topos. We say that  $\mathbf{X}$  is *0-coherent* if and only if the 0-topos (=locale)  $\mathbf{Open}(\mathbf{X})$  is quasicompact. Let  $n \in \mathbf{N}^*$ , and define  $n$ -coherence of  $\infty$ -topoi and their objects recursively as follows.

- An object  $U \in \mathbf{X}$  is  *$n$ -coherent* if and only if the  $\infty$ -topos  $\mathbf{X}_{/U}$  is  $n$ -coherent.
- The  $\infty$ -topos  $\mathbf{X}$  is *locally  $n$ -coherent* if and only if every object  $U \in \mathbf{X}$  admits a cover  $\{V_i \rightarrow U\}_{i \in I}$  in which each  $V_i$  is  $n$ -coherent.

- The  $\infty$ -topos  $\mathbf{X}$  is  $(n + 1)$ -coherent if and only if  $\mathbf{X}$  is locally  $n$ -coherent, and the  $n$ -coherent objects of  $\mathbf{X}$  are closed under finite products.

In particular, if  $\mathbf{X}$  is locally  $n$ -coherent, then  $U \in \mathbf{X}$  is  $(n + 1)$ -coherent if and only if  $U$  is  $n$ -coherent and for any pair  $U', V \in \mathbf{X}_{/U}$  of  $n$ -coherent objects, the fibre product  $U' \times_U V$  is  $n$ -coherent.

One says that an  $\infty$ -topos  $\mathbf{X}$  is *coherent* if and only if  $\mathbf{X}$  is  $n$ -coherent for every  $n \in \mathbb{N}$ , and one says that an object  $U$  of an  $\infty$ -topos  $\mathbf{X}$  is *coherent* if and only if  $\mathbf{X}_{/U}$  is a coherent  $\infty$ -topos. Finally, an  $\infty$ -topos  $\mathbf{X}$  is *locally coherent* if and only if every object  $U \in \mathbf{X}$  admits a cover  $\{V_i \rightarrow U\}_{i \in I}$  in which each  $V_i$  is coherent.

**5.19 Notation.** If  $\mathbf{X}$  is an  $\infty$ -topos, then write  $\mathbf{X}^{coh} \subseteq \mathbf{X}$  for the full subcategory of  $\mathbf{X}$  spanned by the coherent objects and  $\mathbf{X}_{<\infty}^{coh} \subseteq \mathbf{X}$  for the full subcategory of  $\mathbf{X}$  spanned by the truncated coherent objects.

**5.20.** Let  $\mathbf{X}$  be an  $\infty$ -topos and  $U \in \mathbf{X}$ . Then for any integer  $n \geq 0$ , an object  $U' \rightarrow U$  of  $\mathbf{X}_{/U}$  is  $n$ -coherent if and only if  $U'$  is  $n$ -coherent when viewed as an object of  $\mathbf{X}$ . Thus if  $U \in \mathbf{X}_{<\infty}^{coh}$  is a truncated coherent object, then we have a canonical identification

$$(\mathbf{X}_{<\infty}^{coh})_{/U} = (\mathbf{X}_{/U})_{<\infty}^{coh}$$

as full subcategories of  $\mathbf{X}_{/U}$ .

**5.21 Definition.** An  $\infty$ -site  $(X, \tau)$  is said to be *finitary* if and only if  $X$  admits all finite limits, and, for every object  $x \in X$  and every covering sieve  $R \subseteq X_{/x}$ , there is a finite subset  $\{y_i\}_{i \in I} \subseteq R$  that generates a covering sieve.

**5.22 Proposition** ([SAG, Proposition A.3.1.3]). *Let  $(X, \tau)$  be a finitary  $\infty$ -site. Then the  $\infty$ -topos  $\mathbf{Sh}_\tau(X)$  is coherent and locally coherent, and for every object  $x \in X$ , the sheaf  $j(x)$  is a coherent object of  $\mathbf{Sh}_\tau(X)$ , where  $j: X \rightarrow \mathbf{Sh}_\tau(X)$  is the sheafified Yoneda embedding.*

**5.23.** A 1-topos  $\mathbf{X}$  is coherent in the sense of [SGA<sub>4</sub> II, Exposé VI, Definition 2.3] if and only if  $\mathbf{X}$  is equivalent to the 1-topos of sheaves of sets on a small 1-category  $X$  admitting finite limits that is equipped with a finitary Grothendieck topology  $\tau$ . In this case, the corresponding 1-localic  $\infty$ -topos  $\mathbf{Sh}_\tau(X)$  is coherent and locally coherent.

An elementary way to construct a finitary  $\infty$ -site is to make use of an  $\infty$ -categorical analogue of the notion of pretopology on a 1-category.

**5.24 Definition.** An  $\infty$ -presite is a pair  $(X, E)$  consisting of an  $\infty$ -category  $X$  along with a subcategory  $E \subseteq X$  satisfying the following conditions.

- The subcategory  $E$  contains all equivalences of  $X$ .
- The  $\infty$ -category  $X$  admits finite limits, and  $E$  is stable under base change.
- The  $\infty$ -category  $X$  admits finite coproducts, which are universal, and  $E$  is closed under coproducts.

**5.25 Construction.** If  $(X, E)$  is an  $\infty$ -presite, then there exists a topology  $\tau_E$  in which the  $\tau_E$ -covering sieves are generated by finite families  $\{y_i \rightarrow x\}_{i \in I}$  such that  $\coprod_{i \in I} y_i \rightarrow x$  lies in  $E$ . The  $\infty$ -site  $(X, \tau_E)$  is finitary.

**5.26 Example.** By [SAG, Proposition A.7.5.1], if  $X$  is a bounded coherent  $\infty$ -topos, then  $X$  is also locally coherent.

**5.27 Example.** If  $X$  is a coherent scheme, then the  $\infty$ -topos  $X_{\acute{e}t}$  is coherent and locally coherent.

**5.28 Definition.** Let  $X$  and  $Y$  be coherent  $\infty$ -topoi. We say that a geometric morphism  $f_* : X \rightarrow Y$  is *coherent* if and only if, for any coherent object  $F \in Y$ , the object  $f^*F \in X$  is coherent as well. One writes  $\mathbf{Top}_{\infty}^{coh}$  for the subcategory of  $\mathbf{Top}_{\infty}$  whose objects are coherent  $\infty$ -topoi and whose morphisms are coherent geometric morphisms.

**5.29 Example.** If  $X$  and  $Y$  are coherent schemes and  $f : X \rightarrow Y$  is a coherent morphism thereof, then the geometric morphism  $f_* : X_{\acute{e}t} \rightarrow Y_{\acute{e}t}$  is coherent.

### Classification of bounded coherent $\infty$ -topoi via $\infty$ -pretopoi

In this subsection we explain how an  $\infty$ -topos that is both bounded and coherent is *determined* by its truncated coherent objects.

**5.30 Notation.** Denote by  $\mathbf{Top}_{\infty}^{bc} \subset \mathbf{Top}_{\infty}^{coh}$  the full subcategory spanned by those coherent  $\infty$ -topoi that are also bounded, that is, the *bounded coherent*  $\infty$ -topoi

To a large extent, bounded coherent  $\infty$ -topoi function in much the same way as coherent 1-topoi. In particular, any bounded coherent  $\infty$ -topos is, in a canonical fashion, the  $\infty$ -category of sheaves on an  $\infty$ -site with excellent formal properties.

**5.31 Definition.** An  $\infty$ -category  $X$  is said to be an  $\infty$ -pretopos if and only if the following conditions are satisfied.

- ▶ The  $\infty$ -category  $X$  admits finite limits.
- ▶ The  $\infty$ -category  $X$  admits finite coproducts, which are universal and disjoint.
- ▶ Groupoid objects in  $X$  are effective, and their geometric realisations are universal.

If  $X$  and  $Y$  are  $\infty$ -pretopoi, then a functor  $f^* : Y \rightarrow X$  is a *morphism of  $\infty$ -pretopoi* if  $f^*$  preserves finite limits, finite coproducts, and effective epimorphisms. We write  $\mathbf{preTop}_{\infty} \subset \mathbf{Cat}_{\infty, \kappa_1}$  for the subcategory consisting of  $\infty$ -pretopoi and morphisms of  $\infty$ -pretopoi.

**5.32 Example.** If  $X$  is a coherent  $\infty$ -topos, then the full subcategory  $X^{coh} \subseteq X$  spanned by the coherent objects is an  $\infty$ -pretopos [SAG, Corollary A.6.1.7].

The following two useful facts are immediate from the definitions.

**5.33 Lemma.** Let  $\{X_i\}_{i \in I}$  be a collection of  $\infty$ -pretopoi. Then the product  $\prod_{i \in I} X_i$  in  $\mathbf{Cat}_{\infty, \kappa_1}$  is an  $\infty$ -pretopos and for each  $j \in I$  the projection

$$\mathrm{pr}_j : \prod_{i \in I} X_i \rightarrow X_j$$

is a morphism of  $\infty$ -pretopoi.

**5.34 Lemma.** Given morphisms of  $\infty$ -pretopoi  $X \rightarrow Z$  and  $Y \rightarrow Z$ , the pullback  $X \times_Z Y$  in  $\mathbf{Cat}_{\infty, \kappa_1}$  is an  $\infty$ -pretopos, and the projections

$$\mathrm{pr}_1: X \times_Z Y \rightarrow X \quad \text{and} \quad \mathrm{pr}_2: X \times_Z Y \rightarrow Y$$

are morphisms of  $\infty$ -pretopoi.

**5.35 Notation.** If  $X$  is an  $\infty$ -pretopos, then if  $E \subseteq X$  is the collection of effective epimorphisms, then  $(X, E)$  is an  $\infty$ -presite, and we write  $\mathrm{eff} := \tau_E$  for the resulting finitary topology, the *effective epimorphism topology* [SAG, §A.6.2], which is a subcanonical topology [SAG, Corollary A.6.2.6].

**5.36 Definition.** An  $\infty$ -pretopos  $X$  is *bounded* if and only if  $X$  is essentially  $\kappa_0$ -small and every object of  $X$  is truncated. We write  $\mathrm{preTop}_{\infty}^b \subset \mathrm{preTop}_{\infty}$  for the full subcategory spanned by the bounded  $\infty$ -pretopoi.

**5.37 Theorem** ([SAG, Theorem A.7.5.3]). *The constructions  $\mathbf{X} \mapsto \mathbf{X}_{<\infty}^{\mathrm{coh}}$  and  $\mathbf{X} \mapsto \mathbf{Sh}_{\mathrm{eff}}(\mathbf{X})$  are mutually inverse equivalences of  $\infty$ -categories*

$$\mathbf{Top}_{\infty}^{bc} \simeq \mathrm{preTop}_{\infty}^{b, \mathrm{op}}.$$

The following bounded analogue of Lemma 5.33 will also be useful later.

**5.38 Lemma.** *Let  $\{X_i\}_{i \in I}$  be a finite collection bounded of  $\infty$ -pretopoi. Then the  $\infty$ -pretopos given by the product  $\prod_{i \in I} X_i$  in  $\mathbf{Cat}_{\infty, \kappa_1}$  is a bounded  $\infty$ -pretopos.*

*Proof.* For each  $i \in I$  the  $\infty$ -category  $X_i$  is essentially  $\kappa_0$ -small, so the product  $\prod_{i \in I} X_i$  is also essentially  $\kappa_0$ -small. For any integer  $n \geq -2$ , an object  $F \in \prod_{i \in I} X_i$  is  $n$ -truncated if and only if  $\mathrm{pr}_i(F) \in X_i$  is  $n$ -truncated for all  $i \in I$ . Since  $I$  is finite and every object of each of the  $\infty$ -categories  $\{X_i\}_{i \in I}$  is truncated by assumption, every object of the product  $\prod_{i \in I} X_i$  is truncated.  $\square$

We now recall a convenient pretoposic criterion for checking that a morphism of  $\infty$ -topoi is coherent, which is stated as (\*) in the proof of [SAG, Theorem A.7.5.3].

**5.39 Lemma.** *Let  $X$  be a bounded  $\infty$ -pretopos and  $\mathbf{W}$  a coherent  $\infty$ -topos. A geometric morphism  $p_*: \mathbf{W} \rightarrow \mathbf{Sh}_{\mathrm{eff}}(X)$  is coherent if and only if the composite*

$$X \xleftarrow{j} \mathbf{Sh}_{\mathrm{eff}}(X) \xrightarrow{p^*} \mathbf{W}$$

*of the Yoneda embedding with  $p^*$  factors through  $\mathbf{W}^{\mathrm{coh}}$ .*

Combining this with Theorem 5.37=[SAG, Theorem A.7.5.3] we immediately deduce the following.

**5.40 Corollary.** *Let  $p_*: \mathbf{W} \rightarrow X$  be a geometric morphism between coherent  $\infty$ -topoi. If  $X$  is bounded, then  $p_*$  is coherent if and only if  $p^*$  carries  $X_{<\infty}^{\mathrm{coh}}$  to  $\mathbf{W}^{\mathrm{coh}}$ .*



**5.41 Lemma.** *Let  $(W, \tau_W)$  and  $(X, \tau_X)$  be two  $\infty$ -sites, and let  $p^* : (W, \tau_W) \rightarrow (X, \tau_X)$  be a functor that induces a geometric morphism  $p_* : \mathbf{Sh}_{\tau_X}(X) \rightarrow \mathbf{Sh}_{\tau_W}(W)$ . Write  $j_W : W \rightarrow \mathbf{Sh}_{\tau_W}(W)$  for the sheafified Yoneda embedding. If the topology  $\tau_X$  is finitary, then*

$$p^* j_W : W \rightarrow \mathbf{Sh}_{\tau_X}(X)$$

*factors through  $\mathbf{Sh}_{\tau_X}(X)^{coh} \subset \mathbf{Sh}_{\tau_X}(X)$ .*

*Proof.* We have a commutative square

$$\begin{array}{ccc} W & \xrightarrow{p^*} & X \\ j_W \downarrow & & \downarrow j_X \\ \mathbf{Sh}_{\tau_W}(W) & \xrightarrow{p^*} & \mathbf{Sh}_{\tau_X}(X) \end{array}$$

where the vertical functors are sheafified Yoneda embeddings. The claim now follows from the fact that  $j_X : X \rightarrow \mathbf{Sh}_{\tau_X}(X)$  factors through  $\mathbf{Sh}_{\tau_X}(X)^{coh}$ , since the topology  $\tau_X$  is finitary (Proposition 5.22=[SAG, Proposition A.3.1.3]).  $\square$

## Coherence of inverse limits

We now recall that bounded coherent  $\infty$ -topoi and coherent geometric morphisms are stable under inverse limits in  $\mathbf{Top}_\infty$ .

**5.42 Proposition** ([SAG, Proposition A.8.3.1]). *The  $\infty$ -category  $\mathbf{preTop}_\infty^b$  admits filtered colimits and the forgetful functor  $\mathbf{preTop}_\infty^b \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  preserves filtered colimits.*

**5.43 Proposition** ([SAG, Proposition A.8.3.2]). *Let  $X : A \rightarrow \mathbf{preTop}_\infty^b$  be a filtered diagram of bounded  $\infty$ -pretopoi. Then the natural geometric morphism*

$$\mathbf{Sh}_{\text{eff}}(\text{colim}_{\alpha \in A} X_\alpha) \rightarrow \lim_{\alpha \in A^{op}} \mathbf{Sh}_{\text{eff}}(X_\alpha)$$

*is an equivalence in  $\mathbf{Top}_\infty$ .*

The following is immediate from the previous two propositions and Theorem 5.9=[HTT, Theorem 6.3.3.1].

**5.44 Corollary** ([SAG, Corollary A.8.3.3]). *The  $\infty$ -category  $\mathbf{Top}_\infty^{bc}$  admits inverse limits and the inclusion  $\mathbf{Top}_\infty^{bc} \rightarrow \mathbf{Top}_\infty$  and forgetful functor  $\mathbf{Top}_\infty^{bc} \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  preserves inverse limits.*

## Coherence & preservation of filtered colimits

The goal of this subsection is to prove the appropriate  $\infty$ -toposic generalisation of the fact that a coherent geometric morphism of 1-topoi preserves filtered colimits (see Corollary 5.55).

5.45. Since filtered colimits commute with finite limits in an  $\infty$ -topos, for any  $\infty$ -topos  $X$  and integer  $n \geq -2$ , the inclusion  $\tau_{\leq n} X \hookrightarrow X$  preserves filtered colimits. Thus  $\tau_{\leq n} X$  is an  $\omega$ -accessible localisation of  $X$ .

5.46 **Notation.** For  $n \in \mathbb{N}$  we write  $\Delta_{\leq n} \subset \Delta$  for the full subcategory of  $\Delta$  spanned by those totally ordered finite sets of cardinality at most  $n + 1$ .

5.47. Let  $n \in \mathbb{N}$ . Then the inclusion  $\Delta_{\leq n} \hookrightarrow \Delta$  is *n-limit-cofinal* in the following sense: for any  $n$ -category  $C$  and cosimplicial object  $T: \Delta \rightarrow C$ , the inclusion  $\Delta_{\leq n} \hookrightarrow \Delta$  induces an equivalence  $\lim_{\Delta} T \simeq \lim_{\Delta_{\leq n}} T|_{\Delta_{\leq n}}$ .

5.48 **Notation.** For a morphism  $e: U \rightarrow U'$  in an  $\infty$ -category  $X$  with pullbacks, we write  $\check{C}(e): \Delta^{op} \rightarrow X$  for the *Čech nerve* of  $e$ ; this is the simplicial object that carries  $n \in \Delta^{op}$  to the  $(n + 1)$ -fold iterated fibre product  $U \times_{U'} \cdots \times_{U'} U$ .

5.49 **Proposition** ([SAG, Proposition A.6.2.5]). *Let  $X$  be an  $\infty$ -pretopos and  $F$  a presheaf on  $X$ . Then  $F$  is a sheaf with respect to the effective epimorphism topology if and only if  $F$  satisfies the following conditions.*

(5.49.1) *For every finite set  $U_1, \dots, U_n \in X$  of objects of  $X$ , the natural morphism*

$$F(U_1 \sqcup \cdots \sqcup U_n) \rightarrow F(U_1) \times \cdots \times F(U_n)$$

*is an equivalence.*

(5.49.2) *For every effective epimorphism  $e: U \rightarrow U'$  in  $X$ , the induced morphism*

$$F(U') \rightarrow \lim_{\Delta} F(\check{C}(e))$$

*is an equivalence.*

5.50 **Proposition.** *Let  $X$  be an  $\infty$ -pretopos. Then for any integer  $n \geq -2$ , the restriction of the inclusion  $i: \mathbf{Sh}_{\text{eff}}(X) \hookrightarrow \mathbf{P}(X)$  to  $\tau_{\leq n} \mathbf{Sh}_{\text{eff}}(X)$  preserves filtered colimits.*

*Proof.* The claim is trivial if  $n = -2$ , so assume that  $n \geq -1$ . Let  $F: A \rightarrow \tau_{\leq n} \mathbf{Sh}_{\text{eff}}(X)$  be a filtered diagram.

To show that the presheaf  $\text{colim}_{\alpha \in A} i(F_{\alpha})$  is a sheaf for the effective epimorphism topology, it suffices to show that  $\text{colim}_{\alpha \in A} i(F_{\alpha})$  satisfies conditions (5.49.1) and (5.49.2) of Proposition 5.49=[SAG, Proposition A.6.2.5]. Condition (5.49.1) is clear since filtered colimits commute with finite limits in an  $\infty$ -topos.

To prove that  $\text{colim}_{\alpha \in A} i(F_{\alpha})$  satisfies (5.49.2), note that since  $i$  restricts to an inclusion  $\tau_{\leq n} \mathbf{Sh}_{\text{eff}}(X) \hookrightarrow \tau_{\leq n} \mathbf{P}(X)$ , by (5.45) it suffices to show that for every effective epimorphism  $e: U \rightarrow U'$  in  $X$ , the natural morphism

$$(5.51) \quad \text{colim}_{\alpha \in A} \lim_{\Delta} i(F_{\alpha})(\check{C}(e)) \rightarrow \lim_{\Delta} \text{colim}_{\alpha \in A} i(F_{\alpha})(\check{C}(e))$$

is an equivalence, where the limits appearing in (5.51) are computed in  $\tau_{\leq n} \mathbf{P}(X)$ . Since the  $\infty$ -category  $\tau_{\leq n} \mathbf{P}(X)$  is an  $(n + 1)$ -category, (5.47) shows that the morphism (5.51) is equivalent to the natural morphism

$$(5.52) \quad \text{colim}_{\alpha \in A} \lim_{\Delta_{\leq n+1}} i(F_{\alpha})(\check{C}(e)|_{\Delta_{\leq n+1}}) \rightarrow \lim_{\Delta_{\leq n+1}} \text{colim}_{\alpha \in A} i(F_{\alpha})(\check{C}(e)|_{\Delta_{\leq n+1}}).$$

To conclude, note that (5.52) is an equivalence, because filtered colimits commute with finite limits in  $\tau_{\leq n}\mathbf{P}(X)$ .  $\square$

5.53. The conclusions of Proposition 5.49=[SAG, Proposition A.6.2.5] and Proposition 5.50 also hold when  $X$  is a *local*  $\infty$ -pretopos [SAG, Definition A.6.1.1], but in the present work we have no need to consider local  $\infty$ -pretopoi which are not  $\infty$ -pretopoi.

5.54 **Corollary.** *Let  $f^* : Y \rightarrow X$  be a morphism of  $\infty$ -pretopoi. Then for any integer  $n \geq -2$ , the restriction of  $f_* : \mathbf{Sh}_{\text{eff}}(X) \rightarrow \mathbf{Sh}_{\text{eff}}(Y)$  to  $\tau_{\leq n}\mathbf{Sh}_{\text{eff}}(X)$  preserves filtered colimits.*

*Proof.* Note that  $f_* : \mathbf{Sh}_{\text{eff}}(X) \rightarrow \mathbf{Sh}_{\text{eff}}(Y)$  is the composite

$$\mathbf{Sh}_{\text{eff}}(X) \xleftarrow{i} \mathbf{P}(X) \xrightarrow{f_*} \mathbf{P}(Y) \xrightarrow{L} \mathbf{Sh}_{\text{eff}}(Y),$$

where  $i$  is the inclusion,  $f_* : \mathbf{P}(X) \rightarrow \mathbf{P}(Y)$  is pre-composition with  $f^* : Y \rightarrow X$ , and  $L$  is the left adjoint to the inclusion. Now combine Proposition 5.50 and the fact that both  $f_* : \mathbf{P}(X) \rightarrow \mathbf{P}(Y)$  and  $L$  are left adjoints.  $\square$

In light of Theorem 5.37=[SAG, Theorem A.7.5.3], Corollary 5.54 specialises to the following.

5.55 **Corollary.** *Let  $f_* : X \rightarrow Y$  be a coherent geometric morphism between bounded coherent  $\infty$ -topoi. Then for any integer  $n \geq -2$ , the restriction of  $f_*$  to  $\tau_{\leq n}\mathbf{X}$  preserves filtered colimits.*

## Points, Conceptual Completeness, & Deligne Completeness

In this subsection we discuss points of  $\infty$ -topoi as well as the  $\infty$ -toposic generalisations of the Conceptual Completeness Theorem of Makkai–Reyes and Deligne’s Completeness Theorem.

5.56 **Notation.** For an  $\infty$ -topos  $\mathbf{X}$ , we write

$$\text{Pt}(\mathbf{X}) := \text{Fun}_*(\mathbf{S}, \mathbf{X})^{\text{op}} \simeq \text{Fun}^*(\mathbf{X}, \mathbf{S})$$

of the  $\infty$ -category of *points* of  $\mathbf{X}$ .

We note that a morphism  $g_* \rightarrow f_*$  of  $\text{Pt}(\mathbf{X})$  is a natural transformation  $f_* \rightarrow g_*$ . (The morphisms are the ‘geometric transformations’ usually preferred in 1-topos theory.) This choice syncs well with the direction of posets: for instance, when  $P$  is a noetherian poset, one has  $\text{Pt}(\bar{P}) \simeq P$ .

In general, the passage from an  $\infty$ -topos to its  $\infty$ -category of points loses quite a bit of information. However, the  $\infty$ -toposic version of the Conceptual Completeness Theorem of Makkai–Reyes [34, Theorem 9.2] tells us that bounded coherent  $\infty$ -topoi are determined by their  $\infty$ -categories of points.

5.57 **Theorem** ([SAG, Theorem A.9.0.6]). *A geometric morphism  $f_* : X \rightarrow Y$  between bounded coherent  $\infty$ -topoi is an equivalence if and only if  $f_*$  is coherent and the induced functor  $\text{Pt}(f_*) : \text{Pt}(X) \rightarrow \text{Pt}(Y)$  is an equivalence of  $\infty$ -categories.*

**5.58 Definition.** An  $\infty$ -topos  $\mathbf{X}$  has enough points if a morphism  $\phi$  in  $\mathbf{X}$  is an equivalence if and only if for every point  $x_* \in \text{Pt}(\mathbf{X})$  the stalk  $x_*\phi$  is an equivalence.

In classical topos theory, the Deligne Completeness Theorem states that a coherent topos has enough points. This is no longer true in the setting of  $\infty$ -topoi, the main obstruction being that  $\infty$ -connective morphisms in an  $\infty$ -topos need not be equivalences. For this reason the  $\infty$ -categorical version of Deligne's theorem takes place in the setting of  $\infty$ -topoi where  $\infty$ -connective morphisms are equivalences, i.e.,  $\infty$ -topoi in which Whitehead's Theorem is valid.

**5.59 Definition.** Let  $\mathbf{X}$  be an  $\infty$ -topos. An object  $U \in \mathbf{X}$  is *hypercomplete* if  $U$  is local with respect to the class of  $\infty$ -connective morphisms in  $\mathbf{X}$ . We write  $\mathbf{X}^{\text{hyp}} \subset \mathbf{X}$  for the full subcategory spanned by the hypercomplete objects of  $\mathbf{X}$ . An  $\infty$ -topos is *hypercomplete* if  $\mathbf{X}^{\text{hyp}} = \mathbf{X}$ .

**5.60.** The  $\infty$ -category  $\mathbf{X}^{\text{hyp}} \subset \mathbf{X}$  is a left exact localisation of  $\mathbf{X}$ , hence an  $\infty$ -topos [HTT, p. 699]. Moreover, the  $\infty$ -topos  $\mathbf{X}^{\text{hyp}}$  is hypercomplete [HTT, Lemma 6.5.2.12].

The  $\infty$ -topos  $\mathbf{X}^{\text{hyp}}$  is characterised by the following universal property.

**5.61 Proposition** ([HTT, Proposition 6.5.2.13]). *Let  $\mathbf{X}$  be an  $\infty$ -topos. Then for every hypercomplete  $\infty$ -topos  $\mathbf{H}$ , composition with the inclusion  $\mathbf{X}^{\text{hyp}} \subset \mathbf{X}$  induces an equivalence*

$$\text{Fun}_*(\mathbf{H}, \mathbf{X}^{\text{hyp}}) \simeq \text{Fun}_*(\mathbf{H}, \mathbf{X}).$$

Consequently, the assignment  $\mathbf{X} \mapsto \mathbf{X}^{\text{hyp}}$  defines a functor right adjoint to the inclusion of hypercomplete  $\infty$ -topoi into all  $\infty$ -topoi. For this reason we call  $\mathbf{X}^{\text{hyp}}$  the *hypercompletion* of  $\mathbf{X}$ .

**5.62 Example.** An  $\infty$ -topos with enough points is hypercomplete.

**5.63 Example.** Let  $\mathbf{X}$  be a 1-topos with corresponding 1-localic  $\infty$ -topos  $\mathbf{X}'$ . Then  $\mathbf{X}$  has enough points (in the sense of [SGA<sub>4</sub> I, Exposé IV, Définition 6.4.1]) if and only if the hypercomplete  $\infty$ -topos  $(\mathbf{X}')^{\text{hyp}}$  has enough points.

In light of Example 5.62, the following is the correct  $\infty$ -toposic generalisation of Deligne's completeness theorem.

**5.64 Theorem** ( $\infty$ -Categorical Deligne Completeness; [SAG, Proposition A.4.0.5]). *An  $\infty$ -topos that is locally coherent and hypercomplete has enough points.*

We finish this subsection with the observation that the coherence of an  $\infty$ -topos only depends on its hypercompletion.

**5.65 Proposition** ([SAG, Proposition A.2.2.2]). *Let  $\mathbf{X}$  be an  $\infty$ -topos, and let  $L: \mathbf{X} \rightarrow \mathbf{X}^{\text{hyp}}$  be the left adjoint to the inclusion  $\mathbf{X}^{\text{hyp}} \hookrightarrow \mathbf{X}$ . If  $\mathbf{X}$  is locally  $n$ -coherent for all  $n \geq 0$ , then:*

(5.65.1) *The  $\infty$ -topos  $\mathbf{X}^{\text{hyp}}$  is locally  $n$ -coherent for all  $n \geq 0$ .*

(5.65.2) *An object  $U$  of  $\mathbf{X}^{\text{hyp}}$  is coherent if and only if  $U$  is coherent when viewed as an object of  $\mathbf{X}$ .*

(5.65.3) An object  $U \in \mathbf{X}$  is coherent if and only if  $L(U)$  is coherent.

**5.66 Corollary.** Let  $\mathbf{X}$  be an  $\infty$ -topos. If  $\mathbf{X}$  is (locally) coherent, then the hypercompletion  $\mathbf{X}^{hyp}$  of  $\mathbf{X}$  is (locally) coherent.

**5.67 Example.** Let  $\mathbf{X}$  be a bounded coherent  $\infty$ -topos. Then since  $\mathbf{X}$  is also locally coherent (Example 5.26), the hypercompletion  $\mathbf{X}^{hyp}$  of  $\mathbf{X}$  is coherent and locally coherent.

**5.68.** Please observe that for an  $\infty$ -topos  $\mathbf{X}$ , the hypercompletion  $\mathbf{X}^{hyp}$  has enough points if and only if  $\infty$ -connectiveness of morphisms in  $\mathbf{X}$  can be checked on stalks, i.e., a morphism  $\phi$  in  $\mathbf{X}$  is  $\infty$ -connective if and only if for every point  $x_* \in \text{Pt}(\mathbf{X}) = \text{Pt}(\mathbf{X}^{hyp})$  the stalk  $x_*\phi$  is an equivalence in  $\mathbf{S}$ . The Deligne Completeness Theorem (Theorem 5.64=[SAG, Proposition A.4.0.5]) and Corollary 5.66 show that  $\infty$ -connectiveness in a locally coherent  $\infty$ -topos can be checked on stalks.

## Shape theory

We now recall the basics of *shape theory* for  $\infty$ -topoi. The shape is crucial to the study of Stone  $\infty$ -topoi presented in the next subsection, as well as our development of the *stratified shape* in Part III and *stratified étale homotopy type* in Part IV.

**5.69 Definition.** Let  $\mathbf{X}$  be an  $\infty$ -topos, and write  $\Gamma_! : \mathbf{X} \rightarrow \text{Pro}(\mathbf{S})$  for the proëxistent left adjoint of  $\Gamma^* : \mathbf{S} \rightarrow \mathbf{X}$ . The *shape* of  $\mathbf{X}$  is the prospace  $\Pi_\infty(\mathbf{X}) := \Gamma_!(1_{\mathbf{X}})$ . As a left exact accessible functor  $\mathbf{S} \rightarrow \mathbf{S}$ , the prospace  $\Pi_\infty(\mathbf{X})$  is the composite  $\Gamma_*\Gamma^*$ .

The shape defines a functor  $\Pi_\infty : \mathbf{Top}_\infty \rightarrow \text{Pro}(\mathbf{S})$  whose assignment on morphisms is given by sending a geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  with unit  $\eta : \text{id}_{\mathbf{Y}} \rightarrow f_*f^*$  to the morphism of prospaces corresponding to

$$\Gamma_{Y,*}\eta\Gamma_Y^* : \Gamma_{Y,*}\Gamma_Y^* \rightarrow \Gamma_{Y,*}f_*f^*\Gamma_Y^*$$

in  $\text{Pro}(\mathbf{S})^{op} \subset \text{Fun}(\mathbf{S}, \mathbf{S})$ .

**5.70 Definition.** A geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  of  $\infty$ -topoi is a *shape equivalence* if the induced morphism  $\Pi_\infty(f_*) : \Pi_\infty(\mathbf{X}) \rightarrow \Pi_\infty(\mathbf{Y})$  is an equivalence in  $\text{Pro}(\mathbf{S})$ . An  $\infty$ -topos  $\mathbf{X}$  is said to have *trivial shape* if  $\Pi_\infty(\mathbf{X})$  is a terminal object of  $\text{Pro}(\mathbf{S})$ .

**5.71.** Work of Hoyois [25, Proposition 2.6] shows that a geometric morphism  $f_*$  is a shape equivalence if and only if  $f_*$  induces an equivalence of  $\infty$ -categories of (space-valued) torsors.

**5.72.** The shape functor  $\Pi_\infty : \mathbf{Top}_\infty \rightarrow \text{Pro}(\mathbf{S})$  is left adjoint to the extension

$$\lambda : \text{Pro}(\mathbf{S}) \rightarrow \mathbf{Top}_\infty$$

of the fully faithful functor  $\mathbf{S} \hookrightarrow \mathbf{Top}_\infty$  given by  $\Pi \mapsto \mathbf{S}_{/\Pi} \simeq \text{Fun}(\Pi, \mathbf{S})$ . Note, however, that on the level of prospaces,  $\lambda$  is *not* fully faithful.

**5.73 Notation.** Let  $n \geq -2$  be an integer, and write  $\mathbf{S}_{\leq n} \subset \mathbf{S}$  for the full subcategory spanned by those  $\infty$ -groupoids that are  $n$ -truncated. We write  $\tau_{\leq n} : \text{Pro}(\mathbf{S}) \rightarrow \text{Pro}(\mathbf{S}_{\leq n})$  for the proëxtension of the truncation functor  $\tau_{\leq n} : \mathbf{S} \rightarrow \mathbf{S}_{\leq n}$ . We write

$$\Pi_n := \tau_{\leq n} \circ \Pi_\infty : \mathbf{Top}_\infty \rightarrow \text{Pro}(\mathbf{S}_{\leq n}).$$

## Profinite spaces & Stone $\infty$ -topoi

In this subsection we discuss profinite spaces and their relation to  $\infty$ -topoi, as developed in [SAG, Appendix E].

**5.74 Definition.** We write  $\text{mat} : \mathbf{S}_\pi^\wedge \rightarrow \mathbf{S}$  for the right adjoint to  $(-)_\pi^\wedge$  and refer to  $\text{mat}$  as the *materialisation* functor.

**5.75 Definition.** The *profinite shape* functor is the composite

$$\Pi_\infty^\wedge := (-)_\pi^\wedge \circ \Pi_\infty : \mathbf{Top}_\infty \rightarrow \mathbf{S}_\pi^\wedge$$

of the shape functor  $\Pi_\infty$  with the profinite completion functor  $(-)_\pi^\wedge : \mathbf{Pro}(\mathbf{S}) \rightarrow \mathbf{S}_\pi^\wedge$ .

**5.76 Theorem** ([SAG, Theorem E.2.4.1]). *The composite*

$$\lambda_\pi : \mathbf{S}_\pi^\wedge \hookrightarrow \mathbf{Pro}(\mathbf{S}) \xrightarrow{\lambda} \mathbf{Top}_\infty$$

of the inclusion  $\mathbf{S}_\pi^\wedge \subset \mathbf{Pro}(\mathbf{S})$  with the classifying topos functor  $\lambda$  of (5.72) is fully faithful and right adjoint to the profinite shape functor  $\Pi_\infty^\wedge$ .

**5.77 Definition.** An  $\infty$ -topos  $X$  is *Stone*<sup>9</sup> if  $X$  lies in the essential image of  $\lambda_\pi : \mathbf{S}_\pi^\wedge \hookrightarrow \mathbf{Top}_\infty$ . We write  $\mathbf{Top}_\infty^{\text{Stone}} \subset \mathbf{Top}_\infty$  for the full subcategory spanned by the Stone  $\infty$ -topoi. Consequently, the inclusion  $\mathbf{Top}_\infty^{\text{Stone}} \hookrightarrow \mathbf{Top}_\infty$  admits a left adjoint

$$(-)^{\text{Stone}} : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty^{\text{Stone}}$$

which we refer to as the *Stone reflection*.

**5.78 Proposition** ([SAG, Proposition E.3.1.4]). *Let  $X$  and  $Y$  be  $\infty$ -topoi. If  $Y$  is Stone, then the  $\infty$ -category  $\text{Fun}_*(X, Y)$  is an (essentially small)  $\infty$ -groupoid.*

**5.79.** If  $Y$  is a Stone  $\infty$ -topos, then since  $\mathbf{S}$  is Stone and  $\lambda_\pi$  is fully faithful with left adjoint given by the profinite shape, we see that

$$\text{Pt}(Y) \simeq \text{Map}_{\mathbf{Top}_\infty}(\mathbf{S}, Y) \simeq \text{mat } \Pi_\infty^\wedge(Y) .$$

Since Stone  $\infty$ -topoi are bounded and coherent, Conceptual Completeness (Theorem 5.57=[SAG, Theorem A.9.0.6]) implies the following ‘Whitehead theorem’ for profinite spaces.

**5.80 Theorem** (Whitehead’s Theorem for profinite spaces; [SAG, Theorem E.3.1.6]). *The materialisation functor  $\text{mat} : \mathbf{S}_\pi^\wedge \rightarrow \mathbf{S}$  is conservative.*

**5.81 Proposition** ([SAG, Proposition E.4.6.1]). *Let  $n \in \mathbf{N}$ . A morphism  $f$  in  $\mathbf{S}_\pi^\wedge$  is  $n$ -truncated if and only if  $\text{mat}(f)$  is an  $n$ -truncated morphism of  $\mathbf{S}$ .*

Stone  $\infty$ -topoi have a number of useful alternative characterisations. The first is that, under the assumption of bounded coherence, the conclusion of Proposition 5.78=[SAG, Proposition E.3.1.4] actually characterises Stone  $\infty$ -topoi.

<sup>9</sup>Lurie calls these  $\infty$ -topoi *profinite*.

**5.82 Theorem** ([SAG, Theorem E.3.4.1]). *Let  $\mathbf{X}$  be an  $\infty$ -topos. Then  $\mathbf{X}$  is Stone if and only if both of the following conditions are satisfied.*

- *The  $\infty$ -topos  $\mathbf{X}$  is bounded and coherent.*
- *The  $\infty$ -category of points  $\mathrm{Pt}(\mathbf{X})$  of  $\mathbf{X}$  is an  $\infty$ -groupoid.*

The next characterisation is that bounded coherent objects are in fact *lisse*.

**5.83 Recollection.** Let  $\mathbf{X}$  be an  $\infty$ -topos. An object  $F \in \mathbf{X}$  is said to be a *local system* if and only if there exists a cover  $\{U_\alpha\}_{\alpha \in \Lambda}$  of the terminal object of  $\mathbf{X}$  and a corresponding family  $\{K_\alpha\}_{\alpha \in \Lambda}$  of spaces such that for any  $\alpha \in \Lambda$ , one has an equivalence  $F \times U_\alpha \simeq \Gamma_{\mathbf{X}}^* K_\alpha$ .

We say that a local system  $F$  as above is a *lisse sheaf* or *lisse object*<sup>10</sup> if, in addition, the set  $\Lambda$  can be chosen to be finite, and the spaces  $K_\alpha$  can be chosen to be  $\pi$ -finite.

We denote by  $\mathbf{X}^{\mathrm{locsys}} \subseteq \mathbf{X}$  (respectively, by  $\mathbf{X}^{\mathrm{lisse}} \subseteq \mathbf{X}$ ) the full subcategory spanned by the local systems (respectively, the lisse sheaves). Please note that for any geometric morphism of  $\infty$ -topoi  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$ , the pullback  $f^* : \mathbf{Y} \rightarrow \mathbf{X}$  preserves lisse objects.

There is a simple characterisation of lisse sheaves as a single pullback:

**5.84 Lemma** ([SAG, Proposition E.2.7.7]). *Let  $\mathbf{X}$  be an  $\infty$ -topos. Then an object  $F$  of  $\mathbf{X}$  is lisse if and only if there exist: a full subcategory  $G \subset \mathcal{S}_\pi$  spanned by finitely many objects, an essentially unique geometric morphism  $g_* : \mathbf{X} \rightarrow \mathcal{S}_G$ , and an essentially unique equivalence  $F \simeq g^*(I)$ , where  $I$  classifies the inclusion functor  $G \rightarrow \mathcal{S}$ .*

For later use, let us include the following.

**5.85 Lemma.** *For any  $\pi$ -finite space  $G$ , the  $\infty$ -topos  $\mathcal{S}_G$  is cocompact in  $\mathbf{Top}_{\infty}^{\mathrm{bc}}$ . That is, for any inverse system  $\{X_\alpha\}_{\alpha \in \Lambda}$  of bounded coherent  $\infty$ -topoi with limit  $\mathbf{X}$ , the natural functor*

$$\mathrm{Fun}_*(\mathbf{X}, \mathcal{S}_G) \rightarrow \lim_{\alpha \in \Lambda} \mathrm{Fun}_*(X_\alpha, \mathcal{S}_G)$$

*is an equivalence. In other words, the profinite shape functor, restricted to  $\mathbf{Top}_{\infty}^{\mathrm{bc}}$ , preserves inverse limits.*

*Proof.* We have the canonical identification  $\mathrm{Fun}_*(\mathbf{Y}, \mathcal{S}_G) \simeq \mathrm{Map}_{\mathbf{Y}}(1, \Gamma_{\mathbf{Y}}^*(G))$ , and since  $\mathbf{X}_{<\infty}^{\mathrm{coh}}$  is the filtered colimit of the  $\infty$ -categories  $\mathbf{X}_{\alpha, <\infty}^{\mathrm{coh}}$ , it follows that

$$\mathrm{Map}_{\mathbf{X}}(1, \Gamma_{\mathbf{X}}^*(G)) \simeq \mathrm{colim}_{\alpha \in \Lambda^{\mathrm{op}}} \mathrm{Map}_{X_\alpha}(1, \Gamma_{X_\alpha}^*(G)),$$

as desired. □

**5.86 Proposition** ([SAG, Proposition E.3.1.1]). *Let  $\mathbf{X}$  be  $\infty$ -topos. Then  $\mathbf{X}$  is Stone if and only if both of the following conditions are satisfied.*

- *The  $\infty$ -topos  $\mathbf{X}$  is bounded and coherent.*
- *Every truncated coherent object of  $\mathbf{X}$  is lisse.*

<sup>10</sup>Lurie uses the phrase *locally constant constructible*.

**5.87 Corollary** ([SAG, Corollary E.3.1.2]). *Let  $f_* : X \rightarrow Y$  be a geometric morphism between coherent  $\infty$ -topoi. If  $Y$  is Stone, then  $f_*$  is coherent.*

**5.88 Theorem** ([SAG, Theorem E.2.3.2]). *Let  $X$  be an  $\infty$ -topos. Then:*

- ▶ *The  $\infty$ -category  $X^{\text{lisse}}$  is a bounded  $\infty$ -pretopos and the inclusion  $X^{\text{lisse}} \hookrightarrow X$  is a morphism of  $\infty$ -pretopoi.*
- ▶ *The inclusion  $X^{\text{lisse}} \hookrightarrow X$  induces a geometric morphism  $X \rightarrow \mathbf{Sh}_{\text{eff}}(X^{\text{lisse}})$  which exhibits  $\mathbf{Sh}_{\text{eff}}(X^{\text{lisse}})$  as the Stone reflection of  $X$ .*

**5.89 Corollary** ([SAG, Corollary E.2.3.3]). *Let  $f_* : X \rightarrow Y$  be a geometric morphism of  $\infty$ -topoi. The following are equivalent:*

- ▶ *The induced geometric morphism  $f_*^{\text{Stone}} : X^{\text{Stone}} \rightarrow Y^{\text{Stone}}$  is an equivalence of  $\infty$ -topoi.*
- ▶ *The geometric morphism  $f_*$  is a profinite shape equivalence.*
- ▶ *The morphism  $\text{Pt}(f_*)$  is an equivalence of  $\infty$ -groupoids.*
- ▶ *The pullback functor  $f^*$  restricts to an equivalence of  $\infty$ -categories  $Y^{\text{lisse}} \simeq X^{\text{lisse}}$ .*

## 6 Oriented pushouts & oriented fibre products

Deligne [SGA<sub>7</sub>, II, Exposé XIII; 32] (the latter text written by Gérard Laumon) constructed a 1-topos, called the *evanescent* or *vanishing* topos, which he identified as the natural target for the nearby cycles functor. To do so, he identified, in terms of generating sites, the *oriented fibre product* in a double category of 1-topoi (whose existence was proved first by Giraud [16]). In the  $\infty$ -categorical setting, we shall perform an analogous construction in order to describe the link between two strata in a stratified  $\infty$ -topos that satisfies suitable finiteness hypotheses.

### Recollements of higher topoi

**6.1.** If  $X$  is an  $\infty$ -topos, and  $U$  is an open of  $X$ , then the overcategory  $X_{/U}$  is an  $\infty$ -topos, and the forgetful functor  $j_! : X_{/U} \rightarrow X$  admits a right adjoint  $j^*$ , which itself admits a right adjoint  $j_*$ . The functor  $j_*$  is a fully faithful geometric morphism. In this case, we write  $X_{\setminus U}$  for the closed complement, which is the full subcategory of  $X$  spanned by those objects  $F$  such that  $F \times U \simeq U$ . Write  $i_* : X_{\setminus U} \hookrightarrow X$  for the inclusion. In this case,  $X$  is a recollement (0.9) of  $X_{\setminus U}$  and  $X_{/U}$  with gluing functor  $i^* j_*$ , viz.,

$$X \simeq X_{\setminus U} \tilde{\cup}^{i^* j_*} X_{/U}.$$

**6.2.** Let  $X$  be an  $\infty$ -topos, and let  $i_* : Z \hookrightarrow X$  and  $j_* : U \hookrightarrow X$  be geometric morphisms of  $\infty$ -topoi that exhibit  $X$  as the recollement  $Z \tilde{\cup}^{i^* j_*} U$ . Then since  $i^*$  and  $j^*$  are left exact left adjoints, the natural conservative functor

$$(i^*, j^*) : X \rightarrow Z \sqcup U$$



preserves and reflects colimits and finite limits. (Here  $Z \sqcup U$  denotes the coproduct of  $Z$  and  $U$  in  $\mathbf{Top}_\infty$ , which is the *product* of  $Z$  and  $U$  in  $\mathbf{Cat}_{\infty, \kappa_1}$ .) In particular, a morphism  $f$  in  $X$  is:

- ▶ an effective epimorphism if and only if both  $i^*(f)$  and  $j^*(f)$  are effective epimorphisms.
- ▶  $n$ -truncated for some integer  $n \geq -2$  if and only if both  $i^*(f)$  and  $j^*(f)$  are  $n$ -truncated (0.10).

**6.3.** A recollement of  $\infty$ -topoi is tantamount to a geometric morphism of  $\infty$ -topoi  $X \rightarrow \overline{[1]}$ . Indeed, if  $Z$  and  $U$  are  $\infty$ -topoi, and  $\phi: U \rightarrow Z$  is a left exact functor, then the recollement  $X := Z \tilde{\cup}^\phi U$  is an  $\infty$ -topos, and the essentially unique geometric morphisms  $Z \rightarrow S$  and  $U \rightarrow S$  now induce a geometric morphism

$$X \rightarrow S \tilde{\cup}^{\text{id}_S} S \simeq \overline{[1]}.$$

In the other direction, given a geometric morphism  $X \rightarrow \overline{[1]}$ , the closed subtopos  $X_0 := \overline{\{0\}} \times_{\overline{[1]}} X$  and open subtopos  $X_1 := \overline{\{1\}} \times_{\overline{[1]}} X$  of  $X$  form a recollement of  $X$ .

In a strong sense, the entire theory of stratified  $\infty$ -topoi (Definition 9.6) is a generalisation of this observation.

Since  $n$ -localic and bounded  $\infty$ -topoi (Definition 5.11 & Construction 5.16) are each closed under limits in  $\mathbf{Top}_\infty$ , we deduce the following.

**6.4 Lemma.** *Let  $X$  be an  $\infty$ -topos, and let  $i_*: Z \hookrightarrow X$  and  $j_*: U \hookrightarrow X$  be geometric morphisms of  $\infty$ -topoi that exhibit  $X$  as the recollement  $Z \tilde{\cup}^{i_* j_*} U$ . For any  $n \in \mathbf{N}$ , if  $X$  is  $n$ -localic or bounded, then both  $Z$  and  $U$  are each  $n$ -localic or bounded, respectively.*

**6.5 Warning.** We caution, however, that there isn't a simple converse to Lemma 6.4: it is not the case that the recollement of two bounded  $\infty$ -topoi is necessarily bounded. To ensure this, we need a condition on the gluing functor.

**6.6 Definition.** Let  $Z$  and  $U$  be two bounded  $\infty$ -topoi, and let  $\phi: U \rightarrow Z$  be an accessible left exact functor  $\phi: U \rightarrow Z$ . We say that  $\phi$  is a *bounded gluing functor* if and only if the recollement  $X := Z \tilde{\cup}^\phi U$  is bounded.

**6.7 Question.** Do bounded gluing functors admit a simple or useful intrinsic characterisation?

So much for the boundedness of recollements. Let us now turn to coherence (Definition 5.18). We can easily characterise the coherent objects of a coherent recollement.

**6.8 Proposition** ([DAG XIII, Proposition 2.3.22]). *Let  $n \in \mathbf{N}$ , let  $X$  be an  $(n+1)$ -coherent  $\infty$ -topos, and let  $i_*: Z \hookrightarrow X$  and  $j_*: U \hookrightarrow X$  be geometric morphisms of  $\infty$ -topoi that exhibit  $X$  as the recollement  $Z \tilde{\cup}^{i_* j_*} U$ . If  $U$  is 0-coherent, then an object  $F \in X$  is  $n$ -coherent if and only if both  $i^* F$  and  $j^* F$  are  $n$ -coherent. In particular, the  $\infty$ -topoi  $Z$  and  $U$  are  $n$ -coherent.*

**6.9 Warning.** We caution again that there isn't a simple converse to [Proposition 6.8](#): as with boundedness, it is not the case that the recollement of two coherent  $\infty$ -topoi is necessarily coherent.

**6.10 Definition.** Let  $\mathbf{Z}$  and  $\mathbf{U}$  be two coherent  $\infty$ -topoi, and let  $\phi: \mathbf{U} \rightarrow \mathbf{Z}$  be an accessible left exact functor  $\phi: \mathbf{U} \rightarrow \mathbf{Z}$ . We say that  $\phi$  is a *coherent gluing functor* if and only if the recollement  $\mathbf{X} := \mathbf{Z} \tilde{\cup}^\phi \mathbf{U}$  is coherent.

**6.11.** Let  $\mathbf{Z}$  and  $\mathbf{U}$  be two coherent  $\infty$ -topoi, and let  $\phi: \mathbf{U} \rightarrow \mathbf{Z}$  be an accessible left exact functor. Write  $i_*: \mathbf{Z} \hookrightarrow \mathbf{X}$  and  $j_*: \mathbf{U} \hookrightarrow \mathbf{X}$  for the fully faithful functors defining the recollement. Then one can show that the gluing functor  $\phi$  is coherent if the following conditions are satisfied.

- ▶ The functor  $j_*$  is quasicompact in the sense that for any quasicompact object  $F \in \mathbf{X}$ , the object  $j^*F \in \mathbf{U}$  is also quasicompact.
- ▶ For every  $n \in \mathbf{N}$ , every object  $F \in \mathbf{U}$  admits a family  $\{G_\alpha \rightarrow F\}_{\alpha \in \Lambda}$  in which each  $G_\alpha$  is  $n$ -coherent, and the family  $\{\phi(G_\alpha) \rightarrow \phi(F)\}_{\alpha \in \Lambda}$  is a covering in  $\mathbf{Z}$ .

**6.12 Construction.** Let  $\mathbf{Z}$  and  $\mathbf{U}$  be bounded coherent  $\infty$ -topoi, and let  $\phi: \mathbf{U} \rightarrow \mathbf{Z}$  be an accessible left exact functor. Form the recollement

$$\mathbf{X}' := \mathbf{Z} \tilde{\cup}^\phi \mathbf{U},$$

and write  $i_*: \mathbf{Z} \hookrightarrow \mathbf{X}'$  and  $j_*: \mathbf{U} \hookrightarrow \mathbf{X}'$  for the closed and open embeddings. Consider the full subcategory  $X_0 \subseteq \mathbf{X}'$  spanned by those objects  $F$  such that  $i^*F$  and  $j^*F$  are each truncated coherent, so that  $X_0$  is the oriented fibre product (0.7) in  $\mathbf{Cat}_{\infty, \kappa_1}$ :

$$X_0 = \mathbf{Z}_{< \infty}^{coh} \downarrow_{\mathbf{Z}} \mathbf{U}_{< \infty}^{coh}.$$

Then since  $X_0 \subset \mathbf{X}$  is closed under finite limits, finite coproducts, and the formation of geometric realisations of groupoid objects,  $X_0$  is an  $\infty$ -pretopos and the inclusion  $X_0 \hookrightarrow \mathbf{X}$  is a morphism of  $\infty$ -pretopoi ([Definition 5.31](#)). Moreover, by (6.2) every object of  $X_0$  is truncated and by (0.8) the  $\infty$ -category  $X_0$  is essentially  $\kappa_0$ -small, hence  $X_0$  is a bounded  $\infty$ -pretopos ([Definition 5.36](#)). Consequently, we may form the bounded coherent  $\infty$ -topos ([Notation 5.35](#))

$$\mathbf{X} := \mathbf{Sh}_{\text{eff}}(X_0).$$

By [[SAG, Proposition A.6.4.4](#)], the inclusion  $X_0 \hookrightarrow \mathbf{X}'$  extends (essentially uniquely) to a comparison geometric morphism  $r_*: \mathbf{X}' \rightarrow \mathbf{X}$ , which is not in general an equivalence, but restricts to an equivalence  $r^*: \mathbf{X}_{< \infty}^{coh} \simeq X_0$ . The geometric morphisms  $r_*i_*$  and  $r_*j_*$  are each coherent by construction. We therefore call  $\mathbf{X}$  the *bounded coherent recollement*, and we write

$$\mathbf{Z} \tilde{\cup}_{bc}^\phi \mathbf{U} := \mathbf{X}.$$

**6.13 Lemma.** Let  $\mathbf{Z}$  and  $\mathbf{U}$  be bounded coherent  $\infty$ -topoi, and let  $\phi: \mathbf{U} \rightarrow \mathbf{Z}$  be an accessible left exact functor. Then the natural geometric morphism

$$\mathbf{Z} \tilde{\cup}^{i^*r^*r_*j_*} \mathbf{U} \rightarrow \mathbf{Z} \tilde{\cup}_{bc}^\phi \mathbf{U}$$

is an equivalence.

*Proof.* Write  $X := Z \tilde{U}_{bc}^\phi U$ . The object  $j_! 1_U \in Z \tilde{U}^\phi U$ , is the object  $(\emptyset_Z, 1_U, \emptyset_Z \rightarrow \phi(1_U))$ , which is an open in  $X$  as well as an object of the  $\infty$ -pretopos  $X_0$  of **Construction 6.12**. Thus  $j^* r^*$  restricts to an equivalence  $(X_{/j_! 1_U})_{<\infty}^{coh} \simeq U_{<\infty}^{coh}$ , whence the functor  $r_* j_* : U \rightarrow X_{/j_! 1_U}$  is an equivalence. The truncated coherent objects of the closed subtopos  $X_{\setminus j_! 1_U}$  are precisely those of the form  $(F_Z, 1_U, F_Z \rightarrow \phi(1_U))$  for some truncated coherent object  $F_Z$  of  $Z$ . Hence  $i^* r^*$  restricts to an equivalence  $(X_{\setminus j_! 1_U})_{<\infty}^{coh} \simeq Z_{<\infty}^{coh}$ , whence the functor  $i_* r_* : Z \rightarrow X_{\setminus j_! 1_U}$  is an equivalence.  $\square$

**6.14 Lemma.** *Let  $Z$ , and  $U$  be bounded coherent  $\infty$ -topoi, and let  $\phi : U \rightarrow Z$  be a bounded coherent gluing functor. Then  $Z \tilde{U}^\phi U$  is the bounded coherent recollement.*

*Proof.* This follows from **Proposition 6.8**=[DAG XIII, Proposition 2.3.22] combined with **Theorem 5.37**=[SAG, Theorem A.7.5.3].  $\square$

The critical point that we use repeatedly in the sequel is the observation that the bounded coherent recollement depends only upon the restriction of the gluing functor to truncated coherent objects. More precisely, let  $Z$  and  $U$  be bounded coherent  $\infty$ -topoi, and let  $\phi : U \rightarrow Z$  and  $\phi' : U \rightarrow Z$  be two accessible, left exact functors. Let  $\eta : \phi \rightarrow \phi'$  be a natural transformation. Form the bounded coherent recollements  $X$  and  $X'$  of  $\phi$  and  $\phi'$ , respectively. Now  $\eta$  induces a functor  $\eta^* : X \rightarrow X'$ , which preserves limits and colimits. Consequently,  $\eta^*$  is left adjoint to a geometric morphism  $\eta_*$ . Observe that  $\eta_*$  is completely determined by the morphism of  $\infty$ -pretopoi

$$\eta^* : Z_{<\infty}^{coh} \downarrow_{Z, \phi} U_{<\infty}^{coh} \simeq X_{<\infty}^{coh} \rightarrow (X')_{<\infty}^{coh} \simeq Z_{<\infty}^{coh} \downarrow_{Z, \phi'} U_{<\infty}^{coh}.$$

We thus deduce the following.

**6.15 Proposition.** *Let  $Z$  and  $U$  be bounded coherent  $\infty$ -topoi, and let  $\phi : U \rightarrow Z$  and  $\phi' : U \rightarrow Z$  be two accessible, left exact functors. Let  $\eta : \phi \rightarrow \phi'$  be a natural transformation. If  $\eta|_{U_{<\infty}^{coh}}$  is an equivalence, then  $\eta$  induces an equivalence*

$$Z \tilde{U}_{bc}^\phi U \simeq Z \tilde{U}_{bc}^{\phi'} U.$$

**6.16 Question.** The restriction functor  $\text{Fun}^{lex}(U, Z) \rightarrow \text{Fun}^{lex}(U_{<\infty}^{coh}, Z)$  is, as a result of this proposition, fully faithful on bounded coherent gluing functors, but what is the essential image of the bounded coherent gluing functors? It might be helpful to give a simple intrinsic characterisation.

## Oriented squares

To speak of oriented pullbacks of  $\infty$ -topoi without finding ourselves buried under a mass of pernicious details (or unproved claims) about double  $\infty$ -categories or  $(\infty, 2)$ -categories, we express the universal property of the lax pullback in simple terms. The key kind of square we will have to contemplate is the following.

**6.17 Notation.** The data of geometric morphisms  $f_* : X \rightarrow Z$ ,  $g_* : Y \rightarrow Z$ ,  $p_* : W \rightarrow X$ , and  $q_* : W \rightarrow Y$ , along with a (not necessarily invertible) natural transformation

$\tau: g_*q_* \rightarrow f_*p_*$  will be exhibited by the single square

$$(6.18) \quad \begin{array}{ccc} \mathbf{W} & \xrightarrow{q_*} & \mathbf{Y} \\ p_* \downarrow & \swarrow \tau & \downarrow g_* \\ \mathbf{X} & \xrightarrow{f_*} & \mathbf{Z}. \end{array}$$

**6.19 Warning.** Frustratingly, it seems that this convention for writing 2-cells is the *opposite* of what's written in some of the 1-topos theory literature (but it agrees with much of the algebro-geometric literature); we therefore emphasise that our 2-morphisms are natural transformations between the *right* adjoints.

### Oriented pushouts

The oriented fibre product in  $\mathbf{Cat}_{\infty, \kappa_1}$  of a diagram of  $\infty$ -topoi recovers not the oriented fibre product in  $\mathbf{Top}_{\infty}$ , but rather the *oriented pushout* in  $\mathbf{Top}_{\infty}$ . We shall also have to contemplate the oriented pushout in  $\mathbf{Top}_{\infty}^{bc}$ .

**6.20 Construction.** The  $\infty$ -category  $\mathbf{Top}_{\infty}$  is *tensored* over the  $\infty$ -category  $\mathbf{Cat}_{\infty, \kappa_0}$ . Indeed, if  $\mathbf{W}$  an  $\infty$ -topos and  $C$  is a  $\kappa_0$ -small  $\infty$ -category, then the  $\infty$ -category  $\mathbf{Fun}(C, \mathbf{W})$  is an  $\infty$ -topos, and the functor  $C \rightarrow \mathbf{Fun}_*(\mathbf{W}, \mathbf{Fun}(C, \mathbf{W}))$  that carries an object to the right adjoint of evaluation induces an equivalence of  $\infty$ -categories

$$\mathbf{Fun}_*(\mathbf{Fun}(C, \mathbf{W}), \mathbf{Z}) \simeq \mathbf{Fun}(C, \mathbf{Fun}_*(\mathbf{W}, \mathbf{Z}))$$

for any  $\infty$ -topos  $\mathbf{Z}$ .

Let  $\mathbf{W}$ ,  $\mathbf{Z}$ , and  $\mathbf{U}$  be three  $\infty$ -topoi, and let  $p_*: \mathbf{W} \rightarrow \mathbf{Z}$  and  $q_*: \mathbf{W} \rightarrow \mathbf{U}$  be two geometric morphisms. The recollement  $\mathbf{Z} \tilde{\cup}^{p_*, q_*} \mathbf{U}$  can be identified with the oriented fibre product

$$\mathbf{Z} \downarrow_{\mathbf{W}} \mathbf{U}$$

formed in  $\mathbf{Cat}_{\infty, \kappa_1}$  with respect to the *left* adjoints  $p^*$  and  $q^*$ . We note that  $\mathbf{Z} \tilde{\cup}^{p_*, q_*} \mathbf{U}$  is an  $\infty$ -topos. This  $\infty$ -topos enjoys the following universal property: a geometric morphism

$$\omega(f, g, \tau)_*: \mathbf{Z} \tilde{\cup}^{p_*, q_*} \mathbf{U} \rightarrow \mathbf{X}$$

determines and is determined by an oriented square

$$\begin{array}{ccc} \mathbf{W} & \xrightarrow{q_*} & \mathbf{U} \\ p_* \downarrow & \swarrow \tau & \downarrow g_* \\ \mathbf{Z} & \xrightarrow{f_*} & \mathbf{X}. \end{array}$$

This universal property specifies the  $\infty$ -topos  $\mathbf{Z} \tilde{\cup}^{p_*, q_*} \mathbf{U}$  essentially uniquely. We write

$$\mathbf{Z} \tilde{\cup}^{\mathbf{W}} \mathbf{U} := \mathbf{Z} \tilde{\cup}^{p_*, q_*} \mathbf{U},$$

and we call this  $\infty$ -topos the *oriented pushout* of  $p_*$  and  $q_*$ . In this case, we write  $i_*: \mathbf{Z} \hookrightarrow \mathbf{Z} \tilde{\cup}^{\mathbf{W}} \mathbf{U}$  for the closed embedding and  $j_*: \mathbf{U} \hookrightarrow \mathbf{Z} \tilde{\cup}^{\mathbf{W}} \mathbf{U}$  for its open complement.

**6.21 Warning.** If  $Z$ ,  $U$ , and  $W$  are all bounded coherent, and if  $p_*$  and  $q_*$  are both coherent geometric morphisms, [Warning 6.5](#) & [Warning 6.9](#) still apply: we cannot ensure that the oriented pushout  $Z \tilde{U}^\phi U$  is either bounded or coherent.

**6.22 Construction.** Consider an oriented square

$$\begin{array}{ccc} W & \xrightarrow{q_*} & U \\ p_* \downarrow & \swarrow \tau & \downarrow g_* \\ Z & \xrightarrow{f_*} & X \end{array}$$

where all  $\infty$ -topoi are bounded coherent and all geometric morphisms are coherent. For any truncated coherent object  $G \in X$ , the object  $\omega(f, g, \tau)^* G$  is truncated, and the objects  $i^* \omega(f, g, \tau)^* G \simeq f^* G$  and  $j^* \omega(f, g, \tau)^* G \simeq g^* G$  are each truncated coherent, whence  $\omega(f, g, \tau)_*$  factors through the bounded coherent recollement  $Z \tilde{U}_{bc}^{p_*, q^*} U$  ([Construction 6.12](#)) in an essentially unique manner. Consequently, we write

$$Z \tilde{U}_{bc}^W U := Z \tilde{U}_{bc}^{p_*, q^*} U,$$

and call this  $\infty$ -topos the *bounded coherent oriented pushout*. This is the oriented pushout that is correct in  $\mathbf{Top}_{\infty}^{bc}$ . Accordingly, one has an equivalence of  $\infty$ -pretopoi

$$(Z \tilde{U}_{bc}^W U)_{<\infty}^{coh} \simeq Z_{<\infty}^{coh} \downarrow_{W_{<\infty}^{coh}} U_{<\infty}^{coh}.$$

Please observe that by construction, in the square

$$\begin{array}{ccc} W & \xrightarrow{q_*} & U \\ p_* \downarrow & \swarrow \tau & \downarrow j_* \\ Z & \xrightarrow{i_*} & Z \tilde{U}_{bc}^W U, \end{array}$$

the natural *Beck–Chevalley morphism*

$$\beta_\tau : i^* j_* \rightarrow p_* q^*$$

is an equivalence after restriction to  $U_{<\infty}^{coh}$ . A thorough study of Beck–Chevalley morphisms will occupy [Section 8](#).

## Internal homs & path $\infty$ -topoi

Oriented fibre products have the universal property that is dual to that of oriented pushouts. In order to define them, we must identify the *cotensor* of  $\mathbf{Top}_{\infty}$  over  $\mathbf{Cat}_{\infty, \kappa_0}$ , or at least over  $poSet$ . Partly in order to define oriented fibre products of  $\infty$ -topoi now and partly to define the nerve construction for stratified  $\infty$ -topoi later ([Construction 9.23](#)), we recall some facts about the internal hom in  $\infty$ -topoi. The first point to be made about the internal hom is that it doesn't always exist.

**6.23 Recollection.** Recall [SAG, Theorem 21.1.6.11] that an  $\infty$ -topos  $W$  is *exponentiable* if and only if the functor  $- \times W : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty$  admits a right adjoint  $\mathbf{MOR}(W, -)$ . If  $W$  is exponentiable, then for any  $\infty$ -topos  $Z$ , the points of the  $\infty$ -topos  $\mathbf{MOR}(W, Z)$  are precisely the geometric morphisms  $W \rightarrow Z$ . We thus call  $\mathbf{MOR}(W, Z)$  the *mapping  $\infty$ -topos*. Any compactly generated  $\infty$ -topos is exponentiable (and in fact even more is true: see [SAG, Theorem 21.1.6.12]).

In particular, for any spectral topological space  $S$  and any  $\infty$ -topos  $Z$ , there exists a mapping  $\infty$ -topos  $\mathbf{MOR}(\tilde{S}, Z)$ . A point  $x \in S$  induces a geometric morphism

$$d_{x,*} : \mathbf{MOR}(\tilde{S}, Z) \rightarrow \mathbf{MOR}(\tilde{x}, Z) \simeq Z,$$

and the geometric morphism  $\tilde{S} \rightarrow \mathbf{S}$  induces a geometric morphism

$$\Delta_* : Z \simeq \mathbf{MOR}(\mathbf{S}, Z) \rightarrow \mathbf{MOR}(\tilde{S}, Z).$$

**6.24 Example.** If  $P$  is a finite poset, then one can identify  $\mathbf{MOR}(\tilde{P}, -)$  with the unique limit-preserving endofunctor of  $\mathbf{Top}_\infty$  such that, for any small  $\infty$ -category  $C$ , one has

$$\mathbf{MOR}(\tilde{P}, \mathbf{Fun}(C, \mathbf{S})) \simeq \mathbf{Fun}(\mathbf{Fun}(P, C), \mathbf{S})$$

via the natural functor. In particular, if  $P$  and  $Q$  are finite posets, then

$$\mathbf{MOR}(\tilde{P}, \tilde{Q}) \simeq \widetilde{\mathbf{Fun}(P, Q)}.$$

**6.25 Definition.** For any  $\infty$ -topos  $X$ , the  $\infty$ -topos  $\mathbf{MOR}(\widetilde{[1]}, X)$  is called the *path  $\infty$ -topos* of  $X$  [SAG, Definition 21.3.2.3]. We write  $\mathbf{Path}(X) := \mathbf{MOR}(\widetilde{[1]}, X)$ .

**6.26 Construction.** Let  $(X, \tau)$  be a pair consisting of an essentially  $\kappa_0$ -small  $\infty$ -category  $X$  that admits all finite limits along with a Grothendieck topology  $\tau$ . Write  $X := \mathbf{Sh}_\tau(X)$  for the  $\infty$ -topos of sheaves (of spaces) on  $X$  with respect to  $\tau$ . Then it follows from [SAG, Lemma 21.1.6.16 & Theorem 21.3.2.5] that  $\mathbf{Path}(X)$  is naturally equivalent to the  $\infty$ -topos  $\mathbf{Sh}_{\tau'}(\mathbf{Fun}(\Delta^{1,op}, X))$ , where  $\tau'$  is the topology on  $\mathbf{Fun}(\Delta^{1,op}, X)$  generated by the families

$$\{f_i : v_i \rightarrow u\}_{i \in I},$$

where for each  $i \in I$ , the morphism  $f_i : \Delta^1 \times \Delta^{1,op} \rightarrow X$  is of the form

$$(6.27) \quad \begin{array}{ccc} v_{i,0} & \longleftarrow & v_{i,1} \\ f_{i,0} \downarrow & & \downarrow f_{i,1} \\ u_0 & \longleftarrow & u_1 \end{array}$$

in which one of the following holds:

- ▶ the family  $\{f_{i,0} : v_{i,0} \rightarrow u_0\}_{i \in I}$  generates a  $\tau$ -covering sieve, and for any  $i \in I$ , the morphism  $f_{i,1}$  is an equivalence;
- ▶ the family  $\{f_{i,1} : v_{i,1} \rightarrow u_1\}_{i \in I}$  generates a  $\tau$ -covering sieve, and for any  $i \in I$ , the square (6.27) is a pullback square.

When  $X$  is an  $\infty$ -pretopos equipped with the effective epimorphism topology, then  $\text{Fun}(\Delta^{1,op}, X)$  is an  $\infty$ -pretopos and the topology  $\tau'$  is the effective epimorphism topology.

**6.28 Lemma.** *Let  $n \in \mathbb{N}$ , and let  $Z$  be an  $n$ -localic  $\infty$ -topos. Then the path topos  $\text{Path}(Z)$  is  $n$ -localic.*

*Proof.* Write  $Z \simeq \text{Sh}_\tau(Z)$ , where  $Z$  is a small  $n$ -category with finite limits equipped with a Grothendieck topology  $\tau$ . By [Construction 6.26](#) we can write the path topos  $\text{Path}(Z)$  as sheaves on the  $n$ -category  $\text{Fun}(\Delta^{1,op}, Z)$  with respect to a Grothendieck topology  $\tau'$ :

$$\text{Path}(Z) \simeq \text{Sh}_{\tau'}(\text{Fun}(\Delta^{1,op}, Z)).$$

This is  $n$ -localic. □

## Oriented fibre products

We are now ready to construct the oriented fibre product of  $\infty$ -topoi and to relate it to the classical oriented fibre product of 1-topoi ([Lemma 6.41](#)).

**6.29 Definition.** If  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  are two geometric morphisms of  $\infty$ -topoi, then the *oriented fibre product* is the pullback

$$X \tilde{\times}_Z Y := X \times_{\text{MOR}(\overline{\{0\}}, Z)} \text{MOR}(\overline{\{1\}}, Z) \times_{\text{MOR}(\overline{\{1\}}, Z)} Y$$

in  $\mathbf{Top}_\infty$ . We write  $\text{pr}_{1,*} : X \tilde{\times}_Z Y \rightarrow X$  and  $\text{pr}_{2,*} : X \tilde{\times}_Z Y \rightarrow Y$  for the natural geometric morphisms.

Thus a geometric morphism

$$\psi(p, q, \tau)_* : W \rightarrow X \tilde{\times}_Z Y$$

determines and is determined by a square ([6.18](#)). This universal property specifies the  $\infty$ -topos  $X \tilde{\times}_Z Y$  essentially uniquely.

**6.30 Warning.** Please note that this is *not* the oriented/lax pullback in  $\mathbf{Cat}_{\infty, \kappa_1}$ ; we will therefore take pains to express clearly where the oriented fibre product is taking place.

Additionally, in this paper, the symbol  $\tilde{\times}$  is only ever used for the oriented fibre product in  $\mathbf{Top}_\infty$ ; we only use the notation  $X \downarrow_Z Y$  for the oriented fibre product in some  $\mathbf{Cat}_{\infty, \kappa}$  (see [\(0.7\)](#)).

**6.31.** Please observe that since the exponential functor  $\text{Path}(-) : \mathbf{Top}_\infty \rightarrow \mathbf{Top}_\infty$  is a right adjoint and limits in  $\text{Fun}(\Lambda_2^2, \mathbf{Top}_\infty)$  are computed pointwise, the functor

$$\text{Fun}(\Lambda_2^2, \mathbf{Top}_\infty) \rightarrow \mathbf{Top}_\infty$$

given by the formation of the oriented fibre product preserves limits.

**6.32 Example.** When  $Z = \mathbf{S}$ , the oriented fibre product reduces to the product in  $\mathbf{Top}_\infty$ :

$$X \tilde{\times}_{\mathbf{S}} Y \simeq X \times Y.$$

**6.33.** Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be geometric morphisms of  $\infty$ -topoi. Then under the identifications  $X \simeq X \tilde{\times}_S S$  and  $Y \simeq S \tilde{\times}_S Y$ , the projections  $\text{pr}_{1,*} : X \tilde{\times}_Z Y \rightarrow X$  and  $\text{pr}_{2,*} : X \tilde{\times}_Z Y \rightarrow Y$  are equivalent to  $\text{id}_X \tilde{\times}_{\Gamma_{Z,*}} \Gamma_{Y,*}$  and  $\Gamma_{X,*} \tilde{\times}_{\Gamma_{Z,*}} \text{id}_Y$ , respectively (Notation 5.5).

**6.34 Example.** For any  $\infty$ -topos  $X$ , the oriented fibre product  $X \tilde{\times}_X X$  is canonically identified with the path  $\infty$ -topos  $\text{Path}(X)$ .

**6.35.** For any  $\infty$ -topos  $E$ , the functor  $\text{Fun}_*(E, -)^{op} : \mathbf{Top}_\infty \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  commutes with cotensors with  $\mathbf{Cat}_{\infty, \kappa_1}$  (in particular, cotensoring with  $\Delta^1$ ) and pullbacks of  $\infty$ -topoi, hence  $\text{Fun}_*(E, -)^{op}$  carries oriented fibre products in  $\mathbf{Top}_\infty$  to oriented fibre products in  $\mathbf{Cat}_{\infty, \kappa_1}$ .

Specialising to the case  $E = S$ , we deduce the following.

**6.36 Lemma.** *The functor  $\text{Pt} : \mathbf{Top}_\infty \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  carries oriented fibre products in  $\mathbf{Top}_\infty$  to oriented fibre products in  $\mathbf{Cat}_{\infty, \kappa_1}$ . That is, if  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  are geometric morphisms of  $\infty$ -topoi, then the natural functor*

$$\text{Pt}(X \tilde{\times}_Z Y) \rightarrow \text{Pt}(X) \downarrow_{\text{Pt}(Z)} \text{Pt}(Y)$$

is an equivalence.

**6.37 Example.** There is a canonical geometric morphism

$$\psi(\text{pr}_1, \text{pr}_2, \text{id})_* : X \times_Z Y \rightarrow X \tilde{\times}_Z Y.$$

**6.38 Example.** The  $\infty$ -topos  $X \tilde{\times}_Z Z$  is called the *evanescent* (or *vanishing*)  $\infty$ -topos of  $f_*$ , and the natural functor

$$\Psi_{f,*} := \psi(\text{id}_X, f, \text{id})_* : X \rightarrow X \tilde{\times}_Z Z$$

is called the *nearby cycles functor*. Dually, the  $\infty$ -topos  $Z \tilde{\times}_Z Y$  is called the *coëvanescent* (or *covanishing*)  $\infty$ -topos of  $g_*$ , and the natural functor

$$\Psi_*^g := \psi(g, \text{id}_Y, \text{id})_* : Y \rightarrow Z \tilde{\times}_Z Y$$

is called the *conearby cycles functor*.

One observes that the oriented fibre product can be decomposed into fibre products in  $\mathbf{Top}_\infty$  involving the evanescent and coëvanescent  $\infty$ -topoi as follows: one has

$$X \tilde{\times}_Z Y \simeq (X \tilde{\times}_Z Z) \times_Z Y \quad \text{and} \quad X \tilde{\times}_Z Y \simeq X \times_Z (Z \tilde{\times}_Z Y),$$

and, more symmetrically,

$$X \tilde{\times}_Z Y \simeq (X \tilde{\times}_Z Z) \times_{\text{Path}(Z)} (Z \tilde{\times}_Z Y).$$

**6.39 Example.** Keep the notations of Definition 6.29, and let  $p_* : Z \hookrightarrow Z'$  be a fully faithful geometric morphism. Then  $p_*$  induces an equivalence of  $\infty$ -topoi

$$X \tilde{\times}_Z Y \simeq X \tilde{\times}_{Z'} Y.$$

To see this, simply note that  $X \tilde{\times}_Z Y$  and  $X \tilde{\times}_{Z'} Y$  have the same universal property since  $p_*$  is fully faithful. Hence for the purpose of computing oriented fibre products, we may assume that  $Z$  is a presheaf  $\infty$ -topos.



**6.40 Lemma.** *Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be geometric morphisms of  $\infty$ -topoi. If  $X$ ,  $Y$ , and  $Z$  are  $n$ -localic (Definition 5.11), so is the oriented fibre product  $X \bar{\times}_Z Y$ . More generally, if  $X$ ,  $Y$ , and  $Z$  are bounded (Construction 5.16), so is the oriented fibre product  $X \bar{\times}_Z Y$ .*

*Proof.* For the first assertion, by Lemma 6.28 the oriented fibre product is a limit of  $n$ -localic  $\infty$ -topoi, hence  $n$ -localic. The second claim follows from the fact that formation of the oriented fibre product preserves limits (6.31).  $\square$

The 1-toposic oriented fibre product [26; 27; 32; 37; 41] is related to the oriented fibre product of corresponding 1-localic  $\infty$ -topoi via the following easy result.

**6.41 Lemma.** *Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be geometric morphisms of 1-topoi, and write  $X'$ ,  $Y'$ , and  $Z'$  for the corresponding 1-localic  $\infty$ -topoi associated to  $X$ ,  $Y$ , and  $Z$ , respectively. Then the oriented fibre product of 1-topoi  $X \bar{\times}_Z Y$  is canonically equivalent to the 1-topos of 0-truncated objects of  $X' \bar{\times}_{Z'} Y'$ .*

## Generating $\infty$ -sites for oriented fibre products

We now describe a generating  $\infty$ -site for the oriented fibre product in the setting of sheaf  $\infty$ -topoi. This description is adapted from Deligne's. We employ it to deduce that the oriented fibre product of bounded coherent  $\infty$ -topoi and coherent geometric morphisms is coherent (Lemma 6.48). We begin with oriented fibre products of presheaf topoi.

**6.42 Construction.** Let  $X$ ,  $Y$ , and  $Z$  be three essentially  $\kappa_0$ -small  $\infty$ -categories, each of which admit finite limits. Let  $f^* : Z \rightarrow X$  and  $g^* : Z \rightarrow Y$  be left exact functors that induce, via precomposition, geometric morphisms

$$f_* : P(X) \rightarrow P(Z) \quad \text{and} \quad g_* : P(Y) \rightarrow P(Z)$$

on  $\infty$ -categories of presheaves of spaces.

Represent  $f^*$  and  $g^*$  as a cartesian fibration  $m : M \rightarrow \Lambda_2^2$ , so that the fibres over the vertices 0, 1, and 2 are  $X$ ,  $Y$ , and  $Z$ , respectively, and  $m$  is classified by the diagram  $X \leftarrow Z \rightarrow Y$ . Now form the  $\infty$ -category

$$\vec{W}(f, g) := \text{Fun}_{\Lambda_3^2}(\Lambda_2^2, M) \simeq \text{Fun}(\Delta^1, X) \times_{\text{Fun}(\Delta^1, X)} Z \times_{\text{Fun}(\Delta^1, Y)} \text{Fun}(\Delta^1, Y)$$

of sections of  $m$ . Let us write  $K_Y$  for the class of morphisms  $\phi : \Delta^1 \times \Lambda_2^2 \rightarrow M$  in  $\text{Fun}_{\Lambda_3^2}(\Lambda_2^2, M)$  of the form

$$\begin{array}{ccccc} v_X & \longrightarrow & v_Z & \longleftarrow & v_Y \\ \phi_X \downarrow & & \downarrow \phi_Z & & \downarrow \phi_Y \\ u_X & \longrightarrow & u_Z & \longleftarrow & u_Y \end{array}$$

in which  $\phi_X$  is an equivalence, and the diagram above exhibits  $\phi_Y$  as the pullback of  $g^* \phi_Z$ . Dually, let us write  $K_X$  for those morphisms  $\phi$  in which  $\phi_Y$  is an equivalence, and the diagram above exhibits  $\phi_X$  as the pullback of  $f^* \phi_Z$ .

We now define two new  $\infty$ -categories by inverting these morphisms in the  $\infty$ -categorical sense (0.3):

$$\overline{W}(f, g) := K_Y^{-1} \overline{W}(f, g) \quad \text{and} \quad W(f, g) := K_X^{-1} \overline{W}(f, g).$$

6.43. The  $\infty$ -category  $\overline{W}(f, g)$  admits finite limits, which are computed pointwise. The sets  $K_Y$  and  $K_X$  are stable under composition and pullback. It follows that the classes  $K_Y$  and  $K_X$  each give rise to right calculi of fractions on  $\overline{W}(f, g)$  in the sense of Cisinski's book [9, Theorem 7.2.16].

Consequently, the mapping spaces in  $\overline{W}(f, g)$  admit a very simple description: for any objects  $u, v \in \overline{W}(f, g)$ , write

$$A(u, v) \subseteq \overline{W}(f, g)_{/u} \times_{\overline{W}(f, g)} \overline{W}(f, g)_{/v}$$

for the full subcategory spanned by those diagrams  $u \leftarrow w \rightarrow v$  in which the morphism  $u \leftarrow w$  lies in  $K_Y$ . Then one has a natural weak homotopy equivalence

$$\text{Map}_{\overline{W}(f, g)}(u, v) \simeq \text{Ex}^\infty A(u, v).$$

Furthermore, the  $\infty$ -categories  $\overline{W}(f, g)$  and  $W(f, g)$  admit finite limits, and the localisations  $\overline{W}(f, g) \rightarrow \overline{W}(f, g)$  and  $\overline{W}(f, g) \rightarrow W(f, g)$  each preserve finite limits [9, Corollary 7.1.16 & Theorem 7.2.25].

6.44 **Construction.** Keep the notations of **Construction 6.42**. We also have left exact functors  $p^* : X \rightarrow \overline{W}(f, g)$  and  $q^* : Y \rightarrow \overline{W}(f, g)$  defined by the assignments

$$x \mapsto [x \rightarrow 1 \leftarrow 1] \quad \text{and} \quad y \mapsto [1 \rightarrow 1 \leftarrow y].$$

We also regard these left exact functors as landing in  $\overline{W}(f, g)$  and  $W(f, g)$  by composing with the relevant localisations.

There exists a section  $\sigma : Z \rightarrow \overline{W}(f, g)$  of the natural projection that carries  $z$  to the cartesian section  $f^*(z) \rightarrow z \leftarrow g^*(z)$ . We thus have natural transformations

$$p^* f^* \xleftarrow{\theta} \sigma \xrightarrow{\xi} q^* g^*$$

where for any  $z \in Z$ , the components  $\theta_z$  and  $\xi_z$  are given by the diagram

$$\begin{array}{ccccc} f^*(z) & \longrightarrow & 1 & \longleftarrow & 1 \\ \parallel & & \uparrow ! & & \uparrow ! \\ f^*(z) & \longrightarrow & z & \longleftarrow & g^*(z) \\ \downarrow ! & & \downarrow ! & & \parallel \\ 1 & \longrightarrow & 1 & \longleftarrow & g^*(z). \end{array}$$

In particular, note that  $\theta_z \in K_X$  and  $\xi_z \in K_Y$ . Consequently, when we pass to  $\overline{W}(f, g)$ , we obtain a natural transformation  $\theta \xi^{-1} : q^* g^* \rightarrow p^* f^*$ , and this becomes an equivalence upon passage to  $W(f, g)$ .

Now the functors  $p^*$  and  $q^*$ , along with the natural transformation  $\tau^* := \theta\xi^{-1}$ , gives rise to a square

$$(6.45) \quad \begin{array}{ccc} \mathbf{P}(\overline{W}(f, g)) & \xrightarrow{q^*} & \mathbf{P}(Y) \\ p_* \downarrow & \swarrow_{\tau} & \downarrow g_* \\ \mathbf{P}(X) & \xrightarrow{f_*} & \mathbf{P}(Z), \end{array}$$

which in turn gives rise to an identification of the oriented fibre product of presheaf  $\infty$ -topoi, *viz.*

$$\mathbf{P}(X) \tilde{\times}_{\mathbf{P}(Z)} \mathbf{P}(Y) \simeq \mathbf{P}(\overline{W}(f, g)).$$

In the same manner, one obtains an identification of the oriented fibre product of presheaf  $\infty$ -topoi, *viz.*

$$\mathbf{P}(X) \times_{\mathbf{P}(Z)} \mathbf{P}(Y) \simeq \mathbf{P}(W(f, g)).$$

**6.46 Construction.** Let  $(X, \tau_X)$ ,  $(Y, \tau_Y)$ , and  $(Z, \tau_Z)$  be three essentially  $\kappa_0$ -small finitary  $\infty$ -sites (Definition 5.21). Let  $f^* : Z \rightarrow X$  and  $g^* : Z \rightarrow Y$  be left exact functors, and assume that the two functors  $f_* : \mathbf{P}(X) \rightarrow \mathbf{P}(Z)$  and  $g_* : \mathbf{P}(Y) \rightarrow \mathbf{P}(Z)$  descend to geometric morphisms

$$f_* : \mathbf{X} := \mathbf{Sh}_{\tau_X}(X) \rightarrow \mathbf{Sh}_{\tau_Z}(Z) =: \mathbf{Z} \quad \text{and} \quad g_* : \mathbf{Y} := \mathbf{Sh}_{\tau_Y}(Y) \rightarrow \mathbf{Sh}_{\tau_Z}(Z) =: \mathbf{Z}.$$

Define the  $\infty$ -category  $\overline{W}(f, g)$  as in Construction 6.42. Then one has a natural equivalence of  $\infty$ -topoi

$$\mathbf{X} \tilde{\times}_{\mathbf{Z}} \mathbf{Y} \simeq \mathbf{Sh}_{\bar{\tau}}(\overline{W}(f, g)),$$

where  $\bar{\tau}$  is the finitary topology generated by the families  $\{\phi_i : v_i \rightarrow u\}_{i \in I}$ , in which for each  $i \in I$ , the morphism  $\phi_i$  is the image of a morphism of  $\overline{W}(f, g)$  of the form

$$\begin{array}{ccccc} v_{i,X} & \longrightarrow & v_{i,Z} & \longleftarrow & v_{i,Y} \\ \phi_{i,X} \downarrow & & \downarrow \phi_{i,Z} & & \downarrow \phi_{i,Y} \\ u_X & \longrightarrow & u_Z & \longleftarrow & u_Y \end{array}$$

in which one of the following holds:

- ▶ the family  $\{\phi_{i,X} : v_{i,X} \rightarrow u_X\}_{i \in I}$  generates a  $\tau_X$ -covering sieve, and for any  $i \in I$ , the morphisms  $\phi_{i,Z}$  and  $\phi_{i,Y}$  are equivalences;
- ▶ the family  $\{\phi_{i,Y} : v_{i,Y} \rightarrow u_Y\}_{i \in I}$  generates a  $\tau_Y$ -covering sieve, and for any  $i \in I$ , the morphisms  $\phi_{i,Z}$  and  $\phi_{i,X}$  are equivalences.

The topology  $\tau_Z$  is irrelevant here, as we should expect, since  $\mathbf{X} \tilde{\times}_{\mathbf{Z}} \mathbf{Y} \simeq \mathbf{X} \tilde{\times}_{\mathbf{P}(Z)} \mathbf{Y}$  (Example 6.39).

Please observe that the finitary topology  $\tau$  on  $W(f, g)$  generated by these same families produces the usual (unoriented) fibre product of  $\infty$ -topoi, *viz.*,

$$\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} \simeq \mathbf{Sh}_{\tau}(W(f, g)).$$

If each of  $X$ ,  $Y$ , and  $Z$  is an  $\infty$ -pretopos, each of the functors  $f^*$  and  $g^*$  is an  $\infty$ -pretopos morphism, and each of  $\tau_X$ ,  $\tau_Y$ , and  $\tau_Z$  is the effective epimorphism topology, then  $\overline{W}(f, g)$  and  $W(f, g)$  are each  $\infty$ -pretopoi, and  $\overline{\tau}$  and  $\tau$  are each the effective epimorphism topology.

**6.47 Lemma.** *Keep the notations of [Construction 6.46](#) and write  $j_X: X \rightarrow \mathbf{X}$  and  $j_Y: Y \rightarrow \mathbf{Y}$  for the sheafified Yoneda embeddings. Then:*

(6.47.1) *The oriented fibre product  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$  is coherent and locally coherent, and for any objects  $x \in X$  and  $y \in Y$ , the objects  $\mathrm{pr}_1^* j_X(x)$  and  $\mathrm{pr}_2^* j_Y(y)$  are each coherent in  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$ .*

(6.47.2) *The pullback  $\mathbf{X} \times_Z \mathbf{Y}$  is coherent and locally coherent, and for any objects  $x \in X$  and  $y \in Y$ , the objects  $\mathrm{pr}_1^* j_X(x)$  and  $\mathrm{pr}_2^* j_Y(y)$  are each coherent in  $\mathbf{X} \times_Z \mathbf{Y}$ .*

*Proof.* [Proposition 5.22](#)=[[SAG, Proposition A.3.1.3](#)] ensures that the  $\infty$ -topoi  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$  and  $\mathbf{X} \times_Z \mathbf{Y}$  are coherent and locally coherent.

The fact that  $\mathrm{pr}_1^*$  carries  $j_X(X)$  to  $(\mathbf{X} \tilde{\times}_Z \mathbf{Y})^{\mathrm{coh}}$  follows from [Lemma 5.41](#) since  $\mathrm{pr}_1^*$  is induced by the functor

$$(X, \tau_X) \rightarrow (\overline{W}(f, g), \overline{\tau}).$$

The proof that  $\mathrm{pr}_2^*$  carries  $j_Y(Y)$  to  $(\mathbf{X} \tilde{\times}_Z \mathbf{Y})^{\mathrm{coh}}$  is the same.

The proof of (6.47.2) is the same as the proof of (6.47.1), where we replace the  $\infty$ -site  $(\overline{W}(f, g), \overline{\tau})$  by  $(W(f, g), \tau)$ .  $\square$

**6.48 Lemma.** *Let  $f_*: \mathbf{X} \rightarrow \mathbf{Z}$  and  $g_*: \mathbf{Y} \rightarrow \mathbf{Z}$  be coherent geometric morphisms between bounded coherent  $\infty$ -topoi. Then:*

(6.48.1) *The oriented fibre product  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$  is bounded coherent and the projections  $\mathrm{pr}_{1,*}$  and  $\mathrm{pr}_{2,*}$  are coherent.*

(6.48.2) *The pullback  $\mathbf{X} \times_Z \mathbf{Y}$  is bounded coherent and the projections  $\mathrm{pr}_{1,*}$  and  $\mathrm{pr}_{2,*}$  are coherent.*

*Proof.* By [Lemmas 6.40](#) and [6.47](#) the oriented fibre product  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$  is bounded coherent. By [Lemma 6.47](#) and the closure of bounded  $\infty$ -topoi under limits in  $\mathbf{Top}_\infty$ , the pullback  $\mathbf{X} \times_Z \mathbf{Y}$  is also bounded coherent. Using the effective epimorphism topologies on  $X = \mathbf{X}_{<\infty}^{\mathrm{coh}}$ ,  $Y = \mathbf{Y}_{<\infty}^{\mathrm{coh}}$ , and  $Z = \mathbf{Z}_{<\infty}^{\mathrm{coh}}$  and the fact that the effective epimorphism topology on an  $\infty$ -pretopos is subcanonical (see [Notation 5.35](#)), [Lemma 6.47](#) and [Corollary 5.40](#) together imply that  $\mathrm{pr}_{1,*}$  and  $\mathrm{pr}_{2,*}$  are coherent.  $\square$

In the case of [Lemma 6.48](#), the  $\infty$ -topos  $\mathbf{X} \tilde{\times}_Z \mathbf{Y}$  is *determined* by its  $\infty$ -category of points in the following sense.

**6.49 Proposition.** *An oriented square*

$$\begin{array}{ccc} W & \xrightarrow{q_*} & Y \\ P_* \downarrow & \swarrow \tau & \downarrow g_* \\ X & \xrightarrow{f_*} & Z, \end{array}$$

of bounded coherent  $\infty$ -topoi and coherent geometric morphisms is an oriented fibre product square if and only if the induced oriented square

$$\begin{array}{ccc} \mathrm{Pt}(\mathbf{W}) & \xrightarrow{q_*} & \mathrm{Pt}(\mathbf{Y}) \\ p_* \downarrow & \nearrow \tau & \downarrow g_* \\ \mathrm{Pt}(\mathbf{X}) & \xrightarrow{f_*} & \mathrm{Pt}(\mathbf{Z}), \end{array}$$

in  $\mathbf{Cat}_{\infty, \kappa_1}$  exhibits  $\mathrm{Pt}(\mathbf{W})$  as the oriented fibre product  $\mathrm{Pt}(\mathbf{X}) \downarrow_{\mathrm{Pt}(\mathbf{Z})} \mathrm{Pt}(\mathbf{Y})$  (0.7).

*Proof.* This follows from Conceptual Completeness (Theorem 5.57=[SAG, Theorem A.9.o.6]), along with the fact that the functor  $\mathrm{Pt}: \mathbf{Top}_{\infty} \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  preserves oriented fibre product squares (Lemma 6.36).  $\square$

## Compatibility of oriented fibre products & étale geometric morphisms

We turn to the compatibility of oriented fibre products with étale geometric morphisms. Our treatment is inspired by Illusie's discussion [27, Exposé XI, 1.1o(b)]. First we prove what must be a standard fact about the compatibility of ordinary pullbacks and étale geometric morphisms (Lemma 6.51) which we could not locate in the literature.

**6.50 Notation.** Let  $f_*: \mathbf{X} \rightarrow \mathbf{Z}$  and  $g_*: \mathbf{Y} \rightarrow \mathbf{Z}$  be geometric morphisms of  $\infty$ -topoi, and suppose we are given objects  $X \in \mathbf{X}$ ,  $Y \in \mathbf{Y}$ , and  $Z \in \mathbf{Z}$ , along with morphisms  $\phi: X \rightarrow f^*(Z)$  and  $\psi: Y \rightarrow g^*(Z)$ . We write

$$X \times_{\mathbf{Z}} Y := \mathrm{pr}_1^*(X) \times_{\mathrm{pr}_1^* f^*(Z)} \mathrm{pr}_2^*(Y) \in \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$$

for the pullback of  $\mathrm{pr}_1^*(X)$  and  $\mathrm{pr}_2^*(Y)$  over  $\mathrm{pr}_1^* f^*(Z) \simeq \mathrm{pr}_2^* g^*(Z)$  formed in the (un-oriented) pullback topos  $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ .

**6.51 Lemma.** *Keep the notations of Notation 6.50. Then the natural geometric morphism  $p_*: \mathbf{X}_{/X} \times_{\mathbf{Z}/Z} \mathbf{Y}_{/Y} \rightarrow \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$  is étale and  $p_!(1) \simeq X \times_{\mathbf{Z}} Y$ .*

*Proof.* First note that the commutative square

$$\begin{array}{ccccc} (\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/(X \times_{\mathbf{Z}} Y)} & \longrightarrow & (\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/\mathrm{pr}_2^*(Y)} & \longrightarrow & \mathbf{Y}_{/Y} \\ \downarrow & & & & \downarrow \\ (\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/\mathrm{pr}_1^*(X)} & & & & \\ \downarrow & & & & \downarrow \\ \mathbf{X}_{/X} & \longrightarrow & & \longrightarrow & \mathbf{Z}_{/Z} \end{array}$$

defines a geometric morphism  $e_*: (\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/(X \times_{\mathbf{Z}} Y)} \rightarrow \mathbf{X}_{/X} \times_{\mathbf{Z}/Z} \mathbf{Y}_{/Y}$ . We claim that  $e_*$  is an equivalence of  $\infty$ -topoi. Indeed, for any  $\infty$ -topos  $\mathbf{E}$ , consider the commutative

square

$$\begin{array}{ccc} \mathrm{Fun}^*(\mathbf{X}/\mathbf{X} \times_{\mathbf{Z}/\mathbf{Z}} \mathbf{Y}/\mathbf{Y}, \mathbf{E}) & \xrightarrow{\sim} & \mathrm{Fun}^*(\mathbf{X}/\mathbf{X}, \mathbf{E}) \times_{\mathrm{Fun}^*(\mathbf{Z}/\mathbf{Z}, \mathbf{E})} \mathrm{Fun}^*(\mathbf{Y}/\mathbf{Y}, \mathbf{E}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}, \mathbf{E}) & \xrightarrow{\sim} & \mathrm{Fun}^*(\mathbf{X}, \mathbf{E}) \times_{\mathrm{Fun}^*(\mathbf{Z}, \mathbf{E})} \mathrm{Fun}^*(\mathbf{Y}, \mathbf{E}). \end{array}$$

Now it follows from [Recollection 5.6](#)=[[HTT](#), [Corollary 6.3.5.6](#)] that the functor

$$\mathrm{Fun}^*(\mathbf{X}/\mathbf{X} \times_{\mathbf{Z}/\mathbf{Z}} \mathbf{Y}/\mathbf{Y}, \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}, \mathbf{E})$$

is a left fibration whose fibre over an object  $h^*$  is the space

$$\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{X})) \times_{\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{Z}))} \mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_2^*(\mathbf{Y})) \simeq \mathrm{Map}_{\mathbf{E}}(1, h^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})).$$

On the other hand, again by [Recollection 5.6](#)=[[HTT](#), [Corollary 6.3.5.6](#)], the natural geometric morphism  $(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})} \rightarrow \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$  induces a left fibration

$$\mathrm{Fun}^*((\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})}, \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}, \mathbf{E})$$

whose fibre over  $h^*$  is the space  $\mathrm{Map}_{\mathbf{E}}(1, h^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}))$ . Thus the geometric morphism  $e_*$  induces a fibrewise equivalence

$$\mathrm{Fun}^*((\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})_{/(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y})}, \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathbf{X}/\mathbf{X} \times_{\mathbf{Z}/\mathbf{Z}} \mathbf{Y}/\mathbf{Y}, \mathbf{E})$$

of left fibrations over  $\mathrm{Fun}^*(\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}, \mathbf{E})$ . □

Now we turn to the compatibility of oriented fibre products and étale geometric morphisms. We can employ essentially the same reasoning as in [Lemma 6.5.1](#).

**6.52 Lemma.** *Let  $\mathbf{Z}$  be an  $\infty$ -topos, and let  $Z \in \mathbf{Z}$  be an object. Then the natural geometric morphism  $p_* : \mathrm{Path}(\mathbf{Z}/\mathbf{Z}) \rightarrow \mathrm{Path}(\mathbf{Z})$  is étale and  $p_!(1) \simeq \mathrm{pr}_1^*(Z)$ .*

*Proof.* We have two geometric morphisms

$$p_* : \mathrm{Path}(\mathbf{Z})_{/\mathrm{pr}_1^*(Z)} \rightarrow \mathbf{Z}/\mathbf{Z} \quad \text{and} \quad q_* : \mathrm{Path}(\mathbf{Z})_{/\mathrm{pr}_1^*(Z)} \rightarrow \mathrm{Path}(\mathbf{Z})_{/\mathrm{pr}_2^*(Z)} \rightarrow \mathbf{Z}/\mathbf{Z}$$

along with a natural transformation  $\sigma : q_* \rightarrow p_*$ . These furnish us with a geometric morphism

$$e_* : \mathrm{Path}(\mathbf{Z})_{/\mathrm{pr}_1^*(Z)} \rightarrow \mathrm{Path}(\mathbf{Z}/\mathbf{Z})$$

over  $\mathrm{Path}(\mathbf{Z})$ . We claim that  $e_*$  is an equivalence of  $\infty$ -topoi.

First, for any  $\infty$ -topos  $\mathbf{E}$ , consider the commutative square

$$\begin{array}{ccc} \mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}/\mathbf{Z}), \mathbf{E}) & \xrightarrow{\sim} & \mathrm{Fun}(\Delta^1, \mathrm{Fun}^*(\mathbf{Z}/\mathbf{Z}, \mathbf{E})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}), \mathbf{E}) & \xrightarrow{\sim} & \mathrm{Fun}(\Delta^1, \mathrm{Fun}^*(\mathbf{Z}, \mathbf{E})). \end{array}$$

It follows from [HTT, Corollaries 2.1.2.9 & 6.3.5.6] that the functor

$$\mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}/\mathbf{Z}), \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}), \mathbf{E})$$

is a left fibration whose fibre over  $h^*$  is the space

$$\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{Z})) \times_{\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_2^*(\mathbf{Z}))} \mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_2^*(\mathbf{Z})) \simeq \mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{Z})).$$

Here the map  $\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{Z})) \rightarrow \mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_2^*(\mathbf{Z}))$  is induced by the natural transformation  $\hat{\tau}: \mathrm{pr}_1^* \rightarrow \mathrm{pr}_2^*$  adjoint to the defining natural transformation  $\tau: \mathrm{pr}_{2,*} \rightarrow \mathrm{pr}_{1,*}$  of the path  $\infty$ -topos  $\mathrm{Path}(\mathbf{Z})$ .

On the other hand, by [Recollection 5.6](#)=[HTT, [Corollary 6.3.5.6](#)] for any  $\infty$ -topos  $\mathbf{E}$ , the natural geometric morphism  $\mathrm{Path}(\mathbf{Z})/\mathrm{pr}_1^*(\mathbf{Z}) \rightarrow \mathrm{Path}(\mathbf{Z})$  induces a left fibration

$$\mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z})/\mathrm{pr}_1^*(\mathbf{Z}), \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}), \mathbf{E})$$

whose fibre over  $h^*$  is the space  $\mathrm{Map}_{\mathbf{E}}(1, h^* \mathrm{pr}_1^*(\mathbf{Z}))$ . Thus for any  $\infty$ -topos  $\mathbf{E}$ , the geometric morphism  $e_*$  induces a fibrewise equivalence

$$\mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z})/\mathrm{pr}_1^*(\mathbf{Z}), \mathbf{E}) \rightarrow \mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}/\mathbf{Z}), \mathbf{E})$$

of left fibrations over  $\mathrm{Fun}^*(\mathrm{Path}(\mathbf{Z}), \mathbf{E})$ . □

**6.53 Construction.** Let  $f_*: \mathbf{X} \rightarrow \mathbf{Z}$  and  $g_*: \mathbf{Y} \rightarrow \mathbf{Z}$  be geometric morphisms of  $\infty$ -topoi, and let  $X \in \mathbf{X}$ ,  $Y \in \mathbf{Y}$ , and  $Z \in \mathbf{Z}$  be objects, along with morphisms  $\phi: X \rightarrow f^*(Z)$  and  $\psi: Y \rightarrow g^*(Z)$ . Form the oriented fibre product

$$\begin{array}{ccc} \mathbf{X} \tilde{\times}_{\mathbf{Z}} \mathbf{Y} & \xrightarrow{\mathrm{pr}_{2,*}} & \mathbf{Y} \\ \mathrm{pr}_{1,*} \downarrow & \swarrow_{\tau} & \downarrow g_* \\ \mathbf{X} & \xrightarrow{f_*} & \mathbf{Z}. \end{array}$$

Write  $X \tilde{\times}_{\mathbf{Z}} Y$  for the object of  $\mathbf{X} \tilde{\times}_{\mathbf{Z}} \mathbf{Y}$  defined by the pullback square

$$\begin{array}{ccc} X \tilde{\times}_{\mathbf{Z}} Y & \longrightarrow & \mathrm{pr}_2^*(Y) \\ \downarrow \lrcorner & & \downarrow \mathrm{pr}_2^*(\psi) \\ \mathrm{pr}_1^*(X) & \xrightarrow{\hat{\tau}(Z) \circ \mathrm{pr}_1^*(\phi)} & \mathrm{pr}_2^*(g^*(Z)), \end{array}$$

where  $\hat{\tau}: \mathrm{pr}_1^* f^* \rightarrow \mathrm{pr}_2^* g^*$  is the natural transformation adjoint to  $\tau: g_* \mathrm{pr}_{2,*} \rightarrow f_* \mathrm{pr}_{1,*}$ .

[Lemma 6.52](#) and [Lemma 6.51](#) together now imply the following.

**6.54 Proposition.** *Keep the notations of [Construction 6.53](#). Then the natural geometric morphism  $p_*: \mathbf{X}/\mathbf{X} \tilde{\times}_{\mathbf{Z}/\mathbf{Z}} \mathbf{Y}/\mathbf{Y} \rightarrow \mathbf{X} \tilde{\times}_{\mathbf{Z}} \mathbf{Y}$  is étale and  $p_!(1) \simeq \mathrm{pr}_1^*(X \tilde{\times}_{\mathbf{Z}} Y)$ .*

**6.55 Corollary.** *Keep the notations of [Construction 6.53](#). If the morphism  $\mathrm{pr}_2^*(\psi) : \mathrm{pr}_2^*(Y) \rightarrow \mathrm{pr}_2^*g^*(Z)$  is an equivalence, then we have a natural equivalence*

$$(X \tilde{\times}_Z Y)_{/X \tilde{\times}_Z Y} \simeq (X \tilde{\times}_Z Y)_{/ \mathrm{pr}_1^*(X)}.$$

**6.56.** Keep the notation of [Construction 6.53](#) and assume, in addition, that  $X, Y$  and  $Z$  are bounded coherent, the geometric morphisms  $f_*$  and  $g_*$  are coherent, and the objects  $X, Y$ , and  $Z$  are all truncated coherent. Then the object  $X \tilde{\times}_Z Y \in \mathbf{X} \tilde{\times}_Z \mathbf{Y}$  is the image of the object of  $\overline{W}(f, g)$  ([Construction 6.46](#)) given by  $X \rightarrow Z \leftarrow Y$  under the Yoneda embedding  $j : \overline{W}(f, g) \hookrightarrow \mathbf{X} \tilde{\times}_Z \mathbf{Y}$ .

## 7 Local $\infty$ -topoi & localisations

In this section we generalise the basic theory of what are usually called *local geometric morphisms* and *local topoi* to the setting of  $\infty$ -topoi [[SGA<sub>4</sub> II](#), Exposé IV, §8; [29](#), §C.3.6; [30](#)]. The  $\infty$ -toposic theory follows the 1-toposic story very closely; as such, a number of items in this section are likely known to experts.

### Quasi-equivalences

As a precursor, we begin by discussing the  $\infty$ -toposic generalisation of the notion of a *connected geometric morphism* [[29](#), p. 525]. In the homotopical setting, the term ‘connected’ (and its variants) doesn’t seem appropriate. Instead, we elect for the distinct term *quasi-equivalence*.

**7.1 Definition.** A geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  of  $\infty$ -topoi is a *quasi-equivalence* if the pullback functor  $f^*$  is fully faithful.

**7.2.** Every geometric morphism of  $\infty$ -topoi factors as the composite of a quasi-equivalence followed by an *algebraic* geometric morphism, and this factorisation is unique up to (canonical) equivalence [[HTT](#), [Proposition 6.3.6.2](#)].

If  $f_*$  is a quasi-equivalence, then  $f^*$  is fully faithful, whence we deduce the following.

**7.3 Lemma.** *Let  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  be a quasi-equivalence of  $\infty$ -topoi. Then the canonical natural transformation  $\Gamma_{Y,*} \rightarrow \Gamma_{X,*}f^*$  is an equivalence ([Notation 5.5](#)).*

**7.4.** If  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  is a quasi-equivalence of  $\infty$ -topoi, then by composing the canonical natural transformation  $\Gamma_{Y,*} \rightarrow \Gamma_{X,*}f^*$  with  $\Gamma_Y^*$ , [Lemma 7.3](#) ensures that the canonical natural transformation

$$\Gamma_{Y,*}\Gamma_Y^* \rightarrow \Gamma_{X,*}f^*\Gamma_Y^* \simeq \Gamma_{Y,*}f_*f^*\Gamma_Y^*$$

is an equivalence in  $\mathrm{Pro}(\mathcal{S})^{op} \subset \mathrm{Fun}(\mathcal{S}, \mathcal{S})$ , so that  $f_*$  is a shape equivalence ([Definition 5.70](#)).

**7.5.** As noted in [[HTT](#), [Remark 7.1.6.12](#)], an  $\infty$ -topos  $\mathbf{X}$  has trivial shape if and only if the geometric morphism  $\mathbf{X} \rightarrow \mathcal{S}$  is a quasi-equivalence. However, in general a shape equivalence of  $\infty$ -topoi need not be a quasi-equivalence.



## Local $\infty$ -topoi

Now we specialise to local  $\infty$ -topoi.

**7.6 Definition.** A geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  of  $\infty$ -topoi is said to be *coëssential* if  $f_*$  admits a right adjoint  $f^! : \mathbf{Y} \rightarrow \mathbf{X}$ . In this case, the functor  $f^!$  and its left adjoint  $f_*$  define a geometric morphism  $f^! : \mathbf{Y} \rightarrow \mathbf{X}$  called the *centre* of  $f_*$ .

The next definition generalises what are sometimes called *local geometric morphisms* in the 1-topos theory literature [29, §C.3.6; 30]. We instead choose terminology that syncs with the algebro-geometric terminology for local rings and doesn't conflict with other uses of the term 'local' in higher category theory.

**7.7 Definition.** A geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  of  $\infty$ -topoi is said to exhibit  $\mathbf{X}$  as *local over  $\mathbf{Y}$*  if  $f_*$  is both coëssential and a quasi-equivalence.

An  $\infty$ -topos  $\mathbf{X}$  is said to be *local* if  $\mathbf{X}$  is local over  $\mathbf{S}$ . In this case we simply call  $\Gamma^! : \mathbf{S} \rightarrow \mathbf{X}$  the *centre* of  $\mathbf{X}$ .

**7.8.** Please observe that a geometric morphism of  $\infty$ -topoi  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  exhibits  $\mathbf{X}$  as local over  $\mathbf{Y}$  if and only if the functor  $f_*$  admits a *fully faithful* right adjoint  $f^!$ . Equivalently,  $\mathbf{X}$  is local over  $\mathbf{Y}$  if and only if  $f_*$  admits a section  $f^!$  in the  $(\infty, 2)$ -category  $\mathbf{Top}_{\infty}$ .

**7.9.** Let  $\mathbf{X}$  be an  $\infty$ -topos. Note that if the global sections functor  $\Gamma_* : \mathbf{X} \rightarrow \mathbf{S}$  admits a right adjoint  $\Gamma^! : \mathbf{S} \rightarrow \mathbf{X}$ , then  $\Gamma^!$  is automatically fully faithful, whence  $\mathbf{X}$  is local.

Consequently, by the Adjoint Functor Theorem and (7.9), an  $\infty$ -topos  $\mathbf{X}$  is local if and only if the terminal object  $1_{\mathbf{X}} \in \mathbf{X}$  is compact and projective.

**7.10 Definition.** Let  $\mathbf{X}$  and  $\mathbf{Y}$  be local  $\infty$ -topoi with centres  $x_*$  and  $y_*$ , respectively. A geometric morphism  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  is a *local geometric morphism* if  $f_* x_* \simeq y_*$ . Write  $\mathbf{Top}_{\infty}^{loc} \subset \mathbf{Top}_{\infty}$  for the (non-full) subcategory whose objects are local  $\infty$ -topoi and whose morphisms are local geometric morphisms.

If  $\mathbf{X}$  is a local  $\infty$ -topos, then its centre is an initial object of the  $\infty$ -category  $\mathbf{Pt}(\mathbf{X})$ ; in fact, more is true.

**7.11 Notation.** Let  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  and  $f'_* : \mathbf{X}' \rightarrow \mathbf{Y}$  be two geometric morphisms of  $\infty$ -topoi. Write

$$\mathbf{Fun}_{\mathbf{Y},*}(\mathbf{X}, \mathbf{X}') := \mathbf{Fun}_*(\mathbf{X}, \mathbf{X}') \times_{\mathbf{Fun}_*(\mathbf{X}, \mathbf{Y})} \{f_*\}$$

for the  $\infty$ -category of geometric morphisms  $\mathbf{X} \rightarrow \mathbf{X}'$  over  $\mathbf{Y}$ .

**7.12 Lemma.** Let  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  be a geometric morphism that exhibits  $\mathbf{X}$  as local over  $\mathbf{Y}$  with centre  $f^!$ . Then  $f^!$  is a terminal object of the  $\infty$ -category  $\mathbf{Fun}_{\mathbf{Y},*}(\mathbf{Y}, \mathbf{X})$ .

*Proof.* Let  $g_* : \mathbf{Y} \rightarrow \mathbf{X}$  be a geometric morphism over  $\mathbf{Y}$ . Then

$$\begin{aligned} \mathbf{Map}_{\mathbf{Fun}_{\mathbf{Y},*}(\mathbf{Y}, \mathbf{X})}(g_*, f^!) &\simeq \mathbf{Map}_{\mathbf{Fun}_{\mathbf{Y},*}(\mathbf{Y}, \mathbf{Y})}(f_* g_*, \mathbf{id}_{\mathbf{Y}}) \\ &\simeq \mathbf{Map}_{\mathbf{Fun}_{\mathbf{Y},*}(\mathbf{Y}, \mathbf{Y})}(\mathbf{id}_{\mathbf{Y}}, \mathbf{id}_{\mathbf{Y}}) \simeq * . \end{aligned} \quad \square$$

Local  $\infty$ -topoi provide a convenient way to compute stalks as global sections after pulling back to an appropriate local  $\infty$ -topos. The following is immediate.

**7.13 Lemma.** *Let  $p_* : W \rightarrow X$  be a geometric morphism of  $\infty$ -topoi where  $W$  is local with centre  $w_*$ , and let  $x_* := p_* w_*$ . Then  $x^* \simeq \Gamma_{W,*} p^*$ .*

We shall soon see (Definition 7.20 and (7.21)) that for any  $\infty$ -topos  $X$  and any point  $x_* \in \text{Pt}(X)$ , there is a geometric morphism  $p_* : W \rightarrow X$  in which  $W$  is local with centre  $w_*$  and  $x_* \simeq p_* w_*$  (and is, moreover, universal with this property).

## Nearby cycles & localisations

We now show that the evanescent  $\infty$ -topos (Example 6.38) provides a wealth of local  $\infty$ -topoi. Then, following Deligne as well as Peter Johnstone and Ieke Moerdijk [30, Definition 3.1], we use the evanescent topos to construct the localisation of an  $\infty$ -topos at a point.

A site-theoretic proof of the following result (originally stated without proof by Laumon [32, 3.2]) is given in [27, Exposé XI, Proposition 4.4]. The reliance on sites renders the proof given in [27, Exposé XI] inadequate in the context of  $\infty$ -topoi; luckily, however, the work of Emily Riehl and Dominic Verity [44] permit us to employ simple 2-categorical arguments.

**7.14 Proposition.** *Let  $f_* : X \rightarrow Z$  be a geometric morphism of  $\infty$ -topoi. Then:*

(7.14.1) *The nearby cycles functor  $\Psi_{f,*} : X \tilde{\times}_Z Z$  is right adjoint to the projection  $\text{pr}_{1,*} : X \tilde{\times}_Z Z \rightarrow X$ .*

(7.14.2) *The functor  $\Psi_{f,*}$  is fully faithful, hence the geometric morphism  $\text{pr}_{1,*}$  exhibits  $X \tilde{\times}_Z Z$  as local over  $X$  with centre  $\Psi_{f,*}$ .*

*Proof.* Recall that for any  $\infty$ -topos  $E$ , the functor  $\text{Fun}_*(E, -)^{op} : \mathbf{Top}_\infty \rightarrow \mathbf{Cat}_{\infty, \kappa_1}$  carries oriented fibre products in  $\mathbf{Top}_\infty$  to oriented fibre products in  $\mathbf{Cat}_{\infty, \kappa_1}$  (6.35). Thus the proof of [44, Proposition 3.4.6] works perfectly, giving the oriented fibre product in  $\mathbf{Top}_\infty$  the necessary ‘weak universal property’ (as Riehl and Verity call it) to apply [44, Lemma 3.1.7], proving both (7.14.1) and (7.14.2).  $\square$

The dual notion to being a local over an  $\infty$ -topos naturally appears as the property satisfied by the second projection from the coëvanescent topos in the dual to Proposition 7.14.

**7.15 Definition.** A geometric morphism  $f_* : X \rightarrow Y$  of  $\infty$ -topoi exhibits  $X$  as *colocal over  $Y$*  if  $f_*$  is a quasi-equivalence and  $f^*$  admits a left exact left adjoint  $f_! : X \rightarrow Y$ . In this case, the functor  $f^*$  and its left adjoint  $f_!$  define a geometric morphism  $f^* : Y \rightarrow X$  called the *cocentre* of  $f_*$ .

**7.16.** In the setting of 1-topoi, Johnstone [29, Theorem C.3.6.16] uses the term *totally connected* for what we call colocal. Again, such lingo is inapt in our context.

**7.17 Proposition.** *Let  $g_* : Y \rightarrow Z$  be a geometric morphism of  $\infty$ -topoi. Then:*

(7.14.1) The conearby cycles functor  $\Psi_*^g : Y \rightarrow Z \tilde{\times}_Z Y$  is left adjoint to the projection  $\text{pr}_{2,*} : Z \tilde{\times}_Z Y \rightarrow Y$ .

(7.14.2) The functor  $\Psi_*^g \simeq \text{pr}_2^*$  is fully faithful, whence the geometric morphism  $\text{pr}_{2,*}$  exhibits  $Z \tilde{\times}_Z Y$  as colocal over  $Y$  with cocentre  $\Psi_*^g$ .

7.18. A geometric morphism  $f_*$  that exhibits an  $\infty$ -topos as colocal over another will always satisfy the étale projection formula

$$f_!(f^*(X) \times_{f^*(Z)} Y) \simeq X \times_Z f_!(Y)$$

of [HTT, Proposition 6.3.5.11], but the geometric morphism  $f_*$  will almost never be étale as  $f_!$  is conservative if and only if  $f_*$  is an equivalence.

7.19 Example. For any  $\infty$ -topos  $X$  the diagonal functor  $\psi(\text{id}_X, \text{id}_X, \text{id})_*$  is both the nearby and conearby cycles functor

$$X \rightarrow X \tilde{\times}_X X \simeq \text{Path}(X).$$

Combining Propositions 7.14 and 7.17, we deduce that we have a chain of (left exact) adjoints

$$\text{Path}(X) \begin{array}{c} \xleftarrow{\text{pr}_1^*} \\ \xleftarrow{\text{pr}_{1,*}} \xrightarrow{\text{pr}_2^*} \\ \xleftarrow{\text{pr}_2^*} \xrightarrow{\text{pr}_{2,*}} \end{array} X.$$

In particular, the geometric morphisms  $\text{pr}_{1,*}, \text{pr}_{2,*} : \text{Path}(X) \rightarrow X$  are shape equivalences.

Now we define the *localisation* of an  $\infty$ -topos at a point as a  $\infty$ -evanescent topos; for this please recall Notation 5.5.

7.20 Definition. Let  $X$  be an  $\infty$ -topos and  $x_* : S \rightarrow X$  a point of  $X$ . The *localisation* of  $X$  at  $x_*$  is the evanescent  $\infty$ -topos

$$X_{(x)} := \tilde{x} \tilde{\times}_X X.$$

We write  $\ell_{x,*} : X_{(x)} \rightarrow X$  for the second projection  $\text{pr}_{2,*} : \tilde{x} \tilde{\times}_X X \rightarrow X$ .

7.21. Let  $X$  be an  $\infty$ -topos and  $x_*$  a point of  $X$ . By Proposition 7.14, the  $\infty$ -topos  $X_{(x)}$  is local with centre  $\Psi_{x,*} : S \rightarrow X_{(x)}$ . By Lemma 7.13, for every object  $F \in X$  we can compute the stalk at  $x$  via the familiar formula

$$F_x \simeq \Gamma(X_{(x)}; \ell_x^* F).$$

7.22 Notation. Write  $\mathbf{Top}_{\infty,*} := \mathbf{Top}_{\infty,S/}$  for the  $\infty$ -category of pointed  $\infty$ -topoi. The assignment  $(X, x_*) \mapsto X_{(x)}$  defines a functor  $\mathbf{Top}_{\infty,*} \rightarrow \mathbf{Top}_{\infty}^{\text{loc}}$ .

In the other direction, the assignment  $X \mapsto (X, \Gamma^!)$  defines a fully faithful functor  $\mathbf{Top}_{\infty}^{\text{loc}} \hookrightarrow \mathbf{Top}_{\infty,*}$ .

**7.23 Proposition.** *Let  $X$  be a local  $\infty$ -topos with centre  $x_*$ . Then the geometric morphism  $\ell_{x_*}: X_{(x)} \rightarrow X$  is an equivalence.*

*Proof.* Let  $\eta: \text{id}_X \rightarrow x_*\Gamma_{X,*}$  be the unit of the adjunction  $\Gamma_{X,*} \dashv x_*$ . Then the oriented square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \downarrow & \swarrow \eta & \parallel \\ \tilde{x} & \longrightarrow & X \end{array}$$

exhibits  $X$  as the oriented fibre product  $\tilde{x} \tilde{x}_X X$ .  $\square$

**7.24 Corollary.** *The fully faithful functor  $\mathbf{Top}_{\infty}^{\text{loc}} \hookrightarrow \mathbf{Top}_{\infty,*}$  admits a right adjoint given by the assignment  $(X, x_*) \mapsto X_{(x)}$ .*

## Localisation à la Grothendieck–Verdier

In order to get our hands on geometric examples of localised  $\infty$ -topoi, we give another description of  $X_{(x)}$  that is akin to the original (1-toposic) definition of the localisation due to Grothendieck–Verdier [SGA<sub>4</sub> II, Exposé VI, 8.4.2] as a limit over étale neighborhoods of  $x_*$  in  $X$ .

**7.25 Definition.** Let  $(X, x_*)$  be a pointed  $\infty$ -topos. The  $\infty$ -category of neighborhoods of  $x_*$  is the pullback

$$\begin{array}{ccc} \text{Nbd}(x) & \longrightarrow & \mathbf{S}_* \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{x^*} & \mathbf{S} \end{array}$$

formed in  $\mathbf{Cat}_{\infty, \kappa_1}$ .

By [HTT, Corollary 6.3.5.6 & Remark 6.3.5.7] the  $\infty$ -category  $\text{Nbd}(x)$  is equivalent to the full subcategory of  $(\mathbf{Top}_{\infty,*})_{/(X, x_*)}$  spanned by those objects  $(E, e_*) \rightarrow (X, x_*)$  with the property that the geometric morphism  $E \rightarrow X$  is étale.

Please note that  $\text{Nbd}(x)$  is an inverse  $\infty$ -category.

To provide the limit description of the localisation as well as the familiar colimit formula for the stalk at  $x$ , we must speak of limits of diagrams indexed by the (not necessarily  $\kappa_0$ -small)  $\infty$ -category  $\text{Nbd}(x)$ . Happily the exact same cofinality argument given in [SGA<sub>4</sub> II, Exposé IV, 6.8] works in the setting of higher topoi, showing that  $\text{Nbd}(x)$  admits a limit-cofinal  $\kappa_0$ -small subcategory.

**7.26 Construction.** Let  $X$  be a  $\infty$ -topos and  $x_* \in \text{Pt}(X)$ . Then by the Yoneda lemma the stalk functor  $x^*: X \rightarrow \mathbf{S}$  can be computed as the filtered colimit

$$x^* \simeq \text{colim}_{(U, u) \in \text{Nbd}(x)^{\text{op}}} \text{Map}_X(U, -).$$

The assignment  $(U, u) \mapsto X_{/U}$  defines a functor  $E_x: \text{Nbd}(x) \rightarrow \mathbf{Top}_{\infty, /X}$ . Moreover, the natural forgetful functor  $\mathbf{Top}_{\infty, /E_x} \rightarrow \mathbf{Top}_{\infty, /X}$  is a right fibration. We write  $\lim_{(U, u) \in \text{Nbd}(x)} X_{/U}$  for the limit in  $\mathbf{Top}_{\infty, /X}$  (equivalently, in  $\mathbf{Top}_{\infty}$ ) of the diagram  $E_x$ .

By [Recollection 5.6](#)=[[HTT](#), [Corollary 6.3.5.6](#)], specifying a geometric morphism

$$\mathbf{X}' \rightarrow \lim_{(U,u) \in \text{Nbd}(x)} \mathbf{X}/U$$

is equivalent to specifying a geometric morphism  $p_* : \mathbf{X}' \rightarrow \mathbf{X}$  along with a global section

$$\sigma \in \Gamma_{\mathbf{X}',*} \left( \lim_{(U,u) \in \text{Nbd}(x)} P^*U \right) \simeq \lim_{(U,u) \in \text{Nbd}(x)} \Gamma_{\mathbf{X}',*} P^*U.$$

Since  $\mathbf{X}_{(x)}$  is the localisation of  $\mathbf{X}$  at  $x_*$ , we have a natural equivalence  $x^* \simeq \Gamma_{\mathbf{X}_{(x)},*} \ell_x^*$  ([7.21](#)), whence for  $U \in \mathbf{X}$ , we obtain a natural equivalence

$$\lim_{(U,u) \in \text{Nbd}(x)} x^*(U) \simeq \Gamma_{\mathbf{X}_{(x)},*} \left( \lim_{(U,u) \in \text{Nbd}(x)} \ell_x^*(U) \right).$$

The global sections  $u \in x^*(U)$  for  $(U,u) \in \text{Nbd}(x)$  together define a global section  $s \in \lim_{(U,u) \in \text{Nbd}(x)} x^*(U)$ . This furnishes us with a geometric morphism

$$g_* : \mathbf{X}_{(x)} \rightarrow \lim_{(U,u) \in \text{Nbd}(x)} \mathbf{X}/U$$

over  $\mathbf{X}$ .

**7.27 Proposition.** *Let  $\mathbf{X}$  be an  $\infty$ -topos and  $x_*$  a point of  $\mathbf{X}$ . Then the geometric morphism  $g_* : \mathbf{X}_{(x)} \rightarrow \lim_{(U,u) \in \text{Nbd}(x)} \mathbf{X}/U$  of [Construction 7.26](#) is an equivalence.*

*Proof.* We wish to show that  $g_* : \mathbf{X}_{(x)} \rightarrow \lim_{(U,u) \in \text{Nbd}(x)} \mathbf{X}/U$  induces an equivalence

$$\mathbf{Top}_{\infty,/\mathbf{X}_{(x)}} \xrightarrow{\simeq} \mathbf{Top}_{\infty,/\mathbf{E}_x}.$$

Since both projections onto  $\mathbf{Top}_{\infty,/\mathbf{X}}$  are right fibrations, we are reduced to showing that for every object  $p_* : \mathbf{X}' \rightarrow \mathbf{X}$  of  $\mathbf{Top}_{\infty,/\mathbf{X}}$  the induced map on fibres of these right fibrations is an equivalence. By [Recollection 5.6](#)=[[HTT](#), [Corollary 6.3.5.6](#)] the fibre of the right fibration  $\mathbf{Top}_{\infty,/\mathbf{E}_x} \rightarrow \mathbf{Top}_{\infty,/\mathbf{X}}$  over  $p_* : \mathbf{X}' \rightarrow \mathbf{X}$  is given by

$$\{p_*\} \times_{\mathbf{Top}_{\infty,/\mathbf{X}}} \mathbf{Top}_{\infty,/\mathbf{E}_x} \simeq \lim_{(U,u) \in \text{Nbd}(x)} \Gamma_{\mathbf{X}',*} P^*(U),$$

On the other hand,

$$\{p_*\} \times_{\mathbf{Top}_{\infty,/\mathbf{X}}} \mathbf{Top}_{\infty,/\mathbf{X}_{(x)}} \simeq \text{Map}_{\text{Fun}_*(\mathbf{X}',\mathbf{X})}(p_*, x_* \Gamma_{\mathbf{X}',*}) \simeq \text{Map}_{\text{Fun}(\mathbf{X},\mathbf{S})}(x^*, \Gamma_{\mathbf{X}',*} p^*).$$

By the colimit formula for the stalk ([Construction 7.26](#)), we have natural equivalences

$$\begin{aligned} \text{Map}_{\text{Fun}(\mathbf{X},\mathbf{S})}(x^*, \Gamma_{\mathbf{X}',*} p^*) &\simeq \text{Map}_{\text{Fun}(\mathbf{X},\mathbf{S})} \left( \text{colim}_{(U,u) \in \text{Nbd}(x)^{op}} \text{Map}_{\mathbf{X}}(U, -, \Gamma_{\mathbf{X}',*} p^*) \right) \\ &\simeq \lim_{(U,u) \in \text{Nbd}(x)} \Gamma_{\mathbf{X}',*} P^*(U). \end{aligned}$$

Unwinding definitions, we see that the induced map on fibres

$$\{p_*\} \times_{\mathbf{Top}_{\infty,/\mathbf{X}}} \mathbf{Top}_{\infty,/\mathbf{X}_{(x)}} \rightarrow \{p_*\} \times_{\mathbf{Top}_{\infty,/\mathbf{X}}} \mathbf{Top}_{\infty,/\mathbf{E}_x}$$

is an equivalence. □

## Coherence of localisations

In this subsection we use the Grothendieck–Verdier description of the localisation to deduce that  $\mathbf{X}_{(x)}$  is bounded coherent when  $\mathbf{X}$  is. Please note that this is not automatic from [Lemma 6.47](#), as points of bounded coherent  $\infty$ -topoi need not be coherent in general.

**7.28.** Let  $f : U \rightarrow V$  be a morphism between coherent objects of an  $\infty$ -topos  $\mathbf{X}$ . Then the geometric morphism  $f_* : \mathbf{X}_{/U} \rightarrow \mathbf{X}_{/V}$  is coherent.

**7.29 Lemma.** Let  $\mathbf{X}$  be a bounded  $\infty$ -topos and  $U \in \mathbf{X}_{<\infty}$  a truncated object of  $\mathbf{X}$ . Then the over  $\infty$ -topos  $\mathbf{X}_{/U}$  is bounded.

*Proof.* Indeed, if  $U$  is  $n$ -truncated, and if  $\mathbf{X}$  is  $N$ -localic for some  $N \geq n$ , then  $\mathbf{X}_{/U}$  is  $N$ -localic as well. The desired result now follows by exhibiting  $\mathbf{X}$  as an inverse limit of localic  $\infty$ -topoi.  $\square$

**7.30.** Let  $\mathbf{X}$  be a bounded coherent  $\infty$ -topos and  $x_*$  a point of  $\mathbf{X}$ . Then the full subcategory  $\mathrm{Nbd}_{<\infty}^{\mathrm{coh}}(x) \subset \mathrm{Nbd}(x)$  consisting of those neighborhoods  $(U, u)$  such that  $U$  is a truncated coherent object of  $\mathbf{X}$  is limit-cofinal in  $\mathrm{Nbd}(x)$ . Thus [Proposition 7.27](#), [\(7.28\)](#), and [Lemma 7.29](#) together show that

$$\mathbf{X}_{(x)} \simeq \lim_{(U,u) \in \mathrm{Nbd}_{<\infty}^{\mathrm{coh}}(x)} \mathbf{X}_{/U}$$

is an inverse limit in  $\mathbf{Top}_{\infty}$  of bounded coherent  $\infty$ -topoi and coherent geometric morphisms.

We deduce the following.

**7.31 Lemma.** Let  $\mathbf{X}$  be a bounded coherent  $\infty$ -topos and  $x_*$  a point of  $\mathbf{X}$ . Then the localisation  $\mathbf{X}_{(x)}$  is bounded coherent and the geometric morphism  $\ell_{x,*} : \mathbf{X}_{(x)} \rightarrow \mathbf{X}$  is coherent.

## Geometric examples of localisations

**7.32 Example** ([\[30, Example 1.2\(a\)\]](#)). Let  $X$  be a topological space and  $s \in X$  a *special point* in the sense that the only open set of  $X$  containing  $s$  is  $X$  itself. Then it is immediate that the functor  $\bar{X} \rightarrow \mathbf{S}$  given by taking the stalk at  $s$  is equivalent to the global sections functor, so the  $\infty$ -topos  $\bar{X}$  is local with centre  $x_* : \mathbf{S} \rightarrow \bar{X}$ .

**7.33 Subexample** ([\[SGA<sub>4</sub> II, Exposé VI, 8.4.6\]](#)). In particular, when  $X = \mathrm{Spec}(A)^{\mathrm{zar}}$  is the Zariski space of the spectrum of a local ring  $A$ , and  $s = m$  is the maximal ideal, we deduce that the Zariski  $\infty$ -topos of  $A$  is local. Moreover, if  $\phi : A \rightarrow A'$  is a local homomorphism of local rings, then the induced geometric morphism of Zariski  $\infty$ -topoi  $\mathrm{Spec}(A')_{\mathrm{zar}} \rightarrow \mathrm{Spec}(A)_{\mathrm{zar}}$  is a local geometric morphism.

**7.34 Example** ([\[SGA<sub>4</sub> II, Exposé VI, 8.4.4\]](#)). Let  $X$  be a scheme and  $x \in X$ . Then the localisation of the Zariski  $\infty$ -topos of  $X$  at the point  $x$  is the Zariski  $\infty$ -topos of  $O_{X,x}$ .

**7.35 Example.** Let  $X$  be a scheme, and let  $x \rightarrow X$  be a point with image  $x_0 \in X^{zar}$ . Suppose  $x$  is a *geometric point* in the sense that  $\kappa(x)$  is a separable closure of  $\kappa(x_0)$ . Then the localisation of the étale  $\infty$ -topos of  $X$  at  $x$  is the étale  $\infty$ -topos of the *strict localisation*  $X_{(x)} := \text{Spec } O_{X,x_0}^{sh}$ , viz.,

$$(X_{\acute{e}t})_{(x)} \simeq (X_{(x)})_{\acute{e}t}.$$

More generally, for any point  $x \rightarrow X$ , the evanescent  $\infty$ -topos  $x_{\acute{e}t} \bar{\times}_{X_{\acute{e}t}} X_{\acute{e}t}$  can be identified with the étale  $\infty$ -topos of  $X_{(x)} := \text{Spec } A$ , where  $A \supseteq O_{X,x}^h$  is the unramified extension of the henselisation whose residue field is the separable closure of  $\kappa(x_0)$  in  $\kappa(x)$ .

## 8 Beck–Chevalley conditions & gluing squares

The goal of this section is to prove a basechange result for oriented fibre products of bounded coherent  $\infty$ -topoi ([Theorem 8.4](#)). Our result provides a nonabelian refinement of a basechange result of Ofer Gabber [[27](#), Exposé XI, Théorème 2.4] as well as one of Moerdijk and Jacob Vermeulen [[37](#), Theorem 2(i)]. This basechange result is essential to our *décollage* approach to *stratified higher topoi* in [§9](#).

### The Beck–Chevalley transformation & Beck–Chevalley conditions

We begin by recalling the *Beck–Chevalley natural transformation* associated to an oriented square of  $\infty$ -topoi.

**8.1 Definition.** Consider an oriented square of  $\infty$ -topoi and geometric morphisms:

$$(8.2) \quad \begin{array}{ccc} \mathbf{W} & \xrightarrow{q_*} & \mathbf{Y} \\ p_* \downarrow & \lrcorner_{\tau} & \downarrow g_* \\ \mathbf{X} & \xrightarrow{f_*} & \mathbf{Z} \end{array}$$

and the corresponding geometric morphism  $\psi(p, q, \tau)_* : \mathbf{W} \rightarrow \mathbf{X} \bar{\times}_{\mathbf{Z}} \mathbf{Y}$  of [Definition 6.29](#). Write  $\eta : \text{id}_{\mathbf{Y}} \rightarrow q_* q^*$  for the unit and  $\varepsilon : f^* f_* \rightarrow \text{id}_{\mathbf{X}}$  for the counit. The *Beck–Chevalley transformation* is the composition

$$\beta_{\tau} : f^* g_* \xrightarrow{f^* g_* \eta} f^* g_* q_* q^* \xrightarrow{f^* \tau q^*} f^* f_* p_* q^* \xrightarrow{\varepsilon p_* q^*} p_* q^*.$$

We say that the square (8.2) – or equivalently the geometric morphism  $\psi(p, q, \tau)_*$  – satisfies the:

- ▶ *Beck–Chevalley condition* if the natural transformation  $\beta_{\tau}$  is an equivalence.
- ▶ *bounded Beck–Chevalley condition* if for every truncated object  $F \in \mathbf{Y}_{<\infty}$ , the morphism  $\beta_{\tau}(F) : f^* g_*(F) \rightarrow \text{pr}_{1,*} \text{pr}_2^*(F)$  is an equivalence in  $\mathbf{X}$ .

8.3. Please observe that given oriented squares of  $\infty$ -topoi

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & \swarrow \sigma & \downarrow & \swarrow \tau & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array},$$

the Beck–Chevalley morphism of the outer oriented rectangle is equivalent to natural transformation given by the composite of the Beck–Chevalley morphisms

$$\begin{array}{ccccc} X & \longleftarrow & Y & \longleftarrow & Z \\ \downarrow & \nwarrow \beta_\sigma & \downarrow & \nwarrow \beta_\tau & \downarrow \\ X' & \longleftarrow & Y' & \longleftarrow & Z' \end{array}.$$

We now are now prepared to state our basechange result.

**8.4 Theorem.** *Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be coherent geometric morphisms between bounded coherent  $\infty$ -topoi. Then the oriented fibre product square*

$$(8.5) \quad \begin{array}{ccccc} X \tilde{\times}_Z Y & \xrightarrow{\text{pr}_{2,*}} & Y \\ \text{pr}_{1,*} \downarrow & \swarrow \tau & \downarrow g_* \\ X & \xrightarrow{f_*} & Z \end{array}$$

*satisfies the bounded Beck–Chevalley condition.*

By passing to 1-localic  $\infty$ -topoi in [Theorem 8.4](#), we deduce Moerdijk and Vermeulen’s 1-toposic Beck–Chevalley condition [[37](#), Theorem 2(i)].

**8.6 Corollary.** *Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be coherent geometric morphisms between coherent 1-topoi (in the sense of [[SGA<sub>4</sub> II](#), Exposé VI, Définition 2.3]). Then the oriented fibre product square of 1-topoi*

$$\begin{array}{ccccc} X \tilde{\times}_Z Y & \xrightarrow{\text{pr}_{2,*}} & Y \\ \text{pr}_{1,*} \downarrow & \swarrow \tau & \downarrow g_* \\ X & \xrightarrow{f_*} & Z \end{array}$$

*satisfies the Beck–Chevalley condition – i.e., the Beck–Chevalley natural transformation  $f^* g_* \rightarrow \text{pr}_{1,*} \text{pr}_2^*$  is an isomorphism.*

*Proof.* Write  $X'$ ,  $Y'$ , and  $Z'$  for the 1-localic  $\infty$ -topoi associated to  $X$ ,  $Y$ , and  $Z$ , respectively. Then [Theorem 8.4](#) shows that the oriented fibre product square of  $\infty$ -topoi

$$\begin{array}{ccccc} X' \tilde{\times}_{Z'} Y' & \longrightarrow & Y' \\ \downarrow & \swarrow & \downarrow \\ X' & \longrightarrow & Z' \end{array}$$



satisfies the bounded Beck–Chevalley condition. We conclude by restricting to 0-truncated objects and applying [Lemma 6.41](#).  $\square$

In the setting of derived categories, we also immediately deduce Gabber’s result [[27](#), Exposé XI, Théorème 2.4].

**8.7.** If  $g_*$  is a *proper* geometric morphism [[HTT](#), [Definition 7.3.1.4](#)], then effectively by definition one has a Beck–Chevalley condition that does not require bounded coherence hypotheses: given geometric morphisms of  $\infty$ -topoi  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  where  $g_*$  is proper, the corresponding oriented fibre product square satisfies the Beck–Chevalley condition, and  $\mathrm{pr}_{1,*}$  is again proper. This result is not used in the sequel.

The proof of [Theorem 8.4](#) requires a number of preliminaries that will occupy the next few subsections. Our proof is essentially a reinterpretation of the proof of Gabber’s result that Luc Illusie presents in [[27](#), Exposé XI, Théorème 2.4].

## Localisations & the bounded Beck–Chevalley condition

In this subsection we prove the following *bounded* Beck–Chevalley condition for localisations of bounded coherent  $\infty$ -topoi.

**8.8 Proposition.** *Let  $p_* : W \rightarrow X$  be a coherent geometric morphism between bounded coherent  $\infty$ -topoi. Then for any point  $x_*$  of  $X$ , the pullback square*

$$\begin{array}{ccc} \mathbf{S} \tilde{\times}_X W & \longrightarrow & W \\ \downarrow & \lrcorner & \downarrow p_* \\ X_{(x_*)} & \xrightarrow{\ell_{x_*}} & X \end{array}$$

*satisfies the bounded Beck–Chevalley condition.*

To do so, we use the Grothendieck–Verdier description of the localisation ([Proposition 7.27](#)) and the following well-known Beck–Chevalley condition for étale geometric morphisms to reduce the problem to a general result on inverse limits ([Proposition 8.12](#)).

The following is a direct consequence of [Lemma 6.51](#).

**8.9 Lemma.** *Let  $f_* : E \rightarrow X$  and  $p_* : W \rightarrow X$  be geometric morphisms of  $\infty$ -topoi. If  $f_*$  is étale, then the pullback square*

$$\begin{array}{ccc} E \times_X W & \longrightarrow & W \\ \downarrow & \lrcorner & \downarrow g_* \\ E & \xrightarrow{f_*} & X \end{array}$$

*satisfies the Beck–Chevalley condition.*

We fix some useful notation for the next two results.

**8.10 Notation.** Let  $W, X: I \rightarrow \mathbf{Top}_\infty$  be diagrams of  $\infty$ -topoi. For each morphism  $\alpha: j \rightarrow i$  in  $I$ , we write

$$e_{\alpha,*}: W_j \rightarrow W_i \quad \text{and} \quad f_{\alpha,*}: X_j \rightarrow X_i$$

for the transition morphisms. For each  $i \in I$ , we write

$$\xi_{i,*}: \lim_{i \in I} W_i \rightarrow W_i \quad \text{and} \quad \pi_{i,*}: \lim_{i \in I} X_i \rightarrow X_i$$

for the projections.

**8.11 Lemma.** Let  $X: I \rightarrow \mathbf{Top}_\infty^{bc}$  be an inverse system of bounded coherent  $\infty$ -topoi and coherent geometric morphisms. Then for each  $i \in I$ , the natural morphism

$$\operatorname{colim}_{\alpha \in (I_i)^{op}} f_{\alpha,*} f_\alpha^* \rightarrow \pi_{i,*} \pi_i^*$$

in  $\mathbf{Fun}(X_i, X_i)$  is an equivalence.

*Proof.* For simplicity write

$$X' := \lim_{i \in I} X_i, \quad X' := (X')_{<\infty}^{coh}, \quad \text{and} \quad X_j := X_{j,<\infty}^{coh},$$

so that  $X'$  is the filtered colimit  $\operatorname{colim}_{i \in I^{op}} X_i$  in  $\mathbf{preTop}_\infty^b$ . It suffices to show that for all sheaves  $F \in X_i$  and all  $U \in X_i$ , the natural morphism

$$\operatorname{colim}_{\alpha \in (I_i)^{op}} f_{\alpha,*} f_\alpha^*(F)(U) \rightarrow \pi_{i,*} \pi_i^*(F)(U)$$

is an equivalence. This is a straightforward computation:

$$\begin{aligned} \operatorname{colim}_{\alpha \in (I_i)^{op}} f_{\alpha,*} f_\alpha^*(F)(U) &\simeq \operatorname{colim}_{\alpha \in (I_i)^{op}} f_\alpha^*(F)(f_\alpha^*(U)) \\ &\simeq \operatorname{colim}_{[\alpha: j \rightarrow i] \in (I_i)^{op}} \int^{V \in X_j^{op}} \operatorname{Map}_{X_j^{op}}(f_\alpha^*(V), f_\alpha^*(U)) \times F(V) \\ &\simeq \int^{V \in X_i^{op}} \operatorname{colim}_{[\alpha: j \rightarrow i] \in (I_i)^{op}} \operatorname{Map}_{X_j^{op}}(f_\alpha^*(V), f_\alpha^*(U)) \times F(V) \\ &\simeq \int^{V \in X_i^{op}} \operatorname{Map}_{(X')^{op}}(\pi_\alpha^*(V), \pi_\alpha^*(U)) \times F(V) \\ &\simeq \pi_i^*(F)(\pi^*(U)) = \pi_{i,*} \pi_i^*(F)(U). \quad \square \end{aligned}$$

**8.12 Proposition.** Let  $W, X: I \rightarrow \mathbf{Top}_\infty^{bc}$  be two inverse systems of bounded coherent  $\infty$ -topoi and coherent geometric morphisms, and  $p: W \rightarrow X$  a natural transformation, each of whose components  $p_{i,*}: W_i \rightarrow X_i$  is a coherent geometric morphism. If for each morphism  $\alpha: j \rightarrow i$  in  $I$  the square

$$(8.13) \quad \begin{array}{ccc} W_j & \xrightarrow{e_{\alpha,*}} & W_i \\ p_{j,*} \downarrow & & \downarrow p_{i,*} \\ X_j & \xrightarrow{f_{\alpha,*}} & X_i \end{array}$$

satisfies the bounded Beck–Chevalley condition, then for each  $i \in I$  the induced square

$$\begin{array}{ccc} \lim_{i \in I} W_i & \xrightarrow{\xi_{i,*}} & W_i \\ \lim_{i \in I} P_{i,*} \downarrow & & \downarrow P_{i,*} \\ \lim_{i \in I} X_i & \xrightarrow{\pi_{i,*}} & X_i \end{array}$$

satisfies the bounded Beck–Chevalley condition.

*Proof.* For each  $i \in I$ , the forgetful functor  $I_{/i} \rightarrow I$  is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that  $I$  admits a terminal object  $1$  and that  $i = 1$ . Writing  $q_* := \lim_{i \in I} P_{i,*}$ , we see that we have reduced to showing that the square

$$(8.14) \quad \begin{array}{ccc} \lim_{i \in I} W_i & \xrightarrow{\xi_{1,*}} & W_1 \\ q_* \downarrow & & \downarrow P_{1,*} \\ \lim_{i \in I} X_i & \xrightarrow{\pi_{1,*}} & X_1 \end{array}$$

satisfies the bounded Beck–Chevalley condition.

Inverse limits in  $\mathbf{Top}_{\infty}^{bc}$  are computed in  $\mathbf{Cat}_{\infty, \kappa_1}$  (Corollary 5.44=[SAG, Corollary A.8.3.3]), so an object of the limit of a diagram  $Y: I \rightarrow \mathbf{Top}_{\infty}^{bc}$  is specified by a compatible system  $\{U_i\}_{i \in I}$  of objects  $U_i \in Y_i$  along with, for each  $\alpha: j \rightarrow i$  in  $I$ , an equivalence  $\phi_{\alpha}: g_{\alpha,*}(U_j) \simeq U_i$ , where  $g_{\alpha,*}: Y_j \rightarrow Y_i$  is the transition morphism. Thus for  $U \in W_1$  we have

$$q_* \xi_1^*(U) \simeq \{p_{i,*} \xi_{i,*} \xi_1^*(U)\}_{i \in I},$$

and

$$\pi_1^* p_{1,*}(U) \simeq \{\pi_{i,*} \pi_1^* p_{1,*}(U)\}_{i \in I}.$$

It therefore suffices to show that for each  $i \in I$ , the natural morphism

$$\pi_{i,*} \beta: \pi_{i,*} \pi_1^* p_{1,*} \rightarrow \pi_{i,*} q_* \xi_1^* \simeq p_{i,*} \xi_{i,*} \xi_1^*$$

induced by the Beck–Chevalley morphism  $\beta: \pi_1^* p_{1,*} \rightarrow q_* \xi_1^*$  is an equivalence when restricted to  $W_{1, < \infty}$ .

For  $i \in I$ , we simply write  $f_{i,*} := f_{\alpha,*}$  and  $e_{i,*} := e_{\alpha,*}$  if  $\alpha: i \rightarrow 1$ . Note that we have natural transformations

$$\begin{aligned} \pi_{i,*} \pi_1^* p_{1,*} &\simeq \pi_{i,*} \pi_i^* f_i^* p_{1,*} \\ &\simeq \operatorname{colim}_{\alpha \in (I_i)^{op}} f_{\alpha,*} f_{\alpha}^* f_i^* p_{1,*} && \text{(Lemma 8.11)} \\ &\rightarrow \operatorname{colim}_{[\alpha: j \rightarrow i] \in (I_i)^{op}} f_{\alpha,*} p_{j,*} e_{\alpha}^* f_i^* \\ &\simeq \operatorname{colim}_{\alpha \in (I_i)^{op}} p_{i,*} e_{\alpha,*} e_{\alpha}^* e_i^* \end{aligned}$$

in which the third natural transformation is an equivalence when restricted to  $W_{1, < \infty}$  by assumption. In addition, [Lemma 8.11](#) and the fact that  $\xi_i^* f_i^* \simeq \xi_1^*$  give equivalences

$$p_{i,*} \left( \operatorname{colim}_{\alpha \in (I_i)^{op}} e_{\alpha,*} e_{\alpha}^* e_i^* \right) \simeq p_{i,*} \xi_{i,*} \xi_i^* f_i^* \simeq p_{i,*} \xi_{i,*} \xi_1^* .$$

Since  $p_{i,*}$  is coherent,  $p_{i,*}$  commutes with filtered colimits of uniformly truncated objects ([Corollary 5.55](#)). As left exact functors preserve  $n$ -truncatedness for all  $n \geq -2$ , we see that for every truncated object  $U$  of  $W_1$ , the natural morphism

$$\operatorname{colim}_{\alpha \in (I_i)^{op}} p_{i,*} e_{\alpha,*} e_{\alpha}^* e_i^*(U) \rightarrow p_{i,*} \left( \operatorname{colim}_{\alpha \in (I_i)^{op}} e_{\alpha,*} e_{\alpha}^* e_i^*(U) \right)$$

is an equivalence, which provides an equivalence

$$(8.15) \quad \pi_{i,*} \pi_1^* p_{1,*}(U) \simeq p_{i,*} \xi_{i,*} \xi_1^*(U) .$$

To conclude, note that the constructed natural transformation  $\pi_{i,*} \pi_1^* p_{1,*} \rightarrow p_{i,*} \xi_{i,*} \xi_1^*$  giving the equivalence (8.15) is homotopic to  $\pi_{1,*} \beta$ .  $\square$

*Proof of [Proposition 8.8](#).* Combine [Lemma 8.9](#) and [Proposition 8.12](#) (the hypotheses of which are valid by (7.30) and [Corollary 5.44](#)=[[SAG](#), [Corollary A.8.3.3](#)]).  $\square$

## Functoriality of oriented fibre products in oriented diagrams

In this subsection we discuss the functoriality of the oriented fibre product in oriented diagrams of cospans, and we use this additional functoriality to construct some unexpected extra adjoints to the second projection from the oriented fibre product ([Proposition 8.21](#)). In nice cases, this provides a way to check that the Beck–Chevalley morphism is an equivalence after passing to stalks ([Lemma 8.24](#)).

8.16. Suppose that we are given a diagram of  $\infty$ -topoi

$$\begin{array}{ccccc} X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y \\ x_* \downarrow & \eta \swarrow & \downarrow z_* & \nwarrow \theta & \downarrow y_* \\ X' & \xrightarrow{f'_*} & Z' & \xleftarrow{g'_*} & Y' . \end{array}$$

Then by the universal property of the oriented fibre product  $X' \tilde{\times}_{Z'} Y'$ , the diagram

$$\begin{array}{ccccc}
 X \tilde{\times}_Z Y & \xrightarrow{\text{pr}_{2,*}} & Y & & \\
 \text{pr}_{1,*} \downarrow & & \downarrow g_* & \searrow y_* & \\
 & \tau \swarrow & & & Y' \\
 X & \xrightarrow{f_*} & Z & & \\
 x_* \searrow & \eta \swarrow & \downarrow z_* & \swarrow & \downarrow g'_* \\
 & & X' & \xrightarrow{f'_*} & Z'
 \end{array}$$

(functorially) induces a geometric morphism  $X \tilde{\times}_Z Y \rightarrow X' \tilde{\times}_{Z'} Y'$ . We simply denote the geometric morphism by  $x_* \tilde{\times}_{z_*} y_*$ , leaving the natural transformations  $\eta$  and  $\theta$  implicit. Please note that  $x_* \tilde{\times}_{z_*} y_*$  satisfies the obvious relations

$$\text{pr}_{1,*} \circ (x_* \tilde{\times}_{z_*} y_*) \simeq x_* \text{pr}_{1,*} \quad \text{and} \quad \text{pr}_{2,*} \circ (x_* \tilde{\times}_{z_*} y_*) \simeq y_* \text{pr}_{2,*} .$$

The remainder of this subsection focuses on generalising [27, Exposé XI, Proposition 2.3].

8.17. Suppose that we are given a diagram of  $\infty$ -topoi

$$(8.18) \quad \begin{array}{ccccc}
 X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y \\
 x_* \downarrow & & \downarrow z_* & & \downarrow y_* \\
 X' & \xrightarrow{f'_*} & Z' & \xleftarrow{g'_*} & Y'
 \end{array} ,$$

and suppose further that  $x_*$ ,  $y_*$ , and  $z_*$  are coessential with centres  $x^!$ ,  $y^!$ , and  $z^!$ , respectively. Then taking the adjoint squares in the diagram (8.18) with respect to the adjunctions  $x_* \dashv x^!$ ,  $y_* \dashv y^!$ , and  $z_* \dashv z^!$  [HA, Definition 4.7.4.13], we obtain a pair of oriented squares

$$(8.19) \quad \begin{array}{ccccc}
 X' & \xrightarrow{f'_*} & Z' & \xleftarrow{g'_*} & Y' \\
 x^! \downarrow & \swarrow & \downarrow z^! & \swarrow & \downarrow y^! \\
 X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y
 \end{array} .$$

Note that the natural transformation in the left-hand square of (8.19) points in the *wrong* direction to apply (8.16).

8.20. Keep the notations of (8.17), and additionally assume that the natural transformation in the left-hand square of (8.19) is an equivalence, so that  $f_* x^! \simeq z^! f'_*$ . Then by the functoriality of the oriented fibre product in oriented diagrams (8.16), the diagram (8.19) defines a geometric morphism  $x^! \tilde{\times}_{z^!} y^! : X' \tilde{\times}_{Z'} Y' \rightarrow X \tilde{\times}_Z Y$ .

The following is now formal.

**8.21 Proposition.** *With the notations and assumptions of (8.20), the geometric morphism*

$$x_* \tilde{\times}_{z_*} y_* : X \tilde{\times}_Z Y \rightarrow X' \tilde{\times}_{Z'} Y'$$

*is coëssential with centre  $x^! \tilde{\times}_{z^!} y^! : X' \tilde{\times}_{Z'} Y' \rightarrow X \tilde{\times}_Z Y$ .*

We now explain a particular application of **Proposition 8.21** that allows us to deduce that the  $\text{pr}_{2,*} : X \tilde{\times}_Z Y \rightarrow Y$  exhibits  $X \tilde{\times}_Z Y$  as local over  $Y$ .

**8.22.** Let  $f_* : X \rightarrow Z$  be a local geometric morphism of local  $\infty$ -topoi with centres  $x_*$  and  $z_*$ , respectively, and let  $g_* : Y \rightarrow Z$  be a geometric morphism of  $\infty$ -topoi. Then since all of the vertical morphisms in the commutative diagram of  $\infty$ -topoi

$$\begin{array}{ccccc} X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y \\ \Gamma_{X,*} \downarrow & & \downarrow \Gamma_{Z,*} & & \parallel \\ S & \xlongequal{\quad} & S & \xleftarrow{\Gamma_{Y,*}} & Y \end{array}$$

exhibit the top  $\infty$ -topoi as local over the bottom  $\infty$ -topoi, applying the discussion of (8.17), the assumption that  $f_*$  is a local geometric morphism shows that we are in the situation of (8.20). That is to say  $x_*$ ,  $z_*$ , and  $\text{id}_Y$  induce a geometric morphism

$$x_* \tilde{\times}_{z_*} \text{id}_Y : Y \simeq S \tilde{\times}_S Y \rightarrow X \tilde{\times}_Z Y .$$

The following is our generalisation of [27, Exposé XI, Proposition 2.3]. Note that this generalisation is not just  $\infty$ -toposic: in our version we don't need to take stalks.

**8.23 Lemma.** *With the notations of (8.22), the second projection  $\text{pr}_{2,*} : X \tilde{\times}_Z Y \rightarrow Y$  exhibits  $X \tilde{\times}_Z Y$  as local over  $Y$  with centre*

$$x_* \tilde{\times}_{z_*} \text{id}_Y : Y \simeq S \tilde{\times}_S Y \rightarrow X \tilde{\times}_Z Y .$$

*Proof.* The fact that  $\text{pr}_{2,*}$  is coëssential with centre  $x_* \tilde{\times}_{z_*} \text{id}_Y$  is immediate from **Proposition 8.21**, and the full faithfulness of  $x_* \tilde{\times}_{z_*} \text{id}_Y$  follows from the equivalence

$$\text{pr}_{2,*} \circ (x_* \tilde{\times}_{z_*} \text{id}_Y) \simeq \text{id}_Y . \quad \square$$

In the setting of **Lemma 8.23**, we deduce that the Beck–Chevalley morphism becomes an equivalence after taking its stalk at the centre of  $X$ .

**8.24 Lemma.** *Consider an oriented square of  $\infty$ -topoi*

$$\begin{array}{ccc} W & \xrightarrow{q_*} & Y \\ P_* \downarrow & \swarrow \tau & \downarrow g_* \\ X & \xrightarrow{f_*} & Z , \end{array}$$

where  $q_*$  is a quasi-equivalence,  $\mathbf{X}$  and  $\mathbf{Z}$  are local with centres  $x_*$  and  $z_*$ , respectively, and  $f_*$  is a local geometric morphism. Then the natural transformation

$$x^* \beta_r : x^* f^* g_* \rightarrow x^* p_* q^*$$

is an equivalence.

*Proof.* We prove the stronger claim that  $x^* f^* g_* \simeq x^* p_* q^*$  and the space of natural transformations  $x^* f^* g_* \rightarrow x^* p_* q^*$  is contractible. Since  $\mathbf{Z}$  is local we have equivalences

$$x^* f^* g_* \simeq z^* g_* \simeq \Gamma_{Z,*} g_* \simeq \Gamma_{Y,*} .$$

Since  $\mathbf{X}$  is local and  $q_*$  is a quasi-equivalence, applying [Lemma 7.3](#) we have equivalences

$$x^* p_* q^* \simeq \Gamma_{X,*} p_* q^* \simeq \Gamma_{W,*} q^* \simeq \Gamma_{Y,*} .$$

Thus both  $x^* f^* g_*$  and  $x^* p_* q^*$  are equivalent to the global sections functor on  $\mathbf{Y}$ . We are now done since  $\Gamma_{Y,*}$  is corepresented by the terminal object of  $\mathbf{Y}$ .  $\square$

## Proof of the Beck–Chevalley condition for oriented fibre products

This subsection is devoted to the proof of [Theorem 8.4](#).

*Proof of Theorem 8.4.* Write  $\beta : f^* g_* \rightarrow \text{pr}_{1,*} \text{pr}_2^*$  for the Beck–Chevalley natural transformation of the oriented fibre product square (8.5). Notice that since  $\mathbf{X}$  is bounded coherent, left exact functors preserve truncated objects, and morphisms between truncated objects are truncated, (5.68) shows that to prove the claim it suffices to show that for every point  $x_* \in \text{Pt}(\mathbf{X})$  and truncated object  $F \in \mathbf{Y}_{<\infty}$ , the morphism

$$x^* \beta(F) : x^* f^* g_*(F) \rightarrow x^* \text{pr}_{1,*} \text{pr}_2^*(F)$$

is an equivalence in  $\mathbf{S}$ .

Fix a point  $x_* \in \text{Pt}(\mathbf{X})$ , define  $z_* := f_* x_*$ , and let  $\bar{f}_* : \mathbf{X}_{(x)} \rightarrow \mathbf{Z}_{(z)}$  be the induced geometric morphism on localisations. To simplify notation we write  $\mathbf{W} := \mathbf{X} \bar{\times}_{\mathbf{Z}} \mathbf{Y}$ ,  $\mathbf{W}_{(x)} := \mathbf{X}_{(x)} \times_{\mathbf{X}} \mathbf{W}$ , and  $\mathbf{Y}_{(z)} := \mathbf{Z}_{(z)} \times_{\mathbf{Z}} \mathbf{Y}$ . Consider the cube

$$(8.25) \quad \begin{array}{ccccc} & & \mathbf{W} & \xrightarrow{\text{pr}_{2,*}} & \mathbf{Y} \\ & \nearrow \bar{\ell}_{x_*} & \downarrow & & \nearrow \bar{\ell}_{z_*} \\ \mathbf{W}_{(x)} & \xrightarrow{q_*} & \mathbf{Y}_{(z)} & & \downarrow g_* \\ & \downarrow \text{pr}_{1,*} & \downarrow \bar{g}_* & & \downarrow \\ & \mathbf{X} & \xrightarrow{f_*} & \mathbf{Z} & \\ & \downarrow p_* & & \downarrow & \\ \mathbf{X}_{(x)} & \xrightarrow{\bar{f}_*} & \mathbf{Z}_{(z)} & & \downarrow \ell_{z_*} \\ & \nearrow \ell_{x_*} & & \nearrow \ell_{z_*} & \end{array} ,$$

formed by pulling back the back face along the bottom face. In the cube (8.25), the front face is an oriented square, the back face is an oriented fibre product square, all other faces are commutative, and the side faces are pullback squares. Moreover, the cube satisfies the following property:

- (\*) The natural transformation between the right adjoints given by the composite of the back and left faces of (8.25) is equivalent to the natural transformation given by the composite of the front and right faces of (8.25).

We claim that the front face of (8.25) is an oriented fibre product square. To see this, note that by Proposition 7.27, the compatibility of the oriented fibre product with limits (6.31), the compatibility of oriented fibre products with étale geometric morphisms (Proposition 6.54), and Corollary 6.55, we have equivalences

$$\begin{aligned} \mathbf{X}_{(x)} \tilde{\times}_{\mathbf{Z}(z)} \mathbf{Y}_{(z)} &\simeq \lim_{U \in \text{Nbd}(x)} \lim_{V \in \text{Nbd}(z)} (\mathbf{X} \tilde{\times}_Z \mathbf{Y}) / \text{pr}_1^*(U) \\ &\simeq \mathbf{X}_{(x)} \times_X \mathbf{W} = \mathbf{W}_{(x)}. \end{aligned}$$

Applying Lemma 8.23 to the front face of (8.25), we deduce that  $q_* : \mathbf{W}_{(x)} \rightarrow \mathbf{Y}_{(z)}$  exhibits  $\mathbf{W}_{(x)}$  as local over  $\mathbf{Y}_{(z)}$ .

Now we define natural transformations

$$\alpha^R : x^* f^* g_* \rightarrow \Gamma_{\mathbf{X}_{(x)},*} \bar{f}^* \bar{g}_* \bar{\ell}_z^* \quad \text{and} \quad \alpha^L : x^* \text{pr}_{1,*} \text{pr}_2^* \rightarrow \Gamma_{\mathbf{X}_{(x)},*} p_* q^* \bar{\ell}_z^*,$$

which are both equivalences when restricted to  $\mathbf{Y}_{<\infty}$ , as follows. Write  $\beta^R$  for the Beck–Chevalley morphism of the right-hand face of (8.25) and  $\beta^L$  for the Beck–Chevalley morphism of the left-hand face. Since the bottom face of (8.25) commutes, under identification of left adjoints,  $\beta^R$  defines a natural transformation

$$\bar{f}^* \beta^R : \ell_x^* f^* g_* \simeq \bar{f}^* \ell_z^* g_* \rightarrow \bar{f}^* \bar{g}_* \bar{\ell}_z^*.$$

Let  $\alpha^R$  be the composite

$$\alpha^R : x^* f^* g_* \xrightarrow{\sim} \Gamma_{\mathbf{X}_{(x)},*} \bar{f}^* \ell_z^* g_* \xrightarrow{\Gamma_{\mathbf{X}_{(x)},*} \bar{f}^* \beta^R} \Gamma_{\mathbf{X}_{(x)},*} \bar{f}^* \bar{g}_* \bar{\ell}_z^*,$$

where the left-hand equivalence is by Lemma 7.13 and the fact that  $z^* = x^* f^*$ . By Proposition 8.8,  $\beta^R$  is an equivalence when restricted to  $\mathbf{Y}_{<\infty}$ ; therefore  $\alpha^R$  is also an equivalence when restricted to  $\mathbf{Y}_{<\infty}$ . Similarly, since the top face of (8.25) commutes, under identification of left adjoints,  $\beta^L$  defines a natural transformation

$$\beta^L \text{pr}_2^* : \ell_x^* \text{pr}_{1,*} \text{pr}_2^* \rightarrow p_* \bar{\ell}_x^* \text{pr}_2^* \simeq p_* q^* \bar{\ell}_z^*.$$

Let  $\alpha^L$  be the composite

$$\alpha^L : x^* \text{pr}_{1,*} \text{pr}_2^* \xrightarrow{\sim} \Gamma_{\mathbf{X}_{(x)},*} \ell_x^* \text{pr}_{1,*} \text{pr}_2^* \xrightarrow{\Gamma_{\mathbf{X}_{(x)},*} \beta^L \text{pr}_2^*} \Gamma_{\mathbf{X}_{(x)},*} p_* q^* \bar{\ell}_z^*,$$



where the left-hand equivalence is ensured by [Lemma 7.13](#). By [Proposition 8.8](#), the natural transformation  $\beta^L$  is an equivalence when restricted to  $W_{<\infty}$ , so since  $\text{pr}_2^*$  is left exact we see that  $\alpha^L$  is an equivalence when restricted to  $Y_{<\infty}$ .

Write  $\tilde{\beta}: \tilde{f}^* \tilde{g}_* \rightarrow p_* q^*$  for the Beck–Chevalley morphism for the front face of the cube [\(8.25\)](#). Since  $q_*: W_{(x)} \rightarrow Y_{(z)}$  exhibits  $W_{(x)}$  as local over  $Y_{(z)}$ , [Lemma 8.24](#) shows that the natural transformation

$$\Gamma_{X_{(x)},*} \tilde{\beta}: \Gamma_{X_{(x)},*} \tilde{f}^* \tilde{g}_* \rightarrow \Gamma_{X_{(x)},*} p_* q^*$$

is an equivalence. Since  $\alpha^R$  and  $\alpha^L$  are equivalences when restricted to  $Y_{<\infty}$ , to complete the proof it suffices to show that the square

$$\begin{array}{ccc} x^* f^* g_* & \xrightarrow{\alpha^R} & \Gamma_{X_{(x)},*} \tilde{f}^* \tilde{g}_* \tilde{\ell}_z^* \\ x^* \beta \downarrow & & \downarrow \Gamma_{X_{(x)},*} \tilde{\beta} \tilde{\ell}_z^* \\ x^* \text{pr}_{1,*} \text{pr}_2^* & \xrightarrow{\alpha^L} & \Gamma_{X_{(x)},*} p_* q^* \tilde{\ell}_z^* \end{array}$$

commutes. This is immediate from the property  $(*)$  combined with [\(8.3\)](#).  $\square$

## Applications of Beck–Chevalley

**8.26 Example.** Let  $f_*: X \rightarrow Z$  and  $g_*: Y \rightarrow Z$  be geometric morphisms of  $\infty$ -topoi, and assume that  $X$  and  $Y$  are bounded coherent and  $Z$  is Stone. Then by [Corollary 5.87](#) [= [SAG, Corollary E.3.1.2](#)],  $f_*$  and  $g_*$  are automatically coherent. Since  $X \tilde{\times}_Z Y \simeq X \times_Z Y$  ([Proposition 10.1](#)), [Theorem 8.4](#) shows that the (unoriented) pullback square

$$(8.27) \quad \begin{array}{ccc} X \times_Z Y & \xrightarrow{\text{pr}_{2,*}} & Y \\ \text{pr}_{1,*} \downarrow & \lrcorner & \downarrow g_* \\ X & \xrightarrow{f_*} & Z \end{array}$$

satisfies the bounded Beck–Chevalley condition.

**8.28 Subexample.** Set  $Z = \mathbf{S}$  in [Example 8.26](#), so that  $f_* = \Gamma_{X,*}$  and  $g_* = \Gamma_{Y,*}$ . Since left exact functors preserve truncated objects, for any truncated space  $K$  the natural morphism

$$\Gamma_{X,*} \Gamma_X^* \Gamma_{Y,*} \Gamma_Y^*(K) \rightarrow \Gamma_{X,*} \text{pr}_{1,*} \text{pr}_2^* \Gamma_Y^*(K)$$

in  $\mathbf{S}$  is an equivalence. Hence the natural morphism

$$(\Pi_\infty^\wedge(X) \circ \Pi_\infty^\wedge(Y))_\pi^\wedge \rightarrow \Pi_\infty^\wedge(X \times Y)$$

in  $\mathbf{S}_\pi^\wedge$  is an equivalence. In light of [Recollection 4.10](#), we deduce that

$$\Pi_\infty^\wedge(X \times Y) \simeq \Pi_\infty^\wedge(X) \times \Pi_\infty^\wedge(Y).$$

Combining this with [Lemma 5.85](#) we see that the profinite shape  $\Pi_\infty^\wedge: \mathbf{Top}_{\infty}^{bc} \rightarrow \mathbf{S}_\pi^\wedge$  preserves both inverse limits and finite products.

## Gluing squares

We now use the bounded Beck–Chevalley condition for oriented fibre products to study oriented squares that are both oriented fibre product squares and oriented pushouts in the setting of bounded coherent  $\infty$ -topoi. These *gluing squares* are essential to our décollage approach to stratified higher topoi in §9.

**8.29 Definition.** A *gluing square* is an oriented square

$$\begin{array}{ccc} W & \xrightarrow{q_*} & U \\ p_* \downarrow & \swarrow \sigma & \downarrow j_* \\ Z & \xrightarrow{i_*} & X \end{array}$$

in which:

- ▶ every  $\infty$ -topos is bounded coherent;
- ▶ every geometric morphism is coherent;
- ▶ the natural geometric morphism  $Z \overset{W}{\underset{bc}{\cup}} U \rightarrow X$  is an equivalence ([Construction 6.22](#));
- ▶ the natural geometric morphism  $W \rightarrow Z \overset{\tilde{X}}{\times} U$  is an equivalence ([Definition 6.29](#)).

We call the oriented fibre product  $W$  the *link* of the gluing square, or the *deleted tubular neighbourhood* of  $Z$  inside  $X$ .

**8.30 Construction.** Let  $X$  be a bounded coherent  $\infty$ -topos, along with a closed subtopos  $i_* : Z \hookrightarrow X$  and quasicompact open complement  $j_* : U \hookrightarrow X$ . Then we may form the oriented fibre product  $Z \overset{\tilde{X}}{\times} U$ , yielding the square

$$(8.31) \quad \begin{array}{ccc} Z \overset{\tilde{X}}{\times} U & \xrightarrow{\text{pr}_{2,*}} & U \\ \text{pr}_{1,*} \downarrow & \swarrow \tau & \downarrow j_* \\ Z & \xrightarrow{i_*} & X. \end{array}$$

The  $\infty$ -topos  $X$  is the bounded coherent recollement  $Z \overset{\text{pr}_{1,*}, \text{pr}_2^*}{\underset{bc}{\cup}} U$ . Indeed, the bounded Beck–Chevalley condition ([Theorem 8.4](#)) ensures that  $\beta_\tau : i_* j_* \rightarrow \text{pr}_{1,*} \text{pr}_2^*$  is an equivalence after restriction to  $U_{<\infty}^{\text{coh}}$ . So [Proposition 6.15](#) applies, whence (8.31) is a gluing square.

Dually, let  $W$ ,  $Z$ , and  $U$  be bounded coherent  $\infty$ -topoi, and let  $p_* : W \rightarrow Z$  and  $q_* : W \rightarrow U$  be geometric morphisms. Forming the bounded coherent oriented pushout  $X := Z \overset{W}{\underset{bc}{\cup}} U$ , we obtain a square

$$(8.32) \quad \begin{array}{ccc} W & \xrightarrow{q_*} & U \\ p_* \downarrow & \swarrow \sigma & \downarrow j_* \\ Z & \xrightarrow{i_*} & Z \overset{W}{\underset{bc}{\cup}} U. \end{array}$$

We thus obtain a geometric morphism  $\psi(p, q, \sigma)_* : W \rightarrow Z \tilde{\times}_X U$ , and if  $\psi(p, q, \sigma)_*$  is an equivalence, then the square (8.32) is a gluing square.

The full subcategory of  $\text{Fun}(\Delta^1 \times \Delta^1, \mathbf{Top}_{\infty}^{bc})$  spanned by the gluing squares is equivalent to the (non-full) subcategory of  $\text{Fun}(\Delta^1, \mathbf{Cat}_{\infty, \kappa_1})$  whose objects are bounded coherent gluing functors between bounded coherent  $\infty$ -topoi and whose morphisms  $\phi \rightarrow \phi'$  are squares

$$\begin{array}{ccc} U & \xrightarrow{\phi} & Z \\ f_* \downarrow & & \downarrow g_* \\ U' & \xrightarrow{\phi'} & Z' \end{array}$$

in which  $f_*$  and  $g_*$  are coherent geometric morphisms.

**8.33 Warning.** Without some boundedness and coherence hypotheses, the notion of a gluing square would not be apposite: if  $X := [0, 1]$  is the usual closed interval,  $Z := \{0\}$ , and  $U := ]0, 1]$ , then the oriented fibre product  $\tilde{Z} \tilde{\times}_X \tilde{U}$  is the empty topos.

**8.34 Example.** If  $W, Z$ , and  $U$  are profinite spaces, and if  $p: W \rightarrow Z$  and  $q: W \rightarrow U$  are morphisms, then we may form the profinite [1]-stratified space  $X$  corresponding to the profinite spatial décollage  $Z \leftarrow W \rightarrow U$ . Now we form the Stone  $\infty$ -topoi

$$W := \widetilde{W}, \quad Z := \widetilde{Z}, \quad \text{and} \quad U := \widetilde{U},$$

and we form the bounded coherent oriented pushout  $X := Z \widetilde{U}_{bc}^W U$ :

$$\begin{array}{ccc} W & \xrightarrow{q_*} & U \\ p_* \downarrow & \swarrow_{\sigma} & \downarrow j_* \\ Z & \xleftarrow{i_*} & X. \end{array}$$

The natural geometric morphism  $X \rightarrow \widetilde{X}$  is now an equivalence, since  $\widetilde{X}$  is the recollement of  $Z$  and  $U$  along  $p_* q^*$ , and  $\widetilde{X}$  is bounded and coherent. Now we compute

$$Z \tilde{\times}_X U \simeq \text{MOR}_{\widetilde{[1]}}(\widetilde{[1]}, X) \simeq \text{Map}_{\widetilde{[1]}}(\widetilde{[1]}, X) \simeq \widetilde{W} = W.$$

Thus the square above is in fact a gluing square.

## Part III

# Stratified higher topos theory

## 9 Stratified higher topoi

### Higher topoi attached to posets & proposets

9.1. A sheaf on a poset  $P$  (with its Alexandroff topology – [Definition 1.1](#)) is determined by its values on the principal open sets, which coincide with its stalks. Precisely, the assignment  $p \mapsto P_{\geq p}$  is a fully faithful functor  $P \rightarrow \text{Open}(P)^{op}$ , which induces an equivalence

$$\tilde{P} := \text{Sh}(\text{Open}(P)) \simeq \text{Fun}(P, \mathbf{S})$$

([Example 5.4](#)). In particular, the  $\infty$ -topos  $\tilde{P}$  is both 0-localic and Postnikov complete [[SAG](#), [§A.7.2](#)].

9.2. If  $P$  is a finite poset, then  $\tilde{P}$  is a coherent  $\infty$ -topos, and a sheaf  $F$  on  $P$  is  $n$ -coherent if and only if all of the stalks of  $F$  have finite homotopy sets in degrees  $m \leq n$ .

9.3. The assignment  $P \mapsto \tilde{P}$  extends to a functor  $\text{Pro}(\text{poSet}) \rightarrow \text{Top}_\infty$ , which we also denote by  $\mathbf{P} \mapsto \tilde{\mathbf{P}}$ . Thus if  $\mathbf{P} := \{P_\alpha\}_{\alpha \in \Lambda}$  is an inverse system of posets, then

$$\tilde{\mathbf{P}} \simeq \lim_{\alpha \in \Lambda} \tilde{P}_\alpha$$

in  $\text{Top}_\infty$ . That is, by [[HTT](#), [Theorem 6.3.3.1](#)], the  $\infty$ -category  $\tilde{\mathbf{P}}$  can be regarded as the one whose objects are collections  $\{F_\alpha\}_{\alpha \in \Lambda}$  of functors  $F_\alpha : \tilde{P}_\alpha \rightarrow \mathbf{S}$  along with compatible identifications of  $F_\beta$  with the right Kan extension of  $F_\alpha$  along  $P_\alpha \rightarrow P_\beta$  for any morphism  $\alpha \rightarrow \beta$  in  $\Lambda$ . In particular,  $\tilde{\mathbf{P}}$  is 0-localic.

9.4. If  $S$  is a spectral topological space, then  $\tilde{S}$  is coherent, and the  $\infty$ -pretopos  $\tilde{S}_{<\infty}^{coh}$  of truncated coherent objects of  $\tilde{S}$  can be identified with the filtered colimit  $\text{colim}_{P \in \text{FC}(S)^{op}} \tilde{P}_{<\infty}^{coh}$  over the category  $\text{FC}(S)$  of finite constructible stratifications  $S \rightarrow P$ , along the relevant restriction functors.

On the other hand, the 0-topos (locale)  $\text{Open}(S)$  is the limit of the 0-topoi  $\text{Open}(P)$  over  $\text{FC}(S)$ . Thus there is an equivalence of 0-localic  $\infty$ -topoi

$$\tilde{S} \simeq \lim_{P \in \text{FC}(S)} \tilde{P}.$$

In particular, the  $\infty$ -topos  $\tilde{S}$  is compactly generated [[HTT](#), [Proposition 5.5.7.6](#)].

Additionally, please observe that if  $f : S' \rightarrow S$  is a quasicompact continuous map of spectral topological spaces, then the induced geometric morphism  $f_* : \tilde{S}' \rightarrow \tilde{S}$  is coherent.

9.5. If  $S$  is a spectral topological space, then the  $\infty$ -category of points of  $\tilde{S}$  is equivalent to the materialisation of  $S$ , *viz.*,

$$\text{Pt}(\tilde{S}) \simeq \text{mat}(S).$$

Thus the points of  $\tilde{S}$  are precisely the points of  $S$  equipped with the specialisation partial ordering.

## Stratifications over posets

There are a number of ways to describe stratified  $\infty$ -topoi, but let us focus upon the most elementary description – a straightforward generalisation of the notion of a stratified topological space (Definition 1.7).

**9.6 Definition.** For any poset  $P$  and any  $\infty$ -topos  $\mathbf{X}$ , a *stratification of  $\mathbf{X}$  by  $P$*  – or, more briefly, a  *$P$ -stratification of  $\mathbf{X}$*  – is a geometric morphism of  $\infty$ -topoi  $f_* : \mathbf{X} \rightarrow \bar{P}$ . We define the  $\infty$ -category  $\mathbf{StrTop}_{\infty, P}$  of  *$P$ -stratified  $\infty$ -topoi* as the overcategory  $\mathbf{Top}_{\infty, \bar{P}}$ .

We define the  $\infty$ -category  $\mathbf{StrTop}_{\infty}$  of *stratified  $\infty$ -topoi* as the pullback

$$\mathbf{StrTop}_{\infty} := \mathrm{Fun}(\Delta^1, \mathbf{Top}_{\infty}) \times_{\mathrm{Fun}(\Delta^1, \mathbf{Top}_{\infty})} \mathit{poSet} .$$

Since  $\mathbf{Top}_{\infty}$  admits fibre products, the projection  $\mathbf{StrTop}_{\infty} \rightarrow \mathit{poSet}$  is a bicartesian fibration whose fibre over a poset  $P$  may be identified with the  $\infty$ -category  $\mathbf{StrTop}_{\infty, P}$ .

**9.7 Notation.** Let  $P$  be a poset, and let  $\mathbf{X}$  be a  $P$ -stratified  $\infty$ -topos. For any open subset  $U \subseteq P$ , we abuse notation and write  $U$  also for the corresponding open of  $\bar{P}$ , and we write

$$\mathbf{X}_U := \mathbf{X}_{/f^*U} \simeq \mathbf{X} \times_{\bar{P}} \bar{U} \subseteq \mathbf{X}$$

for the corresponding open subtopos. (Here the fibre product is formed in  $\mathbf{Top}_{\infty}$ .) Dually, if  $Z \subseteq P$  is closed, then we write

$$\mathbf{X}_Z := \mathbf{X}_{\setminus f^*(P \setminus Z)} \simeq \mathbf{X} \times_{\bar{P}} \bar{Z} \subseteq \mathbf{X}$$

for the corresponding closed subtopos, so that if  $U$  and  $Z$  are complementary, then one exhibits  $\mathbf{X}$  as a recollement of  $\mathbf{X}_Z$  and  $\mathbf{X}_U$ .

In particular, for any point  $p \in P$ , we write

$$\mathbf{X}_{\geq p} := \mathbf{X}_{P_{\geq p}} \quad \text{and} \quad \mathbf{X}_{> p} := \mathbf{X}_{P_{> p}}$$

as well as

$$\mathbf{X}_{\leq p} := \mathbf{X}_{P_{\leq p}} \quad \text{and} \quad \mathbf{X}_{< p} := \mathbf{X}_{P_{< p}} .$$

More generally, if  $\Sigma \subseteq P$  is any subset, then we write

$$\mathbf{X}_{\Sigma} := \mathbf{X} \times_{\bar{P}} \bar{\Sigma}$$

for the fibre product formed in  $\mathbf{Top}_{\infty}$ . So we define the  $p$ -th *stratum* as the fibre product in  $\mathbf{Top}_{\infty}$ :

$$\mathbf{X}_p := \mathbf{X}_{\geq p} \times_{\mathbf{X}} \mathbf{X}_{\leq p} ,$$

which is an open subtopos of the closed subtopos  $\mathbf{X}_{\leq p} \subseteq \mathbf{X}$  as well as a closed subtopos of the open subtopos  $\mathbf{X}_{\geq p} \subseteq \mathbf{X}$ .

**9.8 Definition.** A stratification  $\mathbf{X} \rightarrow \bar{P}$  of an  $\infty$ -topos  $\mathbf{X}$  is *finite* or *noetherian* if and only if the poset  $P$  is so. We write  $\mathbf{StrTop}_{\infty}^{\mathit{noeth}} \subset \mathbf{StrTop}_{\infty}$  for the full subcategory spanned by the noetherian stratifications.

Let  $P$  be a finite poset. We say that a  $P$ -stratified  $\infty$ -topos  $f_* : \mathbf{X} \rightarrow \tilde{P}$  is *constructible* if and only if for any point  $p \in P$  and any quasicompact open  $V \in \text{Open}(\mathbf{X})$ , the  $\infty$ -topos  $\mathbf{X}_{\geq p} \times_{\mathbf{X}} \mathbf{X}_{/V}$  is coherent. We say that a constructible stratification  $f_* : \mathbf{X} \rightarrow \tilde{P}$  is *coherent constructible* if  $\mathbf{X}$  is a coherent  $\infty$ -topos, and we say that  $f_*$  is *bounded coherent constructible* if  $\mathbf{X}$  is a bounded coherent  $\infty$ -topos. **Proposition 6.8**=[DAG XIII, Proposition 2.3.22] shows that a stratification  $f_* : \mathbf{X} \rightarrow \tilde{P}$  is coherent constructible if and only if  $\mathbf{X}$  is coherent and the geometric morphism  $f_*$  is coherent. We write  $\mathbf{StrTop}_{\infty}^{bcc} \subset \mathbf{StrTop}_{\infty}$  for the subcategory whose objects are bounded coherent constructible stratified  $\infty$ -topoi and whose morphisms are *coherent* geometric stratified morphisms:

$$\mathbf{StrTop}_{\infty}^{bcc} := \text{Fun}(\Delta^1, \mathbf{Top}_{\infty}^{bc}) \times_{\text{Fun}(\Delta^1, \mathbf{Top}_{\infty}^{bc})} \text{poSet}^{fin} .$$

**9.9.** Since  $\tilde{P}$  is 0-localic, it follows that a  $P$ -stratification of an  $\infty$ -topos  $\mathbf{X}$  is tantamount to the data of a morphism of 0-topoi (locales)  $\text{Open}(\mathbf{X}) \rightarrow \text{Open}(P)$ , where  $\text{Open}(\mathbf{X})$  is the 0-topos of  $(-1)$ -truncated objects of  $\mathbf{X}$ , and  $\text{Open}(P) = \text{Open}(\tilde{P})$  is the 0-topos of open subsets of  $P$ . Thus one obtains an equivalence of  $\infty$ -categories

$$\mathbf{StrTop}_{\infty, P} \simeq \mathbf{Top}_{\infty} \times_{\mathbf{Top}_0} \mathbf{Top}_{0, / \text{Open}(P)} .$$

One may speak of a stratification of an  $n$ -topos for any  $n \in \mathbf{N}$  (as well as the  $\infty$ -category  $\mathbf{StrTop}_n$ ), and it is tantamount to a stratification of the corresponding  $n$ -localic  $\infty$ -topos:

$$\mathbf{StrTop}_{n, P} \simeq \mathbf{Top}_n \times_{\mathbf{Top}_0} \mathbf{Top}_{0, / \text{Open}(P)} .$$

**9.10.** Since noetherian posets are sober, the functor  $\text{poSet}^{noeth} \rightarrow \mathbf{Top}_{\infty}$  given by the assignment  $P \mapsto \tilde{P}$  is fully faithful, whence the  $\infty$ -category  $\mathbf{StrTop}_{\infty}^{noeth}$  of noetherian stratified  $\infty$ -topoi can be identified with a *full* subcategory of  $\text{Fun}(\Delta^1, \mathbf{Top}_{\infty})$ .

**9.11 Example.** A  $\{0\}$ -stratified  $\infty$ -topos is nothing more than an  $\infty$ -topos.

**9.12 Example.** Rephrasing (6.3), a  $[1]$ -stratified  $\infty$ -topos  $\mathbf{X} \rightarrow \widetilde{[1]}$  is tantamount to a recollement of  $\infty$ -topoi. If  $\mathbf{X}$  is coherent, the stratification is constructible if and only if the open subtopos  $\mathbf{X}_1$  is quasicompact.

**9.13.** To generalise the previous example, let  $P$  be a poset. We claim that the data of a  $P$ -stratified  $\infty$ -topos determines and is determined by a suitable colax functor from  $P^{op}$  to a double  $\infty$ -category of  $\infty$ -topoi and left exact functors.

To make a precise assertion, we shall say that a locally cocartesian fibration  $X \rightarrow P^{op}$  is *left exact* if each fibre  $X_p$  admits all finite limits, and for any  $p \leq q$  in  $P$ , the functor  $X_q \rightarrow X_p$  is left exact. Now left exact locally cocartesian fibrations  $X \rightarrow P^{op}$  whose fibres are  $\infty$ -topoi organise themselves into an  $\infty$ -category  $\mathbf{LocCocart}_{P^{op}}^{lex, top}$ . Then it seems likely that one can produce an equivalence of  $\infty$ -categories

$$\mathbf{LocCocart}_{P^{op}}^{lex, top} \simeq \mathbf{StrTop}_{\infty, P} ,$$

natural in  $P$ . To prove this would involve a diversion into a simplicial thicket that is unnecessary for our work here; we therefore leave this matter for a later paper.

**9.14 Example.** The  $\infty$ -topos  $\tilde{P}$ , equipped with the identity stratification, is itself is terminal in  $\mathbf{StrTop}_{\infty, P}$ .

**9.15 Example.** If  $P$  is a noetherian poset, and  $\mathbf{TSpc}^{sober}$  denotes the 1-category of sober topological spaces, then the assignment  $W \mapsto \tilde{W}$  is a fully faithful functor

$$\mathbf{TSpc}_{/P}^{sober} \hookrightarrow \mathbf{StrTop}_{\infty, P}.$$

**9.16 Example.** Let  $P$  be a poset, and  $f : \Pi \rightarrow P$  a  $P$ -stratified space (Definition 2.1); i.e.,  $f$  is a conservative functor. In light of the equivalence  $\tilde{P} \simeq \mathbf{Fun}(P, \mathbf{S})$ , let us abuse notation slightly and write

$$\tilde{\Pi} := \mathbf{Fun}(\Pi, \mathbf{S})$$

for the  $\infty$ -topos of functors  $\Pi \rightarrow \mathbf{S}$ ; then right Kan extension along  $f$  is a morphism of  $\infty$ -topoi

$$f_* : \tilde{\Pi} \rightarrow \tilde{P},$$

whence  $\tilde{\Pi}$  is a  $P$ -stratified  $\infty$ -topos. For any point  $p \in P$ , the  $p$ -th stratum of  $\tilde{\Pi}$  is canonically identified the  $\infty$ -topos  $\tilde{\Pi}_p = \mathbf{Fun}(\Pi_p, \mathbf{S})$ .

The assignment  $\Pi \mapsto \tilde{\Pi}$  defines a functor  $\mathbf{Str} \rightarrow \mathbf{StrTop}_{\infty}$  over  $po\mathbf{Set}$ .

**9.17 Subexample.** Let  $P$  be a noetherian poset, and Let  $X$  be a conically  $P$ -stratified topological space Definition A.5.5. Then we obtain the  $P$ -stratified space  $\Pi_{(\infty, 1)}(X; P) := \mathbf{Sing}^P(X)$  and thus the  $P$ -stratified  $\infty$ -topos  $\tilde{\Pi}_{(\infty, 1)}(X; P)$ . If  $X$  is hereditarily paracompact and locally of singular shape, then in light of [HA, §A.4], we may identify the stratum  $\tilde{\Pi}_{(\infty, 1)}(X; P)_p$  over any point  $p \in P$  with the  $\infty$ -category of locally constant sheaves on  $X_p$ , and in light of [HA, §A.9], one may identify  $\tilde{\Pi}_{(\infty, 1)}(X; P)$  with the  $\infty$ -category of *formally constructible* sheaves on  $X$  – i.e., those sheaves whose restrictions to each stratum  $X_p$  are locally constant.

**9.18 Lemma.** *Let  $P$  be a finite poset and  $\Pi$  be a  $\pi$ -finite  $P$ -stratified space. Then the stratification  $\tilde{\Pi} \rightarrow \tilde{P}$  is bounded coherent constructible.*

*Proof.* By definition  $\tilde{\Pi}$  is  $n$ -localic for some  $n \in \mathbf{N}$ . Moreover, the truncated coherent objects of  $\tilde{\Pi}$  are those functors  $\Pi \rightarrow \mathbf{S}$  that are valued in  $\pi$ -finite spaces. One concludes that  $\tilde{\Pi}$  is coherent. Since this is true for  $\Pi$ , it is true for any open therein, whence  $\tilde{\Pi} \rightarrow \tilde{P}$  is constructible.  $\square$

## Toposic décollages

In analogy with the construction of the spatial décollage attached to a stratified space (Construction 4.4), we can attach to a stratified  $\infty$ -topos what we call its (*toposic*) *décollage*. Whereas a stratified  $\infty$ -topos consists of strata that are glued together, its décollage is the result of pulling these strata apart while retaining the linking information necessary to reconstruct the stratified  $\infty$ -topos.

**9.19 Definition.** Let  $P$  be a poset. We say that a functor  $\mathbf{D} : \mathbf{sd}^{op}(P) \rightarrow \mathbf{Top}_{\infty}^{bc}$  is a *décollage over  $P$*  if and only if the following conditions are satisfied.

- If  $p_0, p_1 \in P$  are two points such that  $p_0 < p_1$ , then the square

$$\begin{array}{ccc}
 \mathbf{D}\{p_0, p_1\} & \longrightarrow & \mathbf{D}\{p_1\} \\
 \downarrow & \swarrow & \downarrow j_* \\
 \mathbf{D}\{p_0\} & \xleftarrow{i_*} & \mathbf{D}\{p_0\} \tilde{\cup}_{bc}^{\mathbf{D}\{p_0, p_1\}} \mathbf{D}\{p_1\}
 \end{array}$$

is a gluing square.

- For any string  $\{p_0, \dots, p_m\} \subseteq P$ , the geometric morphism to the fibre product of  $\infty$ -topoi

$$\mathbf{D}\{p_0, \dots, p_m\} \rightarrow \mathbf{D}\{p_0, p_1\} \times_{\mathbf{D}\{p_1\}} \mathbf{D}\{p_1, p_2\} \times_{\mathbf{D}\{p_2\}} \cdots \times_{\mathbf{D}\{p_{m-1}\}} \mathbf{D}\{p_{m-1}, p_m\}$$

is an equivalence.

We write  $\text{D}\acute{e}c_P(\mathbf{Top}_\infty^{bc}) \subseteq \text{Fun}(\text{sd}^{op}(P), \mathbf{Top}_\infty^{bc})$  for the full subcategory spanned by the décollages over  $P$ .

It seems likely that a décollage over  $P$  can be thought of as a suitable  $\infty$ -category internal to  $\mathbf{Top}_\infty^{bc}$  along with a conservative functor to  $P$ . Making such an interpretation precise and helpful is a task that lies outside the scope of this work.

**9.20.** If  $\mathbf{D}: \text{sd}^{op}(P) \rightarrow \mathbf{Top}_\infty^{bc}$  is a décollage over  $P$ , and if  $p, q \in P$  are points with  $p < q$ , then for the sake of typographical brevity, let us here write

$$\mathbf{D}\{p\} \tilde{\cup} \mathbf{D}\{q\} := \mathbf{D}\{p\} \tilde{\cup}_{bc}^{\mathbf{D}\{p, q\}} \mathbf{D}\{q\}.$$

The two conditions of [Definition 9.19](#) together specify, for any string  $\{p_0, \dots, p_m\} \subseteq P$ , an equivalence

$$\mathbf{D}\{p_0, \dots, p_m\} \simeq \mathbf{D}\{p_0\} \tilde{\cup}_{\mathbf{D}\{p_0\} \tilde{\cup} \mathbf{D}\{p_1\}}^{\bar{\chi}} \mathbf{D}\{p_1\} \tilde{\cup}_{\mathbf{D}\{p_1\} \tilde{\cup} \mathbf{D}\{p_2\}}^{\bar{\chi}} \cdots \tilde{\cup}_{\mathbf{D}\{p_{m-1}\} \tilde{\cup} \mathbf{D}\{p_m\}}^{\bar{\chi}} \mathbf{D}\{p_m\},$$

which we will call the *Segal equivalence*.

**9.21 Example.** The terminal object of  $\text{D}\acute{e}c_P(\mathbf{Top}_\infty^{bc})$  is the constant functor  $\text{sd}^{op}(P) \rightarrow \mathbf{Top}_\infty^{bc}$  whose value is the  $\infty$ -topos  $\mathbf{S}$ .

**9.22 Construction.** Consider the 1-category  $J$  of [Construction 4.3](#), whose objects are pairs  $(P, \Sigma)$  consisting of a poset  $P$  and a string  $\Sigma \subseteq P$ , so that the assignment  $(P, \Sigma) \mapsto P$  is a cocartesian fibration  $J \rightarrow \text{poSet}$  whose fibre over a poset  $P$  is the poset  $\text{sd}^{op}(P)$ .

We write

$$\text{Pair}_{\text{poSet}}(J, \mathbf{Top}_\infty^{bc})$$

for the simplicial set over  $\text{poSet}$  defined by the following universal property: for any simplicial set  $K$  over  $\text{poSet}$ , one demands a bijection

$$\text{Mor}_{\text{sSet}/\text{poSet}}(K, \text{Pair}_{\text{poSet}}(J, \mathbf{Top}_\infty^{bc})) \cong \text{Mor}_{\text{sSet}}(K \times_{\text{poSet}} J, \mathbf{Top}_\infty^{bc}),$$



natural in  $K$ . By [HTT, Corollary 3.2.2.13], the functor

$$\text{Pair}_{\text{poSet}}(J, \mathbf{Top}_{\infty}^{bc}) \rightarrow \text{poSet}$$

is a cartesian fibration whose fibre over a poset  $P$  is the  $\infty$ -category  $\text{Fun}(\text{sd}^{op}(P), \mathbf{Top}_{\infty}^{bc})$ . Now let

$$\mathbf{D}\acute{e}\text{c}(\mathbf{Top}_{\infty}^{bc}) \subset \text{Pair}_{\text{poSet}}(J, \mathbf{Top}_{\infty}^{bc})$$

denote the full subcategory spanned by the pairs  $(P, \mathbf{D})$  in which  $\mathbf{D}$  is a toposic décollage over  $P$ . Since  $\mathbf{D}\acute{e}\text{c}(\mathbf{Top}_{\infty}^{bc})$  contains all the cartesian edges, the functor  $\mathbf{D}\acute{e}\text{c}(\mathbf{Top}_{\infty}^{bc}) \rightarrow \text{poSet}$  is a cartesian fibration.

### The nerve of a stratified $\infty$ -topos

**9.23 Construction.** Let  $P$  be a poset, and let  $f_* : \mathbf{X} \rightarrow \tilde{P}$  be a  $P$ -stratified  $\infty$ -topos. Then for any monotonic map  $\phi : Q \rightarrow P$ , we define the  $\infty$ -topos of sections of  $\mathbf{X}$  over  $Q$  as the pullback of  $\infty$ -topoi

$$\text{MOR}_{\tilde{P}}(\tilde{Q}, \mathbf{X}) := \text{MOR}(\tilde{Q}, \mathbf{X}) \times_{\text{MOR}(\tilde{Q}, \tilde{P})} \{\widehat{\phi}\}.$$

The  $\infty$ -topos  $\text{MOR}_{\tilde{P}}(\tilde{Q}, \mathbf{X})$  depends only on the pullback  $\mathbf{X} \times_{\tilde{P}} \tilde{Q}$ :

$$\text{MOR}_{\tilde{P}}(\tilde{Q}, \mathbf{X}) \simeq \text{MOR}_{\tilde{Q}}(\tilde{Q}, \mathbf{X} \times_{\tilde{P}} \tilde{Q}).$$

We thus obtain a functor  $N_P(\mathbf{X}) : \text{sd}^{op}(P) \rightarrow \mathbf{Top}_{\infty}$  that carries a string  $\Sigma \subseteq P$  to the  $\infty$ -topos

$$N_P(\mathbf{X})(\Sigma) := \text{MOR}_{\tilde{P}}(\tilde{\Sigma}, \mathbf{X}).$$

For any string  $\{p_0, \dots, p_m\} \subseteq P$ , we thus obtain an identification

$$N_P(\mathbf{X})\{p_0, \dots, p_m\} \simeq X_{p_0} \tilde{\times}_X X_{p_1} \tilde{\times}_X \cdots \tilde{\times}_X X_{p_m}.$$

In particular, if  $P$  is finite and  $\mathbf{X}$  is bounded coherent constructible (Definition 9.8), then the functor  $N_P(\mathbf{X})$  is a décollage over  $P$ . We call  $N_P(\mathbf{X})$  the *nerve* of the  $P$ -stratified  $\infty$ -topos  $\mathbf{X}$ , and we call  $N : \mathbf{StrTop}_{\infty}^{bcc} \rightarrow \mathbf{D}\acute{e}\text{c}(\mathbf{Top}_{\infty}^{bc})$  over  $\text{poSet}$  the *nerve functor*.

**9.24 Example.** Let  $P$  be a poset, and  $\Pi$  a  $P$ -stratified space. Then one has a identification

$$N_P(\widetilde{\Pi}) \simeq \widehat{N_P(\Pi)},$$

natural in  $P$  and  $\Pi$ , since for any string  $\Sigma \subseteq P$ , one has

$$\text{MOR}_{\tilde{P}}(\tilde{\Sigma}, \widetilde{\Pi}) \simeq \widehat{\text{Map}_P(\Sigma, \Pi)}$$

via the natural morphism.

We now proceed to demonstrate that the nerve is an equivalence of  $\infty$ -categories.

**9.25 Theorem.** For any finite poset  $P$ , the nerve functor  $N_P : \mathbf{StrTop}_{\infty, P}^{bcc} \rightarrow \mathbf{D}\acute{e}\text{c}_P(\mathbf{Top}_{\infty}^{bc})$  is an equivalence of  $\infty$ -categories.

*Proof.* We begin by reducing to the case in which  $P$  is a nonempty, finite, totally ordered set. To make this reduction, we note that  $P \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}(P)} \Sigma$ , whence  $\bar{P}$  is the limit  $\bar{P} \simeq \lim_{\Sigma \in \operatorname{sd}^{op}(P)} \bar{\Sigma}$  in  $\mathbf{Cat}_{\infty, \kappa_1}$  (which is the colimit in  $\mathbf{Top}_{\infty}$ ) and moreover

$$\operatorname{sd}^{op}(P) \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}(P)} \operatorname{sd}^{op}(\Sigma).$$

From this we deduce that

$$\mathbf{StrTop}_{\infty, P}^{bcc} \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}(P)} \mathbf{StrTop}_{\infty, \Sigma}^{bcc} \quad \text{and} \quad \mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{bc}) \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}(P)} \mathbf{D\acute{e}c}_{\Sigma}(\mathbf{Top}_{\infty}^{bc}),$$

which provides our reduction.

Now when  $P = [n] := \{0, \dots, n\}$  is a nonempty totally ordered finite set, we construct an inverse  $U_n : \mathbf{D\acute{e}c}_{[n]}(\mathbf{Top}_{\infty}^{bc}) \rightarrow \mathbf{StrTop}_{\infty, [n]}^{bcc}$  to the nerve functor  $N_n := N_{[n]}$  by forming the iterated bounded coherent oriented pushout:

$$U_n(\mathbf{D}) := \mathbf{D}\{0\} \overset{D\{0,1\}}{\underset{bc}{\cup}} \mathbf{D}\{1\} \overset{D\{1,2\}}{\underset{bc}{\cup}} \dots \overset{D\{n-1,n\}}{\underset{bc}{\cup}} \mathbf{D}\{n\},$$

equipped with its canonical geometric morphism to

$$\widetilde{[n]} \simeq U_n(\mathbf{S}),$$

which is visibly coherent.

The universal properties of the iterated bounded coherent oriented pushout and the iterated oriented pullback provide natural transformations  $U_n N_n \rightarrow \operatorname{id}$  and  $\operatorname{id} \rightarrow N_n U_n$ . We aim to show that these natural transformations are equivalences.

To see that  $U_n$  is an inverse to  $N_n$ , we may induct on  $n$ . The case  $n = 0$  is obvious. Assume now that  $n \geq 1$  and that  $U_{n-1}$  is an inverse to  $N_{n-1}$ . Now if  $X$  is a bounded coherent  $\infty$ -topos with a constructible stratification  $X \rightarrow \widetilde{[n]}$ , then consider the recollement of  $X$  given by  $X_{\leq n-1}$  and  $X_n$ . We thus have a gluing square

$$\begin{array}{ccc} X_{\leq n-1} \overset{\tilde{\chi}_X}{\times} X_n & \xrightarrow{q_*} & X_n \\ p_* \downarrow & \swarrow \sigma & \downarrow j_* \\ X_{\leq n-1} & \xrightarrow{i_*} & X. \end{array}$$

As a result, we compute:

$$U_n N_n(X) \simeq U_{n-1} N_{n-1}(X_{\leq n-1}) \overset{X_{\leq n-1} \overset{\tilde{\chi}_X}{\times} X_n}{\underset{bc}{\cup}} X_n \simeq X_{\leq n-1} \overset{X_{\leq n-1} \overset{\tilde{\chi}_X}{\times} X_n}{\underset{bc}{\cup}} X_n \simeq X,$$

as desired. In the other direction, suppose  $\mathbf{D} : \operatorname{sd}^{op}([n]) \rightarrow \mathbf{Top}_{\infty}^{bc}$  is a toposic décollage. For any  $k \in [n]$ , write  $\widehat{k}$  for  $\{0, \dots, k-1, k+1, \dots, n\} \subset [n]$ ; for any string  $\Sigma \subseteq \widehat{k}$ , the map  $\mathbf{D}(\Sigma) \rightarrow N_n U_n(\mathbf{D})(\Sigma)$  clearly factors via equivalences

$$\mathbf{D}(\Sigma) \simeq (\mathbf{D} | \operatorname{sd}^{op}(\widehat{k}))(\Sigma) \simeq N_{\widehat{k}} U_{\widehat{k}}(\mathbf{D} | \operatorname{sd}^{op}(\widehat{k}))(\Sigma) \simeq N_n U_n(\mathbf{D})(\Sigma),$$

so it remains only to contemplate the case  $\Sigma = [n]$  itself. For this, note that the morphism  $\mathbf{D}([n]) \rightarrow N_n U_n(\mathbf{D})([n])$  is homotopic to the Segal equivalence

$$\mathbf{D}\{0, \dots, n\} \simeq \mathbf{D}\{0\} \overset{\tilde{\chi}}{\underset{U_n(\mathbf{D})}{\cup}} \mathbf{D}\{1\} \overset{\tilde{\chi}}{\underset{U_n(\mathbf{D})}{\cup}} \dots \overset{\tilde{\chi}}{\underset{U_n(\mathbf{D})}{\cup}} \mathbf{D}\{n\},$$

whence our claim.  $\square$

## Stratifications over spectral topological spaces

**9.26 Definition.** For any propoet  $P$ , a  $P$ -stratified  $\infty$ -topos is a morphism of  $\infty$ -topoi  $X \rightarrow \tilde{P}$ . We write  $\mathbf{StrTop}_{\infty, P}$  for the  $\infty$ -category  $\mathbf{Top}_{\infty, \tilde{P}}$  of  $P$ -stratified  $\infty$ -topoi.

We are interested exclusively in the case where  $P$  is a spectral topological space, viewed as a profinite poset. Hence we define

$$\mathbf{StrTop}_{\infty}^{\wedge} := \mathbf{Fun}(\Delta^1, \mathbf{Top}_{\infty}) \times_{\mathbf{Fun}(\Delta^{\mathbb{N}}, \mathbf{Top}_{\infty})} \mathbf{TSp}^{spec},$$

so that the fibre over  $S$  can be identified with  $\mathbf{StrTop}_{\infty, S}$ .

**9.27.** If  $S$  is a spectral topological space, then the  $\infty$ -topos  $\tilde{S}$  of sheaves on  $S$  coincides with the limit of  $\infty$ -topoi  $\lim_{P \in \mathbf{FC}(S)} \tilde{P}$ , so there is no ambiguity in the notation; furthermore, one has

$$\mathbf{StrTop}_{\infty, S}^{\wedge} \simeq \mathbf{Top}_{\infty} \times_{\mathbf{Top}_0} \mathbf{Top}_{0, / \mathbf{Open}(S)},$$

where  $\mathbf{Open}(S)$  is the locale of open subsets of  $S$ .

In the case of stratifications over spectral topological spaces, we employ notations as in [Notation 9.7](#).

**9.28 Notation.** Let  $S$  be a spectral topological space, and let  $X$  be a  $S$ -stratified  $\infty$ -topos. For any open subset  $U \subseteq S$ , we abuse notation and write  $U$  also for the corresponding open of  $\tilde{S}$ , and we write

$$X_U := X_{/f^*U} \simeq X \times_{\tilde{S}} \tilde{U} \subseteq X$$

for the corresponding open subtopos. (Here the fibre product is formed in  $\mathbf{Top}_{\infty}$ .) Dually, if  $Z \subseteq S$  is closed, then we write

$$X_Z := X_{\setminus f^*(S \setminus Z)} \simeq X \times_{\tilde{S}} \tilde{Z} \subseteq X$$

for the corresponding closed subtopos, so that if  $U$  and  $Z$  are complementary, then one exhibits  $X$  as a recollement of  $X_Z$  and  $X_U$ .

More generally, for any subspace  $W \subset S$ , we write

$$X_W := X \times_{\tilde{S}} \tilde{W}.$$

In particular, for any point  $s \in S$  we define the  $s$ -th *stratum* as the fibre product in  $\mathbf{Top}_{\infty}$ :

$$X_s := X \times_{\tilde{S}} \{\tilde{s}\} \subseteq X.$$

The key finiteness condition for stratifications over spectral topological spaces is *bounded coherent constructibility*.

**9.29 Definition.** If  $X$  is an  $\infty$ -topos and  $S$  is a spectral topological space, then a stratification  $f_* : X \rightarrow \tilde{S}$  is said to be *constructible* if and only if, for any quasicompact open  $U \subseteq S$  and any quasicompact open  $V \in \mathbf{Open}(X)$ , the  $\infty$ -topos

$$X_U \times_X X_V \simeq X_{/f^*(U) \times V}$$

is coherent. We say that a constructible stratification  $f_* : X \rightarrow \tilde{S}$  is *coherent constructible* if  $X$  is a coherent  $\infty$ -topos, and we say that  $f_*$  is *bounded coherent constructible* if  $X$  is a bounded coherent  $\infty$ -topos.

**9.30 Lemma.** *Let  $S$  be a spectral topological space and  $f_* : X \rightarrow \bar{S}$  be an  $S$ -stratified  $\infty$ -topos. If  $X$  is coherent, then the stratification  $f_*$  is constructible if and only if  $f_*$  is a coherent geometric morphism.*

*Proof.* If  $f_*$  is coherent, then since quasicompact opens in  $X$  are coherent [SAG, Remark A.2.3.5] and coherent objects of  $X$  are closed under finite products,  $f_*$  is a constructible stratification

For the other direction, assume that  $f_*$  is a constructible stratification. By Corollary 5.40, to show that  $f_*$  is coherent it suffices to show that  $f^*$  carries truncated coherent objects of  $\bar{S}$  to coherent objects of  $X$ . Let  $F \in \bar{S}_{<\infty}^{coh}$  be a truncated coherent object; then there exists a finite constructible stratification  $S \rightarrow P$  such that  $F$  is the pullback of a truncated coherent object of  $\bar{P}$ . Thus, for every point  $p \in P$ , the restriction  $f^*(F)|_{X_p}$  is lisse. By Proposition 6.8=[DAG XIII, Proposition 2.3.22] it follows that  $F$  is coherent.  $\square$

**9.31 Notation.** Let  $S$  be a spectral topological space. We define the  $\infty$ -category of *coherent constructible  $S$ -stratified  $\infty$ -topoi* as the overcategory

$$\mathbf{StrTop}_{\infty, S}^{\wedge, cc} := \mathbf{Top}_{\infty, \bar{S}}^{coh}.$$

We write  $\mathbf{StrTop}_{\infty, S}^{\wedge, bcc} \subset \mathbf{StrTop}_{\infty, S}^{\wedge, cc}$  for the full subcategory spanned by the *bounded coherent constructible  $S$ -stratified  $\infty$ -topoi*.

More generally, we define

$$\mathbf{StrTop}_{\infty}^{\wedge, cc} := \mathbf{Fun}(\Delta^1, \mathbf{Top}_{\infty}^{coh}) \times_{\mathbf{Fun}(\Delta^1, \mathbf{Top}_{\infty}^{coh})} \mathbf{TSp}^{spec},$$

so that the fibre over  $S$  can be identified with  $\mathbf{StrTop}_{\infty, S}^{\wedge, cc}$ . We write  $\mathbf{StrTop}_{\infty}^{\wedge, bcc} \subset \mathbf{StrTop}_{\infty}^{\wedge, cc}$  for the full subcategory spanned by those objects  $X \rightarrow \bar{S}$  where  $X$  is a bounded  $\infty$ -topos.

## The natural stratification of a coherent $\infty$ -topos

It turns out that *any* coherent  $\infty$ -topos  $X$  has a canonical profinite stratification: the 0-topos (=locale)  $\mathbf{Open}(X)$  is the locale of a spectral topological space. This provides a fully faithful embedding of the  $\infty$ -category of coherent  $\infty$ -topoi into that of coherent constructible stratified  $\infty$ -topoi.

To explain this point, let us first recall the equivalence between coherent locales and spectral topological spaces.

**9.32 Recollection.** Let  $A$  be a locale. An object  $a \in A$  is *quasicompact*<sup>11</sup> if and only if for every subset  $S \subset A$  such that  $\coprod_{s \in S} s = a$ , there exists a finite subset  $S_0 \subset S$  such that  $\coprod_{s \in S_0} s = a$ .

One says that  $A$  is *coherent* if and only if the following conditions are satisfied:

- The quasicompact elements of  $A$  form a *sublattice* of  $A$ : the maximal element  $1_A \in A$  is quasicompact and binary products (=meets) of quasicompact elements are quasicompact.

<sup>11</sup>Such elements are sometimes called *finite*; see [28, Chapter II, §3.1].

- The quasicompact elements of  $A$  *generate*  $A$ : every element  $a \in A$  can be written as a coproduct (=join)  $a = \coprod_{s \in S} s$ , where  $S \subset A$  is a subset consisting of quasicompact elements of  $A$ .

A morphism  $A \rightarrow A'$  between coherent locales is *coherent* if and only if the corresponding map of posets  $A' \rightarrow A$  sends quasicompact elements to quasicompact elements.

We write  $\mathbf{Top}_0^{coh}$  for the category of coherent locales and coherent morphisms between them.

**9.33 Example.** Let  $\mathbf{X}$  be an  $\infty$ -topos. Then an open  $U \in \mathbf{Open}(\mathbf{X})$  is a quasicompact element of the locale  $\mathbf{Open}(\mathbf{X})$  if and only if  $U$  is a quasicompact (i.e., 0-coherent) object of the  $\infty$ -topos  $\mathbf{X}$ .

The following three results are immediate from the definitions and [Example 9.33](#).

**9.34 Lemma.** For any 1-coherent  $\infty$ -topos  $\mathbf{X}$ , the locale  $\mathbf{Open}(\mathbf{X})$  is coherent.

**9.35 Lemma.** Let  $f_* : \mathbf{X} \rightarrow \mathbf{Y}$  be a coherent geometric morphism between coherent  $\infty$ -topoi. Then the induced morphism  $\mathbf{Open}(\mathbf{X}) \rightarrow \mathbf{Open}(\mathbf{Y})$  of coherent locales is coherent.

**9.36 Corollary.** Let  $S$  be a spectral topological space and  $f_* : \mathbf{X} \rightarrow \tilde{S}$  an  $S$ -stratified  $\infty$ -topos. If  $\mathbf{X}$  is coherent, then  $f_*$  is a constructible stratification if and only if the induced morphism of coherent locales  $\mathbf{Open}(\mathbf{X}) \rightarrow \mathbf{Open}(S)$  is coherent.

The following classical result is an important recognition principle for coherent locales.

**9.37 Proposition** ([28, Chapter II, §§3.3–3.4]). The functor  $\mathbf{Open} : \mathbf{TSp}^{spec} \rightarrow \mathbf{Top}_0^{coh}$  given by sending a spectral topological space  $S$  to its locale of opens subsets factors through  $\mathbf{Top}_0^{coh}$  and defines an equivalence of categories

$$\mathbf{Open} : \mathbf{TSp}^{spec} \simeq \mathbf{Top}_0^{coh} .$$

**9.38.** The functor  $\mathbf{Open} : \mathbf{TSp}^{spec} \simeq \mathbf{Top}_0^{coh}$  has an explicit inverse  $\mathbf{Top}_0^{coh} \simeq \mathbf{TSp}^{spec}$  given by taking the topological space of *points* of a locale; see [28, Chapter II, §1.3].

**9.39 Notation.** [Lemma 9.35](#) and [Proposition 9.37](#) provide a functor

$$S : \mathbf{Top}_\infty^{coh} \xrightarrow{\mathbf{Open}} \mathbf{Top}_0^{coh} \xrightarrow{\sim} \mathbf{TSp}^{spec} ,$$

which we denote by  $S$ . By definition, the 0-localic reflection of a coherent  $\infty$ -topos  $\mathbf{X}$  is given by the  $\infty$ -topos of sheaves on the spectral topological space  $S(\mathbf{X})$ . Thus  $\mathbf{X}$  comes equipped with a natural  $S(\mathbf{X})$ -stratification  $\mathbf{X} \rightarrow \widetilde{S(\mathbf{X})}$ .

The localisation  $\mathbf{Top}_\infty \rightleftarrows \mathbf{Top}_0$  thus restricts to a localisation  $\mathbf{Top}_\infty^{coh} \rightleftarrows \mathbf{Top}_0^{coh}$ .

**9.40 Lemma.** For any coherent  $\infty$ -topos  $\mathbf{X}$ , the natural stratification  $f_* : \mathbf{X} \rightarrow \widetilde{S(\mathbf{X})}$  is constructible ([Definition 9.29](#)).

*Proof.* Clear from [Corollary 9.36](#) and the fact that  $f_* : \mathbf{X} \rightarrow \widetilde{S(\mathbf{X})}$  induces an equivalence of locales

$$\mathbf{Open}(\mathbf{X}) \simeq \mathbf{Open}(\widetilde{S(\mathbf{X})}) = \mathbf{Open}(S(\mathbf{X})) . \quad \square$$

**9.41.** The source functor  $\mathbf{StrTop}_\infty^{\wedge, cc} \rightarrow \mathbf{Top}_\infty^{coh}$  admits a fully faithful left adjoint, given by the the assignment

$$\mathbf{X} \mapsto [\mathbf{X} \rightarrow \widetilde{\mathbf{S}(\mathbf{X})}].$$

The essential image of this left adjoint is the full subcategory spanned by those coherent constructible stratified  $\infty$ -topoi  $\mathbf{X} \rightarrow \widetilde{\mathbf{S}}$  such that the stratification induces an equivalence of locales  $\text{Open}(\mathbf{X}) \simeq \text{Open}(\mathbf{S})$ .

The source functor  $\mathbf{StrTop}_\infty^{\wedge, cc} \rightarrow \mathbf{Top}_\infty^{coh}$  also admits a fully faithful right adjoint, which carries a coherent  $\infty$ -topos  $\mathbf{X}$  to  $\mathbf{X}$  again, equipped with the essentially unique stratification over  $\mathbf{S} = \{\overline{0}\}$ .

### Stratified spaces & profinite stratified spaces as stratified $\infty$ -topoi

We now extend the functor  $\mathbf{Str}_\pi \rightarrow \mathbf{StrTop}_\infty$  given by  $\Pi \mapsto \widetilde{\Pi}$  to a functor on profinite stratified spaces.

**9.42 Notation.** Denote by  $\lambda: \mathbf{Str} \rightarrow \mathbf{StrTop}_\infty$  the left exact functor over  $po\text{Set}$  that is defined by the assignment  $\Pi \mapsto \widetilde{\Pi}$ . For each poset  $P$ , we consider also the functor on fibres  $\lambda_P: \mathbf{Str}_P \rightarrow \mathbf{StrTop}_{\infty, P}$ .

In light of [Example 9.24](#), if  $P$  is finite, then the diagram

$$\begin{array}{ccc} \mathbf{Str}_{\pi, P} & \xrightarrow{\lambda_P} & \mathbf{StrTop}_{\infty, P}^{bcc} \\ N_P \downarrow \wr & & \downarrow N_P \\ \mathbf{D}\acute{e}c_P(\mathbf{S}_\pi) & \xrightarrow{\lambda_{\{0\}^{\circ-}}} & \mathbf{D}\acute{e}c_P(\mathbf{Top}_\infty^{bc}) \end{array}$$

commutes, where the vertical functors are equivalences ([Definition 2.17](#), (4.8), and [Theorem 9.25](#)).

We now show that the functor  $\lambda$  is fully faithful. We first describe stratified geometric morphisms  $\mathbf{X} \rightarrow \widetilde{\Pi}$  in a more familiar fashion. Let us begin with the case in which the base poset is trivial.

**9.43.** In light of [Recollection 5.6](#)=[[HTT](#), [Corollary 6.3.5.6](#)], if  $\Pi$  is an essentially  $\kappa_0$ -small space, then one has an equivalence

$$\text{Map}_{\text{Pro}(\mathbf{S})}(\Pi_\infty(\mathbf{X}), \Pi) \simeq \text{Fun}_*(\mathbf{X}, \widetilde{\Pi}),$$

where  $\Pi_\infty(\mathbf{X})$  is the shape prospace  $\Gamma_{X, *}, \Gamma_X^*: \mathbf{S} \rightarrow \mathbf{S}$  ([Definition 5.69](#)). In particular,  $\text{Fun}_*(\mathbf{X}, \widetilde{\Pi})$  is an essentially  $\kappa_0$ -small Kan complex.

In this case, one also deduces that if  $\Pi, \Pi'$  are two essentially  $\kappa_0$ -small spaces, then the natural map  $\text{Map}_{\mathbf{S}}(\Pi', \Pi) \rightarrow \text{Map}_{\text{Top}_\infty}(\widetilde{\Pi}', \widetilde{\Pi})$  is an equivalence.

Now we extend this result to the context of  $P$ -stratified  $\infty$ -topoi.

**9.44 Notation.** Let  $P$  be a finite poset, and let  $f_*: \mathbf{X} \rightarrow \widetilde{P}$  and  $g_*: \mathbf{Y} \rightarrow \widetilde{P}$  be  $P$ -stratified  $\infty$ -topoi. Let us write

$$\text{Fun}_{P, *}(\mathbf{X}, \mathbf{Y}) := \text{Fun}_*(\mathbf{X}, \mathbf{Y}) \times_{\text{Fun}_*(\mathbf{X}, \widetilde{P})} \{f_*\}.$$

The mapping space  $\text{Map}_{\text{StrTop}_{\infty, P}}(\mathbf{X}, \mathbf{Y})$  is the interior of  $\text{Fun}_{P, *}(\mathbf{X}, \mathbf{Y})$ .

If  $\mathbf{X}$  and  $\mathbf{Y}$  are bounded coherent and constructibly stratified, then in light of [Theorem 9.25](#), one has an equivalence of  $\infty$ -categories

$$\text{Fun}_{P, *}(\mathbf{X}, \mathbf{Y}) \simeq \int_{\Sigma \in \text{sd}^{op}(P)} \text{Fun}_*(N_P(\mathbf{X})(\Sigma), N_P(\mathbf{Y})(\Sigma)).$$

This implies the following.

**9.45 Proposition.** *If  $P$  is a finite poset, and  $\mathbf{X}$  is a bounded coherent constructible  $P$ -stratified  $\infty$ -topos, then for any  $\pi$ -finite  $P$ -stratified space  $\Pi$ , one has a natural equivalence*

$$\text{Fun}_{P, *}(\mathbf{X}, \widetilde{\Pi}) \simeq \int_{\Sigma \in \text{sd}^{op}(P)} \text{Map}_{\text{Pro}(\mathcal{S})}(\Pi_{\infty}(N_P(\mathbf{X})(\Sigma)), N_P(\Pi)(\Sigma)).$$

Since the right hand side is a  $\kappa_0$ -small limit of  $\kappa_0$ -small Kan complexes, we obtain the following.

**9.46 Corollary.** *If  $P$  is a finite poset, and  $\mathbf{X}$  is a bounded coherent constructible  $P$ -stratified  $\infty$ -topos, then for any  $\pi$ -finite  $P$ -stratified space  $\Pi$ , the  $\infty$ -category  $\text{Fun}_{P, *}(\mathbf{X}, \widetilde{\Pi})$  is an essentially  $\kappa_0$ -small Kan complex.*

Additionally, the full faithfulness of  $\lambda_{\{0\}}$  now implies the following.

**9.47 Corollary.** *For any finite poset and any two  $\pi$ -finite  $P$ -stratified spaces  $\Pi$  and  $\Pi'$ , the functor*

$$\text{Map}_P(\Pi', \Pi) \rightarrow \text{Fun}_{P, *}(\widetilde{\Pi}', \widetilde{\Pi})$$

*is an equivalence. That is, the functor  $\lambda_P$  is a fully faithful functor  $\text{Str}_{\pi, P} \hookrightarrow \text{StrTop}_{\infty, P}^{bcc}$ .*

Finally, we obtain:

**9.48 Corollary.** *If  $P$  is a finite poset, then for any bounded coherent constructible  $P$ -stratified  $\infty$ -topos  $\mathbf{X}$  and any filtered diagram  $\Pi : A \rightarrow \text{Str}$  of  $\pi$ -finite  $P$ -stratified spaces, the natural map*

$$\text{colim}_{\alpha \in A} \text{Map}_{\text{StrTop}_{\infty}}(\mathbf{X}, \widetilde{\Pi}_{\alpha}) \rightarrow \text{Map}_{\text{StrTop}_{\infty}}(\mathbf{X}, \text{colim}_{\alpha \in A} \widetilde{\Pi}_{\alpha})$$

*is an equivalence.*

**9.49.** Please observe that the functor  $\lambda : \text{Str}_{\pi} \hookrightarrow \text{StrTop}_{\infty}^{bcc}$  is left exact. This is because the functor  $\text{poSet}^{fin} \rightarrow \text{Top}_{\infty}^{bc}$  given by  $P \mapsto \widetilde{P}$  is left exact, and for any finite poset  $P$ , the functor  $\lambda_P : \text{Str}_{\pi, P} \rightarrow \text{StrTop}_{\infty, P}^{bcc}$  is left exact, since as a functor  $\text{D}\acute{e}c_P(\mathcal{S}_{\pi}) \rightarrow \text{D}\acute{e}c_P(\text{Top}_{\infty}^{bc})$  it is equivalent to composition with  $\lambda_{\{0\}}$ .

**9.50 Construction.** Since bounded coherent constructible stratified  $\infty$ -topoi are closed under the formation of inverse limits, we can now apply [\(0.6\)](#) and extend  $\lambda$  to a functor

$$\widehat{\lambda} : \text{Str}_{\pi}^{\wedge} \rightarrow \text{StrTop}_{\infty}^{\wedge, bcc}$$

over  $\mathbf{TSp}^{spec}$ , which we write as the assignment  $\mathbf{II} \mapsto \widetilde{\mathbf{II}}$ .

Let us caution that if  $S$  is a spectral topological space and  $\mathbf{II}$  is a profinite  $S$ -stratified space, then although  $S$  determines and is determined by the  $\text{mat}(S)$ -stratified space  $\text{mat}(\mathbf{II})$ , the  $\infty$ -topoi  $\widetilde{\mathbf{II}}$  and  $\widehat{\text{mat}(\mathbf{II})}$  are quite different in general. The latter is always a presheaf  $\infty$ -category, but the former is typically not.

**9.51 Proposition.** *The functor  $\widehat{\lambda}$  is fully faithful. In particular, if  $S$  is a spectral topological space, then we obtain a fully faithful functor  $\mathbf{Str}_{\pi,S}^{\wedge} \hookrightarrow \mathbf{StrTop}_{\infty,S}^{\wedge,bcc}$ .*

*Proof.* First we treat the case in which  $S = P$  is a finite poset. In this case, in light of the equivalences

$$\mathbf{Str}_{\pi,P}^{\wedge} \simeq \mathbf{D}\acute{e}c_P(\mathbf{S}_{\pi}^{\wedge}) \quad \text{and} \quad \mathbf{StrTop}_{\infty,P}^{bcc} \simeq \mathbf{D}\acute{e}c_P(\mathbf{Top}_{\infty}^{bc})$$

of [Construction 4.11](#) and [Theorem 9.25](#), it suffices to prove that the functor

$$\mathbf{D}\acute{e}c_P(\mathbf{S}_{\pi}^{\wedge}) \rightarrow \mathbf{D}\acute{e}c_P(\mathbf{Top}_{\infty}^{bc})$$

given by the objectwise application of  $\widehat{\lambda}_{\{0\}} : \mathbf{Str}_{\pi}^{\wedge} \rightarrow \mathbf{Top}_{\infty}^{bc}$  is fully faithful. This follows as in [Corollary 9.47](#) from the full faithfulness of the functor  $\mathbf{S}_{\pi}^{\wedge} \rightarrow \mathbf{Top}_{\infty}^{bc}$ .

Now for a more general spectral topological space  $S$ , the identifications

$$\mathbf{Str}_{\pi,S}^{\wedge} \simeq \lim_{P \in \text{FC}(S)} \mathbf{Str}_{\pi,P}^{\wedge} \quad \text{and} \quad \mathbf{StrTop}_{\infty,S}^{\wedge,bcc} \simeq \lim_{P \in \text{FC}(S)} \mathbf{StrTop}_{\infty,P}^{bcc},$$

the first of which is [Proposition 3.20](#) and the latter of which is obvious, together complete the proof.  $\square$

**9.52 Proposition.** *Let  $P$  be a finite poset. Then the essential image of the functor*

$$\mathbf{D}\acute{e}c_P(\mathbf{S}_{\pi}^{\wedge}) \rightarrow \mathbf{D}\acute{e}c_P(\mathbf{Top}_{\infty}^{bc})$$

*given by the objectwise application of  $\widehat{\lambda}_{\{0\}} : \mathbf{Str}_{\pi}^{\wedge} \rightarrow \mathbf{Top}_{\infty}^{bc}$  is the the full subcategory  $\mathbf{D}\acute{e}c_P(\mathbf{Top}_{\infty}^{Stone}) \subset \mathbf{D}\acute{e}c_P(\mathbf{Top}_{\infty}^{bc})$  spanned by those décollages over  $P$  that carry each string to a Stone  $\infty$ -topos.*

*Proof.* The only nontrivial point to verify is that indeed  $\widehat{\lambda}$  carries décollages in profinite spaces to décollages in Stone  $\infty$ -topoi. This follows from [Example 8.34](#).  $\square$

The essential image of  $\widehat{\lambda}$  can be characterised as the  $\infty$ -category of *spectral  $\infty$ -topoi*, to which we shall now turn.

## 10 Spectral higher topoi

In this section, we define the notion of a *spectral  $\infty$ -topos*. The idea is that, on one hand, these are the kinds of  $\infty$ -topoi that arise as the étale  $\infty$ -topoi of coherent schemes, and on the other, these will turn out to be precisely the  $\infty$ -topoi that arise as  $\widetilde{\mathbf{II}}$  for some profinite stratified space  $\mathbf{II}$ .

We begin with some preliminary results on the interaction between Stone  $\infty$ -topoi and oriented fibre products.



## Stone $\infty$ -topoi & oriented fibre products

In this subsection we prove two useful facts about oriented fibre products involving Stone  $\infty$ -topoi.

**10.1 Proposition.** *Let  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  be geometric morphisms of  $\infty$ -topoi. If  $Z$  is Stone, then the natural geometric morphism  $X \times_Z Y \rightarrow X \tilde{\times}_Z Y$  is an equivalence.*

*Proof.* It suffices to show that the projections  $\text{pr}_{1,*}, \text{pr}_{2,*} : \text{Path}(Z) \rightarrow Z$  are equivalences. Since  $Z$  is Stone, by [Lemma 6.48](#) the  $\infty$ -topos  $\text{Path}(Z)$  is bounded coherent, and [Theorem 5.82](#)=[[SAG, Theorem E.3.4.1](#)] shows that the  $\infty$ -category  $\text{Pt}(Z)$  is an  $\infty$ -groupoid. Thus

$$\text{Pt}(\text{Path}(Z)) \simeq \text{Fun}(\Delta^1, \text{Pt}(Z))$$

is an  $\infty$ -groupoid as well, and again appealing to [Theorem 5.82](#)=[[SAG, Theorem E.3.4.1](#)] we conclude that  $\text{Path}(Z)$  is Stone. The claim now follows from the fact that  $\text{pr}_{1,*}$  and  $\text{pr}_{2,*}$  are shape equivalences ([Example 7.19](#)).  $\square$

**10.2 Proposition.** *Let  $X$  and  $Y$  be Stone  $\infty$ -topoi,  $Z$  a bounded coherent  $\infty$ -topos, and  $f_* : X \rightarrow Z$  and  $g_* : Y \rightarrow Z$  coherent geometric morphisms. Then the oriented fibre product  $X \tilde{\times}_Z Y$  is a Stone  $\infty$ -topos.*

*Proof.* By [Lemma 6.48](#) the  $\infty$ -topos  $X \tilde{\times}_Z Y$  is bounded coherent, so by [Theorem 5.82](#)=[[SAG, Theorem E.3.4.1](#)] it suffices to prove that the  $\infty$ -category  $\text{Pt}(X \tilde{\times}_Z Y)$  is an  $\infty$ -groupoid. In light of [Lemma 6.36](#) we have  $\text{Pt}(X \tilde{\times}_Z Y) \simeq \text{Pt}(X) \downarrow_{\text{Pt}(Z)} \text{Pt}(Y)$ , so the fact that  $\text{Pt}(X)$  and  $\text{Pt}(Y)$  are  $\infty$ -groupoids implies that the  $\infty$ -category  $\text{Pt}(X \tilde{\times}_Z Y)$  is as well.  $\square$

## Spectral $\infty$ -topoi & toposic décollages

**10.3 Definition.** Let  $S$  be a spectral topological space. An  $S$ -stratified  $\infty$ -topos  $X \rightarrow \tilde{S}$  is a *spectral*  $S$ -stratified  $\infty$ -topos if and only if the following conditions are satisfied.

- ▶ The  $\infty$ -topos  $X$  is bounded and coherent.
- ▶ The stratification by  $S$  is constructible.
- ▶ For every point  $s \in S$ , the stratum  $X_s$  is a Stone  $\infty$ -topos.

We write  $\text{StrTop}_{\infty, S}^{\text{spec}} \subset \text{StrTop}_{\infty, S}^{\wedge, \text{bcc}}$  for the full subcategory spanned by the spectral  $S$ -stratified  $\infty$ -topoi.

More generally, write

$$\text{StrTop}_{\infty}^{\text{spec}} \subset \text{StrTop}_{\infty}^{\wedge, \text{bcc}}$$

for the full subcategory whose objects are spectral  $\infty$ -topoi and whose morphisms are squares

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{S}' & \longrightarrow & \tilde{S} \end{array}$$

of coherent geometric morphisms. We observe that the pullback of a spectral  $\infty$ -topos along the geometric morphism induced by a quasicompact continuous map is again spectral, whence the functor  $\mathbf{StrTop}_{\infty}^{spec} \rightarrow \mathbf{TSpc}^{spec}$  is a cartesian fibration.

**10.4 Example.** Let  $\mathbf{II} \rightarrow S$  be a profinite stratified space (Definition 3.15). Then  $\widetilde{\mathbf{II}}$  is a spectral  $\infty$ -topos, as the fibres  $\widetilde{\mathbf{II}}_s \simeq \widetilde{\mathbf{II}}_s$  are Stone  $\infty$ -topoi.

**10.5.** In Theorem 10.10, we will prove the central  $\infty$ -Categorical Hochster Duality Theorem, which states that every spectral  $\infty$ -topos is of the form  $\widetilde{\mathbf{II}}$  for some profinite stratified space.

**10.6 Example.** Let  $X$  be a coherent scheme. Write  $X^{zar}$  for its underlying Zariski spectral topological space, and let  $X_{\acute{e}t}$  denote its coherent, 1-localic étale  $\infty$ -topos. The identification  $\text{Open}(X_{\acute{e}t}) \cong \text{Open}(X^{zar})$  provides a canonical  $X^{zar}$ -stratification of  $X_{\acute{e}t}$ . For any point  $x \in X^{zar}$ , the stratum  $(X_{\acute{e}t})_x$  is identified with  $(\text{Spec } \kappa(x))_{\acute{e}t}$ , which is the Stone  $\infty$ -topos  $\widetilde{BG}_{\kappa(x)}$ . Consequently  $X_{\acute{e}t}$  is a spectral  $\infty$ -topos.

**10.7 Proposition.** *Let  $S$  be a spectral topological space, and let  $\mathbf{X}$  be a bounded coherent constructible  $S$ -stratified  $\infty$ -topos. Then  $\mathbf{X}$  is spectral if and only if the functor*

$$\text{Pt}(\mathbf{X}) \rightarrow \text{Pt}(\widetilde{S}) \simeq \text{mat}(S)$$

*exhibits  $\text{Pt}(\mathbf{X})$  as a  $\text{mat}(S)$ -stratified space.*

*Proof.* This follows directly from Theorem 5.82=[SAG, Theorem E.3.4.1].  $\square$

**10.8.** Let  $P$  be a finite poset. We now consider the nerve of a spectral  $P$ -stratified  $\infty$ -topos  $\mathbf{X} \rightarrow \widetilde{P}$ . Since each stratum  $\mathbf{X}_p$  is Stone, it follows from Proposition 10.2 that for any string  $\{p_0, \dots, p_n\} \subseteq P$ , the value

$$N_P(\mathbf{X})\{p_0, \dots, p_n\} \simeq \mathbf{X}_{p_0} \bar{\times}_X \mathbf{X}_{p_1} \bar{\times}_X \cdots \bar{\times}_X \mathbf{X}_{p_n}$$

is a Stone  $\infty$ -topos. Consequently, we deduce that the equivalence

$$N_P : \mathbf{StrTop}_{\infty, P}^{bcc} \simeq \mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{bc})$$

restricts to an equivalence between the  $\infty$ -category of spectral  $P$ -stratified  $\infty$ -topoi and the full subcategory  $\mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{Stone}) \subset \mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{bc})$  spanned by those décollages over  $P$  that carry each string to a Stone  $\infty$ -topos.

**10.9 Lemma.** *Let  $P$  be a finite poset. Then the nerve equivalence*

$$N_P : \mathbf{StrTop}_{\infty, P}^{bcc} \simeq \mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{bc})$$

*restricts to an equivalence  $\mathbf{StrTop}_{\infty, P}^{spec} \simeq \mathbf{D\acute{e}c}_P(\mathbf{Top}_{\infty}^{Stone})$ .*

## Hochster duality for higher topoi

In (1.12) we described Hochster duality as a cube of dualities: the equivalence of 1-categories between profinite posets and spectral topological spaces restricts on one hand to an equivalence of 1-categories between profinite sets and Stone spaces, and on the other to an equivalence of 1-categories between finite posets and finite topological spaces. Our objective now is to exhibit the analogous cube for higher topoi:

$$\begin{array}{ccccc}
 & & \mathbf{S}_\pi & \xrightarrow{\sim} & \mathbf{Top}_\infty^{fin} \\
 & \swarrow & \uparrow & & \swarrow \\
 \mathbf{S}_\pi^\wedge & \xrightarrow{\sim} & \mathbf{Top}_\infty^{Stone} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{Str}_\pi & \xrightarrow{\sim} & \mathbf{StrTop}_\infty^{fin} \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathbf{Str}_\pi^\wedge & \xrightarrow{\sim} & \mathbf{StrTop}_\infty^{spec} & & 
 \end{array}$$

where the vertical fully faithful functors are given by equipping an object with the trivial stratification. The top face of this cube was established by Lurie [SAG, Appendix E]. We must now address the bottom face, more precisely the equivalence  $\mathbf{Str}_\pi^\wedge \simeq \mathbf{StrTop}_\infty^{spec}$ .

**10.10 Theorem** ( $\infty$ -Categorical Hochster Duality). *Let  $S$  be a spectral topological space. Then the functor*

$$\widehat{\lambda}_S : \mathbf{Str}_{\pi,S}^\wedge \rightarrow \mathbf{StrTop}_{\infty,S}^{spec}$$

*given by the assignment  $\Pi \mapsto \widehat{\Pi}$  is an equivalence of  $\infty$ -categories. Consequently, the functor*

$$\widehat{\lambda} : \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{StrTop}_\infty^{spec}$$

*is an equivalence of  $\infty$ -categories.*

*Proof.* Since  $\widehat{\lambda}$  is fully faithful (Proposition 9.51) and preserves inverse limits, it suffices to prove that for any finite poset  $P$ , the fully faithful functor  $\widehat{\lambda} : \mathbf{Str}_{\pi,P}^\wedge \hookrightarrow \mathbf{StrTop}_{\infty,P}^{spec}$  is essentially surjective.

This now follows from the conjunction of Lemma 10.9 and Proposition 9.52.  $\square$

The back face of the cube is now just a restriction of the front face: we define  $\mathbf{Top}_\infty^{fin}$  as the full subcategory of  $\mathbf{Top}_\infty^{Stone}$  spanned by the essential image of the fully faithful functor  $\mathbf{S}_\pi \hookrightarrow \mathbf{Top}_\infty^{Stone}$  given by  $\Pi \mapsto \mathbf{S}_{/\Pi} \simeq \text{Fun}(\Pi, \mathbf{S})$ . Then  $\mathbf{StrTop}_\infty^{fin}$  is the  $\infty$ -category of bounded coherent constructible  $\infty$ -topoi over a finite poset  $P$  such that for every point  $p \in P$ , the  $\infty$ -topos  $X_p$  is in  $\mathbf{Top}_\infty^{fin}$ .

## Constructible sheaves

The truncated coherent objects of a Stone  $\infty$ -topos are exactly the *lisse sheaves* (Recollection 5.83). This turns out to be a defining property of Stone  $\infty$ -topoi (Proposition 5.86=[SAG, Proposition E.3.1.1]). In the same manner, the truncated coherent objects of a spectral  $\infty$ -topos are exactly the *constructible sheaves*, to which we now turn.

**10.11 Definition.** Let  $P$  be a noetherian poset and  $X$  a  $P$ -stratified  $\infty$ -topos. An object  $F \in X$  is said to be *formally constructible* (or *formally  $P$ -constructible* if disambiguation is called for) if and only if, for any point  $p \in P$ , the restriction  $F|_{X_p} := e_p^*F \in X_p$  is a local system, where  $e_{p,*} : X_p \hookrightarrow X$  is the inclusion of the  $p$ -th stratum.

We say that  $F$  is *constructible* (or  *$P$ -constructible*) if and only if the following pair of conditions is satisfied:

- ▶ The object  $F$  is formally constructible.
- ▶ For any point  $p \in P$ , the restriction  $F|_{X_p} \in X_p$  is lisse.

**10.12.** This notion of constructibility depends upon the whole structure of the stratified  $\infty$ -topos, not only upon the underlying  $\infty$ -topos.

**10.13.** For any noetherian poset  $P$  and  $P$ -stratified  $\infty$ -topos  $X \rightarrow \tilde{P}$ , the  $\infty$ -category of constructible sheaves on  $X$  is given by the pullback of  $\infty$ -categories:

$$\begin{array}{ccc} \mathbf{X}^{P\text{-constr}} & \longrightarrow & \prod_{p \in P} \mathbf{X}_p^{\text{lisse}} \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{X} & \xrightarrow{\prod_{p \in P} e_p^*} & \prod_{p \in P} \mathbf{X}_p \end{array} ,$$

where here  $\prod_{p \in P} \mathbf{X}_p$  is the product in  $\mathbf{Cat}_{\infty, \kappa_1}$ . **Lemmas 5.33** and **5.34** now show that  $\mathbf{X}^{P\text{-constr}}$  is an  $\infty$ -pretopos (**Definition 5.31**) and the inclusion  $\mathbf{X}^{P\text{-constr}} \hookrightarrow \mathbf{X}$  is a morphism of  $\infty$ -pretopoi.

**10.14.** If  $P$  is a nonnoetherian poset, **Definition 10.11** is insufficient, and one needs to assume also the following convergence condition:

- ▶ for any ideal  $A \subseteq P$ , if we write  $i_{A,*} : X_A \hookrightarrow X$  for the closed immersion, then the natural morphism

$$i_A^*F \rightarrow \lim_{p \in A^{op}} i_{p,*} i_p^*F$$

is an equivalence, where  $i_{p,*} : X_{\leq p} \hookrightarrow X_A$  is the inclusion of the closed subtopos.

This condition is automatically satisfied for noetherian stratifications, which are our sole concern in this text.

The pullback of a geometric morphism of  $\infty$ -topoi preserves lisse objects (see **Recollection 5.83**); in the same manner, the pullback of a morphism of stratified  $\infty$ -topoi preserves constructible objects.

**10.15 Lemma.** Let  $f : P' \rightarrow P$  be a morphism of noetherian posets, and let  $X' \rightarrow \tilde{P}'$  and  $X \rightarrow \tilde{P}$  be stratified  $\infty$ -topoi. Then for any geometric morphism  $q_* : X' \rightarrow X$  over  $f_* : \tilde{P}' \rightarrow \tilde{P}$ , the pullback  $q^* : X \rightarrow X'$  sends  $P$ -constructible objects of  $X$  to  $P'$ -constructible objects of  $X'$ . Hence  $q^*$  restricts to a morphism of  $\infty$ -pretopoi

$$q^* : \mathbf{X}^{P\text{-constr}} \rightarrow (\mathbf{X}')^{P'\text{-constr}} .$$

*Proof.* Let  $F \in \mathbf{X}^{P\text{-constr}}$  be a  $P$ -constructible object of  $\mathbf{X}$ . Then for any point  $p \in P'$ , the restriction  $F|_{\mathbf{X}_{f(p)}}$  is lisse, so since the pullback in a geometric morphism preserves lisse objects, we see that  $q^*(F)|_{\mathbf{X}'_p}$  is lisse. Hence  $q^*(F)$  is  $P'$ -constructible.

The fact that  $q^* : \mathbf{X}^{P\text{-constr}} \rightarrow (\mathbf{X}')^{P'\text{-constr}}$  is a morphism of  $\infty$ -pretopoi is immediate from (10.13).  $\square$

**10.16 Proposition.** *Let  $P$  be a finite poset and  $\mathbf{X} \rightarrow \tilde{P}$  a  $P$ -stratified  $\infty$ -topos. Then the  $\infty$ -pretopos  $\mathbf{X}^{P\text{-constr}}$  is bounded (Definition 5.36).*

*Proof.* If  $P = \emptyset$ , then the claim is obvious, so assume that  $P$  is nonempty. We prove the claim by induction on the rank of  $P$ .

In the base case where  $P$  has rank 0,  $P$  is discrete, so  $\mathbf{X}$  is finite the coproduct of  $\infty$ -topoi  $\coprod_{p \in P} \mathbf{X}_p$  (which is the product  $\prod_{p \in P} \mathbf{X}_p$  in  $\text{Cat}_{\infty, \kappa_1}$ ). Thus  $\mathbf{X}^{P\text{-constr}}$  is the product of  $\infty$ -categories:

$$\mathbf{X}^{P\text{-constr}} = \prod_{p \in P} \mathbf{X}_p^{\text{lisse}}.$$

By Theorem 5.88=[SAG, Theorem E.2.3.2], for all  $p \in P$  the  $\infty$ -pretopos  $\mathbf{X}_p^{\text{lisse}}$  is bounded; the finiteness of  $P$  and Lemma 5.38 now show that  $\mathbf{X}^{P\text{-constr}}$  is also bounded.

For the induction step, let  $n \geq 0$  be a natural number and assume that the claim holds for all finite posets  $P$  of rank  $n$  and  $P$ -stratified  $\infty$ -topoi  $\mathbf{X} \rightarrow \tilde{P}$ . Let  $P$  be a finite poset of rank  $n + 1$ , and write  $M \subset P$  for the full subposet spanned by the minimal elements of  $P$ . Then  $M$  is discrete and closed in  $P$ . Write  $U := P \setminus M$  for the open complement of  $M$  in  $P$ . Then  $U$  is a poset of rank  $n$ . Moreover, since  $\tilde{P}$  is the recollement of  $\tilde{M}$  and  $\tilde{U}$ , the  $P$ -stratified  $\infty$ -topos  $\mathbf{X}$  is the recollement of  $\mathbf{X}_M$  and  $\mathbf{X}_U$ . An object  $F \in \mathbf{X}$  is  $P$ -constructible if and only if  $F|_{\mathbf{X}_M}$  and  $F|_{\mathbf{X}_U}$  are both constructible, from which we deduce that  $\mathbf{X}^{P\text{-constr}}$  is the oriented fibre product of  $\infty$ -categories

$$\mathbf{X}^{P\text{-constr}} \simeq \mathbf{X}_M^{M\text{-constr}} \downarrow_{\mathbf{X}_M} \mathbf{X}_U^{U\text{-constr}}.$$

Since  $M$  is a poset of rank 0 and  $U$  is a poset of rank  $n$ , by the induction hypothesis both  $\mathbf{X}_M^{M\text{-constr}}$  and  $\mathbf{X}_U^{U\text{-constr}}$  are bounded  $\infty$ -pretopoi. To conclude that the  $\infty$ -pretopos  $\mathbf{X}^{P\text{-constr}}$  is a bounded, note that by (6.2) every object of  $\mathbf{X}^{P\text{-constr}}$  is truncated and by (0.8) the  $\infty$ -category  $\mathbf{X}^{P\text{-constr}}$  is essentially  $\kappa_0$ -small.  $\square$

**10.17 Definition.** Let  $S$  be a spectral topological space and  $\mathbf{X}$  an  $S$ -stratified  $\infty$ -topos. We say that an object  $F \in \mathbf{X}$  is *formally constructible* (or *formally  $S$ -constructible*) if and only if there exist a finite poset  $P$  and a constructible stratification  $S \rightarrow P$  of posets such that  $F$  is formally  $P$ -constructible. We say that  $F$  is *constructible* (or  *$S$ -constructible*) if and only if there exist a poset  $P$  and a finite constructible stratification  $S \rightarrow P$  of posets such that  $F$  is  $P$ -constructible.

For any spectral topological space  $S$  and any  $S$ -stratified  $\infty$ -topos  $\mathbf{X} \rightarrow \tilde{S}$ , we denote by  $\mathbf{X}^{S\text{-fconstr}} \subseteq \mathbf{X}$  (respectively, by  $\mathbf{X}^{S\text{-constr}} \subseteq \mathbf{X}$ ) the full subcategory spanned by the formally constructible objects (respectively, the constructible objects).

**10.18.** For any spectral topological space  $S$  and  $S$ -stratified  $\infty$ -topos  $\mathbf{X} \rightarrow \tilde{S}$ , the  $\infty$ -category of constructible sheaves on  $\mathbf{X}$  is thus a filtered colimit of  $\infty$ -categories:

$$\mathbf{X}^{S\text{-constr}} \simeq \text{colim}_{P \in \text{FC}(S)^{\text{op}}} \mathbf{X}^{P\text{-constr}}.$$

Thus [Lemma 10.15](#) and [Proposition 10.16](#) combined with [Proposition 5.42](#)=[SAG, [Proposition A.8.3.1](#)] show that  $\mathbf{X}^{S\text{-constr}}$  is a bounded  $\infty$ -pretopos. Moreover, [\(10.13\)](#) shows that the inclusion  $\mathbf{X}^{S\text{-constr}} \hookrightarrow \mathbf{X}$  is a morphism of  $\infty$ -pretopoi.

From [Lemma 10.15](#) we now immediately deduce the following.

**10.19 Lemma.** *Let  $f : S' \rightarrow S$  be a quasicompact continuous map of spectral topological spaces, and let  $\mathbf{X}' \rightarrow \widetilde{S}'$  and  $\mathbf{X} \rightarrow \widetilde{S}$  be stratified  $\infty$ -topoi. Then for any geometric morphism  $q_* : \mathbf{X}' \rightarrow \mathbf{X}$  over  $f_* : \widetilde{S}' \rightarrow \widetilde{S}$ , the pullback  $q^* : \mathbf{X} \rightarrow \mathbf{X}'$  sends  $S$ -constructible objects of  $\mathbf{X}$  to  $S'$ -constructible objects of  $\mathbf{X}'$ . Hence  $q^*$  restricts to a morphism of  $\infty$ -pretopoi*

$$q^* : \mathbf{X}^{S\text{-constr}} \rightarrow (\mathbf{X}')^{S'\text{-constr}} .$$

We now turn to the relationship between coherence and constructibility in  $\infty$ -topoi stratified by a spectral topological space.

**10.20 Lemma.** *Let  $S$  be a spectral topological space, and let  $\mathbf{X}$  be an  $S$ -stratified  $\infty$ -topos. Then an object  $F$  of  $\mathbf{X}$  is constructible if and only if, for every point  $s \in S$ , there exists a constructible subset  $W \subseteq S$  containing  $s$  such that  $F|_{X_W}$  is lisse.*

*Proof.* The ‘only if’ direction is clear. Conversely, assume that every point of  $S$  is contained in such a constructible set. Hence the collection  $\{W_\alpha\}_{\alpha \in \Lambda}$  of constructible subsets of  $S$  such that  $F|_{X_{W_\alpha}}$  is lisse is a cover of  $S$  by constructible subsets. Since the constructible topology on  $S$  is quasicompact, it follows that there exists a finite subcover  $\{W_\alpha\}_{\alpha \in \Lambda'}$ . Select a finite constructible stratification  $S \rightarrow P$  of  $S$  such that for every  $p \in P$ , there exists an  $\alpha \in \Lambda'$  such that the stratum  $S_p \subseteq W_\alpha$ . Now  $F$  is  $P$ -constructible.  $\square$

**10.21 Lemma.** *Let  $S$  be a spectral topological space, and  $\mathbf{X} \rightarrow \widetilde{S}$  a coherent coherent constructible  $S$ -stratified  $\infty$ -topos. Then every constructible object of  $\mathbf{X}$  is truncated and coherent. If  $\mathbf{X}$  is also bounded and every truncated and coherent object of  $\mathbf{X}$  is constructible, then  $\mathbf{X}$  is spectral.*

*Proof.* For the first statement, let  $F \in \mathbf{X}^{S\text{-constr}}$ , and let  $S \rightarrow P$  be a finite constructible stratification such that for every point  $p \in P$ , the restriction  $F|_{X_p}$  is lisse. By [Proposition 6.8](#)=[DAG XIII, [Proposition 2.3.22](#)] it follows that  $F$  is coherent. If each  $F|_{X_p}$  is  $N$ -truncated, then  $F$  is  $N$ -truncated.

For the second statement, if every truncated coherent object of  $\mathbf{X}$  is constructible and  $\mathbf{X}$  is bounded, then  $\mathbf{X} \simeq \mathbf{Sh}_{\text{eff}}(\mathbf{X}^{S\text{-constr}})$ . For any point  $s \in S$ , one thus has an equivalence  $\mathbf{X}_s \simeq \mathbf{Sh}_{\text{eff}}(\mathbf{X}_s^{\text{lisse}})$ , which is a Stone  $\infty$ -topos. Thus  $\mathbf{X}$  is spectral.  $\square$

**10.22 Proposition.** *If  $S$  is a spectral topological space, and  $\mathbf{X}$  is a spectral  $S$ -stratified  $\infty$ -topos  $\mathbf{X}$ , then every truncated and coherent object of  $\mathbf{X}$  is constructible.*

*Proof.* Let  $F$  be a truncated coherent object of  $\mathbf{X}$ , and  $s \in S$  a point. We wish to show that there exists a constructible subset of  $W \subset S$  containing  $s$  such that  $F|_{X_W}$  is lisse ([Lemma 10.20](#)). Passing to the closure of  $s$ , it suffices to assume that  $S$  is irreducible, and  $s$  is its generic point.

Since  $F|_{X_s}$  is lisse, it follows from [Lemma 5.84](#)=[SAG, [Proposition E.2.7.7](#)] that there exists a full subcategory  $E \subset \mathcal{S}_\pi$  spanned by finitely many  $\pi$ -finite spaces and

a unique geometric morphism  $g_* : X_s \rightarrow \mathcal{S}_{/IE}$  and an equivalence  $\varepsilon_s : F|X_s \simeq g^*(I)$ , where  $I$  is the inclusion functor  $E \hookrightarrow \mathcal{S}$ . Now since  $\mathcal{S}_{/IE}$  is cocompact as an object of  $\mathbf{Top}_\infty^{bc}$  (Lemma 5.85) and  $X_s$  is identified with the limit  $\lim_W X_W$  over constructible subsets  $W \subset S$  containing  $s$ , it follows that for some such  $W$ , one may factor  $g_*$  through a geometric morphism  $g_{W,*} : X_W \rightarrow \mathcal{S}_{/IE}$ . Now since  $X_{s,<\infty}^{coh} \simeq \text{colim}_W X_{W,<\infty}^{coh}$ , we shrink  $W$  as needed to ensure that there exists an equivalence  $\varepsilon_s : F|X_W \simeq g_W^*(I)$ , and conclude that  $F$  is lisse on  $W$ .  $\square$

**10.23 Example.** If  $X$  is a coherent scheme, then the truncated coherent objects of  $X_{\acute{e}t}$  are precisely the constructible sheaves of spaces. This is the nonabelian analogue of the well-known result that for a finite ring  $\Lambda$ , the compact objects of  $\mathbf{Sh}_{\acute{e}t}(X; \mathbf{D}(\Lambda))$  coincide with the bounded derived  $\infty$ -category of constructible  $\Lambda$ -sheaves.

We have shown that the  $\infty$ -category  $\mathbf{Str}_\pi^\wedge$  of profinite stratified spaces is equivalent to the  $\infty$ -category  $\mathbf{StrTop}_\infty^{spec}$ , which is in turn a full subcategory of  $\mathbf{StrTop}_\infty^{\wedge,bcc}$  of bounded coherent constructible stratified  $\infty$ -topoi. This last  $\infty$ -category is a *non-full* subcategory of  $\mathbf{StrTop}_\infty^\wedge$ , however. Just as how every geometric morphism between Stone  $\infty$ -topoi is coherent (Corollary 5.87=[SAG, Corollary E.3.1.2]), the subcategory  $\mathbf{StrTop}_\infty^{spec} \subset \mathbf{StrTop}_\infty^\wedge$  is full, as we shall now explain.

**10.24 Proposition.** *Let  $f : S' \rightarrow S$  be a quasicompact continuous map of spectral topological spaces, let  $X' \rightarrow S'$  be a coherent constructible stratified  $\infty$ -topos, and let  $X \rightarrow \tilde{S}$  be a spectral  $\infty$ -topos. Then any geometric morphism  $q_* : X' \rightarrow X$  over  $f_* : \tilde{S}' \rightarrow \tilde{S}$  is coherent.*

*Proof.* By Corollary 5.40 it suffices to show that if  $F \in X$  is truncated and coherent, then  $p^*F$  is coherent. By Proposition 10.22

$$X^{S\text{-constr}} = X_{<\infty}^{coh},$$

so the claim now follows from the facts that  $q^*$  preserves constructibility (Lemma 10.19) and the  $S'$ -constructible objects of  $X'$  are truncated coherent (Lemma 10.21).  $\square$

**10.25 Corollary.** *The subcategory  $\mathbf{StrTop}_\infty^{spec} \subset \mathbf{StrTop}_\infty^\wedge$  is full.*

**10.26 Construction.** Let  $S$  be a spectral topological space, and  $X$  an  $S$ -stratified  $\infty$ -topos. By [SAG, Proposition A.6.4.4], the fully faithful inclusion  $X^{S\text{-constr}} \hookrightarrow X$  of  $\infty$ -pretopoi extends (essentially uniquely) to a geometric morphism  $X \rightarrow \mathbf{Sh}_{\text{eff}}(X^{S\text{-constr}})$  over  $\tilde{S}$ . By construction, the  $S$ -stratified  $\infty$ -topos

$$X^{S\text{-spec}} := \mathbf{Sh}_{\text{eff}}(X^{S\text{-constr}})$$

is spectral. Furthermore,  $X^{S\text{-spec}}$  is the universal spectral  $S$ -stratified  $\infty$ -topos receiving a geometric morphism over  $\tilde{S}$  from  $X$ . Thus the assignment

$$X \mapsto X^{S\text{-spec}}$$

provides a relative left adjoint to the inclusion  $\mathbf{StrTop}_\infty^{spec} \hookrightarrow \mathbf{StrTop}_\infty^\wedge$  over  $\mathbf{TSpc}^{spec}$ , which we call the *spectrification*. This is the stratified analogue of the Stone reflection (Theorem 5.88).

**10.27 Example.** When  $S = [n]$ , the spectrification of a bounded coherent  $\infty$ -topos  $\mathbf{X}$  equipped with a constructible stratification by  $[n]$  can be identified as an iterated bounded coherent oriented pushout:

$$\mathbf{X}^{[n]\text{-spec}} \simeq \mathbf{X}_0^{\text{Stone}} \mathop{\cup}_{bc}^{(\mathbf{X}_0 \times_{\mathbf{X}} \mathbf{X}_1)^{\text{Stone}}} \dots \mathop{\cup}_{bc}^{(\mathbf{X}_{n-1} \times_{\mathbf{X}} \mathbf{X}_n)^{\text{Stone}}} \mathbf{X}_n^{\text{Stone}} .$$

**10.28 Construction.** Thanks to the existence of the spectrification functor, we deduce the forgetful functor  $\mathbf{StrTop}_{\infty}^{\text{spec}} \rightarrow \mathbf{TSp}^{\text{spec}}$  is a cocartesian fibration (as well as a cartesian fibration): for any quasicompact continuous map  $f : S' \rightarrow S$  and any spectral  $S'$ -stratified  $\infty$ -topos  $\mathbf{X}$ , the stratified geometric morphism  $\mathbf{X} \rightarrow \mathbf{X}^{S\text{-spec}}$  is a cocartesian edge over  $f$ .

**10.29 Lemma.** *Let  $S$  be a spectral topological space. Then the natural functor*

$$\mathbf{StrTop}_{\infty, S}^{\text{spec}} \rightarrow \lim_{P \in \text{FC}(S)} \mathbf{StrTop}_{\infty, P}^{\text{spec}}$$

*is an equivalence of  $\infty$ -categories.*

*Proof.* The formation of the limit is an inverse. □

## 11 Profinite stratified shape

### The definition of the profinite stratified shape

**11.1 Construction.** We have constructed ([Theorem 10.10](#)) an equivalence of  $\infty$ -categories  $\widehat{\lambda} : \mathbf{Str}_{\pi}^{\wedge} \simeq \mathbf{StrTop}_{\infty}^{\text{spec}}$  over  $\mathbf{TSp}^{\text{spec}}$ , given by the assignment  $\mathbf{\Pi} \mapsto \widetilde{\mathbf{\Pi}}$ . The further inclusion  $\mathbf{StrTop}_{\infty}^{\text{spec}} \hookrightarrow \mathbf{StrTop}_{\infty}^{\wedge}$  admits a left adjoint, given by spectrification ([Construction 10.26](#)). We therefore obtain an adjunction

$$\Pi_{(\infty, 1)}^{\wedge} : \mathbf{StrTop}_{\infty}^{\wedge} \rightleftarrows \mathbf{Str}_{\pi}^{\wedge} : \widehat{\lambda}$$

in which the left adjoint carries a stratified  $\infty$ -topos  $\mathbf{X} \rightarrow \widetilde{S}$  to the profinite  $S$ -stratified space that as a left exact accessible functor  $\mathbf{Str}_{\pi} \rightarrow \mathbf{S}$  is given by

$$\mathbf{\Pi} \mapsto \text{Map}_{\mathbf{StrTop}_{\infty}^{\wedge}}(\mathbf{X}, \widetilde{\mathbf{\Pi}}) .$$

Over any spectral topological space  $S$ , we obtain an adjunction

$$\Pi_{(\infty, 1)}^{S, \wedge} : \mathbf{StrTop}_{\infty, S}^{\wedge} \rightleftarrows \mathbf{Str}_{\pi, S}^{\wedge} : \widehat{\lambda}_S$$

over  $S$ .

**11.2 Example.** For any spectral topological space  $S$  and any profinite  $S$ -stratified space  $\mathbf{\Pi}$ , we have  $\Pi_{(\infty, 1)}^{S, \wedge}(\widetilde{\mathbf{\Pi}}) \simeq \mathbf{\Pi}$ .

**11.3 Example.** The functor  $\Pi_{(\infty, 1)}^{(0), \wedge}$  is the profinite shape of [Definition 5.75](#).



**11.4 Definition.** Let  $S$  be a spectral topological space, and let  $\mathbf{X} \rightarrow \tilde{S}$  be an  $S$ -stratified  $\infty$ -topos. Then we call the profinite  $S$ -stratified space  $\Pi_{(\infty,1)}^{S,\wedge}(\mathbf{X})$  the  *$S$ -stratified homotopy type* of  $\mathbf{X}$ .

**11.5.** Since left adjoints compose, if  $\eta: S' \rightarrow S$  is a quasicompact continuous map of spectral topological spaces, then there is a natural equivalence

$$\eta! \Pi_{(\infty,1)}^{S',\wedge} \simeq \Pi_{(\infty,1)}^{S,\wedge}.$$

**11.6 Example.** For any bounded coherent constructible  $S$ -stratified  $\infty$ -topos  $\mathbf{X}$ , the homotopy type  $\Pi_{\infty}^{\wedge}(\mathbf{X})$  is the classifying profinite space of the profinite  $\infty$ -category  $\Pi_{(\infty,1)}^{S,\wedge}(\mathbf{X})$ ; thus the stratification on  $\mathbf{X}$  gives rise to a delocalisation of its homotopy type.

Combining  $\infty$ -Categorical Hochster Duality ([Theorem 10.10](#)) with [Proposition 10.22](#) we deduce the *Exodromy Equivalence* stated as [Theorem B](#) in the introduction.

**11.7 Theorem** (Exodromy Equivalence for Stratified  $\infty$ -Topoi). *Let  $S$  be a spectral topological space and  $\mathbf{X}$  an  $S$ -stratified  $\infty$ -topos. Then the counit  $\tilde{\Pi}_{(\infty,1)}^{S,\wedge}(\mathbf{X}) \rightarrow \mathbf{X}$  of the adjunction to profinite stratified spaces restricts to an equivalence*

$$\mathrm{Fun}(\Pi_{(\infty,1)}^{S,\wedge}(\mathbf{X}), \mathcal{S}_{\pi}) \simeq \mathbf{X}^{S\text{-constr}}.$$

## Points & materialisation

We now provide a stratified refinement of [\(5.79\)](#), which allows us to prove a ‘Whitehead Theorem’ for profinite stratified spaces, and effectively speak of  *$n$ -truncated* profinite stratified spaces via materialisation.

**11.8.** Let  $S$  be a spectral topological space, and let  $\mathbf{X}$  be an  $S$ -stratified  $\infty$ -topos. The  $\infty$ -category of points of  $\mathbf{X}$  is

$$\mathrm{Pt}(\mathbf{X}) = \mathrm{Fun}_{*}(\mathcal{S}, \mathbf{X})^{op} \simeq \mathrm{Fun}_{\mathrm{StrTop}_{\infty}^{\wedge,*}}(\{\overline{0}\}, \mathbf{X})^{op}.$$

Since  $\Pi_{(\infty,1)}^{\wedge}(\{\overline{0}\}) \simeq *$ , applying  $\Pi_{(\infty,1)}^{\wedge}$  yields a natural functor

$$\mathrm{Pt}(\mathbf{X}) \rightarrow \mathrm{Fun}_{\mathrm{Str}_{\pi}^{\wedge}}(*, \Pi_{(\infty,1)}^{\wedge}(\mathbf{X})) \simeq \mathrm{mat} \Pi_{(\infty,1)}^{\wedge}(\mathbf{X}).$$

In the case where  $\mathbf{X}$  is a spectral  $\infty$ -topos, then  $\infty$ -Categorical Hochster Duality ([Theorem 10.10](#)) implies the following stratified refinement of [\(5.79\)](#).

**11.9 Lemma.** *If  $\mathbf{X}$  is a spectral  $\infty$ -topos, then the natural morphism*

$$\mathrm{Pt}(\mathbf{X}) \rightarrow \mathrm{mat} \Pi_{(\infty,1)}^{\wedge}(\mathbf{X})$$

*of stratified spaces is an equivalence.*

Now we can deduce a stratified refinement of Whitehead’s Theorem for profinite spaces ([Theorem 5.80](#)=[[SAG](#), [Theorem E.3.1.6](#)]).

**11.10 Theorem** (Profinite Stratified Whitehead Theorem). *The materialisation functor  $\text{mat} : \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{Str}$  is conservative.*

*Proof.* Let  $f : \mathbf{II} \rightarrow \mathbf{II}'$  be a morphism in  $\mathbf{Str}_\pi^\wedge$  and assume that  $\text{mat}(f)$  is an equivalence in  $\mathbf{Str}$ . Write  $f_* : \widetilde{\mathbf{II}} \rightarrow \widetilde{\mathbf{II}'}$  for the induced morphism of spectral  $\infty$ -topoi. From [Lemma 11.9](#) we deduce that

$$\text{Pt}(f_*) : \text{Pt}(\widetilde{\mathbf{II}}) \rightarrow \text{Pt}(\widetilde{\mathbf{II}'})$$

is an equivalence of  $\infty$ -categories. Conceptual Completeness ([Theorem 5.57](#)=[[SAG, Theorem A.9.o.6](#)]) implies that  $f_*$  is an equivalence of  $\infty$ -topoi. The full faithfulness of the functor  $\mathbf{II} \mapsto \widetilde{\mathbf{II}}$  completes the proof.  $\square$

We can employ the Profinite Stratified Whitehead Theorem to study the Postnikov tower of profinite stratified spaces.

**11.11 Definition.** Let  $n \in \mathbf{N}$ . A profinite stratified space  $\mathbf{II} \rightarrow S$  is said to be *n-truncated* if and only if it can be exhibited as an inverse limit of finite *n*-truncated spaces. Equivalently, if we extend  $h_n : \mathbf{Str}_\pi \rightarrow \mathbf{Str}_\pi^\wedge$  to an inverse-limit preserving functor  $h_n : \mathbf{Str}_\pi^\wedge \rightarrow \mathbf{Str}_\pi^\wedge$ , then an *n*-truncated profinite space is one in the essential image of  $h_n$ .

We write  $(\mathbf{Str}_\pi^\wedge)_{\leq n} \subset \mathbf{Str}_\pi^\wedge$  for the full subcategory spanned by the *n*-truncated profinite stratified spaces.

**11.12 Lemma.** *Let  $n \in \mathbf{N}$ , and let  $S$  be a spectral topological space. Then a profinite stratified space  $\mathbf{II} \rightarrow S$  is *n*-truncated if and only if, for all  $s, t \in \text{mat}(S)$  with  $s \leq t$ , the induced morphism*

$$N_{\text{mat}(S)}(\mathbf{II})\{s, t\} \rightarrow \mathbf{II}_s \times \mathbf{II}_t$$

*is an  $(n - 1)$ -truncated morphism of  $S_\pi^\wedge$ .*

*Proof.* If  $\mathbf{II}$  is exhibited as a sequence  $\{\Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  of  $\pi$ -finite *n*-truncated stratified spaces, then express  $s$  and  $t$  as sequences  $\{s_\alpha\}_{\alpha \in \Lambda}$  and  $\{t_\alpha\}_{\alpha \in \Lambda}$  of points. So the sequence

$$\{N_{P_\alpha}(\Pi_\alpha)\{s_\alpha, t_\alpha\} \rightarrow \Pi_{s_\alpha} \times \Pi_{t_\alpha}\}_{\alpha \in \Lambda},$$

which exhibits the morphism  $N_{\text{mat}(S)}(\mathbf{II})\{s, t\} \rightarrow \mathbf{II}_s \times \mathbf{II}_t$  of  $S_\pi^\wedge$ , is  $(n - 1)$ -truncated.

Conversely, assume that  $\mathbf{II}$  is exhibited as a sequence  $\{\Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  of  $\pi$ -finite stratified spaces, and that for any  $s, t \in \text{mat}(S)$  with  $s \leq t$ , the morphism  $N_{\text{mat}(S)}(\mathbf{II})\{s, t\} \rightarrow \mathbf{II}_s \times \mathbf{II}_t$  of  $S_\pi^\wedge$  is  $(n - 1)$ -truncated. Now consider  $h_n \mathbf{II} := \{h_n \Pi_\alpha \rightarrow P_\alpha\}_{\alpha \in \Lambda}$  and the natural morphism  $\mathbf{II} \rightarrow h_n \mathbf{II}$ . To see that this morphism is an equivalence, we may pass to the materialisation by [Theorem 11.10](#), where it is obvious.  $\square$

**11.13 Lemma.** *Let  $n \in \mathbf{N}$ . A profinite stratified space  $\mathbf{II} \rightarrow S$  is *n*-truncated if and only if  $\text{mat}(\mathbf{II}) \in \mathbf{Str}$  is *n*-truncated in the sense of [Definition 2.11](#).*

*Proof.* For  $s, t \in \text{mat}(S)$  with  $s \leq t$ , we have

$$\text{mat}(N_{\text{mat}(S)}(\mathbf{II})\{s, t\}) \simeq N_{\text{mat}(S)}(\text{mat}(\mathbf{II}))\{s, t\}.$$

By [Proposition 5.81](#)=[[SAG, Proposition E.4.6.1](#)] and the fact that materialisation is a right adjoint, we see that a profinite stratified space  $\mathbf{II}$  is  $n$ -truncated if and only if the morphism

$$N_{\text{mat}(S)}(\text{mat}(\mathbf{II}))\{s, t\} \rightarrow \text{mat}(\mathbf{II})_s \times \text{mat}(\mathbf{II})_t$$

is an  $(n - 1)$ -truncated morphism of spaces, which is true if and only if  $\text{mat}(\mathbf{II})$  is  $n$ -truncated in the sense of [Definition 2.11](#).  $\square$

**11.14.** Combining the preceding with ordinary Stone Duality between profinite sets and Stone topological spaces, the functor  $\text{Pt} : (\mathbf{Str}_\pi^\wedge)_{\leq 1} \rightarrow \mathbf{Cat}_1$  factors through a fully faithful functor  $(\mathbf{Str}_\pi^\wedge)_{\leq 1} \hookrightarrow \mathbf{Cat}(\mathbf{TSpc}^{\text{Stone}})$  from the 2-category of 1-truncated profinite stratified spaces to the 2-category of category objects in the category of Stone topological spaces. The essential image of this functor is spanned by the *layered* category objects – i.e., the ones in which every endomorphism is an isomorphism.

### Stratified homotopy types via décollages

To identify the functor  $\Pi_{(\infty, 1)}^\wedge$  in terms of the usual homotopy type  $\Pi_\infty^\wedge$ , we can pass to the décollage over  $P$ .

**11.15 Construction.** Let  $P$  be a finite poset. Let us consider the functor

$$\widehat{\lambda}_P^{\text{déc}} : \mathbf{Déc}_P(\mathbf{S}_\pi^\wedge) \rightarrow \mathbf{Déc}_P(\mathbf{Top}_\infty^{bc})$$

given by composition with  $\lambda_{\{0\}}$ , so that a profinite spatial décollage  $D : \text{sd}^{op}(P) \rightarrow \mathbf{S}_\pi^\wedge$  is carried to the toposic décollage  $\Sigma \mapsto \widehat{D}(\Sigma)$ . We have seen ([Proposition 9.52](#)) that this is a fully faithful functor whose essential image is  $\mathbf{Déc}_P(\mathbf{Top}_\infty^{\text{Stone}})$ .

In the other direction, let us consider the functor

$$\Pi_\infty^{\text{predéc}, P, \wedge} : \mathbf{Déc}_P(\mathbf{Top}_\infty^{bc}) \rightarrow \text{Fun}(\text{sd}^{op}(P), \mathbf{S}_\pi^\wedge)$$

given by composition with the profinite shape functor  $\Pi_\infty^\wedge$ , so that a toposic décollage  $D : \text{sd}^{op}(P) \rightarrow \mathbf{Top}_\infty^{bc}$  is carried to the functor  $\Sigma \mapsto \Pi_\infty^\wedge D(\Sigma)$ . We can then compose this with the Segalification functor – that is, the left adjoint to the fully faithful functor  $\mathbf{Déc}_P(\mathbf{S}_\pi^\wedge) \hookrightarrow \text{Fun}(\text{sd}^{op}(P), \mathbf{S}_\pi^\wedge)$  – to obtain a functor

$$\Pi_\infty^{\text{déc}, P, \wedge} : \mathbf{Déc}_P(\mathbf{Top}_\infty^{bc}) \rightarrow \mathbf{Déc}_P(\mathbf{S}_\pi^\wedge)$$

that is left adjoint to  $\widehat{\lambda}_P^{\text{déc}}$ .

The difficulty here is that the functor  $\Pi_\infty^{\text{predéc}, P, \wedge}$  is very inexplicit, because it involves Segalification. To address this, we have the following.

**11.16 Theorem.** *Let  $P$  be a finite poset. If  $\mathbf{X} \rightarrow \widetilde{P}$  is a spectral  $P$ -stratified  $\infty$ -topos, then the functor  $\Sigma \mapsto \Pi_\infty^\wedge D(\Sigma)$  is already a profinite spatial décollage; that is, the Segalification morphism*

$$\Pi_\infty^{\text{predéc}, P, \wedge}(\mathbf{X}) \rightarrow \Pi_\infty^{\text{déc}, P, \wedge}(\mathbf{X})$$

*is an equivalence in  $\text{Fun}(\text{sd}^{op}(P), \mathbf{S}_\pi^\wedge)$ .*

*Proof.* It suffices to prove that for every string  $\Sigma := \{p_0, \dots, p_n\} \subset P$ , the natural morphism

$$f_\Sigma: \Pi_\infty^\wedge(\mathbf{X}_{p_0} \tilde{\times}_X \cdots \tilde{\times}_X \mathbf{X}_{p_n}) = \Pi_\infty^\wedge \text{Mor}_{\tilde{P}}(\tilde{\Sigma}, \mathbf{X}) \rightarrow \text{Map}_P(\Sigma, \Pi_{(\infty,1)}^\wedge(\mathbf{X}))$$

in  $\mathbf{S}_\pi^\wedge$  an equivalence. By Whitehead's Theorem for profinite spaces ([Theorem 5.80](#)=[SAG, [Theorem E.3.1.6](#)]), it suffices to prove that the materialisation  $\text{mat}(f_\Sigma)$  is an equivalence. Since  $\mathbf{X}$  is spectral, we have a natural equivalence

$$\text{mat} \Pi_\infty^\wedge(\mathbf{X}_{p_0} \tilde{\times}_X \cdots \tilde{\times}_X \mathbf{X}_{p_n}) \simeq \text{Pt}(\mathbf{X}_{p_0} \tilde{\times}_X \cdots \tilde{\times}_X \mathbf{X}_{p_n}).$$

Similarly, since  $\Sigma$  is constant as a proobject and  $\mathbf{X}$  is spectral, by Whitehead's Theorem for profinite stratified spaces ([Theorem 11.10](#)) we have natural equivalences

$$\begin{aligned} \text{mat} \text{Map}_P(\Sigma, \Pi_{(\infty,1)}^\wedge(\mathbf{X})) &\simeq \text{Map}_P(\Sigma, \text{mat} \Pi_{(\infty,1)}^\wedge(\mathbf{X})) \\ &\simeq \text{Map}_P(\Sigma, \text{Pt}(\mathbf{X})). \end{aligned}$$

By the universal property of the iterated oriented fibre product  $\mathbf{X}_{p_0} \tilde{\times}_X \cdots \tilde{\times}_X \mathbf{X}_{p_n}$ , we have a natural identification

$$(11.17) \quad \text{Map}_P(\Sigma, \text{Pt}(\mathbf{X})) \simeq \text{Pt}(\mathbf{X}_{p_0} \tilde{\times}_X \cdots \tilde{\times}_X \mathbf{X}_{p_n}).$$

To complete the proof, note that the materialisation  $\text{mat}(f_\Sigma)$  is equivalent to the morphism (11.17).  $\square$

**11.18 Example.** Let  $P$  be a finite poset, and let  $\mathbf{X} \rightarrow \tilde{P}$  be a spectral  $P$ -stratified  $\infty$ -topos. It follows from [Theorem 11.16](#) that, for any point  $p \in P$ , the  $p$ -th stratum  $\Pi_{(\infty,1)}^{P,\wedge}(\mathbf{X})_p$  is equivalent to the homotopy type  $\Pi_\infty^\wedge(\mathbf{X}_p)$ .

**11.19 Example.** Let  $P$  be a finite poset, and let  $\mathbf{X} \rightarrow \tilde{P}$  be a spectral  $P$ -stratified  $\infty$ -topos. It follows from [Theorem 11.16](#) that, for any points  $p, q \in P$  with  $p < q$ , the link  $\text{Map}_P(\{p, q\}, \Pi_{(\infty,1)}^{P,\wedge}(\mathbf{X}))$  between the  $p$ -th and  $q$ -th strata of  $\Pi_{(\infty,1)}^{P,\wedge}(\mathbf{X})$  is equivalent to the homotopy type  $\Pi_\infty^\wedge(\mathbf{X}_p \tilde{\times}_X \mathbf{X}_q)$  of the link.

**11.20 Example.** Let  $P$  be a finite poset, and  $\mathbf{X}$  a spectral  $P$ -stratified  $\infty$ -topos. For any points  $p, q \in P$  with  $p \leq q$ , write

$$i_{pq,*} : \mathbf{X}_p \hookrightarrow \mathbf{X}_{\{p,q\}} \quad \text{and} \quad j_{pq,*} : \mathbf{X}_q \hookrightarrow \mathbf{X}_{\{p,q\}}$$

for the closed and open immersions of strata, respectively. Then the Beck–Chevalley Theorem ([Theorem 8.4](#)) ensures that the décollage

$$\Pi_\infty^{\text{déc}, P, \wedge}(\mathbf{X}) : \text{sd}^{\text{op}}(P) \rightarrow \mathbf{S}_\pi^\wedge$$

carries any string  $\{p_0, \dots, p_n\} \subseteq P$  to the profinite space  $\mathbf{S}_\pi^\wedge \rightarrow \mathbf{S}$  given by the composite

$$\Gamma_{\mathbf{X}_{p_0},*} i_{p_0 p_1}^* j_{p_0 p_1,*} i_{p_1 p_2}^* j_{p_1 p_2,*} \cdots i_{p_{n-1} p_n}^* j_{p_{n-1} p_n,*} \Gamma_{\mathbf{X}_{p_n}}^* .$$

## Van Kampen theorem

If  $P$  is a poset and  $\eta: P \rightarrow \{0\}$  then the ‘invert everything’ functor  $\Pi \mapsto \Pi^+ \simeq \eta_! \Pi$  from  $P$ -stratified spaces to spaces, regarded as a functor from spatial décollages to spaces, is given by the formation of the colimit. That is, if  $\Pi \rightarrow P$  is a  $P$ -stratified space, then one has

$$\Pi^+ \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}^{op}(P)} N_P(\Pi)(\Sigma).$$

The ‘invert everything’ functor extends to a functor  $\mathbf{\Pi} \mapsto \mathbf{\Pi}^+$  from profinite  $P$ -stratified spaces to profinite spaces, and this formula is precisely the same in that context. The compatibility (11.5) therefore provides a colimit description of the homotopy type of a stratified  $\infty$ -topos:

**11.21 Proposition** (van Kampen Theorem). *Let  $P$  be a finite poset, and let  $\mathbf{X} \rightarrow \tilde{P}$  be a spectral  $P$ -stratified  $\infty$ -topos. Then the homotopy type of  $\mathbf{X}$  is equivalent to the colimit*

$$\Pi_{\infty}^{\wedge}(\mathbf{X}) \simeq \operatorname{colim}_{\Sigma \in \operatorname{sd}^{op}(P)} \Pi_{\infty}^{\wedge}(N_P(\mathbf{X})(\Sigma))$$

in profinite spaces.

**11.22 Example.** If  $\mathbf{X}$  is a spectral  $\infty$ -topos exhibited as a recollement  $\mathbf{Z} \overset{\phi}{\cup} \mathbf{U}$  of Stone  $\infty$ -topoi  $\mathbf{Z}$  and  $\mathbf{U}$ , then one has the formula

$$\Pi_{\infty}^{\wedge}(\mathbf{X}) \simeq \Pi_{\infty}^{\wedge}(\mathbf{Z}) \cup^{J_{\infty}^{\wedge}(\mathbf{Z} \tilde{\times}_{\mathbf{X}} \mathbf{U})} \Pi_{\infty}^{\wedge}(\mathbf{U})$$

in profinite spaces.

**11.23 Example.** Let  $n \in \mathbf{N}$ , and let  $\mathbf{X} \rightarrow \widetilde{[n]}$  be a spectral  $[n]$ -stratified  $\infty$ -topos. Then  $\Pi_{\infty}^{\wedge}(\mathbf{X})$  can be exhibited as the colimit of a punctured  $(n+1)$ -cube  $\operatorname{sd}^{op}([n]) \rightarrow \mathbf{S}_{\pi}^{\wedge}$  given by

$$\{p_0, \dots, p_k\} \mapsto \Pi_{\infty}^{\wedge}(\mathbf{X}_{p_0} \tilde{\times}_{\mathbf{X}} \mathbf{X}_{p_1} \tilde{\times}_{\mathbf{X}} \dots \tilde{\times}_{\mathbf{X}} \mathbf{X}_{p_k}).$$

## Part IV

# Stratified étale homotopy theory

## 12 Aide-mémoire on the étale homotopy type

In this section we recall how to situate the *étale homotopy type* of Artin–Mazur–Friedlander in the  $\infty$ -categorical setting, as well as some example computations of the étale homotopy type.

### Definition

From an  $\infty$ -categorical perspective, there are *a priori* four étale shaps to contemplate:

**12.1 Definition.** Let  $X$  be a coherent scheme  $X$ . We define:

- ▶ the shape  $\Pi_{\infty}^{\acute{e}t}(X) := \Pi_{\infty}(X_{\acute{e}t})$  of the 1-localic étale  $\infty$ -topos,
- ▶ the shape  $\Pi_{\infty}^{\acute{e}t, hyp}(X) := \Pi_{\infty}(X_{\acute{e}t}^{hyp})$  of the hypercomplete étale  $\infty$ -topos,
- ▶ the homotopy type  $\Pi_{\infty}^{\acute{e}t, \wedge}(X) := \Pi_{\infty}^{\wedge}(X_{\acute{e}t})$  of the 1-localic  $\infty$ -topos, and
- ▶ the homotopy type  $\Pi_{\infty}^{\acute{e}t, hyp, \wedge}(X) := \Pi_{\infty}^{\wedge}(X_{\acute{e}t}^{hyp})$  of the hypercomplete étale  $\infty$ -topos.

It is a formal matter that the last two of these are always equivalent:

**12.2.** Since truncated objects of an  $\infty$ -topos are hypercomplete, for any  $\infty$ -topos  $\mathbf{X}$  the inclusion  $\mathbf{X}^{hyp} \hookrightarrow \mathbf{X}$  induces an equivalence  $\Pi_{\infty}^{\wedge}(\mathbf{X}^{hyp}) \simeq \Pi_{\infty}^{\wedge}(\mathbf{X})$ , whence we obtain an equivalence  $\Pi_{\infty}^{\acute{e}t, hyp, \wedge}(X) \simeq \Pi_{\infty}^{\acute{e}t, \wedge}(X)$ .

For a locally noetherian scheme  $X$ , Artin–Mazur [3, §9] constructed a proobject in the homotopy category of spaces called the *étale homotopy type* of  $X$ . Friedlander [15, §4] later refined this construction, producing a proobject in simplicial sets called the *étale topological type* of  $X$  whose image in  $\text{Pro}(h\mathbf{S})$  agrees with the étale homotopy type of Artin–Mazur [15, Proposition 4.5]. Hoyois [25, §5] has shown that Friedlander’s étale topological type corepresents the shape of the hypercomplete étale  $\infty$ -topos of  $X$ :

**12.3 Theorem** ([25, Corollary 5.6]). *Let  $X$  be a locally noetherian scheme. Then the étale topological type of  $X$ , as defined by Friedlander, corepresents the shape  $\Pi_{\infty}^{\acute{e}t, hyp}(X)$  of the hypercomplete étale  $\infty$ -topos.*

In certain cases, the étale topological type is already profinite.

**12.4 Theorem** ([3, Theorem 11.1; 15, Theorem 7.3]). *Let  $X$  be a connected noetherian scheme that is geometrically unibranch. Then  $\Pi_{\infty}^{\acute{e}t, hyp}(X)$  is profinite, that is,*

$$\Pi_{\infty}^{\acute{e}t, hyp}(X) \simeq \Pi_{\infty}^{\acute{e}t, hyp, \wedge}(X) \simeq \Pi_{\infty}^{\acute{e}t, \wedge}(X).$$

**12.5 Question.** Let  $X$  be a connected noetherian scheme that is geometrically unibranch. Even in simple cases, we do not at this point have a very good understanding of the kind of information that is contained in the étale shape  $\Pi_{\infty}^{\acute{e}t}(X)$  but not in the other invariants. In this paper, we are content to focus our attention on the profinite homotopy types (and their stratified variants, of course).

## Examples

**12.6 Example.** For any field  $k$  one has a noncanonical identification

$$\Pi_{\infty}^{\acute{e}t,\wedge}(\mathrm{Spec} k) \simeq BG_k,$$

where  $G_k$  is the absolute Galois group of  $k$ .

**12.7 Example.** Let  $k$  be an algebraically closed field of characteristic 0 and

$$C = \mathrm{Spec}(k[x, y]/(y^2 - x^3 - x^2))$$

the nodal cubic. Then one has a noncanonical identification  $\Pi_{\infty}^{\acute{e}t}(C) \simeq BZ$ , whereas the étale homotopy type is given by  $\Pi_{\infty}^{\acute{e}t,\wedge}(C) \simeq B\widehat{Z}$ .

**12.8 Example.** If  $C$  is a smooth irreducible curve over a field  $k$  with Euler characteristic  $\chi(C) < 2$ , then we have a noncanonical identification  $\Pi_{\infty}^{\acute{e}t,\wedge}(C) \simeq B\pi_1^{\acute{e}t}(C)$ .

**12.9 Example** (see [Theorem 13.13](#)). We have an equivalence  $\Pi_{\infty}^{\acute{e}t,\wedge}(\mathbf{P}_C^1) \simeq (S^2)_{\pi}^{\wedge}$ , where  $S^2$  denotes the 2-sphere.

**12.10 Example** ([24, Theorem 1]). Let  $k$  be an algebraically closed field of characteristic  $p > 0$  and let  $X$  be a smooth  $k$ -variety. Then  $\Pi_{\infty}^{\acute{e}t, hyp}(X) \simeq *$  if and only if  $X$  is isomorphic to  $\mathrm{Spec} k$ .

## 13 Stratified étale homotopy types

### Definition

In this subsection we define a stratified refinement of the étale homotopy type and provide a number of example computations of the *stratified étale homotopy type*.

**13.1 Notation.** Recall that for a coherent scheme  $X$ , we let  $\mathrm{FC}(X)$  denote the 1-category of nondegenerate, finite, constructible stratifications of the spectral topological space  $X^{\mathrm{zar}}$ . We abuse notation and write merely  $P$  for an object  $X^{\mathrm{zar}} \rightarrow P$  of this category, leaving the structure morphism implicit. The 1-category  $\mathrm{FC}(X)$  is, up to equivalence, a poset in which  $P \leq Q$  if and only if  $P$  refines  $Q$ ; that is,  $P \leq Q$  if and only if  $X^{\mathrm{zar}} \rightarrow Q$  factors through  $X^{\mathrm{zar}} \rightarrow P$ . The spectral topological space  $X^{\mathrm{zar}}$  corresponds under Hochster Duality to the profinite poset  $\{P\}_{P \in \mathrm{FC}(X)}$ .

We write  $\Phi_X$  for the set of *filters* on  $\mathrm{FC}(X)$  – i.e., open subsets that are inverse as 1-categories – equipped with the partial ordering given by inclusion. One has a natural injection  $\mathrm{FC}(X)^{op} \hookrightarrow \Phi_X$  that carries  $P$  to the *principal filter*  $F_{X, \geq P}$ .

**13.2 Notation.** We write  $\mathbf{Sch}$  for the 1-category of *coherent* schemes.

**13.3 Definition.** Let  $X$  be a coherent scheme. Then we write

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X) := \Pi_{(\infty,1)}^{X^{zar},\wedge}(X_{\acute{e}t}).$$

We call this the *stratified étale homotopy type* of  $X$ . This is a functor  $\Pi_{(\infty,1)}^{\acute{e}t,\wedge} : \mathbf{Sch} \rightarrow \mathbf{Str}_{\pi}^{\wedge}$ .

More generally, if  $\Lambda \in \Phi_X$  is a filter, then  $\Lambda$  is an inverse system of finite posets, and we have a constructible stratification  $p : X^{zar} \rightarrow \Lambda$ . We may therefore define

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; \Lambda) := \Pi_{(\infty,1)}^{\Lambda,\wedge}(X_{\acute{e}t}).$$

**13.4.** We obtain a diagram

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; -) : \Phi_X^{op} \rightarrow \mathbf{Str}_{\pi}^{\wedge}$$

of localisations.

In particular, for any nondegenerate, finite, constructible stratification  $P \in \mathbf{FC}(X)$ , we have

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; P) := \Pi_{(\infty,1)}^{P,\wedge}(X_{\acute{e}t}) \simeq \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; F_{X,\geq P}).$$

**13.5 Construction.** Let  $X$  be a coherent scheme. The  $X^{zar}$ -stratified  $\infty$ -topos  $X_{\acute{e}t}$  is spectral. Our  $\infty$ -Categorical Hochster Duality Theorem [Theorem 10.10](#) implies that  $X_{\acute{e}t} \simeq \mathbf{Sh}_{\text{eff}}(\overline{\Pi}_{(\infty,1)}^{\acute{e}t,\wedge}(X))$ , and thus

$$X_{\acute{e}t}^{constr} \simeq \mathbf{Fun}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X), \mathbf{S}_{\pi}).$$

Here, at last, is the *exodromy equivalence*. If  $X$  and  $Y$  are coherent schemes, then the natural map

$$\mathbf{Map}_{\mathbf{Top}_{\infty}}(X_{\acute{e}t}, Y_{\acute{e}t}) \rightarrow \mathbf{Map}_{\mathbf{Str}_{\pi}^{\wedge}}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X), \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(Y))$$

is an equivalence.

We also have an equivalence

$$\mathbf{mat}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X)) \simeq \mathbf{Pt}(X_{\acute{e}t}),$$

and this  $\infty$ -category can be described in the following manner: an object is geometric points  $x \rightarrow X$ , and for any geometric points  $x \rightarrow X$  and  $y \rightarrow X$ , the space  $\mathbf{Map}_{\mathbf{Pt}(X_{\acute{e}t})}(x, y)$  is identified with the space of points  $\mathbf{Pt}(x \tilde{\times}_X y) \simeq \mathbf{mat}(\Pi_{\infty}^{\wedge}(x \tilde{\times}_X y))$ . This is a discrete space whose components are specialisations  $x \leftarrow y$ . In other words,  $\mathbf{mat}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X))$  can be identified with the 1-category  $\mathbf{Gal}(X)$  from the Introduction preceding [Theorem A](#).

In particular, the profinite stratified space  $\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X)$  is 1-truncated; that is, it is a profinite 1-category, and so in light of [\(11.14\)](#), it can be considered as a category object in the category of Stone topological spaces. The topology on  $\mathbf{Gal}(X)$  is precisely the one described in the introduction.



**13.6 Example.** If  $X$  is any (coherent) scheme, we may consider  $X$  with its trivial  $\{0\}$ -stratification. In this case, one recovers the usual étale homotopy type: one has a canonical equivalence

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; \{0\}) \simeq \Pi_{\infty}^{\acute{e}t,\wedge}(X) .$$

**13.7 Example.** Let  $S = \text{Spec } A$  be a trait with closed point  $s$  and generic point  $\eta$ . Then  $S^{\text{zar}} \cong [1]$ , so  $S_{\acute{e}t}$  is a spectral  $\infty$ -topos that is naturally  $[1]$ -stratified, with closed stratum  $s_{\acute{e}t}$  and open stratum  $\eta$ .

Write  $S^h$  and  $S^{sh}$  for the spectra of the henselisation  $A^h$  and the strict henselisation  $A^{sh}$  of  $A$ , and write  $\eta^h$  and  $\eta^{sh}$  for the spectra of the fraction field  $K^h$  of  $A^h$  and the fraction field  $K^{sh}$  of  $A^{sh}$ .

In this case, please observe that the evanescent  $\infty$ -topos  $s_{\acute{e}t} \bar{\times}_{S_{\acute{e}t}} S_{\acute{e}t}$  can be identified with the étale  $\infty$ -topos  $S_{\acute{e}t}^h$  (Example 7.35), and the oriented fibre product  $s_{\acute{e}t} \bar{\times}_{S_{\acute{e}t}} \eta_{\acute{e}t}$  can be identified with the étale  $\infty$ -topos  $\eta_{\acute{e}t}^h$ .

Now we have the following (noncanonical) identifications of profinite spaces:

$$\Pi_{\infty}^{\wedge}(\eta) \simeq BG_K, \quad \Pi_{\infty}^{\wedge}(\eta^h) \simeq BD_A, \quad \Pi_{\infty}^{\wedge}(\eta^{sh}) \simeq BI_A, \quad \text{and} \quad \Pi_{\infty}^{\wedge}(S^h) \simeq BG_k,$$

where  $G_K$  and  $G_k$  are the absolute Galois groups of  $K$  and  $k$ , the subgroup  $D_A \subseteq G_K$  is the decomposition group, and  $I_A \subseteq D_A$  is the inertia group.

We thus identify, noncanonically, the corresponding profinite décollage  $N_{[1]}(\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(S))$  over  $[1]$  as the functor  $\text{sd}^{op}([1]) \rightarrow \text{Pro}(S_{\pi}^{\wedge})$  given by the diagram

$$BG_k \leftarrow BD_A \rightarrow BG_K .$$

**13.8 Construction.** More generally, a nonempty closed subset  $Z \subset X$  with dense, quasi-compact open complement  $U \subset X$  specifies a nondegenerate constructible stratification  $X^{\text{zar}} \rightarrow [1]$ , and we may – in an overindulgence of abusive notation – write

$$\Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; Z) := \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; [1]) ,$$

which is a profinite  $[1]$ -stratified  $\infty$ -category. Its décollage is the functor  $\text{sd}^{op}([1]) \rightarrow \text{Pro}(S_{\pi}^{\wedge})$  given by the diagram

$$\Pi_{\infty}^{\acute{e}t,\wedge}(Z) \leftarrow \Pi_{\infty}^{\wedge}(Z_{\acute{e}t} \bar{\times}_{X_{\acute{e}t}} U_{\acute{e}t}) \rightarrow \Pi_{\infty}^{\acute{e}t,\wedge}(U) .$$

(Note that any subscheme structure on  $Z$  will do, as nilimmersions don't affect the étale  $\infty$ -topos.) The profinite space  $\Pi_{\infty}^{\wedge}(Z_{\acute{e}t} \bar{\times}_{X_{\acute{e}t}} U_{\acute{e}t})$  is the *deleted tubular neighbourhood* of  $\Pi_{\infty}^{\acute{e}t,\wedge}(Z)$  in  $\Pi_{\infty}^{\acute{e}t,\wedge}(X)$ .

When  $Z = \{z\}$  with  $\kappa(z)$  separably closed, one may identify the deleted tubular neighbourhood of  $\Pi_{\infty}^{\acute{e}t,\wedge}(Z) \simeq *$  in  $\Pi_{\infty}^{\acute{e}t,\wedge}(X)$  with the étale homotopy type of the punctured Milnor ball  $X_{(z)}^{sh} \setminus \{z\}$ .

When  $X$  is a curve over a field  $k$  and  $Z = \{z\}$  is a rational point, we obtain an identification of the deleted tubular neighbourhood with the classifying space of 'the' profinite decomposition group  $D_z \subseteq \pi_1^{\acute{e}t}(U)$ . More generally, we may regard the deleted tubular neighbourhood  $\Pi_{\infty}^{\wedge}(Z_{\acute{e}t} \bar{\times}_{X_{\acute{e}t}} U_{\acute{e}t})$  as a kind of generalised 'decomposition homotopy type'.

In any case, our van Kampen theorem ([Proposition 11.21](#)) exhibits an equivalence of profinite spaces

$$\Pi_{\infty}^{\acute{e}t, \wedge}(X) \simeq \Pi_{\infty}^{\acute{e}t, \wedge}(Z) \cup^{\Pi_{\infty}^{\wedge}(Z_{\acute{e}t} \bar{X}_{X_{\acute{e}t}} U_{\acute{e}t})} \Pi_{\infty}^{\acute{e}t, \wedge}(U),$$

which functions in the same manner as Friedlander’s van Kampen theorem [[15](#), [Proposition 15.6](#)].

**13.9 Question.** Cox [[10](#); [11](#)] also developed a deleted tubular neighbourhood for schemes, which is what appears in Friedlander’s formulation of van Kampen. One is tempted to believe, therefore, that Cox’s deleted tubular neighbourhood and our toposic version have, at the very least, equivalent profinite homotopy types. At this point, unfortunately, we do not know.

### Stratified Riemann Existence Theorem

We now use the Artin Comparison Theorem to prove a stratified refinement of the Riemann Existence Theorem of Artin–Mazur–Friedlander, giving a comparison between étale and analytic stratified homotopy types for schemes of finite type over the complex numbers.

**13.10 Notation.** Write  $\mathbf{C}$  for the field of complex numbers and  $\mathbf{Sch}_{\mathbf{C}}^{ft}$  for category of schemes of finite type over  $\mathbf{C}$  and finite type morphisms between them. We write

$$(-)^{an} : \mathbf{Sch}_{\mathbf{C}}^{ft} \rightarrow \mathbf{TSpc}$$

for the *analytification* functor, which carries a scheme  $X$  of finite type over  $\mathbf{C}$  to  $X(\mathbf{C})$  equipped with the complex analytic topology.

**13.11 Recollection.** Let  $X$  be a scheme finite type over  $\mathbf{C}$ . In [[SGA<sub>4</sub> III](#), Exposé XI, 4.0], Artin defines a natural geometric morphism of 1-topoi  $\varepsilon_* : \tau_{\leq 0} \widetilde{X}^{an} \rightarrow \tau_{\leq 0} X_{\acute{e}t}$  from the 1-topos of sheaves of sets on  $X^{an}$  to the 1-topos of sheaves of sets on the étale site of  $X$ . The geometric morphism  $\varepsilon_*$  extends to a natural geometric morphism of 1-localic  $\infty$ -topoi

$$\varepsilon_* : \widetilde{X}^{an} \rightarrow X_{\acute{e}t}.$$

The naturality can be encoded as a functor  $\mathbf{Sch}_{\mathbf{C}}^{ft} \rightarrow \mathbf{Fun}(\Delta^1, \mathbf{Top}_{\infty})$ , so that if  $f : X \rightarrow Y$  is a finite type morphism of finite type  $\mathbf{C}$ -schemes, then one has an equivalence  $f_*^{\acute{e}t} \varepsilon_* \simeq \varepsilon_* f_*^{an}$ .

Additionally, we have the critical basechange result of Artin. A straightforward Postnikov argument permits us to reformulate Artin’s theorem as follows.

**13.12 Theorem** (Artin Comparison; [[SGA<sub>4</sub> III](#), Exposé XVI, Théorème 4.1]). *Let  $f : X \rightarrow Y$  be a finite type morphism of finite type  $\mathbf{C}$ -schemes, and let  $F \in X_{\acute{e}t}$  be a constructible sheaf. Then the natural morphism*

$$\varepsilon_*^* f_*^{\acute{e}t} F \rightarrow f_*^{an} \varepsilon_*^* F$$

*is an equivalence.*

Artin–Mazur [3, Theorem 12.9] and later Friedlander [15, Theorem 8.6] proved versions of the Riemann existence theorem that, in light of [25, Corollary 5.6], asserts that  $\widetilde{X}^{an}$  and  $X_{\acute{e}t}$  have the same profinite étale homotopy type when regarded as proobjects of the homotopy category of spaces. One may refine the Artin–Mazur equivalence to an equivalence of proobjects in the  $\infty$ -category of spaces (cf. [8, Proposition 4.12]). Indeed, the Théorème de Comparaison [SGA<sub>4</sub> III, Exposé XI, Théorèmes 4.3 & 4.4] can be employed to provide an equivalence between the  $\infty$ -category of lisse sheaves of spaces on  $X$  and that of lisse sheaves of spaces on  $X^{an}$ , whence we obtain the following.

**13.13 Theorem (Riemann Existence).** *Let  $X$  be a scheme finite type over  $\mathbf{C}$ . Then  $\varepsilon^*$  induces an equivalence  $X_{\acute{e}t}^{lisse} \simeq (\widetilde{X}^{an})^{lisse}$  of  $\infty$ -categories of lisse sheaves. Consequently,  $\varepsilon_*$  induces an equivalence*

$$\Pi_{\infty}^{\wedge}(X^{an}) \simeq \Pi_{\infty}^{\wedge}(\widetilde{X}^{an}) \simeq \Pi_{\infty}^{\wedge}(X_{\acute{e}t})$$

of profinite homotopy types.

We now promote this to a *stratified* Riemann Existence Theorem.

**13.14 Construction.** If  $X$  is a scheme of finite type over  $\mathbf{C}$ , then the topological space  $X^{an}$  admits the evident profinite stratification  $X^{an} \rightarrow X^{zar}$ , and  $\varepsilon_*$  is an  $X^{zar}$ -stratified geometric morphism.

If  $X^{zar} \rightarrow P$  is a finite stratification (automatically constructible), then the topological space  $X^{an}$  also inherits a stratification  $X^{an} \rightarrow P$ , which is conical.

On each stratum  $X_p$ , the functor  $\varepsilon^*$  restricts to a functor (in fact an equivalence)  $X_{p,\acute{e}t}^{lisse} \rightarrow (\widetilde{X}_p^{an})^{lisse}$ , whence we obtain a functor  $\varepsilon^* : X_{\acute{e}t}^{P-constr} \rightarrow (\widetilde{X}^{an})^{P-constr}$ , which in turn induces a  $P$ -stratified geometric morphism

$$\varepsilon_*^{constr} : \mathbf{Sh}_{\text{eff}}((\widetilde{X}^{an})^{P-constr}) \rightarrow \mathbf{Sh}_{\text{eff}}(X_{\acute{e}t}^{P-constr})$$

of spectral  $P$ -stratified  $\infty$ -topoi.

Please note that we also have the exodromy equivalence for stratified topological spaces, which provides

$$\Pi_{(\infty,1)}^{\wedge}(X^{an}; P) \simeq \Pi_{(\infty,1)}^{\wedge}(\mathbf{Sh}_{\text{eff}}((\widetilde{X}^{an})^{P-constr}); P)$$

(Subexample 9.17).

**13.15 Proposition (Stratified Riemann Existence).** *Let  $X$  be a scheme of finite type over  $\mathbf{C}$ , and let  $X^{zar} \rightarrow P$  be a finite stratification. Then the geometric morphism  $\varepsilon_*^{constr}$  is an equivalence. Consequently,  $\varepsilon_*$  induces an equivalence*

$$\Pi_{(\infty,1)}^{\wedge}(X^{an}; P) \simeq \Pi_{(\infty,1)}^{\wedge}(\widetilde{X}^{an}; P) \simeq \Pi_{(\infty,1)}^{\acute{e}t,\wedge}(X; P)$$

of profinite  $P$ -stratified homotopy types.

*Proof.* On strata,  $\varepsilon_*^{constr}$  is an equivalence by the Riemann Existence Theorem. For any point  $p \in P$ , let us write  $X_p^L$  for the Stone  $\infty$ -topos  $\mathbf{Sh}_{\text{eff}}((\widetilde{X}_p^{an})^{lisse}) \simeq \mathbf{Sh}_{\text{eff}}(X_{p,\acute{e}t}^{lisse})$ . For

any points  $p < q$ , the geometric morphism  $\varepsilon_*^{constr}$  on the link from  $p$  to  $q$  is a geometric morphism of Stone  $\infty$ -topoi

$$X_p^L \tilde{\times}_{\mathbf{Sh}_{\text{eff}}((\overline{X^{an}})^{P\text{-constr}})} X_q^L \rightarrow X_p^L \tilde{\times}_{\mathbf{Sh}_{\text{eff}}(X_{\acute{e}t}^{P\text{-constr}})} X_q^L.$$

To see that this is an equivalence, since the oriented fibre product is invariant under localisations of the corner (Example 6.39), we may assume that  $P = \{p, q\}$ , in which case  $\mathbf{Sh}_{\text{eff}}((\overline{X^{an}})^{P\text{-constr}})$  and  $\mathbf{Sh}_{\text{eff}}(X_{\acute{e}t}^{P\text{-constr}})$  are each bounded coherent recollements of  $X_p^L$  and  $X_q^L$ . Therefore it suffices to prove that the gluing functors coincide on truncated coherent objects. That is, one needs to confirm that the natural transformation

$$\varepsilon^* i_{\ast}^{\acute{e}t, \ast} j_{\ast}^{\acute{e}t} \rightarrow i_{\ast}^{an, \ast} j_{\ast}^{an} \varepsilon^*$$

is an equivalence when restricted to  $(X_q^L)_{< \infty}^{coh} \simeq X_{q, \acute{e}t}^{lisse}$ . This now follows from the Artin Comparison Theorem (Theorem 13.12) and naturality of  $\varepsilon^*$ .  $\square$

Passing to the limit over finite stratifications, we obtain the following.

**13.16 Corollary.** *Let  $X$  be a scheme of finite type over  $\mathbf{C}$ . Then  $\varepsilon_*$  induces an equivalence*

$$\Pi_{(\infty, 1)}^{\wedge}(\overline{X^{an}}; X^{zar}) \simeq \Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X).$$

## 14 Topologically rigid schemes & reconstruction of absolute schemes

We have shown that the étale  $\infty$ -topos  $X_{\acute{e}t}$  of a coherent scheme  $X$  can be reconstructed from the profinite  $\infty$ -category  $\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X)$ . Following Grothendieck's *Brief an Faltings* [18, (8)], we can ask to what extent  $X$  itself can be recovered from  $X_{\acute{e}t}$ . We first note that there are three easily-spotted obstacles to the *conservativity* of the functor  $X \mapsto X_{\acute{e}t}$ .

- ▶ One must restrict attention to schemes over a base with suitable finiteness conditions: for example, a nontrivial extension  $\Omega \subset \Omega'$  of algebraically closed fields will give an equivalence of étale  $\infty$ -topoi (which are of course each trivial).
- ▶ The base must be sufficiently small: over  $\mathbf{C}$ , for example, any two smooth proper curves of the same genus have equivalent étale  $\infty$ -topoi.
- ▶ One must account for universal homeomorphisms: for example, the normalisation of the cuspidal cubic induces an equivalence of étale  $\infty$ -topoi. In fact, any universal homeomorphism induces an equivalence of étale  $\infty$ -topoi; this is the *invariance topologique* of the étale  $\infty$ -topos [SGA<sub>1</sub>, Exposé IX, 4.10] and [SGA<sub>4</sub> II, Exposé VIII, 1.1].

The first two points compel us to impose serious finiteness conditions on our schemes, and this last point compels us to consider the  $\infty$ -category obtained from the 1-category  $\mathbf{Sch}$  of coherent schemes by inverting universal homeomorphisms. Fortunately, it is not necessary to do something excessively abstract: there is a 1-categorical colocalisation that performs this function; this is the *topological rigidification* functor.<sup>12</sup>

<sup>12</sup>The contents of this section appeared around a decade ago in a preprint of the first-named author [5].

## Topological rigidity & seminormality

**14.1 Lemma.** *The following are equivalent for a coherent scheme  $X$ .*

- ▶ Any universal homeomorphism  $X' \rightarrow X$  in which  $X'$  is reduced is an isomorphism.
- ▶ Any universal homeomorphism  $X' \rightarrow X$  admits a section.

*Proof.* If  $X$  satisfies the first condition, then for any universal homeomorphism  $X' \rightarrow X$ , the composite  $X'_{red} \rightarrow X$  is an isomorphism. Conversely, a section of a universal homeomorphism  $X' \rightarrow X$  is a nilimmersion, hence an isomorphism if  $X'$  is reduced.  $\square$

**14.2 Definition.** A coherent scheme  $X$  is said to be *topologically rigid* – or, in the language of [45, B.1], *absolutely weakly normal* – if  $X$  satisfies the equivalent conditions of the previous Lemma 14.1. Denote by  $\text{Sch}_{trig} \subset \text{Sch}$  the full subcategory spanned by the topologically rigid schemes.

The related geometric notion is that of *seminormality*. In effect, a scheme is seminormal if and only if all non-normalities are maximally transverse:

**14.3 Definition.** A universal homeomorphism  $f : Y' \rightarrow Y$  of schemes is said to be *arithmetically trivial*<sup>13</sup> if for any point  $y' \in Y'$ , if  $y = f(y')$ , then the purely inseparable field extension  $\kappa(y) \hookrightarrow \kappa(y')$  is an isomorphism. A coherent scheme  $Y$  is said to be *seminormal* [47] if any arithmetically trivial universal homeomorphism  $Y' \rightarrow Y$  with  $Y'$  reduced is an isomorphism.

**14.4 Example.** ▶ Seminormal schemes are reduced, and reduced normal schemes are seminormal.

- ▶ A schemes in characteristic 0 is topologically rigid if and only if it is seminormal.
- ▶ Let  $k$  be a field. Then  $\text{Spec } k$  is always seminormal, but is topologically rigid if and only if  $k$  is perfect.
- ▶ Let  $k$  be a field. Then a nodal curve (e.g.,  $y^2 = x^2$  or  $y^2 = x^3 + x^2$ ) over  $k$  is seminormal.
- ▶ Let  $k$  be a field. Then for any nonnegative integer  $n$ , the union of the  $n$  coordinate axes in  $A_k^n$  is a seminormal scheme.
- ▶ Let  $X$  be a reduced normal scheme. If, for any generic point  $\eta$  of  $X$ , the field  $\kappa(\eta)$  is perfect, then  $X$  is topologically rigid.
- ▶ Let  $X$  be a reduced scheme whose set of irreducible components is locally finite. Let  $R_X$  denote the quasicoherent  $O_X$ -algebra of rational functions on  $X$  [EGA<sub>1</sub><sup>\*</sup>, 8.3.3]. Then the following are equivalent (cf. [33, 1.4 and 1.7]).

- (a) The scheme  $X$  is seminormal.

<sup>13</sup>Traverso [49; 17] uses the word ‘quasi-isomorphism’ for such universal homeomorphisms.

- (b) For every point  $x \in X$ , the local scheme  $\text{Spec } O_{X,x}$  is seminormal.
- (c) For every point  $x \in X$ , and for any integral element  $f \in R_{X,x}$  over  $O_{X,x}$ , the conductor of  $O_{X,x}$  in  $O_{X,x}[f]$  is a radical ideal of  $O_{X,x}[f]$ .
- (d) For every point  $x \in X$ , if  $f \in R_{X,x}$  is an integral element with the property that both  $f^u, f^v \in O_{X,x}$  for  $u, v \in \mathbf{N}$  coprime, then  $f \in O_{X,x}$ .
- (e) For every point  $x \in X$ , if  $f \in R_{X,x}$  is an integral element with the property that both  $f^2, f^3 \in O_{X,x}$ , then  $f \in O_{X,x}$ .

- Let  $X$  be a seminormal scheme whose set of irreducible components is locally finite; then  $X$  is topologically rigid if and only if, for every point  $x \in X$  and every integral element  $f \in R_{X,x}^{perf}$  over  $O_{X,x}$  with the property that there is a prime number  $p$  such that  $f^p \in O_{X,x}$  and  $pf \in O_{X,x}$ , one has  $f \in O_{X,x}$  [53, Theorem 1].

**14.5 Counterexample.** In effect, a scheme can fail to be topologically rigid for two reasons. First, there is the failure of seminormality.

- Let  $k$  be a field. Cuspidal curves (e.g.,  $y^2 = x^3$ ) are not seminormal.
- Let  $k$  be a field. Then for any nonnegative integer  $n$ , the scheme comprised of  $m$  lines intersecting in one point in  $A_k^n$  is not seminormal if  $m > n$ .

Then there is the failure of each residue field to be perfect.

- Let  $k$  be a field of characteristic  $p > 0$  and  $n$  a positive integer. Then consider the subring

$$A := k[x^{p^n}, x^j y : 0 \leq j < p^n] \subset k[x, y].$$

Since the induced morphism  $A_k^2 \rightarrow \text{Spec } A$  is a nontrivial universal homeomorphism, it follows that  $\text{Spec } A$  is not topologically rigid.

- An  $F_p$ -scheme is topologically rigid only if it is perfect. In particular, if  $X$  is of finite type over a perfect field  $k$ , then  $X$  is topologically rigid only if  $X$  is étale over  $\text{Spec } k$ .
- If  $k$  is a perfect field of characteristic  $p$ , then no smooth  $k$ -scheme of positive dimension is topologically rigid, but any of them is seminormal.

## Topological rigidification

Our aim is now to show that  $\mathbf{Sch}_{trig}$  is the desired colocalisation: we will show that the inclusion  $\mathbf{Sch}_{trig} \hookrightarrow \mathbf{Sch}$  admits a right adjoint  $X \mapsto X_{trig}$  in which the counit  $X_{trig} \rightarrow X$  is a universal homeomorphism. We first need to check that inverse limits of universal homeomorphisms are universal homeomorphisms.

**14.6 Lemma.** *Let  $X$  be a scheme. Let  $\Lambda$  be an inverse category, and  $W : \Lambda \rightarrow \mathbf{Sch}/_X$  a diagram of  $X$ -schemes such that for any object  $\alpha \in \Lambda$ , the structure morphism  $p_\alpha : W_\alpha \rightarrow X$  is a universal homeomorphism. Then the natural morphism*

$$p : W' := \lim_{\alpha \in \Lambda^{op}} W_\alpha \rightarrow X$$

is a universal homeomorphism.

*Proof.* All the bonding morphisms  $W_\alpha \rightarrow W_\beta$  are universal homeomorphisms. It follows from [EGA<sub>IV</sub> III, 8.3.8(i)] that  $p$  is surjective. For any field  $k$ , the diagram  $W'(k): \Lambda^{op} \rightarrow \mathbf{Set}$  is a diagram of injections, whence for any  $\alpha \in \Lambda^{op}$ , the map  $W'(k) \rightarrow W_\alpha(k)$  is an injection; thus  $p$  is a universal injection. It thus remains to show that  $p$  is integral. Since  $W'$  is a diagram of affine  $X$ -schemes, it is enough to observe that the filtered colimit  $\operatorname{colim}_{\alpha \in \Lambda} P_{\alpha,*} \mathcal{O}_{W_\alpha}$  is an integral  $\mathcal{O}_X$ -algebra.  $\square$

**14.7 Proposition.** *The inclusion  $\mathbf{Sch}_{\text{trig}} \hookrightarrow \mathbf{Sch}$  admits a right adjoint, and the counit  $X_{\text{trig}} \rightarrow X$  is a universal homeomorphism.*

*Proof.* For any coherent scheme  $X$ , let  $U_X \subset \mathbf{Sch}/X$  be the full subcategory spanned by the universal homeomorphisms  $p: Y \rightarrow X$ . Limit-cofinal therein is the full subcategory  $U_X^f$  spanned by the finite universal homeomorphisms. Hence the limit of this diagram of  $X$ -schemes exists and is a universal homeomorphism  $\varepsilon: X_{\text{trig}} \rightarrow X$ . Any universal homeomorphism  $Y \rightarrow X_{\text{trig}}$  admits a section, whence  $X_{\text{trig}}$  is topologically rigid. Moreover, if  $Z$  is topologically rigid, then for any morphism  $f: Z \rightarrow X$ , the pull-back  $Z \cong Z \times_X X_{\text{trig}} \rightarrow X_{\text{trig}}$  provides an inverse to the natural map  $\operatorname{Mor}(Z, X_{\text{trig}}) \rightarrow \operatorname{Mor}(Z, X)$ , whence  $\varepsilon$  is a colocalisation of  $\mathbf{Sch}$  relative to  $\mathbf{Sch}_{\text{trig}}$ .  $\square$

**14.8 Corollary.** *The  $\infty$ -category obtained from the 1-category  $\mathbf{Sch}$  by inverting universal homeomorphisms is equivalent to  $\mathbf{Sch}_{\text{trig}}$ .*

**14.9 Definition.** We call the right adjoint  $X \mapsto X_{\text{trig}}$  the *topological rigidification functor* or the *absolute weak normalisation*.

**14.10.** David Rydh [45, Appendix B] presented an alternative description of this functor: if  $X$  is a reduced coherent scheme whose set of irreducible components is finite, or, respectively, an affine scheme, then one may form ‘the’ *absolute integral closure*  $\overline{X}$  of  $X$  [2] or, respectively, ‘the’ *total integral closure*  $\overline{X}$  of  $X$  [14; 21]. In either case, one can show that  $X_{\text{trig}}$  is isomorphic to the weak normalisation of  $X$  (in the sense of Andreotti–Bombieri [1, Teorema 2]) under  $\overline{X} \rightarrow X$ .

**14.11 Example.** Let  $p$  be a prime number. If  $X$  is a reduced scheme of characteristic  $p$ , the topological rigidification and the perfection

$$X_{\text{perf}} := \lim \left( \dots \xrightarrow{\phi_X} X \xrightarrow{\phi_X} X \right)$$

(where  $\phi_X$  is the absolute Frobenius [SGA<sub>5</sub>, XIV=XV §1]) coincide [7, Lemma 3.8]. Said differently, reduced schemes in characteristic  $p$  are topologically rigid if and only if they are perfect.

## Grothendieck’s conjecture

**14.12 Definition.** By an *absolute scheme*, we shall mean a topologically rigid scheme  $X$  such that there exists a universal homeomorphism  $X \rightarrow Y$  to a coherent scheme  $Y$  essentially of finite type over  $\operatorname{Spec} \mathbf{Z}$ . Denote by  $\mathbf{Sch}_{\text{abs}} \subset \mathbf{Sch}_{\text{trig}}$  the full subcategory spanned by the absolute schemes.

Here is the ‘tantalising conjecture’ of Grothendieck in his letter to Faltings [18, p. 7]:

**14.13 Conjecture.** *The functor*

$$\mathbf{Sch}_{abs} \rightarrow \mathbf{Top}_{\infty, /(\mathrm{Spec} Z)_{\acute{e}t}}$$

*given by the assignment  $X \mapsto X_{\acute{e}t}$  is fully faithful. That is, if  $X$  and  $Y$  are absolute schemes, then the natural map*

$$\mathrm{Mor}(X, Y) \rightarrow \mathrm{Map}_{\mathbf{Top}_{\infty, /(\mathrm{Spec} Z)_{\acute{e}t}}}(X_{\acute{e}t}, Y_{\acute{e}t})$$

*is an equivalence.*

From this conjecture we may deduce a stratified anabelian result:

**14.14 Corollary.** *Assume [Conjecture 14.13](#); then the functor*

$$\mathbf{Sch}_{abs} \rightarrow (\mathbf{Str}_{\pi}^{\wedge})_{/\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(\mathrm{Spec} Z)}$$

*given by the assignment  $X \mapsto \Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X)$  is fully faithful. That is, if  $X$  and  $Y$  are absolute schemes, then the natural map*

$$\mathrm{Mor}(X, Y) \rightarrow \mathrm{Map}_{/\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(\mathrm{Spec} Z)}(\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X), \Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(Y))$$

*is an equivalence.*

An early paper of Voevodsky [52] provides a proof of [Conjecture 14.13](#) for normal absolute schemes in characteristic 0.

**14.15 Theorem** ([52, Theorem 3.1]). *Let  $k$  be a finitely generated field of characteristic 0, and write  $\mathbf{Sch}_{norm, k}$  for the category of normal, reduced  $k$ -schemes of finite type over  $k$ . Then the functor*

$$\mathbf{Sch}_{norm, k} \rightarrow \mathbf{Top}_{\infty, /(\mathrm{Spec} k)_{\acute{e}t}}$$

*given by the assignment  $X \mapsto X_{\acute{e}t}$  is fully faithful.*

Voevodsky also claims that his proof – with some modifications – will work when  $k$  is a finitely generated field of characteristic  $p$  and of transcendence degree  $\geq 1$ .

In any case, Voevodsky’s result implies for us a strong reconstruction theorem for these schemes:

**14.16 Theorem.** *If  $k$  is a finitely generated field of characteristic 0, then for any reduced normal  $k$ -schemes  $X$  and  $Y$  of finite type, morphisms  $X \rightarrow Y$  can thus be identified with functors of profinite  $\infty$ -categories with  $G_k$  actions. That is, the natural map*

$$\mathrm{Mor}_k(X, Y) \rightarrow \mathrm{Mor}_{BG_k}(\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X), \Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(Y))$$

*is an equivalence.*

Thus if  $X$  and  $Y$  are normal  $k$ -schemes of finite type, and if  $\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X)$  and  $\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(Y)$  are equivalent as profinite  $\infty$ -categories with  $G_k$  actions, then  $X$  and  $Y$  are isomorphic.

The  $\infty$ -category  $\Pi_{(\infty, 1)}^{\acute{e}t, \wedge}(X)$  is 1-truncated and therefore can be identified with the topological category  $\mathrm{Gal}(X)$  from the introduction ([Construction 13.5](#)). Thus the category of reduced normal  $k$ -schemes of finite type can be embedded in the category of profinite categories with an action of  $G_k$ , as asserted in [Theorem A](#).



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