

ON LEFT AND RIGHT MODEL CATEGORIES AND LEFT AND RIGHT BOUSFIELD LOCALIZATIONS

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Abstract

We verify the existence of left Bousfield localizations and of enriched left Bousfield localizations, and we prove a collection of useful technical results characterizing certain fibrations of (enriched) left Bousfield localizations. We also use such Bousfield localizations to construct a number of new model categories, including models for the homotopy limit of right Quillen presheaves, for Postnikov towers in model categories, and for presheaves valued in a symmetric monoidal model category satisfying a homotopy-coherent descent condition. We then verify the existence of right Bousfield localizations of right model categories, and we apply this to construct a model of the homotopy limit of a left Quillen presheaf as a right model category.

Introduction

A class of maps H in a model category \mathbf{M} specifies a class of H -local objects, which are those objects X with the property that the morphism $\mathbf{R}\mathrm{Mor}_{\mathbf{M}}(f, X)$ is a weak equivalence of simplicial sets for any $f \in H$. The left Bousfield localization of \mathbf{M} with respect to H is a model for the homotopy theory of H -local objects. Similarly, if \mathbf{M} is enriched over a symmetric monoidal model category \mathbf{V} , the class H specifies a class of (H/\mathbf{V}) -local objects, which are those objects X with the property that the morphism $\mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{M}}^{\mathbf{V}}(f, X)$ of derived mapping objects is a weak equivalence of \mathbf{V} for any $f \in H$. The \mathbf{V} -enriched left Bousfield localization of \mathbf{M} is a model for the homotopy theory of (H/\mathbf{V}) -local objects.

The (enriched) left Bousfield localization is described as a new “ H -local” model category structure on the underlying category of \mathbf{M} . The H -local cofibrations of the (enriched) left Bousfield localization are precisely those of \mathbf{M} , the H -local fibrant objects are the (enriched) H -local objects that are fibrant in \mathbf{M} , and the H -local weak equivalences are those morphisms f of \mathbf{M} such that $\mathbf{R}\mathrm{Mor}_{\mathbf{M}}(f, X)$ (resp., $\mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{M}}^{\mathbf{V}}(f, X)$) is a weak equivalence. This is enough to specify H -local fibrations,

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but it can be difficult to get explicit control over them. Luckily, it is frequently possible to characterize some of the H -local fibrations as fibrations that are in addition homotopy pullbacks of fibrations between H -local fibrant objects (4.30).

The (enriched) Bousfield localization gives an effective way of constructing new model categories from old. In particular, we use it to construct models for the homotopy limit of a right Quillen presheaf (4.38) and for presheaves valued in a symmetric monoidal model category satisfying a homotopy-coherent descent condition (4.56).

The right Bousfield localization — or colocalization — of a model category \mathbf{M} with respect to a set K of *objects* is a model for the homotopy theory generated by K — i.e., of objects that can be written as a homotopy colimit of objects of K . Unfortunately, the right Bousfield localization need not exist as a model category unless \mathbf{M} is right proper. This is a rather severe limitation, as many operations on model categories — such as left Bousfield localization — tend to destroy right properness, and as many interesting model categories are not right proper. Fortunately, right Bousfield localizations of these model categories do exist as right model categories (5.13).

The right Bousfield localization gives another method of constructing new model categories. In particular, we use it to construct models for the homotopy limit of a left Quillen presheaf 5.25. We use both left and right Bousfield localizations to construct for Postnikov towers in model categories (5.31 and 5.49)

Plan

In the first section, we give a brief review of the general theory of left and right model categories. This section includes a discussion of properness in model categories, and the notions of combinatorial and tractable model categories. The section ends with a discussion of symmetric monoidal structures.

The next section contains J. Smith’s existence theorem for combinatorial model categories. This material is mostly well-known. There one may find two familiar but important examples: model structures on diagram categories and model structures on section categories.

We then turn to the Reedy model structures. After a very brief reprise of well-known facts about the Reedy model structure, we give a very useful little criterion to determine whether composition with a morphism of Reedy categories determines a left or right Quillen functor. We then give three easy inheritance results, and the section concludes with a somewhat more difficult inheritance result, providing conditions under which the Reedy model structure on diagrams valued in a symmetric monoidal model category is itself symmetric monoidal.

In the fourth section, we define the left Bousfield localization and give the well-known existence theorem due to Smith. Following this, we continue with a small collection of results that permit one to cope with the fact that left Bousfield localization ruins right properness, as well as a characterization of a certain class of H -local fibrations. We give three simple applications of the technique of left Bousfield localization: Dugger’s presentation theorem, the existence of homotopy images, and the construction of homotopy limits of diagrams of model categories. We then describe the enriched left Bousfield localization and prove an existence theorem, and we give an application of the enriched localization: the existence of local model structures on presheaves valued in symmetric monoidal model categories.

In the final section, we show that the right Bousfield localization of a model category \mathbf{M} naturally exists instead as a right model category. This result holds with no properness assumptions on \mathbf{M} . As an application of the right Bousfield localization, we produce a good model for the homotopy limit of left Quillen presheaves. Finally, we discuss Postnikov towers in various contexts using both left and right Bousfield localizations.

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Notations

All universes \mathbf{X} will by assumption contain natural numbers objects $\mathbf{N} \in \mathbf{X}$. For any universe \mathbf{X} , denote by $\mathbf{Set}_{\mathbf{X}}$ the \mathbf{X} -category of \mathbf{X} -small sets, and denote by $\mathbf{Cat}_{\mathbf{X}}$ the \mathbf{X} -category of \mathbf{X} -small categories.

Denote by Δ the \mathbf{X} -category of \mathbf{X} -small totally ordered finite sets, viewed as a full subcategory of $\mathbf{Cat}_{\mathbf{X}}$, for some universe \mathbf{X} . The category Δ is essentially \mathbf{X} -small and is essentially independent of the universe \mathbf{X} ; in fact, the full subcategory comprised of the objects

$$\mathbf{p} := [0 \longrightarrow 1 \longrightarrow \dots \longrightarrow p]$$

is a skeletal subcategory.

Suppose \mathbf{X} a universe. For any \mathbf{X} -category E and any \mathbf{X} -small category A , let E^A denote the category of functors $A \rightarrow E$, and let $E(A) := E^{A^{\text{op}}}$ denote the category of presheaves $A^{\text{op}} \rightarrow E$. Write cE for E^{Δ} , and write sE for $E(\Delta)$.

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1. A taxonomy of homotopy theory

It is necessary to establish some general terminology for categories with weak equivalences and various bits of extra structure. This terminology includes such arcane

and baroque concepts as structured homotopical categories and model categories. Most readers can and should skip this section upon a first reading, returning as needed.

1.0. Suppose \mathbf{X} a universe.

Structured homotopical categories

Here we define the general notion of structured homotopical categories. Structured homotopical categories contain lluf subcategories of cofibrations, fibrations, and weak equivalences, satisfying the “easy” conditions on model categories.

Definition 1.1. Suppose $(\mathbf{E}, w\mathbf{E})$ a homotopical \mathbf{X} -category [8, 33.1] equipped with two lluf subcategories $\text{cof } \mathbf{E}$ and $\text{fib } \mathbf{E}$.

(1.1.1) Morphisms of $\text{cof } \mathbf{E}$ (respectively, of $\text{fib } \mathbf{E}$) are called *cofibrations* (resp., *fibrations*).

(1.1.2) Morphisms of $w \text{cof } \mathbf{E} := w\mathbf{E} \cap \text{cof } \mathbf{E}$ (respectively, of $w \text{fib } \mathbf{E} := w\mathbf{E} \cap \text{fib } \mathbf{E}$) are called *trivial cofibrations* (resp., *trivial fibrations*).

(1.1.3) Objects X of \mathbf{E} such that the morphism $\emptyset \rightarrow X$ (respectively, the morphism $X \rightarrow \star$) is an element of $\text{cof } \mathbf{E}$ (resp., of $\text{fib } \mathbf{E}$) are called *cofibrant* (resp. *fibrant*); the full subcategory comprised of all such objects will be denoted \mathbf{E}_c (resp., \mathbf{E}_f).

(1.1.4) In the context of a functor $\mathbf{C} \rightarrow \mathbf{E}$, a morphism (respectively, an object) of \mathbf{C} will be called a *\mathbf{E} -weak equivalence*, a *\mathbf{E} -cofibration*, or a *\mathbf{E} -fibration* (resp., *\mathbf{E} -cofibrant* or *\mathbf{E} -fibrant*) if its image under $\mathbf{C} \rightarrow \mathbf{E}$ is a weak equivalence, a cofibration, or a fibration (resp., cofibrant or fibrant) in \mathbf{E} , respectively. The full subcategory of \mathbf{C} comprised of all \mathbf{E} -cofibrant (respectively, \mathbf{E} -fibrant) objects will be denoted $\mathbf{C}_{\mathbf{E},c}$ (resp., $\mathbf{C}_{\mathbf{E},f}$).

(1.1.5) One says that $(\mathbf{E}, w\mathbf{E}, \text{cof } \mathbf{E}, \text{fib } \mathbf{E})$ is a *structured homotopical \mathbf{X} -category* if the following axioms hold.

(1.1.5.1) The category \mathbf{E} contains all limits and colimits.

(1.1.5.2) The subcategories $\text{cof } \mathbf{E}$ and $\text{fib } \mathbf{E}$ are closed under retracts.

(1.1.5.3) The set $\text{cof } \mathbf{E}$ is closed under pushouts by arbitrary morphisms; the set $\text{fib } \mathbf{E}$ is closed under pullbacks by arbitrary morphisms.

Lemma 1.2. *The data $(\mathbf{E}, w\mathbf{E}, \text{cof } \mathbf{E}, \text{fib } \mathbf{E})$ is a structured homotopical \mathbf{X} -category if and only if the data $(\mathbf{E}^{\text{op}}, w(\mathbf{E}^{\text{op}}), \text{cof}(\mathbf{E}^{\text{op}}), \text{fib}(\mathbf{E}^{\text{op}}))$ is as well, wherein*

$$w(\mathbf{E}^{\text{op}}) := (w\mathbf{E})^{\text{op}} \quad \text{cof}(\mathbf{E}^{\text{op}}) := (\text{fib } \mathbf{E})^{\text{op}} \quad \text{fib}(\mathbf{E}^{\text{op}}) := (\text{cof } \mathbf{E})^{\text{op}}.$$

1.3. One commonly refers to \mathbf{E} alone as a structured homotopical category, omitting the explicit reference to the data of $w\mathbf{E}$, $\text{cof } \mathbf{E}$, and $\text{fib } \mathbf{E}$.

Left and right model categories

Left and right model categories are structured homotopical categories that, like model categories, include lifting and factorization axioms, but only for particular morphisms. Following the definition, we turn to a sequence of standard results from the homotopy theory of model categories, suitably altered to apply to left and right

model categories. We learned of nearly all of the following ideas and results from M. Spitzweck and his thesis [19].

Definition 1.4. Suppose \mathbf{C} and \mathbf{E} two structured homotopical \mathbf{X} -categories.

(1.4.1) An adjunction

$$F_{\mathbf{C}} : \mathbf{E} \rightleftarrows \mathbf{C} : U_{\mathbf{C}}$$

is a *left \mathbf{E} -model \mathbf{X} -category* if the following axioms hold.

- (1.4.1.1) The right adjoint $U_{\mathbf{C}}$ preserves fibrations and trivial fibrations.
- (1.4.1.2) Any cofibration of \mathbf{C} with \mathbf{E} -cofibrant domain is an \mathbf{E} -cofibration.
- (1.4.1.3) The initial object \emptyset of \mathbf{C} is \mathbf{E} -cofibrant.
- (1.4.1.4) In \mathbf{C} , any cofibration has the left lifting property with respect to any trivial fibration, and any fibration has the right lifting property with respect to any trivial cofibration with \mathbf{E} -cofibrant domain.
- (1.4.1.5) There exist functorial factorizations of any morphism of \mathbf{C} into a cofibration followed by a trivial fibration and of any morphism of \mathbf{C} with \mathbf{E} -cofibrant domain into a trivial cofibration followed by a fibration.

(1.4.2) An adjunction

$$F_{\mathbf{C}} : \mathbf{C} \rightleftarrows \mathbf{E} : U_{\mathbf{C}}$$

is a *right \mathbf{E} -model \mathbf{X} -category* if the corresponding adjunction

$$U_{\mathbf{C}}^{\text{op}} : \mathbf{E}^{\text{op}} \rightleftarrows \mathbf{C}^{\text{op}} : F_{\mathbf{C}}^{\text{op}}$$

is a left \mathbf{E} -model \mathbf{X} -category.

(1.4.3) One says that \mathbf{C} is a(n) *(absolute) left model \mathbf{X} -category* if the identity adjunction

$$\mathbf{C} \rightleftarrows \mathbf{C}$$

is a left \mathbf{C} -model category.

(1.4.4) One says that \mathbf{C} is a(n) *(absolute) right model \mathbf{X} -category* if \mathbf{C}^{op} is a left model \mathbf{X} -category.

(1.4.5) One says that \mathbf{C} is a *model \mathbf{X} -category* if it is both a left and right model \mathbf{X} -category.

1.5. In unambiguous contexts, one refers to \mathbf{C} alone as the \mathbf{E} -left model or \mathbf{E} -right model \mathbf{X} -category, omitting explicit mention of the adjunction.

Lemma 1.6. *The following are equivalent for a structured homotopical category \mathbf{C} .*

- (1.6.1) \mathbf{C} is a left \star -model category.
- (1.6.2) \mathbf{C} is a right \star -model category.
- (1.6.3) \mathbf{C} is a model category.

Proof. To be a left \star -model category is exactly to have the lifting and factorization axioms with no conditions on the source of the morphism, hence to be a right model category as well. The dual assertion follows as usual. \square

Lemma 1.7. *Suppose \mathbf{E} a structured homotopical category, \mathbf{C} a left (respectively, right) \mathbf{E} -model \mathbf{X} -category.*

- (1.7.1) *A morphism $i : K \rightarrow L$ (resp., a morphism i with \mathbf{E} -fibrant codomain L) has the left lifting property with respect to every trivial fibration (resp., every trivial fibration with \mathbf{E} -fibrant codomain) if and only if i is a cofibration.*
- (1.7.2) *Any morphism $i : K \rightarrow L$ with \mathbf{E} -cofibrant domain K (resp., any morphism i) has the left lifting property with respect to every fibration if and only if i is a trivial cofibration.*
- (1.7.3) *Any morphism $p : Y \rightarrow X$ with \mathbf{E} -cofibrant domain Y (resp., any morphism p) has the right lifting property with respect to every trivial cofibration with \mathbf{E} -cofibrant domain (resp., every trivial cofibration) if and only if p is a fibration.*
- (1.7.4) *A morphism $p : Y \rightarrow X$ (resp., a morphism p with \mathbf{E} -fibrant codomain) has the right lifting property with respect to every cofibration if and only if p is a trivial fibration.*

Proof. This follows immediately from the appropriate factorization axioms along with the retract argument. \square

Corollary 1.8. (1.8.1) *If \mathbf{C} is a left model \mathbf{X} -category, a morphism $p : Y \rightarrow X$ satisfies the right lifting property with respect to the trivial cofibrations with cofibrant domains if and only if there exists a trivial fibration $Y' \rightarrow Y$ such that the composite morphism $Y' \rightarrow X$ is a fibration.*

- (1.8.2) *Dually, if \mathbf{C} is a right model \mathbf{X} -category, a morphism $i : K \rightarrow L$ satisfies the left lifting property with respect to the trivial fibrations with fibrant codomains if and only if there exists a trivial cofibration $L \rightarrow L'$ such that the composite morphism $K \rightarrow L'$ is a cofibration.*

Proof. The assertions are dual, so it is enough to prove the first. Morphisms satisfying a right lifting property are of course closed under composition. Conversely, suppose $Y \rightarrow X$ a morphism, $Y' \rightarrow Y$ a trivial fibration such that the composition $Y' \rightarrow X$ satisfies the left lifting property with respect to a trivial cofibration $K \rightarrow L$ is a trivial cofibration with cofibrant domain K . Then for any diagram

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X, \end{array}$$

there is a lift to a diagram

$$\begin{array}{ccc} & & Y' \\ & \nearrow & \downarrow \\ K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X. \end{array}$$

By assumption there is a lift of the exterior quadrilateral, and this provides a lift of the interior square as well. \square

Proposition 1.9 ([19, Proposition 2.4]). *Suppose \mathbf{E} a structured homotopical \mathbf{X} -category, and suppose \mathbf{C} a left \mathbf{E} -model \mathbf{X} -category. Suppose $f, g : B \rightarrow X$ two maps in \mathbf{C} .*

- (1.9.1) *Suppose $f \stackrel{l}{\sim} g$; then $h \circ f \stackrel{l}{\sim} h \circ g$ for any morphism $h : X \rightarrow Y$ of \mathbf{C} .*
- (1.9.2) *Dually, suppose $f \stackrel{r}{\sim} g$; then $f \circ k \stackrel{r}{\sim} g \circ k$ for any morphism $k : A \rightarrow B$ of \mathbf{C} .*
- (1.9.3) *Suppose B cofibrant, and suppose $h : X \rightarrow Y$ any morphism of $\mathbf{C}_{\mathbf{E},c}$; then $f \stackrel{r}{\sim} g$ only if $h \circ f \stackrel{r}{\sim} h \circ g$.*
- (1.9.4) *Dually, suppose X fibrant, and suppose $k : A \rightarrow B$ any morphism of $\mathbf{C}_{\mathbf{E},c}$; then $f \stackrel{l}{\sim} g$ only if $f \circ h \stackrel{l}{\sim} g \circ h$.*
- (1.9.5) *If B is cofibrant, then left homotopy is an equivalence relation on $\text{Mor}(B, X)$.*
- (1.9.6) *If B is cofibrant and X is \mathbf{E} -cofibrant, then $f \stackrel{l}{\sim} g$ only if $f \stackrel{r}{\sim} g$.*
- (1.9.7) *Dually, if X is fibrant and B is cofibrant \mathbf{E} -cofibrant, then $f \stackrel{r}{\sim} g$ only if $f \stackrel{l}{\sim} g$.*
- (1.9.8) *If B is cofibrant, X is cofibrant \mathbf{E} -cofibrant, and $h : X \rightarrow Y$ is either a trivial fibration or a weak equivalence between fibrant objects, then h induces a bijection*

$$(\text{Mor}(B, X) / \stackrel{l}{\sim}) \rightarrow (\text{Mor}(B, Y) / \stackrel{l}{\sim}).$$

- (1.9.9) *Dually, if A is \mathbf{E} -cofibrant, X is fibrant and \mathbf{E} -cofibrant, and $k : A \rightarrow B$ is either a trivial cofibration with A \mathbf{D} -cofibrant or a weak equivalence between cofibrant objects, then k induces a bijection*

$$(\text{Mor}(B, X) / \stackrel{r}{\sim}) \rightarrow (\text{Mor}(A, X) / \stackrel{r}{\sim}).$$

The obvious dual statements for right model categories and \mathbf{E} -right model categories also hold.

Corollary 1.10. *Suppose \mathbf{E} a left (respectively, right) model \mathbf{X} -category, \mathbf{C} a left \mathbf{E} -model (resp., right \mathbf{E} -model) \mathbf{X} -category. Then $\text{Ho } \mathbf{C}$ is an \mathbf{X} -category.*

Definition 1.11. *Suppose \mathbf{E} a left (respectively, right) model \mathbf{X} -category, \mathbf{C} a left \mathbf{E} -model (resp., right \mathbf{E} -model) \mathbf{X} -category; suppose h a homotopy class of morphisms of \mathbf{C} . Then a morphism f of \mathbf{C} is said to be a *representative* of h if the images of f and h are isomorphic as objects of the arrow category $(\text{Ho } \mathbf{M})^1$.*

Definition 1.12. *Suppose \mathbf{E} a homotopical \mathbf{X} -category, and suppose \mathbf{C} and \mathbf{C}' two left (respectively, right) model \mathbf{E} -categories.*

- (1.12.1) *An adjunction*

$$F : \mathbf{C} \rightleftarrows \mathbf{C}' : U$$

is a Quillen adjunction if U preserves fibrations and trivial fibrations (resp., if F preserves cofibrations and trivial cofibrations).

(1.12.2) Suppose

$$F : \mathbf{C} \rightleftarrows \mathbf{C}' : U$$

a Quillen adjunction. Then the *left derived functor* $\mathbf{L}F$ of F is the right Kan extension of the composite

$$\mathbf{C} \xrightarrow{F} \mathbf{C}' \longrightarrow \mathrm{Ho} \mathbf{C}'$$

along the functor $\mathbf{C} \rightarrow \mathrm{Ho} \mathbf{C}$, and, dually, the *right derived functor* $\mathbf{R}U$ of U is the left Kan extension of the composite

$$\mathbf{C}' \xrightarrow{U} \mathbf{C} \longrightarrow \mathrm{Ho} \mathbf{C}$$

along the functor $\mathbf{C}' \rightarrow \mathrm{Ho} \mathbf{C}'$.

Proposition 1.13 ([19, p. 12]). *Suppose \mathbf{E} a structured homotopical category, and suppose \mathbf{C} and \mathbf{C}' two left or right model \mathbf{E} -categories, and suppose*

$$F : \mathbf{C} \rightleftarrows \mathbf{C}' : U$$

a Quillen adjunction. Then the derived functors

$$\mathbf{L}F : \mathrm{Ho} \mathbf{C} \rightleftarrows \mathrm{Ho} \mathbf{C}' : \mathbf{R}U$$

exist and form an adjunction.

Properness

The suitable left and right properness conditions for left or right model categories are slightly more restrictive than the usual conditions for model categories.

1.14. Suppose \mathbf{E} a structured homotopical \mathbf{X} -category, and suppose \mathbf{C} a left (respectively, right) \mathbf{E} -model \mathbf{X} -category.

Definition 1.15. One says that \mathbf{C} is *left* (resp., *right*) \mathbf{E} -*proper* if pushouts (resp., pullbacks) of elements of $w(\mathbf{C}_{\mathbf{E},c})$ (resp., of $w(\mathbf{C}_{\mathbf{E},f})$) along cofibrations (resp., fibrations) are weak equivalences. One says that \mathbf{C} is *right* (resp., *left*) *proper* if pullbacks (resp., pushouts) of weak equivalences along fibrations (resp., cofibrations) are weak equivalences.

1.16. When $\mathbf{E} \cong \star$, the condition of left (resp., right) \mathbf{E} -properness reduces to the classical notion of left (resp., right) properness of model categories.

Lemma 1.17. *In the absolute case, when $\mathbf{E} = \mathbf{C}$, and the adjunction is the identity, \mathbf{C} is automatically left (resp., right) \mathbf{C} -proper.*

Proof. This is Reedy's observation [16, Theorem B] or [9, Proposition 13.1.2]. \square

Definition 1.18. (1.18.1) A pushout diagram

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

in \mathbf{C} in which $K \rightarrow L$ is a cofibration is called an *admissible pushout diagram*,

and the morphism $L \rightarrow X$ is called the *admissible pushout* of $K \rightarrow Y$ along $K \rightarrow L$.

(1.18.2) Dually, a pullback diagram

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

in \mathbf{C} in which $Y \rightarrow X$ is a fibration is called an *admissible pullback diagram*, and the morphism $K \rightarrow Y$ is called the *admissible pullback* of $L \rightarrow X$ along $Y \rightarrow X$.

Proposition 1.19. (1.19.1) *If \mathbf{C} is left (respectively, right) \mathbf{E} -proper, admissible pushouts (resp., pullbacks) of \mathbf{E} -cofibrant (resp., \mathbf{E} -fibrant) objects are homotopy pushouts (resp., pullbacks).*

(1.19.2) *If \mathbf{C} is right (respectively, left) proper, admissible pullbacks (resp., pushouts) are homotopy pullbacks (resp., pushouts).*

Proof. We discuss the case of left \mathbf{E} -model categories; the case of right \mathbf{E} -model categories is of course dual.

Suppose

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

a pushout of \mathbf{E} -cofibrant objects, in which $K \rightarrow L$ is a cofibration. The claim is that X is isomorphic to the homotopy pushout $L \sqcup^{h,K} Y$ in $\text{Ho } \mathbf{C}$. To demonstrate this, choose a functorial factorization of every morphism into a cofibration followed by a trivial fibration. Using this, replace K cofibrantly by an object K' , factor the composite $K' \rightarrow Y$ as a cofibration $K' \rightarrow Y'$ followed by a trivial fibration $Y' \rightarrow Y$, and factor the composite $K' \rightarrow L$ as a cofibration $K' \rightarrow L'$ followed by a trivial fibration $L' \rightarrow L$. By the standard argument, the pushout $X' := L' \sqcup^{K'} Y'$ is isomorphic to the desired homotopy pushout in $\text{Ho } \mathbf{C}$.

Now form also the pushout of the left face, $L'' := K \sqcup^{K'} L'$ as well as the pushout $X'' := L'' \sqcup^{K'} Y'$. Thus we have the diagram

$$\begin{array}{ccccc} & & K & \longrightarrow & Y \\ & & \downarrow & & \downarrow \\ & & K' & \longrightarrow & Y' \\ & & \downarrow & & \downarrow \\ & & L' & \longrightarrow & X' \\ & & \downarrow & & \downarrow \\ & & L'' & \longrightarrow & X'' \\ & & \downarrow & & \downarrow \\ L & \longrightarrow & & & X \end{array}$$

By left \mathbf{E} -properness, the morphism $X' \rightarrow X''$ is a weak equivalence, since it is the

admissible pushout of $Y' \rightarrow Y$ along $Y' \rightarrow X'$. It now suffices to show that the morphism $X'' \rightarrow X$ is a weak equivalence.

Now factor $K \rightarrow Y$ as a cofibration $K \rightarrow Z$ followed by a trivial fibration $Z \rightarrow Y$, and form the associated pushouts $W'' := L'' \sqcup^K Z$ and $W := L \sqcup^K Z$:

$$\begin{array}{ccccc} K & \longrightarrow & Z & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ L'' & \longrightarrow & W'' & \longrightarrow & X'' \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & W & \longrightarrow & X. \end{array}$$

Now left \mathbf{E} -properness implies that $W'' \rightarrow X''$, $W \rightarrow X$, and $W'' \rightarrow W$ are weak equivalences; hence it follows that $X'' \rightarrow X$ is a weak equivalence, as desired.

The second part is almost dual to the first, save only the observation that in order to perform the needed fibrant replacements, one must first make an \mathbf{E} -cofibrant replacement. Indeed, suppose

$$\begin{array}{ccc} K & \longrightarrow & Y \\ \downarrow & & \downarrow \\ L & \longrightarrow & X \end{array}$$

a pullback square, in which $Y \rightarrow X$ is a fibration. The aim is to show that this is in fact a homotopy pullback square.

Without loss of generality, one may assume that L is \mathbf{E} -cofibrant, for if not, it can be replaced \mathbf{E} -cofibrantly, and right properness guarantees that the resulting pullback along $Y \rightarrow X$ is weakly equivalent to K .

Let $X' \rightarrow X$ be a trivial fibration with X' \mathbf{E} -cofibrant, form the pullbacks $Y' := X' \times_X Y$ and $L' := X' \times_X L$, and let $L' \rightarrow L''$ be a trivial fibration with L' \mathbf{E} -cofibrant. Now form the pullback $K' := L' \times_{X'} Y'$:

$$\begin{array}{ccccc} K & \xrightarrow{\quad} & Y & & \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ & K' & \longrightarrow & Y' & \\ \downarrow & \downarrow & & \downarrow & \\ & L' & \longrightarrow & X' & \\ \downarrow & \swarrow & \downarrow & \searrow & \downarrow \\ L & \xrightarrow{\quad} & X & & \end{array}$$

One can again assume that L' is \mathbf{E} -cofibrant, and now the dual of the previous argument applies to show that K' is the desired homotopy pullback. But $K' \rightarrow K$ is the pullback of the trivial fibration $L' \rightarrow L$, hence a trivial fibration itself. \square

Combinatorial and tractable model categories

Combinatorial model categories are those whose homotopy theory is controlled by the homotopy theory of a small subcategory of presentable objects. A variety of algebraic applications require that the sets of (trivial) cofibrations can be generated

(as a saturated set) by a given small set of (trivial) cofibrations with cofibrant domain. This leads to the notion of tractable model categories. Many of the results here have satisfactory proofs in print; the first section of [1] in particular is a very nice reference.¹

Notation 1.20. For any \mathbf{X} -small regular cardinal λ and any λ -accessible \mathbf{X} -category C , denote by C_λ the full subcategory of C spanned by the λ -presentable objects, i.e., those objects that corepresent a functor that commutes with all λ -filtered colimits.

Definition 1.21. Suppose \mathbf{E} a homotopical \mathbf{X} -category, and suppose \mathbf{C} a left (respectively, right) \mathbf{E} -model \mathbf{X} -category. Suppose, in addition, that λ is a regular \mathbf{X} -small cardinal.

(1.21.1) One says that \mathbf{C} is λ -tractable if the underlying \mathbf{X} -category of \mathbf{C} is locally λ -presentable, and if there exist \mathbf{X} -small sets I and J of morphisms of $C_\lambda \cap C_{\mathbf{E},c}$ (resp., of C_λ) such that the following hold.

(1.21.1.1) A morphism (resp., a morphism with \mathbf{E} -fibrant codomain) satisfies the right lifting property with respect to I if and only if it is a trivial fibration.

(1.21.1.2) A morphism satisfies the right lifting property with respect to J if and only if it is a fibration.

(1.21.2) Suppose $\mathbf{E} \cong \star$, so that \mathbf{C} a model category. Then one says that \mathbf{C} is λ -tractable if its underlying left model category is so, and one says that \mathbf{C} is λ -combinatorial if its underlying \mathbf{X} -category is locally λ -presentable, and if there exist \mathbf{X} -small sets I and J of morphisms of C_λ such that the following hold.

(1.21.2.1) A morphism satisfies the right lifting property with respect to I if and only if it is a trivial fibration.

(1.21.2.2) A morphism satisfies the right lifting property with respect to J if and only if it is a fibration.

One says that \mathbf{C} is \mathbf{X} -tractable or \mathbf{X} -combinatorial if \mathbf{C} is a model category just in case there exists a regular \mathbf{X} -small cardinal λ for which it is λ -tractable.

Definition 1.22. Suppose \mathbf{E} a homotopical \mathbf{X} -category. An \mathbf{X} -small full subcategory \mathbf{E}_0 of \mathbf{E} is *homotopy λ -generating* if every object of \mathbf{E} is weakly equivalent to a λ -filtered homotopy colimit of objects of \mathbf{E}_0 . The subcategory \mathbf{E}_0 is said to be *homotopy \mathbf{X} -generating* if it is λ -generating for some regular \mathbf{X} -small cardinal λ .

1.23. Observe that for a model category \mathbf{C} , the condition of λ -tractability amounts to λ -combinatoriality plus the condition that I and J can each be chosen to have cofibrant sources.

Observe that a model category whose underlying right model category is \mathbf{X} -tractable need not even be \mathbf{X} -combinatorial itself. For a related example, see 5.21. The lemma below, the transfinite small object argument 1.25, is critical to the construction of all combinatorial model categories in this work.

¹I would like to thank M. Spitzweck for suggesting this paper; this exposition has benefitted greatly from his recommendation.

Notation 1.24. Suppose C an \mathbf{X} -category, and suppose I an \mathbf{X} -small set of morphisms of C . Denote by $\text{inj } I$ the set of all morphisms with the right lifting property with respect to I , denote by $\text{cof } I$ the set of all morphisms with the left lifting property with respect to $\text{inj } I$, and denote by $\text{cell } I$ the set of all transfinite compositions of pushouts of morphisms of I .

Lemma 1.25 (Transfinite small object argument, [1, Proposition 1.3]). *Suppose λ a regular \mathbf{X} -small cardinal, C a locally λ -presentable \mathbf{X} -category, and I an \mathbf{X} -small set of morphisms of C_λ .*

- (1.25.1) *There is an accessible functorial factorization of every morphism f as $p \circ i$, wherein $p \in \text{inj } I$, and $i \in \text{cell } I$.*
- (1.25.2) *A morphism $q \in \text{inj } I$ if and only if it has the right lifting property with respect to all retracts of morphisms of $\text{cell } I$.*
- (1.25.3) *A morphism j is a retract of morphisms of $\text{cell } I$ if and only $j \in \text{cof } I$.*

Proof. Suppose κ a regular cardinal strictly greater than λ . For any morphism $f : X \rightarrow Y$, consider the set (I/f) of squares

$$\begin{array}{ccc} K & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ L & \longrightarrow & Y, \end{array}$$

where $i \in I$, and let $K_{(I/f)} \rightarrow L_{(I/f)}$ be the coproduct $\coprod_{i \in (I/f)} i$. Define a section P of $d_1 : C^2 \rightarrow C^1$ by

$$Pf := [X \rightarrow X \sqcup^{L_{(I/f)}} K_{(I/f)} \rightarrow Y]$$

for any morphism $f : X \rightarrow Y$. For any regular cardinal α , set $P^\alpha := \text{colim}_{\beta < \alpha} P^\beta$. This provides a functorial factorization P^κ with the required properties.

The remaining parts follow from the existence of this factorization and the retract argument. \square

Symmetric monoidal model categories and enrichments

We present a thorough, if somewhat terse, review of the basic theory of symmetric monoidal and enriched model categories.

1.26. Again suppose \mathbf{X} a universe.

Definition 1.27. (1.27.1) Suppose \mathbf{D} , \mathbf{E} , and \mathbf{F} model \mathbf{X} -categories. Suppose

$$\otimes : \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{F} \quad \underline{\text{Mor}} : \mathbf{E}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{D} \quad \text{mor} : \mathbf{D}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{E}$$

form an adjunction of two variables. Then $(\otimes, \underline{\text{Mor}}, \text{mor})$ is a *Quillen adjunction of two variables* if the following axiom holds.

- (1.27.1.1) (Pushout-product axiom) For any pair of cofibrations $c : Q \rightarrow R$ of \mathbf{D} and $d : S \rightarrow T$ of \mathbf{E} , the *pushout-product*

$$c \square d : (Q \otimes T) \sqcup^{(Q \otimes S)} (R \otimes S) \rightarrow R \otimes T$$

is a cofibration of \mathbf{F} that is trivial if either f or g is.²

- (1.27.2) A *symmetric monoidal model \mathbf{X} -category* is a symmetric monoidal closed \mathbf{X} -category $(\mathbf{V}, \otimes_{\mathbf{V}}, \mathbf{1}_{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{V}})$ [2, Definitions 6.1.1–3], equipped with a model structure such that the following axioms hold.

- (1.27.2.1) The tuple $(\otimes_{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{V}})$ is a Quillen adjunction of two variables.

- (1.27.2.2) (Unit axiom) For any object A , the canonical morphism

$$Q_{\mathbf{V}} \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \rightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{V}} A \rightarrow A$$

is a weak equivalence for some cofibrant replacement $Q\mathbf{1}_{\mathbf{V}} \rightarrow \mathbf{1}_{\mathbf{V}}$.

- (1.27.3) An *internal closed model category* is a symmetric monoidal model category \mathbf{V} in which $\otimes_{\mathbf{V}}$ is the categorical (cartesian) product \times .

- (1.27.4) If \mathbf{V} is a symmetric monoidal model \mathbf{X} -category, then a *model \mathbf{V} -category* is a tensored and cotensored \mathbf{V} -category $(\mathbf{C}, \underline{\text{Mor}}_{\mathbf{C}}^{\mathbf{V}}, \otimes_{\mathbf{C}}^{\mathbf{V}}, \text{mor}_{\mathbf{C}}^{\mathbf{V}})$ [2, Definitions 6.2.1, 6.5.1], equipped with a model structure on the underlying \mathbf{X} -category of \mathbf{C} such that the following axioms hold.

- (1.27.4.1) The tuple $(\otimes_{\mathbf{C}}^{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{C}}^{\mathbf{V}}, \text{mor}_{\mathbf{C}}^{\mathbf{V}})$ is a Quillen adjunction of two variables.

- (1.27.4.2) (Unit axiom) For any object X of \mathbf{C} , the canonical morphism

$$Q_{\mathbf{V}} \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{C}}^{\mathbf{V}} X \rightarrow \mathbf{1}_{\mathbf{V}} \otimes_{\mathbf{C}}^{\mathbf{V}} X \rightarrow X$$

is a weak equivalence for some cofibrant replacement $Q\mathbf{1}_{\mathbf{V}} \rightarrow \mathbf{1}_{\mathbf{V}}$.

- (1.27.5) A *simplicial model \mathbf{X} -category* $(\mathbf{M}, \text{Map}_{\mathbf{M}}, \otimes_{\mathbf{M}}, \text{mor}_{\mathbf{M}})$ is a model $s\text{Set}_{\mathbf{X}}$ -category.

- (1.27.6) Suppose \mathbf{V} a symmetric monoidal model category and \mathbf{C} and \mathbf{C}' model \mathbf{V} -categories.

- (1.27.6.1) A left (respectively, right) \mathbf{V} -adjoint $F : \mathbf{C} \rightarrow \mathbf{C}'$ [2, Definition 6.7.1] whose underlying functor F_0 is a left (resp., right) Quillen functor will be called a *left* (resp., *right*) *Quillen \mathbf{V} -functor*.

- (1.27.6.2) If F_0 is in addition a Quillen equivalence, then F will be called a *left* (resp., *right*) *Quillen \mathbf{V} -equivalence*.

Notation 1.28. Of course the sub- and superscripts on \otimes , $\mathbf{1}$, and $\underline{\text{Mor}}$ will be dropped if no confusion can result, and by the standard harmless abuse, we will refer to \mathbf{V} alone as the symmetric monoidal model category.

Lemma 1.29 ([10, Lemma 4.2.2]). *The following are equivalent for three model \mathbf{X} -categories \mathbf{D} , \mathbf{E} , and \mathbf{F} and an adjunction of two variables*

$$\otimes : \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{F} \quad \underline{\text{Mor}} : \mathbf{E}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{D} \quad \text{mor} : \mathbf{D}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{E}.$$

²If S and T are sets of morphisms, it will be convenient to denote by $S \square T$ the set of morphisms of the form $f \square g$ for $f \in S$ and $g \in T$.

(1.29.1) The tuple $(\otimes, \underline{\text{Mor}}, \text{mor})$ is a Quillen adjunction of two variables.

(1.29.2) For any cofibration $d : S \rightarrow T$ of \mathbf{E} and any fibration $b : V \rightarrow U$ of \mathbf{F} , the morphism

$$\underline{\text{Mor}}_{\square}(d, b) : \underline{\text{Mor}}(T, V) \rightarrow \underline{\text{Mor}}(S, V) \times_{\underline{\text{Mor}}(S, U)} \underline{\text{Mor}}(T, U)$$

is a fibration that is trivial if either d or b is.

(1.29.3) For any cofibration $c : Q \rightarrow R$ of \mathbf{D} and any fibration $b : V \rightarrow U$ of \mathbf{F} , the morphism

$$\text{mor}_{\square}(c, b) : \text{mor}(R, V) \rightarrow \text{mor}(Q, V) \times_{\text{mor}(Q, U)} \text{mor}(R, U)$$

is a fibration that is trivial if either c or b is.

Lemma 1.30 ([10, Lemma 4.2.4]). *Suppose \mathbf{D} an \mathbf{X} -cofibrantly generated model \mathbf{X} -category, with generating cofibrations $I_{\mathbf{D}}$ and generating trivial cofibrations $J_{\mathbf{D}}$. Suppose \mathbf{E} an \mathbf{X} -cofibrantly generated model \mathbf{X} -category, with generating cofibrations $I_{\mathbf{E}}$ and generating trivial cofibrations $J_{\mathbf{E}}$. Suppose \mathbf{F} a model \mathbf{X} -category, and suppose*

$$\otimes : \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{F} \quad \underline{\text{Mor}} : \mathbf{E}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{D} \quad \text{mor} : \mathbf{D}^{\text{op}} \times \mathbf{F} \rightarrow \mathbf{E}$$

form an adjunction of two variables. Then $(\otimes, \underline{\text{Mor}}, \text{mor})$ is a Quillen adjunction of two variables if and only if the following conditions hold.

(1.30.1) $I_{\mathbf{D}} \square I_{\mathbf{E}}$ consists only of cofibrations.

(1.30.2) $I_{\mathbf{D}} \square J_{\mathbf{E}}$ consists only of weak equivalences.

(1.30.3) $J_{\mathbf{D}} \square I_{\mathbf{E}}$ consists only of weak equivalences.

Lemma 1.31. *Suppose \mathbf{V} and \mathbf{C} symmetric monoidal model \mathbf{X} -categories, wherein the unit $\mathbf{1}_{\mathbf{C}}$ is cofibrant. Then a model \mathbf{V} -category structure on \mathbf{C} is equivalent to a Quillen adjunction*

$$\text{real} : \mathbf{V} \rightleftarrows \mathbf{C} : \Pi$$

in which the left adjoint real is symmetric monoidal.

Proof. Suppose $(\otimes_{\mathbf{C}}^{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{C}}^{\mathbf{V}}, \text{mor}_{\mathbf{C}}^{\mathbf{V}})$ a model \mathbf{V} -category structure on \mathbf{C} . Set

$$\text{real} := - \otimes_{\mathbf{C}}^{\mathbf{V}} \mathbf{1}_{\mathbf{C}} \quad \Pi := \underline{\text{Mor}}_{\mathbf{C}}^{\mathbf{V}}(\mathbf{1}_{\mathbf{C}}, -).$$

For any objects K and L of \mathbf{V} , one verifies easily that the objects $(K \otimes_{\mathbf{V}} L) \otimes_{\mathbf{C}}^{\mathbf{V}} \mathbf{1}_{\mathbf{C}}$ and $(K \otimes_{\mathbf{C}}^{\mathbf{V}} \mathbf{1}_{\mathbf{C}}) \otimes_{\mathbf{V}} (L \otimes_{\mathbf{C}}^{\mathbf{V}} \mathbf{1}_{\mathbf{C}})$ corepresent the same functor. Since $\mathbf{1}_{\mathbf{C}}$ is cofibrant, the pushout-product axiom implies that real is left Quillen.

On the other hand, suppose

$$\text{real} : \mathbf{V} \rightleftarrows \mathbf{C} : \Pi$$

a Quillen adjunction in which the left adjoint real is symmetric monoidal. For K and object of \mathbf{V} , and X and Y objects of \mathbf{C} , set

$$\begin{aligned} \text{Mor}_{\mathbf{C}}^{\mathbf{V}}(X, Y) &:= \Pi \underline{\text{Mor}}_{\mathbf{C}}(X, Y), \\ K \otimes_{\mathbf{C}}^{\mathbf{V}} X &:= \text{real}(K) \otimes_{\mathbf{C}} X, \\ \text{mor}_{\mathbf{C}}^{\mathbf{V}}(K, Y) &:= \underline{\text{Mor}}_{\mathbf{C}}(\text{real}(K), Y). \end{aligned}$$

The pushout-product axiom for $\otimes_{\mathbf{C}}$ implies the pushout-product axiom for $\otimes_{\mathbf{C}}^{\mathbf{V}}$.

These two definitions are inverse to one another. \square

Lemma 1.32. *Suppose \mathbf{C} a simplicial, internal model \mathbf{X} -category in which the terminal object \star is cofibrant. Then for any \mathbf{X} -small set S , the object $\text{real}(S)$ is canonically isomorphic to the copower $S \cdot \star$.*

Proof. The functor real is a left adjoint and thus respects copowers; it is symmetric monoidal and thus preserves terminal objects. \square

Proposition 1.33. *Suppose \mathbf{V} internal. Then for any object Y of \mathbf{V} , the comma category (\mathbf{V}/Y) is a \mathbf{V} -model category with*

$$\begin{aligned} Z \otimes_{(\mathbf{V}/Y)}^{\mathbf{V}} X &:= Z \times X \\ \text{mor}_{(\mathbf{V}/Y)}^{\mathbf{V}}(Z, X') &:= Y \times_{\underline{\text{Mor}}_{\mathbf{V}}(Z, Y)} \underline{\text{Mor}}_{\mathbf{V}}(Z, X') \\ \underline{\text{Mor}}_{(\mathbf{V}/Y)}^{\mathbf{V}}(X, X') &:= \star \times_{\underline{\text{Mor}}_{\mathbf{V}}(X, Y)} \underline{\text{Mor}}_{\mathbf{V}}(X, X') \end{aligned}$$

for any object Z of \mathbf{V} and any objects X and X' of (\mathbf{V}/Y) .

Proof. A morphism of the comma category (\mathbf{V}/Y) is a cofibration, fibration, or weak equivalence if and only if its image under the forgetful functor $(\mathbf{V}/Y) \rightarrow \mathbf{V}$ is so; hence the pushout-product and unit axioms for (\mathbf{V}/Y) follow directly from those for \mathbf{V} . \square

1.34. We will be interested in localizing enriched model categories using derived mapping objects in lieu of the homotopy function complexes. The result will have a universal property that is rather different from the ordinary left Bousfield localization. Suppose $(\mathbf{V}, \otimes_{\mathbf{V}}, \mathbf{1}_{\mathbf{V}}, \underline{\text{Mor}}_{\mathbf{V}})$ a symmetric monoidal model \mathbf{X} -category, and suppose \mathbf{C} a model \mathbf{V} -category.

Definition 1.35. (1.35.1) The left Kan extension (if it exists) of the composite

$$\mathbf{C}^{\text{op}} \times \mathbf{C} \longrightarrow \mathbf{V} \longrightarrow \text{Ho } \mathbf{V}$$

along the localization functor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \text{Ho}(\mathbf{C}^{\text{op}} \times \mathbf{C})$ is the *derived mapping object functor*, denoted $\mathbf{R} \underline{\text{Mor}}_{\mathbf{C}}^{\mathbf{V}}$.

(1.35.2) Suppose q and r cofibrant and fibrant replacement functors for \mathbf{C} ; then the (q, r) -*derived mapping object functor* is the functor

$$\begin{aligned} \mathbf{R} \underline{\text{Mor}}_{\mathbf{C}, q, r} : \mathbf{C}^{\text{op}} \times \mathbf{C} &\longrightarrow \text{Ho } \mathbf{V} \\ (A, B) &\longmapsto \underline{\text{Mor}}_{\mathbf{C}}(qA, rB). \end{aligned}$$

Proposition 1.36. *Suppose q and r cofibrant and fibrant replacement functors for \mathbf{C} , respectively. Then the derived mapping object functor $\mathbf{R} \underline{\text{Mor}}_{\mathbf{C}}$ exists, and, up to isomorphism of functors, $\mathbf{R} \underline{\text{Mor}}_{\mathbf{C}, q, r}$ factors through it.*

Proof. This is an immediate consequence of [9, Proposition 8.4.8]. \square

Lemma 1.37. *The following are equivalent for a morphism $A \rightarrow B$ of \mathbf{C} .*

(1.37.1) *The morphism $A \rightarrow B$ is a weak equivalence.*

(1.37.2) *For any fibrant object Z of \mathbf{C} , the induced morphism*

$$\mathbf{R} \underline{\text{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R} \underline{\text{Mor}}_{\mathbf{C}}(A, Z)$$

is an isomorphism of $\text{Ho } \mathbf{V}$.

(1.37.3) For any cofibrant object X of \mathbf{C} , the induced morphism

$$\mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{C}}(X, A) \rightarrow \mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{C}}(X, B)$$

is an isomorphism of $\mathrm{Ho}\mathbf{V}$.

Proof. That (1.37.1) implies both (1.37.2) and (1.37.3) follows from the pushout-product axiom.

To show that (1.37.1) follows from (1.37.2), suppose $A \rightarrow B$ a morphism of \mathbf{C} such that for any fibrant object Z of \mathbf{C} , the induced morphism

$$\mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R}\underline{\mathrm{Mor}}_{\mathbf{C}}(A, Z)$$

is an isomorphism of $\mathrm{Ho}\mathbf{V}$; one may clearly assume that A and B are cofibrant, so that the morphism

$$\underline{\mathrm{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \underline{\mathrm{Mor}}_{\mathbf{C}}(A, Z)$$

is a weak equivalence of \mathbf{V} . Applying $\mathbf{R}\mathrm{Mor}_{\mathbf{V}}(1_{\mathbf{V}}, -)$ yields an isomorphism

$$\mathbf{R}\mathrm{Mor}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(A, Z)$$

of $\mathrm{Ho}s\mathrm{Set}_{\mathbf{X}}$ for any fibrant object Z of \mathbf{C} . Now apply [9, 17.7.7].

The same argument, mutatis mutandis, shows that (1.37.1) follows from (1.37.3). \square

2. Smith's theorem

The result

J. Smith's insight is that the transfinite small object argument and the solution set condition on weak equivalences together provide a good recognition principle for combinatorial model categories. In effect, one requires only two-thirds of the data normally required to produce cofibrantly generated model structures.

2.1. Suppose C an accessible \mathbf{X} -category. Recall that a subcategory $D \subset C$ that is itself accessible is said to be *accessibly embedded* if there exists an \mathbf{X} -small regular cardinal λ such that D and C are each λ -accessible, and D is closed under λ -filtered colimits in C . Recall that the inverse image of an accessibly embedded accessible subcategory of an accessible category under an accessible functor is itself an accessibly embedded accessible subcategory!

Proposition 2.2 (Smith, [1, Theorem 1.7 and Propositions 1.15 and 1.19]). *Suppose C a locally \mathbf{X} -presentable \mathbf{X} -category, W an accessibly embedded, accessible subcategory of $C(\mathbf{1})$, and I an \mathbf{X} -small set of morphisms of C . Suppose in addition that the following conditions are satisfied.*

(2.2.1) *W satisfies the two-out-of-three axiom.*

(2.2.2) *The set $\mathrm{inj} I$ is contained in W .*

(2.2.3) *The intersection $W \cap \mathrm{cof} I$ is closed under pushouts and transfinite composition.*

Then C is a combinatorial model category with weak equivalences W , cofibrations $\mathrm{cof} I$, and fibrations $\mathrm{inj}(W \cap \mathrm{cof} I)$.

Proof. The transfinite small object argument 1.25 and retract arguments apply once one constructs an \mathbf{X} -small set J such that $\text{cof } J = W \cap \text{cof } I$. The following pair of lemmas complete the proof. \square

Lemma 2.3 (Smith, [1, Lemma 1.8]). *Under the hypotheses of proposition 2.2, suppose $J \subset W \cap \text{cof } I$ a set such that any commutative square*

$$\begin{array}{ccc} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ L & \longrightarrow & N \end{array}$$

in which $[K \rightarrow L] \in I$ and $[M \rightarrow N] \in W$ can be factored as a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & M' & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & N' & \longrightarrow & N, \end{array}$$

in which $[M' \rightarrow N'] \in J$. Then $\text{cof } J = W \cap \text{cof } I$.

Proof. To prove this, one need only factor any element of W as an element of cell J followed by an element of $\text{inj } I$. The result then follows from the retract argument.

Suppose κ an \mathbf{X} -small regular cardinal with the property that every codomain of I is κ -presentable. For any morphism $[f : X \rightarrow Y] \in W$, consider the set (I/f) of squares

$$\begin{array}{ccc} K & \longrightarrow & X \\ i \downarrow & & \downarrow f \\ L & \longrightarrow & Y, \end{array}$$

where $i \in I$; for each such square choose an element $j_{(i,f)} \in J$ and a factorization

$$\begin{array}{ccccc} K & \longrightarrow & M(i) & \longrightarrow & X \\ i \downarrow & & \downarrow j_{(i,f)} & & \downarrow f \\ L & \longrightarrow & N(i) & \longrightarrow & Y, \end{array}$$

and let $M_{(I/f)} \rightarrow N_{(I/f)}$ be the coproduct $\coprod_{i \in (I/f)} j_{(i,f)}$. Define an endomorphism Q of (W/Y) by

$$Qf := [X \sqcup^{N_{(I/f)}} M_{(I/f)} \rightarrow Y]$$

for any morphism $[f : X \rightarrow Y] \in W$. For any regular cardinal α , let Q^α be the colimit $\text{colim}_{\beta < \alpha} Q^\beta$. This provides, for any morphism $[f : X \rightarrow Y] \in W$, a factorization

$$X \longrightarrow Q^\kappa f \longrightarrow Y$$

with the desired properties. \square

Lemma 2.4 (Smith, [1, Lemma 1.9]). *Under the hypotheses of proposition 2.2, an \mathbf{X} -small set J satisfying the conditions of lemma 2.3 can be found.*

Proof. Since W is an accessibly embedded accessible subcategory of $\mathbf{C}(\mathbf{1})$, it follows that for any morphism $[i : K \rightarrow L] \in I$, there exists an \mathbf{X} -small subset $W(i) \subset W$ such that for any commutative square

$$\begin{array}{ccc} K & \longrightarrow & M \\ i \downarrow & & \downarrow \\ L & \longrightarrow & N \end{array}$$

in which $[M \rightarrow N] \in W$, there exist a morphism $[P \rightarrow Q] \in W(i)$ and a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & P & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & Q & \longrightarrow & N. \end{array}$$

It thus suffices to find, for every square of the type on the left, an element of $W \cap \text{cof } I$ factoring it.

For every element $[i : K \rightarrow L] \in I$, every element $[w : P \rightarrow Q] \in W(i)$, and every commutative square

$$\begin{array}{ccc} K & \longrightarrow & P \\ i \downarrow & & \downarrow \\ L & \longrightarrow & Q, \end{array}$$

factor the morphism $L \sqcup^K P \rightarrow Q$ as an element of $[L \sqcup^K P \rightarrow R] \in \text{cell } I$ followed by an element of $[R \rightarrow Q] \in \text{inj } I$; this yields a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & P & \xlongequal{\quad} & P \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & R & \longrightarrow & Q \end{array}$$

factoring the original square, in which $[P \rightarrow R] \in W \cap \text{cof } I$. □

Proposition 2.5 (Smith, [4, Propositions 7.1-3]). *Suppose \mathbf{C} an \mathbf{X} -combinatorial model \mathbf{X} -category. For any sufficient large \mathbf{X} -small regular cardinal κ , the following hold.*

- (2.5.1) *There exists a κ -accessible functorial factorization $\mathbf{C}(\mathbf{1}) \rightarrow \mathbf{C}(\mathbf{2})$ of each morphism into a cofibration followed by a trivial fibration.*
- (2.5.2) *There exists a κ -accessible functorial factorization $\mathbf{C}(\mathbf{1}) \rightarrow \mathbf{C}(\mathbf{2})$ of each morphism into a trivial cofibration followed by a fibration.*
- (2.5.3) *There exists a κ -accessible cofibrant replacement functor.*
- (2.5.4) *There exists a κ -accessible fibrant replacement functor.*
- (2.5.5) *Arbitrary κ -filtered colimits preserve weak equivalences.*
- (2.5.6) *Arbitrary κ -filtered colimits in \mathbf{C} are homotopy colimits.*
- (2.5.7) *The set of weak equivalences $w\mathbf{C}$ form a κ -accessibly embedded, κ -accessible subcategory of \mathbf{C}^1 .*

Proof. Observe that (2.5.1) and (2.5.2) (and therefore (2.5.3) and (2.5.4) as well) follow directly from the transfinite small object argument 1.25.

To verify (2.5.5) — and therefore (2.5.6) —, fix κ , an \mathbf{X} -small regular cardinal for which: (a) there are κ -accessible functorial factorizations of each kind, (b) there is an \mathbf{X} -small set I of generating cofibrations with κ -presentable domains and codomains, and (c) the full subcategory of I -tuples of surjective morphisms is a κ -accessibly embedded, κ -accessible subcategory of $\mathbf{Set}_{\mathbf{X}}(I \cdot \mathbf{1})$. Suppose A an \mathbf{X} -small κ -filtered category, and suppose $F \rightarrow G$ an objectwise weak equivalence in \mathbf{C}^A . The κ -accessible functorial factorizations in \mathbf{C} permit one to give a κ -accessible factorization of $F \rightarrow G$ into an objectwise trivial cofibration $F \rightarrow H$ followed by an objectwise fibration (which is therefore an objectwise trivial fibration) $H \rightarrow G$. Hence the morphism $\operatorname{colim} H \rightarrow \operatorname{colim} G$ is a fibration, and it remains only to show that it is also a trivial fibration; for this one need only show that for any morphism $f : K \rightarrow L$ and element of I and any diagram

$$\begin{array}{ccc} K & \longrightarrow & \operatorname{colim} F \\ \downarrow & & \downarrow \\ L & \longrightarrow & \operatorname{colim} G, \end{array}$$

a lift $L \rightarrow \operatorname{colim} F$ exists. This follows from the κ -presentability of K and L .

To verify (2.5.7), let us note that it follows from the existence of a κ -accessible functorial factorization that it suffices to verify that the full subcategory of $\mathbf{C}(\mathbf{1})$ comprised of trivial fibrations is a κ -accessibly embedded, κ -accessible subcategory. For this, consider the functor

$$\operatorname{Mor}_{\mathbf{C}, \square} : \mathbf{C}^{\mathbf{1}} \longrightarrow \mathbf{Set}_{\mathbf{X}}(I \cdot \mathbf{1})$$

$$[Y \rightarrow X] \mapsto ([\operatorname{Mor}_{\mathbf{C}}(L, Y) \rightarrow \operatorname{Mor}_{\mathbf{C}}(L, X) \times_{\operatorname{Mor}_{\mathbf{C}}(K, X)} \operatorname{Mor}_{\mathbf{C}}(K, Y)])_{[K \rightarrow L] \in I},$$

where $\mathbf{Set}_{\mathbf{X}}(I \cdot \mathbf{1})$ is the category of presheaves on the copower category

$$I \cdot \mathbf{1} := \coprod_{i \in I} \mathbf{1}.$$

Since the domains and codomains of I are κ -presentable, this is a κ -accessible functor. The trivial fibrations are by definition the inverse image of the full subcategory of I -tuples of surjective morphisms under $\operatorname{Mor}_{\mathbf{C}, \square}$. \square

Corollary 2.6. *Any \mathbf{X} -combinatorial model \mathbf{X} -category satisfies the hypotheses of 2.2.*

Corollary 2.7. *An \mathbf{X} -combinatorial model \mathbf{X} -category \mathbf{C} is \mathbf{X} -tractable if and only if the \mathbf{X} -small set I of generating cofibrations can be chosen with cofibrant domains.*

Proof. Suppose I is an \mathbf{X} -small set of generating cofibrations with cofibrant domains, and suppose J an \mathbf{X} -small set of trivial cofibrations satisfying the conditions of 2.3. To give another such \mathbf{X} -small set of trivial cofibrations with cofibrant domains, it

suffices to show that any commutative square

$$\begin{array}{ccc} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ L & \longrightarrow & N \end{array}$$

in which $[K \rightarrow L] \in I$ and $[M \rightarrow N] \in J$ can be factored as a commutative diagram

$$\begin{array}{ccccc} K & \longrightarrow & M' & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & N' & \longrightarrow & N, \end{array}$$

in which M' is cofibrant and $[M' \rightarrow N'] \in W \cap \text{cof } I$. To construct this factorization, factor the morphism $K \rightarrow M$ as a cofibration $K \rightarrow M'$ followed by a weak equivalence $M' \rightarrow M$. Then factor the morphism $L \sqcup^K M' \rightarrow N$ as a cofibration $L \sqcup^K M' \rightarrow N'$ followed by a weak equivalence $N' \rightarrow N$. Then the composite $M' \rightarrow N'$ is a trivial cofibration providing the desired factorization. \square

Corollary 2.8. *A left proper \mathbf{X} -combinatorial model \mathbf{X} -category \mathbf{C} is \mathbf{X} -tractable if and only if the \mathbf{X} -small set J of generating trivial cofibrations can be chosen with cofibrant domains.*

Proof. Let I be an \mathbf{X} -small set of generating cofibrations. For every element $[i : K \rightarrow L] \in I$, let $K' \rightarrow K$ be a cofibrant replacement of K , and factor the composite $K' \rightarrow L$ into a cofibration $i' : K' \rightarrow L'$ followed by a trivial fibration $L' \rightarrow L$. Let I' be the resulting \mathbf{X} -small set of morphisms i' , all of which have cofibrant domains. We claim that $I'' := I' \cup J$ is also a set of generating cofibrations for \mathbf{C} .

Note that $\text{inj } I'' \subset \text{inj } J = \text{fib } \mathbf{C}$. Since a fibration of \mathbf{C} has the right lifting property with respect to i if and only if it has the right lifting property with respect to i' [9, Proposition 13.2.1], it follows that $\text{inj } I = \text{inj } I'$. One now applies Smith's theorem 2.2 to the set I' with the set of weak equivalences of \mathbf{C} , and the model structure guaranteed by Smith's theorem coincides with that of \mathbf{C} . \square

2.9. We do not know a single example of a left proper combinatorial model \mathbf{X} -category that is not also tractable, but at the same time we are unable to prove that left properness guarantees that the generating cofibrations can be chosen to have cofibrant domains. Such a guarantee seems not at all implausible, however: let I' be the resulting \mathbf{X} -small set of morphisms described in the proof of the corollary above. By adding suitable maps to I' , it may be possible to construct an \mathbf{X} -small set I'' of generating cofibrations with cofibrant domains. As in the proof above, a fibration of \mathbf{C} has the right lifting property with respect to i if and only if it has the right lifting property with respect to i' , so it would follow that $\text{inj } I = \text{inj } I''$, if only one could show that $\text{inj } I''$ contained only fibrations of \mathbf{C} . Unfortunately, we have not managed to find a way to add enough cofibrations to ensure this.

2.10. Suppose \mathbf{C} an \mathbf{X} -combinatorial model \mathbf{X} -category and \mathbf{D} a locally \mathbf{X} -presentable

category equipped with an adjunction

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : U.$$

We now discuss circumstances under which the model structure on \mathbf{C} may be lifted to \mathbf{D} .

Definition 2.11. (2.11.1) A morphism $f : X \rightarrow Y$ of \mathbf{D} is said to be a *projective weak equivalence* (respectively, a *projective fibration*, a *projective trivial fibration*) if $Uf : UX \rightarrow UY$ is a weak equivalence (resp., fibration, trivial fibration).

(2.11.2) A morphism $f : X \rightarrow Y$ of \mathbf{D} is said to be a *projective cofibration* if it satisfies the left lifting property with respect to any projective trivial fibration; f is said to be a *projective trivial cofibration* if it is, in addition, a projective weak equivalence.

(2.11.3) If the projective weak equivalences, projective cofibrations, and projective fibrations define a model structure on \mathbf{D} , then one calls this model structure the *projective model structure*.

Lemma 2.12. *Suppose that in \mathbf{D} , transfinite compositions and pushouts of projective trivial cofibrations of \mathbf{D} are projective weak equivalences. Then the projective model structure on \mathbf{D} exists; it is \mathbf{X} -combinatorial, and it is \mathbf{X} -tractable if \mathbf{C} is. Furthermore the adjunction (F, U) is a Quillen adjunction.*

Proof. The full accessible inverse image of an accessibly embedded accessible full subcategory is again an accessibly embedded accessible full subcategory; hence the projective weak equivalences are an accessibly embedded accessible subcategory of $\mathbf{D}(\mathbf{1})$. Choose now an \mathbf{X} -small set I of \mathbf{C} (respectively, \mathbf{C}_c) of generating cofibrations.

One now applies the recognition lemma 2.2 to the set W of projective weak equivalences and the \mathbf{X} -small set FI . It is clear that $\text{inj } I \subset W$, and by assumption it follows that $W \cap \text{cof } I$ is closed under pushouts and transfinite compositions. One now verifies easily that the fibrations are the projective ones and that the adjunction (F, U) is a Quillen adjunction.

Since F is left Quillen, the set FI has cofibrant domains if I does. \square

Application I: Model structures on diagram categories

Suppose \mathbf{X} a universe, K an \mathbf{X} -small category, and \mathbf{C} an \mathbf{X} -combinatorial (respectively, \mathbf{X} -tractable) model \mathbf{X} -category. The category $\mathbf{C}(K)$ of \mathbf{C} -valued presheaves on K has two \mathbf{X} -combinatorial (resp. \mathbf{X} -tractable) model structures, to which we now turn.

Definition 2.13. A morphism $X \rightarrow Y$ of \mathbf{C} -valued presheaves on K is a *projective weak equivalence* or *projective fibration* if, for any object k of K , the morphism $X_k \rightarrow Y_k$ is a weak equivalence or fibration of \mathbf{C} .

Theorem 2.14. *The category $\mathbf{C}(K)$ of \mathbf{C} -valued presheaves on K admits an \mathbf{X} -combinatorial (resp., \mathbf{X} -tractable) model structure — the projective model structure $\mathbf{C}(K)_{\text{proj}}$ —, in which the weak equivalences and fibrations are the projective weak equivalences and fibrations.*

Proof. Consider the functor $e : \text{Obj } K \rightarrow K$, which induces an adjunction

$$e_! : \mathbf{C}(\text{Obj } K) \rightleftarrows \mathbf{C}(K) : e^*.$$

The condition of 2.12 follows from the observation that e^* preserves all colimits. \square

Definition 2.15. A morphism $X \rightarrow Y$ of \mathbf{C} -valued presheaves on K is an *injective weak equivalence* or *injective cofibration* if, for any object k of K , the morphism $X_k \rightarrow Y_k$ is a weak equivalence or cofibration of \mathbf{C} .

Theorem 2.16. *The category $\mathbf{C}(K)$ of \mathbf{C} -valued presheaves on K admits an \mathbf{X} -combinatorial model structure — the injective model structure $\mathbf{C}(K)_{\text{inj}}$ —, in which the weak equivalences and cofibrations are the injective weak equivalences and cofibrations.*

Proof. Suppose κ an \mathbf{X} -small regular cardinal such that K is κ -small, \mathbf{C} is locally κ -presentable, and a set of generating cofibrations $I_{\mathbf{C}}$ for \mathbf{C} can be chosen from \mathbf{C}_{κ} (resp., from $\mathbf{C}_{\kappa} \cap \mathbf{C}_{\lambda}$); without loss of generality, we may assume that $I_{\mathbf{C}}$ is the \mathbf{X} -small set of all cofibrations in \mathbf{C}_{κ} (resp., in $\mathbf{C}_{\kappa} \cap \mathbf{C}_{\lambda}$). Denote by $I_{\mathbf{C}(K)}$ the set of injective cofibrations between κ -presentable objects of $\mathbf{C}(K)$ (resp., between κ -presentable objects of $\mathbf{C}(K)$ that are in addition objectwise cofibrant). This set contains a generating set of cofibrations for the projective model structure, so it follows that $\text{inj } I_{\mathbf{C}(K)} \subset W$.

The claim is now that any injective cofibration can be written as a retract of transfinite composition of pushouts of elements of $I_{\mathbf{C}(K)}$. This point follows from a cardinality argument, which proceeds almost exactly as for $s\mathbf{Set}$ -functors from an $s\mathbf{Set}$ -category to a simplicial model category. For this cardinality argument we refer to [14, A.2.8.2, A.3.3.3, and A.3.3.15-17] of J. Lurie, whose proofs and exposition we are unable to improve upon.

Since colimits are formed objectwise, it follows that the injective trivial cofibrations are closed under pushouts and transfinite composition. \square

Proposition 2.17. *The identity functor is a Quillen equivalence*

$$\mathbf{C}(K)_{\text{proj}} \rightleftarrows \mathbf{C}(K)_{\text{inj}}.$$

Proof. Projective cofibrations are injective cofibrations, and projective and injective weak equivalences are the same. \square

Proposition 2.18. *If \mathbf{C} is left or right proper, then so are $\mathbf{C}(K)_{\text{proj}}$ and $\mathbf{C}(K)_{\text{inj}}$.*

Proof. Pullbacks and pushouts are defined objectwise; hence it suffices to note that in both model structures, weak equivalences are defined objectwise, and any cofibration or fibration is in particular an objectwise cofibration or fibration. \square

Proposition 2.19. *A functor $f : K \rightarrow L$ induces Quillen adjunctions*

$$f_! : \mathbf{C}(K)_{\text{proj}} \rightleftarrows \mathbf{C}(L)_{\text{proj}} : f^* \quad \text{and} \quad f^* : \mathbf{C}(L)_{\text{inj}} \rightleftarrows \mathbf{C}(K)_{\text{inj}} : f_*,$$

which are of course equivalences of categories if f is an equivalence of categories.

Proof. Clearly f^* preserves objectwise weak equivalences, objectwise cofibrations, and objectwise fibrations. \square

Application II: Model structures on section categories

Left and right Quillen presheaves are diagrams of model categories. Here we give model structures on their categories of sections, analogous to the injective and projective model structures on diagram categories above.

2.20. Suppose here \mathbf{X} a universe.

Definition 2.21. Suppose K an \mathbf{X} -small category.

(2.21.1) A *left* (respectively, *right*) *Quillen presheaf* on K is a functor³

$$\mathbf{F} : K^{\text{op}} \rightarrow \mathbf{Cat}_{\mathbf{Y}}$$

for some universe \mathbf{Y} with $\mathbf{X} \in \mathbf{Y}$ such that for every $k \in \text{Obj } K$, the category \mathbf{F}_k is a model \mathbf{X} -category, and for every morphism $f : \ell \rightarrow k$ of K , the induced functor $f^* : \mathbf{F}_k \rightarrow \mathbf{F}_\ell$ is left (resp., right) Quillen.

(2.21.2) A left or right Quillen presheaf \mathbf{F} on K is said to be *\mathbf{X} -combinatorial* (respectively, *\mathbf{X} -tractable*, *left proper*, *right proper*, ...) if for every $k \in \text{Obj } K$, the model \mathbf{X} -category \mathbf{F}_k is so.

(2.21.3) A *left* (respectively, *right*) *morphism* $\Theta : \mathbf{F} \rightarrow \mathbf{G}$ of *left* (resp., *right*) *Quillen presheaves* is a pseudomorphism of functors $K^{\text{op}} \rightarrow \mathbf{Cat}_{\mathbf{Y}}$ such that for any $k \in \text{Obj } K$, the functor $\Theta_k : \mathbf{F}_k \rightarrow \mathbf{G}_k$ is left (resp., right) Quillen.

(2.21.4) A *left* (respectively, *right*) *section* X of a left (resp., right) Quillen presheaf \mathbf{F} is a tuple $(X, \phi) = ((X_k)_{k \in \text{Obj } K}, (\phi_f)_{f \in \text{Obj } K(1)})$ comprised of an object $X = (X_k)_{k \in \text{Obj } K}$ of $\prod_{k \in \text{Obj } K} \mathbf{F}_k$ and a morphism $\phi_f : f^* X_\ell \rightarrow X_k$ (resp., $\phi_f : X_k \rightarrow f^* X_\ell$), one for each morphism $[f : \ell \rightarrow k] \in K$, such that for any composable pair

$$[m \xrightarrow{g} \ell \xrightarrow{f} k] \in K,$$

one has the commutative triangle

$$\begin{array}{ccc} & g^* X_\ell & \\ g^* \phi_f \nearrow & & \searrow \phi_g \\ (f \circ g)^* X_k & \xrightarrow{\phi_{f \circ g}} & X_m \end{array} \quad (\text{resp., } \begin{array}{ccc} & X_m & \xrightarrow{\phi_{f \circ g}} & (f \circ g)^* X_k \\ & \searrow \phi_g & & \nearrow g^* \phi_f \\ & g^* X_\ell & & \end{array}).$$

(2.21.5) A *morphism of left* (respectively, *right*) *sections* $r : (X, \phi) \rightarrow (Y, \psi)$ is a morphism $r : X \rightarrow Y$ of $\prod_{k \in \text{Obj } K} \mathbf{F}_k$ such that the diagram

$$\begin{array}{ccc} f^* X_k & \xrightarrow{f^* r_k} & f^* Y_k \\ \phi_f \downarrow & & \downarrow \psi_f \\ X_\ell & \xrightarrow{r_\ell} & Y_\ell \end{array}$$

³In practice, of course, one is usually presented with a pseudofunctor, rather than a functor. Well-known rectification results allow one to replace such a pseudofunctor with a pseudoequivalent functor, and all the model structures can be lifted along this pseudoequivalence.

in \mathbf{F}_ℓ (resp., the diagram

$$\begin{array}{ccc} X_k & \xrightarrow{r_k} & Y_k \\ \phi_f \downarrow & & \downarrow \psi_f \\ f^* X_\ell & \xrightarrow{f^* r_\ell} & f^* Y_\ell \end{array}$$

in \mathbf{F}_k) commutes for any morphism $f : \ell \rightarrow k$ of K . These morphisms clearly compose, to give a category $\text{Sect}^L \mathbf{F}$ (resp., $\text{Sect}^R \mathbf{F}$) of left (resp., right) sections of \mathbf{F} .

2.22. The category $\text{Sect}^L \mathbf{F}$ (respectively, $\text{Sect}^R \mathbf{F}$) is a model for the lax (resp., colax) limit of the diagram \mathbf{F} of categories. Observe that for any left Quillen presheaf \mathbf{F} , there is a corresponding right Quillen presheaf \mathbf{F}^\perp , and dually, for any right Quillen presheaf \mathbf{G} , there is a corresponding left Quillen presheaf ${}^\perp \mathbf{G}$. The section categories of these Quillen presheaves are related by *isomorphisms* of categories

$$\text{Sect}^L(\mathbf{F}) \cong \text{Sect}^R(\mathbf{F}^\perp) \quad \text{and} \quad \text{Sect}^R(\mathbf{G}) \cong \text{Sect}^L({}^\perp \mathbf{G}).$$

Lemma 2.23. *A left (respectively, right) morphism $\Theta : \mathbf{F} \rightarrow \mathbf{G}$ of left (resp., right) Quillen presheaves on an \mathbf{X} -small category K induces a left adjoint*

$$\Theta_! : \text{Sect}^L \mathbf{F} \rightarrow \text{Sect}^L \mathbf{G}$$

(resp., a right adjoint

$$\Theta_* : \text{Sect}^L \mathbf{F} \rightarrow \text{Sect}^L \mathbf{G} \quad).$$

Proof. If Θ is a left morphism of left Quillen presheaves, then define $\Theta_!$ by the formula

$$\Theta_!(X, \phi) := ((\Theta_k X_k)_{k \in \text{Obj } K}, ((\Theta \phi_f) \circ \theta_f)_{f \in \text{Obj}(K^1)})$$

for any left section $(X, \phi) = ((X_k)_{k \in \text{Obj } K}, (\phi_f)_{f \in \text{Obj}(K^1)})$, in which the morphism

$$\theta_f : f^* \Theta_k X_k \rightarrow \Theta_\ell f^* X_k$$

is the structural isomorphism of the pseudomorphism Θ .

Its right adjoint

$$\Theta^* : \text{Sect}^L \mathbf{G} \rightarrow \text{Sect}^L \mathbf{F}$$

is defined by the formula

$$\Theta^*(Y, \psi) := ((H_k Y_k)_{k \in \text{Obj } K}, (\eta_f)_{f \in \text{Obj}(K^1)})$$

for any left section $(Y, \psi) = ((Y_k)_{k \in \text{Obj } K}, (\psi_f)_{f \in \text{Obj}(K^1)})$, in which the morphism

$$\eta_f : f^* H_k Y_k \rightarrow H_\ell f^* Y_k$$

is the morphism adjoint to the composite

$$\Theta_\ell f^* H_k Y_k \xrightarrow{\theta_f} f^* \Theta_k H_k Y_k \xrightarrow{f^* c} f^* H_k,$$

where $c : \Theta_k H_k Y_k \rightarrow Y_k$ is the counit of the adjunction (Θ_k, H_k) .

The corresponding statement for right morphisms follows by duality. \square

2.24. The previous lemma suggests that the most natural model structure on the category of left (respectively, right) sections of a left (resp., right) Quillen presheaf \mathbf{F} is an injective (resp., projective) one, in which the weak equivalences and cofibrations (resp., fibrations) are defined objectwise. This idea is borne out by the observation that these model categories can be thought of as good models for the $(\infty, 1)$ -categorical lax limit (resp., $(\infty, 1)$ -categorical colax limit) of \mathbf{F} .

Lemma 2.25. *If \mathbf{F} is a left (respectively, right) \mathbf{X} -combinatorial Quillen presheaf on an \mathbf{X} -small category K , then the category $\text{Sect}^L \mathbf{F}$ (resp., $\text{Sect}^R \mathbf{F}$) is locally \mathbf{X} -presentable.*

Proof. It is a simple matter to verify that $\text{Sect}^L \mathbf{F}$ and $\text{Sect}^R \mathbf{F}$ are complete and cocomplete. The category of left sections is the lax limit of \mathbf{F} , and the category of right sections is a colax limit of \mathbf{F} ; so the result follows from the fact that the 2-category of \mathbf{X} -accessible categories is closed under arbitrary \mathbf{X} -small weighted bilimits in which all functors are accessible [15, Theorem 5.1.6]. \square

Lemma 2.26. *Suppose $a : L \rightarrow K$ a functor of \mathbf{X} -small categories. If \mathbf{F} is a left (respectively, right) \mathbf{X} -combinatorial Quillen presheaf on an \mathbf{X} -small category L , then there is a string $(a_!, a^*, a_*)$ of adjoints*

$$a_!, a_* : \text{Sect}^L(\mathbf{F} \circ a) \rightleftarrows \text{Sect}^L \mathbf{F} : a^* \quad (\text{resp.}, a_!, a_* : \text{Sect}^R(\mathbf{F} \circ a) \rightleftarrows \text{Sect}^R \mathbf{F} : a^*).$$

Proof. If \mathbf{F} is a left Quillen presheaf, then the functor

$$a^* : \text{Sect}^L(\mathbf{F} \circ a) \rightarrow \text{Sect}^L \mathbf{F}$$

is simply given by the formula

$$a^*(X, \phi) := ((X_{a(k)})_{k \in \text{Obj } K}, (\phi_{a(f)})_{f \in \text{Obj } K(1)}).$$

Since this functor commutes with all limits and colimits, the existence of its left and right adjoints follows from the usual adjoint functor theorems. \square

Definition 2.27. Suppose K an \mathbf{X} -small category, \mathbf{F} a right Quillen presheaf on K . A morphism $X \rightarrow Y$ of right sections of \mathbf{F} is a *projective weak equivalence* or *projective fibration* if, for any object k of K , the morphism $X_k \rightarrow Y_k$ is a weak equivalence or fibration of \mathbf{F}_k .

Theorem 2.28. *The category $\text{Sect}^R \mathbf{F}$ of right sections of an \mathbf{X} -combinatorial right Quillen presheaf \mathbf{F} (respectively, of an \mathbf{X} -tractable right Quillen presheaf \mathbf{F}) on an \mathbf{X} -small category K has an \mathbf{X} -combinatorial (resp., \mathbf{X} -tractable) model structure — the projective model structure $\text{Sect}_{\text{proj}}^R \mathbf{F}$ —, in which the weak equivalences and fibrations are the projective weak equivalences and fibrations.*

Proof. Consider the functor $e : \text{Obj } K \rightarrow K$, which induces an adjunction

$$e_! : \prod_{k \in \text{Obj } K} \mathbf{F}_k \rightleftarrows \text{Sect}^R \mathbf{F} : e^*.$$

The condition of 2.12 follows from the observation that e^* preserves all colimits. \square

Definition 2.29. Suppose K an \mathbf{X} -small category, \mathbf{F} a left Quillen presheaf on K . A morphism $X \rightarrow Y$ of left sections of \mathbf{F} is an *injective weak equivalence* or *injective cofibration* if, for any object k of K , the morphism $X_k \rightarrow Y_k$ is a weak equivalence or cofibration of \mathbf{F}_k .

Theorem 2.30. *The category $\text{Sect}^L \mathbf{F}$ of left sections of an \mathbf{X} -combinatorial (respectively, \mathbf{X} -tractable) left Quillen presheaf \mathbf{F} on an \mathbf{X} -small category K has an \mathbf{X} -combinatorial (resp., \mathbf{X} -tractable) model structure — the injective model structure $\text{Sect}_{\text{inj}}^L \mathbf{F}$ —, in which the weak equivalences and cofibrations are the injective weak equivalences and cofibrations.*

Proof. Suppose κ an \mathbf{X} -small regular cardinal such that K is κ -small, each \mathbf{F}_k is locally κ -presentable, and a set of generating cofibrations $I_{\mathbf{F}_k}$ for each \mathbf{F}_k can be chosen from $\mathbf{F}_{k,\kappa}$ (resp., from $\mathbf{F}_{k,\kappa} \cap \mathbf{F}_{k,c}$); without loss of generality, we may assume that $I_{\mathbf{F}_k}$ is the \mathbf{X} -small set of all cofibrations in $\mathbf{F}_{k,\kappa}$ (resp., in $\mathbf{F}_{k,\kappa} \cap \mathbf{F}_{k,c}$). Denote by $I_{\text{Sect}^L \mathbf{F}}$ the set of injective cofibrations between κ -presentable objects of $\text{Sect}^L \mathbf{F}$ (resp., between κ -presentable objects of $\text{Sect}^L \mathbf{F}$ that are in addition objectwise cofibrant). This set contains a generating set of cofibrations for the projective model structure, so it follows that $\text{inj } I_{\text{Sect}^L \mathbf{F}} \subset W$.

The argument given for the existence of the injective model structure on presheaf categories applies almost verbatim here to demonstrate that any injective cofibration can be written as a retract of transfinite composition of pushouts of elements of $I_{\text{Sect}^L \mathbf{F}}$.

Since colimits are formed objectwise, it follows that the injective trivial cofibrations are closed under pushouts and transfinite composition. \square

Proposition 2.31. *Suppose K an \mathbf{X} -small category, \mathbf{F} an \mathbf{X} -combinatorial left (respectively, right) Quillen presheaf on K . If each \mathbf{F}_k is left or right proper, then so is $\text{Sect}_{\text{inj}}^L \mathbf{F}$ (resp., $\text{Sect}_{\text{proj}}^R \mathbf{F}$).*

Proof. Pullbacks and pushouts are defined objectwise; hence it suffices to note that in both model structures, weak equivalences are defined objectwise, and any cofibration or fibration is in particular an objectwise cofibration or fibration. \square

Proposition 2.32. *A left (resp., right) morphism $\Theta : \mathbf{F} \rightarrow \mathbf{G}$ of \mathbf{X} -combinatorial left (resp., right) Quillen presheaves on an \mathbf{X} -small category K induces a Quillen adjunction*

$$\Theta_! : \text{Sect}_{\text{inj}}^L \mathbf{F} \rightleftarrows \text{Sect}_{\text{inj}}^L \mathbf{G} : \Theta^* \quad (\text{resp.,} \quad \Theta^* : \text{Sect}_{\text{proj}}^R \mathbf{G} \rightleftarrows \text{Sect}_{\text{proj}}^R \mathbf{F} : \Theta_*) ,$$

which is a Quillen equivalence if each Θ_k is a Quillen equivalence.

Proof. Clearly $\Theta_!$ (resp., Θ_*) preserves objectwise weak equivalences and objectwise cofibrations (resp., fibrations). \square

2.33. Note that the model categories constructed in this subsection are strictly more general than those of the previous subsection. Indeed, if K is an \mathbf{X} -small category and \mathbf{C} is a model \mathbf{X} -category, then the presheaf category $\mathbf{C}(K)$ is equivalent to the section category of the constant presheaf of categories on K at \mathbf{C} . This presheaf is both a left and right Quillen presheaf, so one reconstructs the injective and projective model structures, respectively.

3. Reedy model structures

Inverse, direct, and Reedy categories

Suppose \mathbf{E} a structured homotopical category. Then if A is a Reedy category, the category $\mathbf{E}(A)$ of presheaves $A^{\text{op}} \rightarrow \mathbf{E}$ is also a structured homotopical category. If \mathbf{C} a left (respectively, right) \mathbf{E} -model category, the category $\mathbf{C}(A)$ of presheaves $A^{\text{op}} \rightarrow \mathbf{C}$ has a Reedy left (resp., right) $\mathbf{E}(A)$ -model structure. We begin by reviewing some definitions and results analogous to [10, §5.1]. The proofs of the results below are similar to the proofs of the classical results for model categories, save only that one must periodically insert the phrases “with cofibrant domain” (resp., “with fibrant codomain”). We will therefore leave the proofs as an exercise.⁴

3.1. Suppose \mathbf{X} a universe, \mathbf{E} a structured homotopical \mathbf{X} -category, \mathbf{C} a left (respectively, right) model \mathbf{X} -category.

Definition 3.2. Suppose A an \mathbf{X} -small category, λ an \mathbf{X} -small ordinal.

(3.2.1) For any \mathbf{X} -small category A , a functor $d : A \rightarrow \lambda$ is called a *linear extension* of A if it reflects identities, that is, if a morphism f of A is an identity if and only if $d(f)$ is.

(3.2.2) An \mathbf{X} -small category A is said to be a *direct category* if there exists a linear extension $d : A \rightarrow \lambda$.

(3.2.3) An \mathbf{X} -small category A is said to be an *inverse category* if A^{op} is a direct category.

Suppose now C any \mathbf{X} -complete and \mathbf{X} -cocomplete \mathbf{X} -category.

(3.2.4) Suppose A an \mathbf{X} -small inverse category, α an object of A .

(3.2.4.1) The *latching category* at α is the full subcategory $\partial(A^{\text{op}}/\alpha)$ of the category (A^{op}/α) consisting of the nonidentity morphisms $\beta \rightarrow \alpha$. There are two forgetful functors:

$$F_\alpha : (A^{\text{op}}/\alpha) \rightarrow A^{\text{op}} \quad \text{and} \quad \partial F_\alpha : \partial(A^{\text{op}}/\alpha) \rightarrow A^{\text{op}}.$$

(3.2.4.2) The *latching functor* L_α for C is the composite functor

$$C(A) \xrightarrow{\partial F_\alpha^*} C(\partial(A^{\text{op}}/\alpha)) \xrightarrow{\text{colim}} C,$$

and the image of a diagram $X : A \rightarrow C$ is called the *latching object* $L_\alpha X$ of X at α .

(3.2.5) Suppose A an \mathbf{X} -small direct category, α and object of A .

(3.2.5.1) The *matching category* at α is the opposite category $\partial(\alpha/A^{\text{op}}) := (\partial(A/\alpha))^{\text{op}}$ of the latching category at α for A^{op} . There are two forgetful functors:

$$F^\alpha : (\alpha/A^{\text{op}}) \rightarrow A^{\text{op}} \quad \text{and} \quad \partial F^\alpha : \partial(\alpha/A^{\text{op}}) \rightarrow A^{\text{op}}.$$

⁴Alternatively, see [19, Propositions 2.6 and 2.7], where the case of left model categories is addressed.

(3.2.5.2) The *matching functor* M^α for C is the composite functor

$$C(A) \xrightarrow{\partial F^{\alpha,*}} C(\partial(A/\alpha)) \xrightarrow{\text{lim}} C,$$

and the image of a diagram $X : A \rightarrow C$ is called the *matching object* $M^\alpha X$ of X at α .

Proposition 3.3. (3.3.1) *For any \mathbf{X} -small inverse category A , the functor category $\mathbf{E}(A)$ has its projective structured homotopical structure, in which the weak equivalences and fibrations are defined objectwise, and a morphism $X \rightarrow Y$ is a cofibration if and only if for any object α of A , the induced morphism*

$$X_\alpha \amalg^{L_\alpha X} L_\alpha Y \rightarrow Y_\alpha$$

is so.

(3.3.2) *For any \mathbf{X} -small direct category A , the functor category $\mathbf{E}(A)$ has its injective structured homotopical structure, in which the weak equivalences and cofibrations are defined objectwise, and a morphism $X \rightarrow Y$ is a fibration if and only if for any object α of A , the induced morphism*

$$X_\alpha \rightarrow M^\alpha X \times_{M^\alpha Y} Y_\alpha$$

is so.

Theorem 3.4. (3.4.1) *For any \mathbf{X} -small inverse category A , the projective structured homotopical structure on the functor category $\mathbf{C}(A)$ is a left (resp., right) $\mathbf{E}(A)$ -model structure.*

(3.4.2) *For any \mathbf{X} -small direct category A , the injective structured homotopical structure on the functor category $\mathbf{C}(A)$ is a left (resp., right) $\mathbf{E}(A)$ -model structure.*

Proof. This is [10, Theorem 5.1.3], mutatis mutandis. \square

Proposition 3.5. (3.5.1) *For any \mathbf{X} -small inverse category A , a morphism $X \rightarrow Y$ with $\mathbf{E}(A)$ -cofibrant domain X (resp., a morphism $X \rightarrow Y$) of the functor category $\mathbf{C}(A)$ is a trivial cofibration in the projective left (resp., right) model structure if and only if for any object α of A , the induced morphism*

$$X_\alpha \amalg^{L_\alpha X} L_\alpha Y \rightarrow Y_\alpha$$

is so.

(3.5.2) *For any \mathbf{X} -small direct category A , a morphism $X \rightarrow Y$ (resp., a morphism $X \rightarrow Y$ with $\mathbf{E}(A)$ -fibrant codomain Y) of the functor category $\mathbf{C}(A)$ is a trivial fibration in the injective left (resp., right) model structure if and only if for any object α of A , the induced morphism*

$$X_\alpha \rightarrow M^\alpha X \times_{M^\alpha Y} Y_\alpha$$

is so.

Proof. This is [10, Theorem 5.1.3], mutatis mutandis. \square

Proposition 3.6. *Suppose $f : A \rightarrow B$ a functor of \mathbf{X} -small categories.*

(3.6.1) If A and B are inverse categories, then the adjunction

$$f_! : \mathbf{C}(A) \rightleftarrows \mathbf{C}(B) : f^*$$

is a Quillen adjunction between the projective left (resp., right) model categories.

(3.6.2) If A and B are direct categories, then the adjunction

$$f^* : \mathbf{C}(B) \rightleftarrows \mathbf{C}(A) : f_*$$

is a Quillen adjunction between the injective left (resp., right) model categories.

Proof. The functor f^* preserves any properties that are defined objectwise. \square

Definition 3.7. An \mathbf{X} -small Reedy category consists of the following data:

(3.7.A) an \mathbf{X} -small category A ,

(3.7.B) two lluf subcategories A^\rightarrow and A^\leftarrow of A , and

(3.7.C) a unique factorization of every morphism into a morphism of A^\leftarrow followed by a morphism of A^\rightarrow .

These data are subject to the following condition: there exist an ordinal λ and two linear extensions $A^\rightarrow \rightarrow \lambda$ and $(A^\leftarrow)^{\text{op}} \rightarrow \lambda$ such that the diagram

$$\begin{array}{ccc} & A^\rightarrow & \\ \text{Obj } A & \nearrow & \searrow \\ & & \lambda \\ & A^{\leftarrow \text{op}} & \nearrow \end{array}$$

commutes. Write i^\rightarrow (respectively, i^\leftarrow) for the inclusion $A^\rightarrow \rightarrow A$ (resp., for the inclusion $A^\leftarrow \rightarrow A$).

3.8. In other words, a Reedy category consists of a category with a degree function on its objects, so that any morphism can be factored in a functorial fashion as a morphism that decreases the degree followed by a morphism that increases the degree.

Lemma 3.9. If A is an \mathbf{X} -small Reedy category, then A^{op} is as well, with $(A^{\text{op}})^\rightarrow := (A^\leftarrow)^{\text{op}}$ and $(A^{\text{op}})^\leftarrow := (A^\rightarrow)^{\text{op}}$.

Proof. The unique factorization for A will work for A^{op} . \square

Lemma 3.10. Suppose A an \mathbf{X} -small Reedy category, C an arbitrary category, and $A \rightarrow C$ a fully faithful functor.

(3.10.1) For any object γ of C , the slice category (A/γ) is a Reedy category, wherein $(A/\gamma)^\rightarrow$ (respectively, $(A/\gamma)^\leftarrow$) is the lluf subcategory consisting of those morphisms mapping to A^\rightarrow (resp., to A^\leftarrow) under the obvious forgetful functor $(A/\gamma) \rightarrow A$.

(3.10.2) For any object γ of C , the slice category (γ/A) is a Reedy category, wherein $(\gamma/A)^\rightarrow$ (respectively, $(\gamma/A)^\leftarrow$) is the lluf subcategory consisting of those morphisms mapping to A^\rightarrow (resp., to A^\leftarrow) under the obvious forgetful functor $(\gamma/A) \rightarrow A$.

Proof. By the previous lemma, it suffices to show that (A/γ) is a Reedy category. It is clear that the composites $(A/\gamma)^\rightarrow \rightarrow A^\rightarrow \rightarrow \lambda$ and $(A/\gamma)^{\leftarrow, \text{op}} \rightarrow A^{\leftarrow, \text{op}} \rightarrow \lambda$ are linear extensions. The unique factorization for A gives a unique factorization for (A/γ) . \square

Lemma 3.11. *Suppose A an \mathbf{X} -small Reedy category. Then the diagram category $\mathbf{E}(A)$ has its Reedy structured homotopical structure, in which a morphism $\phi : X \rightarrow Y$ is a weak equivalence, cofibration, or fibration if and only if both $i^{\rightarrow, *}\phi$ in $\mathbf{E}(A^\rightarrow)$ and $i^{\leftarrow, *}\phi$ in $\mathbf{E}(A^\leftarrow)$ are so.*

Theorem 3.12. *Suppose A an \mathbf{X} -small Reedy category. Then the diagram category $\mathbf{C}(A)$ has its Reedy left (resp., right) $\mathbf{E}(A)$ -model structure, in which a morphism $\phi : X \rightarrow Y$ is a weak equivalence, cofibration, or fibration if and only if both $i^{\rightarrow, *}\phi$ in $\mathbf{C}(A^\rightarrow)$ and $i^{\leftarrow, *}\phi$ in $\mathbf{C}(A^\leftarrow)$ are so.*

Proof. This is [16, Theorem A], [10, Theorem 5.2.5], and [9, Theorems 15.3.4 and 15.3.15], *mutatis mutandis*. \square

3.13. Note in particular that the weak equivalences are the objectwise weak equivalences.

Lemma 3.14. *The Reedy model structure is functorial in the left (resp., right) model category; that is, suppose A an \mathbf{X} -small Reedy category, \mathbf{C} and \mathbf{D} left (resp., right) model \mathbf{X} -categories, and $F : \mathbf{C} \rightarrow \mathbf{D}$ a left Quillen functor. Then the induced functor $\mathbf{C}(A) \rightarrow \mathbf{D}(A)$ — which will also be denoted F — is left Quillen as well.*

Proof. Since F is a left adjoint, it commutes with all latching functors. \square

Theorem 3.15 (Hirschhorn, [9, Theorem 15.5.2]). *The category $A \times A$ has a natural Reedy category structure, for which the Reedy structured homotopical structure on $\mathbf{E}(A \times A)$ and $\mathbf{C}(A \times A)$ coincides with the “Reedy-Reedy” model structure on $\mathbf{E}(A)(A)$ and $\mathbf{E}(A)(A)$.*

Left and right fibrations of Reedy categories

We now address the question of the functoriality of the Reedy model structure in the Reedy category. For simplicity let us restrict attention to the case of a model category \mathbf{M} . That is, we will describe the circumstances under which a functor $A \rightarrow B$ induces a Quillen adjunction between $\mathbf{M}(A)$ and $\mathbf{M}(B)$.

Definition 3.16. Suppose A and B \mathbf{X} -small Reedy categories.

(3.16.1) A *morphism* $f : A \twoheadrightarrow B$ is a strictly commutative diagram of functors

$$\begin{array}{ccc} A^\rightarrow & \longrightarrow & B^\rightarrow \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \\ \uparrow & & \uparrow \\ A^\leftarrow & \longrightarrow & B^\leftarrow. \end{array}$$

(3.16.2) A morphism $f : A \twoheadrightarrow B$ is a *left fibration* if for any model \mathbf{X} -category \mathbf{M} , the adjunction

$$f^\star : \mathbf{M}(B) \rightleftarrows \mathbf{M}(A) : f_\star$$

is a Quillen adjunction. If $B = \star$, then one says that A is *left fibrant*.

(3.16.3) A morphism $f : A \twoheadrightarrow B$ is a *right fibration* if for any model \mathbf{X} -category \mathbf{M} , the adjunction

$$f_! : \mathbf{M}(A) \rightleftarrows \mathbf{M}(B) : f^\star$$

is a Quillen adjunction. If $B = \star$, then one says that A is *right fibrant*.

3.17. A Reedy model category is thus left (respectively, right) fibrant if and only if it has fibrant (resp., cofibrant) constants in the sense of Hirschhorn [9, Definition 15.10.1]. The notion of a left or right fibration is merely a relative version of Hirschhorn's concepts.

The Reedy model structure lives between the injective model structure and projective model structure on $\mathbf{M}(A)$, if they exist. That is, the identity functor induces a right Quillen functor in the direction $\mathbf{M}(A)_{\text{Reedy}} \rightarrow \mathbf{M}(A)_{\text{proj}}$ and a left Quillen functor in the direction $\mathbf{M}(A)_{\text{Reedy}} \rightarrow \mathbf{M}(A)_{\text{inj}}$. If A is direct (respectively, inverse), then the former (resp., latter) of these is an isomorphism of model categories. If A is left fibrant (respectively, right fibrant), then the fact that the constant functor is right (resp., left) Quillen is an indication that the Reedy model structure is closer to the projective (resp., injective) model structure.

Lemma 3.18. *If A and B are \mathbf{X} -small direct (respectively, inverse) categories, any morphism $f : A \twoheadrightarrow B$ is a left (resp., right) fibration.*

Proof. Immediate from 3.6. □

Lemma 3.19. *For any \mathbf{X} -small Reedy categories A and B , a morphism $f : A \twoheadrightarrow B$ is a left fibration if and only if the functor $f^{\text{op}} : A^{\text{op}} \twoheadrightarrow B^{\text{op}}$ is a right fibration.*

Proof. This follows from 3.9. □

Lemma 3.20. *For any \mathbf{X} -small Reedy categories A and B , a morphism $f : A \twoheadrightarrow B$ is a left (respectively, right) fibration if and only if the functor $f^\leftarrow : A^\leftarrow \twoheadrightarrow B^\leftarrow$ (resp., the functor $f^\rightarrow : A^\rightarrow \twoheadrightarrow B^\rightarrow$) is so.*

Proof. By the previous lemma, it suffices to prove the statement for left fibrations. Since Reedy cofibrations and fibrations are defined by restriction to the direct and

inverse subcategories, it follows that f is a left fibration if and only if f^\rightarrow and f^\leftarrow are left fibrations. But f^\rightarrow is automatically a left fibration by 3.18. \square

Lemma 3.21. *For any \mathbf{X} -small Reedy categories A and B , a morphism $f : A \rightarrow B$ is a left (respectively, right) fibration if and only if for any object β of B , the Reedy category (f/β) (resp., (β/f)) is left (resp., right) fibrant.*

Proof. It suffices to prove the statement for right fibrations, and by the previous lemma, it suffices to assume that A and B are direct categories. Now f is a right fibration if and only if, for any model category \mathbf{M} and any (trivial) cofibration $\phi : X \rightarrow Y$ of $\mathbf{M}(A)$, the induced morphism $f_! \phi : f_! X \rightarrow f_! Y$ is a (trivial) cofibration of $\mathbf{M}(B)$. But (trivial) cofibrations are defined objectwise; hence this is in turn equivalent to the assertion that for any model category \mathbf{M} , any (trivial) cofibration $\phi : X \rightarrow Y$ of $\mathbf{M}(A)$, and any object β of B , the morphism

$$f_! \phi_\beta : (f_! X)_\beta = \operatorname{colim}_{\alpha \in (f \circ p / \beta)} X_\alpha \rightarrow \operatorname{colim}_{\alpha \in (f \circ p / \beta)} Y_\alpha = (f_! Y)_\beta$$

is a (trivial) cofibration of \mathbf{M} . This is precisely the statement that the adjunction

$$\operatorname{colim} : \mathbf{M}(\beta/f) \rightleftarrows \mathbf{M} : \operatorname{const}$$

is a Quillen adjunction, i.e., that (β/f) is right fibrant. \square

Theorem 3.22. *For any \mathbf{X} -small Reedy categories A and B , a morphism $f : A \rightarrow B$ is a left (respectively, right) fibration if and only if for any object α of A and any morphism $f(\alpha) \rightarrow \beta$ (resp., $\beta \rightarrow f(\alpha)$) of B , the nerve of the category $\partial(\alpha/(f^\leftarrow/\beta))$ (resp., of the category $\partial((\beta/f^\rightarrow)/\alpha)$) is either empty or connected.*

Proof. This now follows from the previous lemma and Hirschhorn's necessary and sufficient condition for a Reedy category to have (co)fibrant constants [9, Proposition 15.10.2(1) and Corollary 15.10.5]. \square

Corollary 3.23. *Suppose A an \mathbf{X} -small Reedy category, C an arbitrary category with all finite products (respectively, finite coproducts), $A \rightarrow C$ a fully faithful functor. Suppose that for any object γ of C , the Reedy category (A/γ) (resp., (γ/A)) is left (resp., right) fibrant. Then for any morphism $\gamma \rightarrow \gamma'$ of C , the forgetful functor $(A/\gamma) \rightarrow (A/\gamma')$ (resp., $(\gamma'/A) \rightarrow (\gamma/A)$) is a left (resp., right) fibration.*

Proof. Again it suffices to prove the assertion for left fibrations. Using the characterization of the theorem, one sees that the forgetful functor $(A/\gamma) \rightarrow (A/\gamma')$ is a left fibration if and only if for any object α of (A/γ') , the Reedy category $(A/(\alpha \times \gamma'))$ is left fibrant. \square

Lemmata of inheritance

Let us now reiterate some familiar but nevertheless useful facts on the subject of the Reedy model structure. In particular, it inherits many good formal properties of \mathbf{M} (3.24, 3.33, 3.36, and 3.37).

Suppose A an \mathbf{X} -small Reedy category, \mathbf{E} a structured homotopical \mathbf{X} -category, \mathbf{C} a left (respectively, right) \mathbf{E} -model \mathbf{X} -category.

Lemma 3.24. *If \mathbf{C} is left \mathbf{E} -proper or right proper (resp., left proper or right \mathbf{E} -proper), then the Reedy left (resp., right) $\mathbf{E}(A)$ -model category $\mathbf{C}(A)$ is left $\mathbf{E}(A)$ -proper or right proper (resp., left proper or right $\mathbf{E}(A)$ -proper).*

Proof. This follows immediately from the observation that the Reedy weak equivalences, cofibrations, and fibrations are in particular objectwise weak equivalences, cofibrations, and fibrations. \square

3.25. The Reedy model structure on $\mathbf{C}(A)$ is frequently compatible with a natural symmetric monoidal structure, which arises from the use of objects $y_{\mathbf{C}}(\alpha)$ that represent evaluation at an object α of A . The category $\mathbf{C}(A)$ of \mathbf{C} -valued presheaves Y on A comprise the representable $\mathbf{Set}_{\mathbf{X}}(A)$ -valued presheaves on C , whose value on an object X of \mathbf{C} is the presheaf that assigns to any object α of A the morphisms in $\mathbf{C}(A)$ from a presheaf $y(\alpha) \boxtimes X$ to Y . Extending this correspondence to all presheaves on A in the usual fashion, one arrives at a fundamental adjunction of two variables (3.28) on $\mathbf{C}(A)$ with \mathbf{C} over $\mathbf{Set}_{\mathbf{X}}(A)$.

Notation 3.26. Suppose X an object of \mathbf{C} , and $Y : A^{\text{op}} \rightarrow \mathbf{C}$ a presheaf.

(3.26.1) Write $\text{mor}_{\mathbf{C}(A), \boxtimes}(-, Y)$ for the right Kan extension $a_* Y$ of Y along the opposite $a : A^{\text{op}} \rightarrow \mathbf{Set}_{\mathbf{X}}(A)^{\text{op}}$ of the Yoneda embedding.

(3.26.2) The copower functor

$$\begin{aligned} \mathbf{Set}_{\mathbf{X}} &\longrightarrow \mathbf{C} \\ S &\longmapsto S \cdot X \end{aligned}$$

induces the functor $- \boxtimes_{\mathbf{C}(A)} X : \mathbf{Set}_{\mathbf{X}}(A) \rightarrow \mathbf{C}(A)$.

(3.26.3) The object X corepresents a functor $\mathbf{C} \rightarrow \mathbf{Set}_{\mathbf{X}}$ and thus induces a functor $\text{Mor}_{\mathbf{C}(A), \boxtimes}^s(X, -) : \mathbf{C}(A) \rightarrow \mathbf{Set}_{\mathbf{X}}(A)$.

Lemma 3.27. *For any presheaf $K : A^{\text{op}} \rightarrow \mathbf{Set}_{\mathbf{X}}$ and any presheaf $Y : A^{\text{op}} \rightarrow \mathbf{C}$, there is an isomorphism*

$$\text{mor}_{\mathbf{C}(A), \boxtimes}(K, Y) \cong \int_{\alpha \in A} \text{mor}(K_{\alpha}, Y_{\alpha}).$$

Proof. This is the usual end formula for right Kan extensions. \square

Lemma 3.28. *The triple $(\boxtimes_{\mathbf{C}(A)}, \text{mor}_{\mathbf{C}(A), \boxtimes}, \text{Mor}_{\mathbf{C}(A), \boxtimes}^s)$ is an adjunction of two variables: for any presheaf $K : A^{\text{op}} \rightarrow \mathbf{Set}_{\mathbf{X}}$, any object X of \mathbf{C} , and any presheaf $Y : A^{\text{op}} \rightarrow \mathbf{C}$, there are natural isomorphisms*

$$\begin{aligned} \text{Mor}_{\mathbf{C}}(X, \text{mor}_{\mathbf{C}(A), \boxtimes}(K, Y)) &\cong \text{Mor}_{\mathbf{C}(A)}(K \boxtimes_{\mathbf{C}(A)} X, Y) \\ &\cong \text{Mor}_{\mathbf{Set}_{\mathbf{X}}(A)}(K, \text{Mor}_{\mathbf{C}(A), \boxtimes}^s(X, Y)). \end{aligned}$$

Proof. This follows from the relevant universal properties. \square

Notation 3.29. Write $y : A \rightarrow \mathbf{Set}_{\mathbf{X}}(A)$ for the Yoneda embedding.

Corollary 3.30. *Suppose $Y : A^{\text{op}} \rightarrow \mathbf{C}$ a presheaf; then for any object α of A , there is a natural isomorphism*

$$Y_{\alpha} \cong \text{mor}_{\mathbf{C}(A), \boxtimes}(y(\alpha), Y).$$

Corollary 3.31. *Suppose $Y : A^{\text{op}} \rightarrow \mathbf{C}$ a presheaf; then for any object α of A , there is a presheaf $\partial y(\alpha) : A^{\text{op}} \rightarrow \mathbf{C}$ and a natural isomorphism*

$$M^\alpha Y \cong \text{mor}_{\mathbf{C}(A), \square}(\partial y(\alpha), Y).$$

Proof. Set

$$\partial y(\alpha) := \text{colim}_{\alpha' \in (\alpha/A^{\leftarrow})} y(\alpha').$$

Then one shows easily that $\text{mor}_{\mathbf{C}(A), \square}(\partial y(\alpha), Y)$ is the desired limit. \square

Notation 3.32. For any set K of morphisms of \mathbf{C} , write

$$\Lambda \square K := \{(y(\alpha) \square X) \sqcup^{\partial y(\alpha) \square X} (\partial y(\alpha) \square Y) \rightarrow y(\alpha) \square Y \mid \alpha \in A, [X \rightarrow Y] \in K\}.$$

Lemma 3.33. *For any \mathbf{X} -small Reedy category A , the Reedy left (resp., right) $\mathbf{E}(A)$ -model category $\mathbf{C}(A)$ is \mathbf{X} -combinatorial if \mathbf{C} is.*

Proof. Since \mathbf{C} is \mathbf{X} -combinatorial, it is possible to choose \mathbf{X} -small sets of generating cofibrations and generating trivial cofibrations $I_{\mathbf{C}}$ and $J_{\mathbf{C}}$ such that the domains and codomains of $I_{\mathbf{C}}$ (respectively, of $J_{\mathbf{C}}$) are small with respect to $I_{\mathbf{C}}$ (resp., to $J_{\mathbf{C}}$); then $\Lambda \square I_{\mathbf{C}}$ and $\Lambda \square J_{\mathbf{C}}$ are \mathbf{X} -small sets of generating cofibrations and generating trivial cofibrations of the Reedy model structure on $\mathbf{C}(A)$ [9, Theorem 15.6.27]. Local presentability is inherited by functor categories; hence $\mathbf{C}(A)$ is \mathbf{X} -combinatorial. \square

Lemma 3.34. *For any \mathbf{X} -small Reedy category A , the Reedy model category $\mathbf{C}(A)$ is \mathbf{X} -tractable if \mathbf{C} is.*

Proof. We claim that if $X \rightarrow Y$ is a cofibration with cofibrant source in \mathbf{C} , then $(y(\alpha) \square X) \sqcup^{\partial y(\alpha) \square X} (\partial y(\alpha) \square Y)$ is cofibrant. Suppose that $T \rightarrow S$ is an objectwise trivial fibration. Then by adjunction, a morphism $y(\alpha) \square X \rightarrow S$ has a lifting if and only if the diagram

$$\begin{array}{ccc} & T_\alpha & \\ & \downarrow & \\ X & \longrightarrow & S_\alpha \end{array}$$

has a lifting. It follows from the cofibrancy of X and [9, Proposition 15.3.11] that $y(\alpha) \square X$ is cofibrant in the Reedy model structure on $\mathbf{C}(A)$. It is easy to see by a similar argument that $\partial y(\alpha) \square X \rightarrow \partial y(\alpha) \square Y$ is a cofibration, so

$$y(\alpha) \square X \rightarrow (y(\alpha) \square X) \sqcup^{\partial y(\alpha) \square X} (\partial y(\alpha) \square Y)$$

is a cofibration, whence follows the claim, and thus the lemma. \square

Reedy diagrams in a symmetric monoidal model category

Suppose now A an \mathbf{X} -small Reedy category and $(\mathbf{M}, \otimes_{\mathbf{M}}, \underline{\text{Mor}}_{\mathbf{M}}^{\mathbf{M}})$ a symmetric monoidal model \mathbf{X} -category.

Notation 3.35. Suppose X and Y objects of $\mathbf{M}(A)$ and Z an object of \mathbf{M} . Set

$$\begin{aligned}\underline{\text{Mor}}_{\mathbf{M}(A)}^{\mathbf{M}}(X, Y) &:= \int_{\alpha \in A^{\text{op}}} \underline{\text{Mor}}_{\mathbf{M}}(X_{\alpha}, Y_{\alpha}), \\ (Z \otimes_{\mathbf{M}(A)}^{\mathbf{M}} X)_{\alpha} &:= Z \otimes_{\mathbf{M}} X_{\alpha}, \\ \text{mor}_{\mathbf{M}(A)}^{\mathbf{M}}(Z, Y)_{\alpha} &:= \underline{\text{Mor}}_{\mathbf{M}}(Z, Y_{\alpha}),\end{aligned}$$

for any object α of A . This gives $\mathbf{M}(A)$ the structure of an \mathbf{M} -category.

Lemma 3.36. *With the \mathbf{M} -structure of 3.35, the Reedy model category $\mathbf{M}(A)$ is an \mathbf{M} -model category.*

Proof. To verify the pushout-product axiom, suppose $f : Z \rightarrow Z'$ a cofibration of \mathbf{M} , and $i : X \rightarrow Y$ a cofibration of $\mathbf{M}(A)$; then for any object α of A , the morphism

$$\begin{array}{c} ((Z \otimes Y) \sqcup^{Z \otimes X} (Z' \otimes X))_{\alpha} \sqcup^{M^{\alpha}((Z \otimes Y) \sqcup^{Z \otimes X} (Z' \otimes X))} M^{\alpha}(Z' \otimes Y) \\ \downarrow \\ (Z' \otimes Y)_{\alpha} \end{array}$$

is isomorphic to the morphism

$$(Z \otimes Y_{\alpha}) \sqcup^{Z \otimes (X_{\alpha} \sqcup^{M^{\alpha} X} M^{\alpha} Y)} (Z' \otimes (X_{\alpha} \sqcup^{M^{\alpha} X} M^{\alpha} Y)) \rightarrow Z' \otimes Y_{\alpha},$$

which, by the pushout-product axiom for \mathbf{M} , is a cofibration that is trivial if either f or i is. \square

Corollary 3.37. *If, in addition, \mathbf{M} is a model \mathbf{V} -category for some symmetric monoidal model \mathbf{X} -category \mathbf{V} , then $\mathbf{M}(A)$ is also.*

Lemma 3.38. *There is a functor $y_{\mathbf{M}} : A \rightarrow \mathbf{M}(A)$ such that for any object α of A and any $Y : A^{\text{op}} \rightarrow \mathbf{M}$, there is a canonical isomorphism*

$$\underline{\text{Mor}}_{\mathbf{M}(A)}(y_{\mathbf{M}}(\alpha), Y) \cong Y_{\alpha}.$$

Moreover, if the unit $1_{\mathbf{M}}$ for the symmetric monoidal structure on \mathbf{M} is cofibrant, then for every such object α , $y_{\mathbf{M}}(\alpha)$ is cofibrant.

Proof. Set $y_{\mathbf{M}}(\alpha) := y(\alpha) \boxplus 1_{\mathbf{M}}$. The first part of the result now follows from the enriched Yoneda lemma. \square

Corollary 3.39. *If $F : \mathbf{M} \rightarrow \mathbf{N}$ is a left Quillen functor of symmetric monoidal model \mathbf{X} -categories such that $F(1_{\mathbf{M}}) \cong 1_{\mathbf{N}}$,⁵ then $F(y_{\mathbf{M}}(\alpha)) \cong y_{\mathbf{N}}(\alpha)$ for every object α of A .*

Corollary 3.40. *For any object α of A , there is a simplicial object $\partial y_{\mathbf{M}}(\alpha)$ of \mathbf{M} — which is cofibrant if $1_{\mathbf{M}}$ is — such that for any $Y : A^{\text{op}} \rightarrow \mathbf{M}$, there is a canonical isomorphism*

$$\underline{\text{Mor}}_{\mathbf{M}(A)}(\partial y_{\mathbf{M}}(\alpha), Y) \cong M^{\alpha} Y.$$

⁵Note that one need *not* assume that F itself is symmetric monoidal.

3.41. The exterior tensor product $\mathbf{M}(A) \times \mathbf{M}(A) \rightarrow \mathbf{M}(A \times A)$ is part of a Quillen adjunction of two variables. In order to see this, we quote the following result of Hirschhorn.

Notation 3.42. Denote by

$$\begin{aligned} \boxtimes_{\mathbf{M}(A)} : \mathbf{M}(A) \times \mathbf{M}(A) &\longrightarrow \mathbf{M}(A \times A) \\ \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)} : \mathbf{M}(A)^{\text{op}} \times \mathbf{M}(A \times A) &\longrightarrow \mathbf{M}(A) \\ \text{mor}_{\boxtimes, \mathbf{M}(A)} : \mathbf{M}(A)^{\text{op}} \times \mathbf{M}(A \times A) &\longrightarrow \mathbf{M}(A) \end{aligned}$$

the functors defined by the formulæ

$$(X \boxtimes_{\mathbf{M}(A)} Y)_{(\alpha, \alpha')} := X_{\alpha} \otimes_{\mathbf{M}} Y_{\alpha'}, \quad (3.42.1)$$

$$\underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(Y, F)_{\alpha} := \underline{\text{Mor}}_{\mathbf{M}(A \times A)}^{\mathbf{M}}((y_{\mathbf{M}}(\alpha) \boxtimes_{\mathbf{M}(A)} Y), F), \quad (3.42.2)$$

$$\text{mor}_{\boxtimes, \mathbf{M}(A)}(X, F)_{\alpha} := \underline{\text{Mor}}_{\mathbf{M}(A \times A)}^{\mathbf{M}}((X \boxtimes_{\mathbf{M}(A)} y_{\mathbf{M}}(\alpha)), F), \quad (3.42.3)$$

for any objects X and Y of $\mathbf{M}(A)$, any $F : A^{\text{op}} \times A^{\text{op}} \rightarrow \mathbf{M}$, and any objects α, α' of A .

Proposition 3.43. *The triple $(\boxtimes, \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}, \text{mor}_{\boxtimes, \mathbf{M}(A)})$ is an adjunction of two variables from $\mathbf{M}(A) \times \mathbf{M}(A)$ to $\mathbf{M}(A \times A)$.*

Proof. This is an easy consequence of the Fubini theorem for ends and the representability properties of $y_{\mathbf{M}}(\alpha)$. \square

Lemma 3.44. *For any pair of objects α and β of A , there is a canonical isomorphism*

$$y_{\mathbf{M}}(\alpha) \boxtimes y_{\mathbf{M}}(\beta) \cong y_{\mathbf{M}}(\alpha, \beta)$$

in the category $\mathbf{M}(A \times A)$.

Proof. This follows from the observation that for any \mathbf{X} -small sets S and T , there is a canonical isomorphism

$$(S \cdot \mathbf{1}) \otimes (T \cdot \mathbf{1}) \cong (S \times T) \cdot \mathbf{1}$$

in \mathbf{M} . \square

Corollary 3.45. *For any object α of A and any $F : A^{\text{op}} \times A^{\text{op}} \rightarrow \mathbf{M}$, there is a canonical isomorphism*

$$\underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(y(\alpha), F) \cong F(\alpha, -)$$

in $\mathbf{M}(A)$.

Corollary 3.46. *For any object β of A , any object X of $\mathbf{M}(A)$, and any functor $F : A^{\text{op}} \times A^{\text{op}} \rightarrow \mathbf{M}$, there is a canonical isomorphism*

$$\underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(X, F)_{\beta} \cong \underline{\text{Mor}}_{\mathbf{M}(A)}^{\mathbf{M}}(X, F(-, \beta))$$

in \mathbf{M} .

Corollary 3.47. *For any object β of A , any object X of $\mathbf{M}(A)$, and any functor $F : A^{\text{op}} \times A^{\text{op}} \rightarrow \mathbf{M}$, there is a canonical isomorphism*

$$M^\beta \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(X, F) \cong \underline{\text{Mor}}_{\mathbf{M}(A)}^{\mathbf{M}}(X, M^{(-, \beta)} F)$$

in \mathbf{M} .

Proposition 3.48. *The adjunction of two variables $(\boxtimes, \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}, \text{mor}_{\boxtimes, \mathbf{M}(A)})$ is a Quillen adjunction of two variables.*

Proof. Now suppose $i : X \rightarrow Y$ a cofibration of $\mathbf{M}(A)$, and suppose $p : F \rightarrow G$ a (trivial) fibration of $\mathbf{M}(A)$. Now by 3.15, for any object β of A , p induces a (trivial) fibration

$$F(-, \beta) \rightarrow M^{(-, \beta)}(p)$$

of $\mathbf{M}(A)$, where

$$M^{(-, \beta)}(p) := M^{(-, \beta)} F \times_{M^{(-, \beta)} G} G(-, \beta).$$

Since $\mathbf{M}(A)$ is a model \mathbf{M} -category, it follows that the induced morphism

$$\begin{array}{c} \underline{\text{Mor}}_{\mathbf{M}(A)}(Y, F(-, \beta)) \\ \downarrow \\ (\underline{\text{Mor}}_{\mathbf{M}(A)}(Y, M^{(-, \beta)}(p))) \times_{\underline{\text{Mor}}_{\mathbf{M}(A)}(X, M^{(-, \beta)}(p))} \underline{\text{Mor}}_{\mathbf{M}(A)}(X, F(-, \beta)) \end{array}$$

is a fibration of \mathbf{M} , which is trivial if either i or p is. But this morphism is isomorphic to the morphism $U_\beta \rightarrow M^\beta U \times_{M^\beta V} V_\beta$, wherein

$$U := \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(Y, F);$$

$$V := \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(X, F) \times_{\underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(X, G)} \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(Y, G).$$

Hence $U \rightarrow V$ is a fibration of $\mathbf{M}(A)$, which is trivial if i or p is. \square

Proposition 3.49. *Suppose*

$$F : \mathbf{M}(A \times A) \rightleftarrows \mathbf{M}(A) : U$$

a Quillen adjunction. Then the triple $(\otimes_{\mathbf{M}(A), F}, \underline{\text{Mor}}_{\mathbf{M}(A), F}, \text{mor}_{\mathbf{M}(A), F})$, defined by the formulæ

$$X \otimes_{\mathbf{M}(A), F} Y := F(X \boxtimes_{\mathbf{M}(A)} Y), \quad (3.49.1)$$

$$\underline{\text{Mor}}_{\mathbf{M}(A), F}(Y, Z) := \underline{\text{Mor}}_{\boxtimes, \mathbf{M}(A)}(Y, UZ), \quad (3.49.2)$$

$$\text{mor}_{\mathbf{M}(A), F}(X, Z)_\alpha := \text{mor}_{\boxtimes, \mathbf{M}(A)}(X, UZ), \quad (3.49.3)$$

for any objects X, Y , and Z of $\mathbf{M}(A)$, is a Quillen adjunction of two variables from $\mathbf{M}(A) \times \mathbf{M}(A)$ to $\mathbf{M}(A)$.

Proof. Suppose $i : U \rightarrow V$ and $j : X \rightarrow Y$ cofibrations of $\mathbf{M}(A)$. Then $i \square_{\mathbf{M}(A), F} j = F(i \square_{\boxtimes, \mathbf{M}(A)} j)$, which is a cofibration that is trivial if either i or j is. \square

Corollary 3.50. *Suppose A monoidal, with a structure*

$$\circ : A \times A \rightarrow A$$

that defines a right fibration of Reedy categories. Then the Day convolution product

$$\otimes_{\mathbf{M}(A), \circ} := \circ!(- \boxtimes_{\mathbf{M}(A)} -)$$

is part of a Quillen adjunction of two variables on $\mathbf{M}(A)$.

Theorem 3.51. *If A is left fibrant and the morphisms of A^\leftarrow are epimorphisms, then the diagonal symmetric monoidal structure given by*

$$(X \otimes_{\mathbf{M}(A)}^{\mathbf{M}(A)} Y)_\alpha := X_\alpha \otimes Y_\alpha, \quad (3.51.1)$$

$$\underline{\mathrm{Mor}}_{\mathbf{M}(A)}^{\mathbf{M}(A)}(X, Y)_\alpha := \underline{\mathrm{Mor}}_{\mathbf{M}(A)}^{\mathbf{M}(A)}(y_{\mathbf{M}(A)}(\alpha) \otimes_{\mathbf{M}(A)}^{\mathbf{M}(A)} X, Y), \quad (3.51.2)$$

for any objects X and Y of $\mathbf{M}(A)$ and an object α of A , gives $\mathbf{M}(A)$ the structure of a symmetric monoidal model category.

Proof. The unit axiom follows from the fact that the constant functor is symmetric monoidal and preserves cofibrant objects and equivalences.

It now suffices to show that the diagonal functor $\Delta : A \rightarrow A \times A$ is a left fibration. This is equivalent to showing, for any objects α, β, γ of A , and any pair of morphisms $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ of A^\leftarrow , that the nerve of the category $\partial(\alpha/(\Delta^\leftarrow/(\beta, \gamma)))$ is either empty or connected. Since A is left fibrant, the nerve of the category $\partial(\alpha/A^\leftarrow)$ is either empty or connected. Hence if

$$\begin{array}{ccc} & (\delta, \delta) & \\ \nearrow & & \searrow \\ (\alpha, \alpha) & \longrightarrow & (\beta, \gamma) \\ \searrow & & \nearrow \\ & (\epsilon, \epsilon) & \end{array}$$

is a commutative diagram of $A^\leftarrow \times A^\leftarrow$, then there exists a zig-zag of morphisms of $A^\leftarrow \times A^\leftarrow$ connecting (δ, δ) to (ϵ, ϵ) *under* (α, α) . To see that these morphisms are morphisms over (β, γ) as well, we can, without loss of generality, suppose that there is a morphism $(\delta, \delta) \rightarrow (\epsilon, \epsilon)$ of $((\alpha, \alpha)/(A^\leftarrow \times A^\leftarrow))$. Hence the left half and the exterior square of the diagram

$$\begin{array}{ccc} & (\delta, \delta) & \\ \nearrow & \downarrow & \searrow \\ (\alpha, \alpha) & & (\beta, \gamma) \\ \searrow & \downarrow & \nearrow \\ & (\epsilon, \epsilon) & \end{array}$$

commute. But since $(\alpha, \alpha) \rightarrow (\epsilon, \epsilon)$ is an epimorphism, it follows that the left half of this diagram commutes as well. \square

Example 3.52. For any \mathbf{X} -small simplicial set or category K , the category $\mathbf{M}(\Delta/K)$ is symmetric monoidal with the diagonal symmetric monoidal structure.

Mapping spaces and resolutions

After briefly reviewing the hammock localization, we show how the Mor-spaces therein can be computed using cosimplicial and simplicial resolutions. The standard references on the hammock localization are the triple of papers [5], [6], and [7] of W. G. Dwyer and D. Kan. A more modern treatment can be found in [8].

3.53. Suppose \mathbf{X} a universe, $(\mathbf{C}, w\mathbf{C})$ a \mathbf{Y} -small homotopical \mathbf{X} -category.

Definition 3.54. Let \mathbf{O} denote the category of totally ordered finite sets, and let \mathbf{F} denote the category of finite sets; denote by $|-| : \mathbf{O} \rightarrow \mathbf{F}$ the obvious forgetful functor. Consider the set $|\mathbf{1}| = \{0, 1\}$, and define the category \mathbf{T} of *types* as the slice category $(\mathbf{O}/|\mathbf{1}|)$.

3.55. Thus a type (\mathbf{p}, s) consists of a totally ordered finite set $\mathbf{p} = [0 \rightarrow 1 \rightarrow \dots \rightarrow p] \in \mathbf{O}$ with a choice of a partition, given by a map $s : |p| \rightarrow |\mathbf{1}|$ in \mathbf{F} , and a morphism of types is a morphism in \mathbf{O} that respects the partition.

Lemma 3.56. *The category \mathbf{T} is monoidal under the obvious concatenation product \circ .*

Definition 3.57. Suppose (\mathbf{p}, s) a type.

(3.57.1) Define a category $Z_{\mathbf{p},s}$ whose objects are the natural numbers $-1, 0, 1, \dots, p$, and whose morphisms are freely generated by the morphisms between consecutive numbers:

$$\text{Mor}_{Z_{\mathbf{p},s}}(a-1, a) := \begin{cases} \emptyset & \text{if } s(a) = 0; \\ \{f_a\} & \text{if } s(a) = 1, \end{cases}$$

and

$$\text{Mor}_{Z_{\mathbf{p},s}}(a, a-1) := \begin{cases} \{f_a\} & \text{if } s(a) = 0; \\ \emptyset & \text{if } s(a) = 1. \end{cases}$$

Hence f_a is always the unique morphism between $a-1$ and a , and it goes down if $s(a) = 0$, and up if $s(a) = 1$.

(3.57.2) For any type (\mathbf{p}, s) , the category $Z_{\mathbf{p},s}$ contains two lluf subcategories $L_{\mathbf{p},s}$ and $R_{\mathbf{p},s}$ whose morphisms are the morphisms of $Z_{\mathbf{p},s}$ to the left or to the right, respectively; i.e., the morphisms are given by

$$\text{Mor}_{L_{\mathbf{p},s}}(a, b) := \begin{cases} \emptyset & \text{if } a < b; \\ \text{Mor}_{Z_{\mathbf{p},s}}(a, b) & \text{if } a \geq b, \end{cases}$$

and

$$\text{Mor}_{R_{\mathbf{p},s}}(a, b) := \begin{cases} \text{Mor}_{Z_{\mathbf{p},s}}(a, b) & \text{if } a \leq b; \\ \emptyset & \text{if } a > b. \end{cases}$$

Lemma 3.58. *There is a canonical isomorphism of categories*

$$Z_{(\mathbf{p},s) \circ (\mathbf{p}',s')} \cong Z_{(\mathbf{p},s)} \vee Z_{(\mathbf{p}',s')},$$

where the object $p \in Z_{(\mathbf{p},s)}$ is identified with $-1 \in Z_{(\mathbf{p}',s')}$.

3.59. Observe that we do not attach to a morphism of types a functor between the corresponding categories. Functors out of the $Z_{\mathbf{p},s}$ will actually be *covariant* in the type (\mathbf{p}, s) .

Definition 3.60. (3.60.1) Suppose X and Y objects of \mathbf{C} . A *restricted zig-zag from X to Y of type (p, s)* is a functor $A : Z_{\mathbf{p},s} \rightarrow \mathbf{C}$ such that $X = A(-1)$, $Y = A(p)$, and the composite functor $L_{\mathbf{p},s} \rightarrow \mathbf{C}$ factors through the inclusion $w\mathbf{C} \rightarrow \mathbf{C}$.

(3.60.2) A *morphism of restricted zig-zags from X to Y of type (\mathbf{p}, s)* is simply a morphism of functors all of whose components are in $w\mathbf{C}$, and whose components at -1 and p are the identities on X and Y , respectively. Hence we have the *category $w\text{Mor}_{\mathbf{C}}^{\mathbf{p},s}(X, Y)$ of restricted zig-zags from X to Y of type (\mathbf{p}, s)* .

(3.60.3) One has the obvious composition

$$w\text{Mor}_{\mathbf{C}}^{\mathbf{p},s}(X, Y) \times w\text{Mor}_{\mathbf{C}}^{\mathbf{p}',s'}(Y, Z) \rightarrow w\text{Mor}_{\mathbf{C}}^{(\mathbf{p},s) \circ (\mathbf{p}',s')}(X, Z)$$

given by concatenation.

(3.60.4) A morphism

$$t : (\mathbf{p}, s) \rightarrow (\mathbf{p}', s')$$

of types gives rise to a functor

$$t_! : w\text{Mor}_{\mathbf{C}}^{\mathbf{p},s}(X, Y) \rightarrow w\text{Mor}_{\mathbf{C}}^{\mathbf{p}',s'}(X, Y)$$

that sends a restricted zig-zag A of type (\mathbf{p}, s) to a restricted zig-zag $t_!A$ of type (\mathbf{p}', s') , where for any the morphism $t_!A(f_{a'})$ is the composite of all the morphisms $A(f_a)$ such that $t(a) = a'$ (where of course this is taken to be the identity if there are no such morphisms).

(3.60.5) This is compatible with composition and hence defines a functor

$$w\text{Mor}_{\mathbf{C}}^{(-)}(X, Y) : \mathbf{T} \rightarrow \mathbf{Cat}_{\mathbf{Y}}.$$

(3.60.6) The *hammock localization* of $(\mathbf{C}, w\mathbf{C})$ is the $s\mathbf{Set}_{\mathbf{Y}}$ -category $L^H(\mathbf{C}, w\mathbf{C})$ whose objects are exactly those of \mathbf{C} , with

$$\text{Mor}_{L^H(\mathbf{C}, w\mathbf{C})}(X, Y) = \text{colim}_{(\mathbf{p},s) \in \mathbf{T}} \nu_{\bullet}(w\text{Mor}_{\mathbf{C}}^{\mathbf{p},s}(X, Y))$$

for any objects X and Y . The composition is given by the concatenation.

Notation 3.61. Write $\mathbf{R}\text{Mor}_{(\mathbf{C}, w\mathbf{C})}$ for the simplicial mapping space functor

$$\begin{aligned} (w\mathbf{C}^{-1}\mathbf{C})^{\text{op}} \times w\mathbf{C}^{-1}\mathbf{C} &\longrightarrow \text{Ho } s\mathbf{Set}_{\mathbf{Y}} \\ (X, Y) &\longmapsto \text{Mor}_{L(\mathbf{C}, w\mathbf{C})}(X, Y); \end{aligned}$$

the subscripts may omitted when the context is sufficiently clear.

3.62. Suppose now that \mathbf{C} is a left or right model or model category. In this case, one can compute $\mathbf{R}\text{Mor}_{\mathbf{C}}(X, Y)$ without using the entire colimit over types. To this end,

(3.62.A) let $\mathbf{p}_L := \mathbf{3}$ and $s_L : |\mathbf{3}| \rightarrow |\mathbf{1}|$ be the map $(0, 1, 2, 3) \mapsto (0, 1, 0, 1)$;

(3.62.B) let $\mathbf{p}_R := \mathbf{3}$ and $s_R : |\mathbf{3}| \rightarrow |\mathbf{1}|$ be the map $(0, 1, 2, 3) \mapsto (1, 0, 1, 0)$; and

(3.62.C) let $\mathbf{p}_M := \mathbf{2}$ and $s_M : |\mathbf{2}| \rightarrow |\mathbf{1}|$ be the map $(0, 1, 2) \mapsto (0, 1, 0)$.

Now one can consider the full subcategories

(3.62.D) $w \text{Mor}_{\mathbf{C}}^L(X, Y) \subset w \text{Mor}_{\mathbf{C}}^{(\mathbf{p}_L, s_L)}(X, Y)$, comprised of functors $Z_{\mathbf{p}_L, s_L} \rightarrow \mathbf{C}$ in which f_0 and f_3 are trivial fibrations, and f_2 is a trivial cofibration;

(3.62.E) $w \text{Mor}_{\mathbf{C}}^R(X, Y) \subset w \text{Mor}_{\mathbf{C}}^{(\mathbf{p}_R, s_R)}(X, Y)$, comprised of functors $Z_{\mathbf{p}_R, s_R} \rightarrow \mathbf{C}$ in which f_0 and f_3 are trivial cofibrations, and f_1 is a trivial fibration;

(3.62.F) $w \text{Mor}_{\mathbf{C}}^M(X, Y) \subset w \text{Mor}_{\mathbf{C}}^{(\mathbf{p}_M, s_M)}(X, Y)$ consisting of functors $Z_{\mathbf{p}_M, s_M} \rightarrow \mathbf{C}$ in which f_0 is a trivial fibration, and f_2 is a trivial cofibration.

Another strategy is to use cosimplicial and simplicial resolutions, as in the following definition.

Definition 3.63. A functor $\Gamma^\bullet : \mathbf{C} \rightarrow (c\mathbf{C})_c$ (respectively, a functor $\Lambda_\bullet : \mathbf{C} \rightarrow (s\mathbf{C})_f$) is a *cosimplicial* (resp., *simplicial*) *resolution functor* if there is a chain of objectwise weak equivalences $\text{const}^\bullet \rightarrow r \text{const}^\bullet \leftarrow \Gamma^\bullet$ (resp., of objectwise weak equivalences $\Lambda_\bullet \leftarrow q \text{const}_\bullet \rightarrow \text{const}_\bullet$), where the functor const^\bullet (resp., const_\bullet) assigns to any object X of \mathbf{C} the constant cosimplicial (resp., simplicial) object at X .

Scholium 3.64 (Dwyer-Kan). *Suppose \mathbf{C} a left model, right model, or model category, and let $D = L, R, \text{ or } M$, accordingly. Suppose $Q_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}_c$ a cofibrant replacement functor, $R_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}_f$ a fibrant replacement functor, $\Gamma_{\mathbf{C}}^\bullet : \mathbf{C} \rightarrow (c\mathbf{C})_c$ a cosimplicial resolution functor, and $\Lambda_{\mathbf{C}, \bullet} : \mathbf{C} \rightarrow (s\mathbf{C})_f$ a simplicial resolution functor; then there are natural weak equivalences of the simplicial sets*

$$\begin{array}{ccc}
 \text{Mor}_{\mathbf{C}}(\Gamma_{\mathbf{C}}^\bullet X, R_{\mathbf{C}} Y) & & \text{Mor}_{\mathbf{C}}(Q_{\mathbf{C}} X, \Lambda_{\mathbf{C}, \bullet} Y) \\
 & \searrow & \swarrow \\
 & \text{diag Mor}_{\mathbf{C}}(\Gamma_{\mathbf{C}}^\bullet X, \Lambda_{\mathbf{C}, \bullet} Y) & \\
 & \uparrow & \\
 \text{hocolim}_{(p,q) \in \Delta^{\text{op}} \times \Delta^{\text{op}}} \text{Mor}_{\mathbf{M}}(\Gamma_{\mathbf{C}}^p X, \Lambda_{\mathbf{C}, q} Y) & & \\
 & \downarrow & \\
 \nu_\bullet w \text{Mor}_{\mathbf{C}}^D(X, Y) & & \\
 & \downarrow & \\
 \text{Mor}_{L^H \mathbf{C}}(X, Y). & &
 \end{array}$$

Corollary 3.65. *If \mathbf{C} is a left or right model category, then for any objects X and Y of \mathbf{C} , the simplicial set $\mathbf{R} \text{Mor}_{\mathbf{C}}(X, Y)$ is homotopically \mathbf{X} -small; that is, it has the homotopy type of an \mathbf{X} -small simplicial set.*

Corollary 3.66 ([9, 17.7.7]). *The following are equivalent for a morphism $A \rightarrow B$ of a right or left model category \mathbf{C} .*

(3.66.1) *The morphism $A \rightarrow B$ is a weak equivalence.*

(3.66.2) For any fibrant object Z of \mathbf{C} , the induced morphism

$$\mathbf{R}\mathrm{Mor}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(A, Z)$$

is an isomorphism of $\mathrm{Ho}\,s\mathrm{Set}_{\mathbf{X}}$.

(3.66.3) For any cofibrant object X of \mathbf{M} , the induced morphism

$$\mathbf{R}\mathrm{Mor}_{\mathbf{C}}(X, A) \rightarrow \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(X, B)$$

is an isomorphism of $\mathrm{Ho}\,s\mathrm{Set}_{\mathbf{X}}$.

4. (Enriched) left Bousfield localization

Definition and existence of left Bousfield localizations

Here we review some highlights from the general theory of left Bousfield localization. Since the published references do not include a proof of the existence theorem of J. Smith, I include it solely for convenience of reference, with the caveat that the result should in no way be construed as mine.

4.1. Suppose \mathbf{X} a universe, \mathbf{M} a model \mathbf{X} -category.

Definition 4.2. Suppose H a set of homotopy classes of morphisms of \mathbf{M} .

(4.2.1) A *left Bousfield localization of \mathbf{M} with respect to H* is a model \mathbf{X} -category $L_H\mathbf{M}$, equipped with a left Quillen functor $\mathbf{M} \rightarrow L_H\mathbf{M}$ that is initial among left Quillen functors $F : \mathbf{M} \rightarrow \mathbf{N}$ to model \mathbf{X} -categories \mathbf{N} with the property that for any f representing a class in H , $\mathbf{L}F(f)$ is an isomorphism of $\mathrm{Ho}\,\mathbf{N}$.

(4.2.2) An object Z of \mathbf{M} is *H -local* if for any morphism $A \rightarrow B$ representing an element of H , the morphism

$$\mathbf{R}\mathrm{Mor}_{\mathbf{M}}(B, Z) \rightarrow \mathbf{R}\mathrm{Mor}_{\mathbf{M}}(A, Z)$$

is an isomorphism of $\mathrm{Ho}\,s\mathrm{Set}_{\mathbf{X}}$.

(4.2.3) A morphism $A \rightarrow B$ of \mathbf{M} is an *H -local equivalence* if for any H -local object Z , the morphism

$$\mathbf{R}\mathrm{Mor}_{\mathbf{M}}(B, Z) \rightarrow \mathbf{R}\mathrm{Mor}_{\mathbf{M}}(A, Z)$$

is an isomorphism of $\mathrm{Ho}\,s\mathrm{Set}_{\mathbf{X}}$.

Lemma 4.3. *When it exists, the left Bousfield localization of \mathbf{M} with respect to H is unique up to a unique isomorphism of model \mathbf{X} -categories.*

Proof. Initial objects are essentially unique. □

4.4. Left Bousfield localizations of left proper, \mathbf{X} -combinatorial model \mathbf{X} -categories with respect to \mathbf{X} -small sets of homotopy classes of morphisms are guaranteed to exist, as we shall now demonstrate. For the remainder of this section, suppose H a set of homotopy classes of morphisms of \mathbf{M} . The characterization and existence of left Bousfield localizations are the central objectives of the next few results. It is a familiar fact that if a model structure exists on \mathbf{M} with the same cofibrations whose weak equivalences are the H -local weak equivalences, then this is the left Bousfield

localization. The central point is thus to determine the existence of such a model structure. Smith's existence theorem 4.7 hinges on the recognition principle 2.2 and the following pair of technical lemmata.

Lemma 4.5. *If \mathbf{M} is \mathbf{X} -combinatorial, and H is \mathbf{X} -small, then the set of H -local objects of \mathbf{M} comprise an accessibly embedded, accessible subcategory of $\mathbf{M}(\mathbf{1})$.*

Proof. Choose an accessible fibrant replacement functor $r_{\mathbf{M}}$ for \mathbf{M} , a functorial cosimplicial resolution functor $\Gamma_{\mathbf{M}}^{\bullet} : \mathbf{M} \rightarrow (c\mathbf{M})_c$, and an \mathbf{X} -small set S of representatives for all and only the homotopy classes of H . Then the functor

$$\begin{aligned} \mathbf{M} &\longrightarrow s\mathbf{Set}_{\mathbf{X}}(\mathbf{1}) \\ Z &\longmapsto \coprod_{f \in S} f_{\Gamma_{\mathbf{M}}^{\bullet}, r_{\mathbf{M}}}^{\star}(Z) \end{aligned}$$

is accessible, where

$$f_{\Gamma_{\mathbf{M}}^{\bullet}, r_{\mathbf{M}}}^{\star}(Z) : \text{Mor}_{\mathbf{M}}(\Gamma_{\mathbf{M}}^{\bullet} Y, r_{\mathbf{M}} Z) \rightarrow \text{Mor}_{\mathbf{M}}(\Gamma_{\mathbf{M}}^{\bullet} X, r_{\mathbf{M}} Z)$$

is the morphism of simplicial sets induced by $f : X \rightarrow Y$. Since the full subcategory of $s\mathbf{Set}_{\mathbf{X}}(\mathbf{1})$ comprised of weak equivalences is accessibly embedded and accessible, the full subcategory of H -local objects is also accessibly embedded and accessible. \square

Lemma 4.6. *If \mathbf{M} is \mathbf{X} -combinatorial, and H is \mathbf{X} -small, then the set of H -local equivalences of \mathbf{M} comprise an accessibly embedded, accessible subcategory of $\mathbf{M}^{\mathbf{1}}$.*

Proof. This follows from the previous lemma and the fact that for sufficiently large regular \mathbf{X} -small cardinals κ , the H -local equivalences of \mathbf{M} are closed under κ -filtered colimits. To show the latter point, choose κ so that κ -filtered colimits are homotopy colimits. Then for any H -local object Z , a colimit $\text{colim } A \rightarrow \text{colim } B$ of a κ -filtered diagram of H -local equivalences is a weak equivalence because the morphism

$$\mathbf{R} \text{Mor}_{\mathbf{M}}(\text{colim } B, Z) \rightarrow \mathbf{R} \text{Mor}_{\mathbf{M}}(\text{colim } A, Z)$$

is a homotopy limit of weak equivalences in $s\mathbf{Set}_{\mathbf{X}}$, hence a weak equivalence. \square

Theorem 4.7 (Smith, [18]). *If \mathbf{M} is left proper and \mathbf{X} -combinatorial, and H is an \mathbf{X} -small set of homotopy classes of morphisms of \mathbf{M} , the left Bousfield localization $L_H \mathbf{M}$ of \mathbf{M} along any set representing H exists and satisfies the following conditions.*

(4.7.1) *The model category $L_H \mathbf{M}$ is left proper and \mathbf{X} -combinatorial.*

(4.7.2) *As a category, $L_H \mathbf{M}$ is simply \mathbf{M} .*

(4.7.3) *The cofibrations of $L_H \mathbf{M}$ are exactly those of \mathbf{M} .*

(4.7.4) *The fibrant objects of $L_H \mathbf{M}$ are the fibrant H -local objects Z of \mathbf{M} .*

(4.7.5) *The weak equivalences of $L_H \mathbf{M}$ are the H -local equivalences.*

Proof. The aim is to guarantee that a cofibrantly generated model structure on \mathbf{M} exists satisfying conditions (4.7.3)–(4.7.5) using 2.2. The combinatoriality is then automatic, and the universal property and the left properness are then verified in [9, Theorem 3.3.19 and Proposition 3.4.4].

Fix an \mathbf{X} -small set $I_{\mathbf{M}}$ of generating cofibrations of \mathbf{M} , and let $wL_H \mathbf{M}$ denote the set of the weak equivalences described in (4.7.5). By 4.6, we can now apply 2.2:

observe that since $I_{\mathbf{M}}$ -injectives are trivial fibrations of \mathbf{M} , they are in particular weak equivalences of \mathbf{M} , and hence are among the elements of $wL_H\mathbf{M}$.

It thus remains only to show that pushouts and transfinite compositions of morphisms of $\text{cof } \mathbf{M} \cap wL_H\mathbf{M}$ are H -local weak equivalences. Suppose first that $K \rightarrow L$ a cofibration in $wL_H\mathbf{M}$, and suppose

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ K' & \longrightarrow & L' \end{array}$$

a pushout diagram in \mathbf{M} . Note that by the left properness of \mathbf{M} , this pushout is in fact a homotopy pushout; thus the statement that $K' \rightarrow L'$ is an element of $wL_H\mathbf{M}$ is equivalent to the assertion that, for any H -local object Z , the diagram

$$\begin{array}{ccc} \mathbf{R} \text{Mor}_{\mathbf{M}}(K', Z) & \longrightarrow & \mathbf{R} \text{Mor}_{\mathbf{M}}(K', Z) \\ \downarrow & & \downarrow \\ \mathbf{R} \text{Mor}_{\mathbf{M}}(L, Z) & \longrightarrow & \mathbf{R} \text{Mor}_{\mathbf{M}}(K, Z) \end{array}$$

is a homotopy pullback diagram in $s\mathbf{Set}$, and this follows immediately from the fact that $\mathbf{R} \text{Mor}_{\mathbf{M}}(L, Z) \rightarrow \mathbf{R} \text{Mor}_{\mathbf{M}}(K, Z)$ is a weak equivalence. Since κ -filtered colimits are homotopy colimits for κ sufficiently large, it follows that a transfinite composition of elements of $\text{cof } \mathbf{M} \cap wL_H\mathbf{M}$ is an morphism of $wL_H\mathbf{M}$. \square

Definition 4.8. If \mathbf{M} is left proper and \mathbf{X} -combinatorial, and if H is an \mathbf{X} -small set of homotopy classes of morphisms of \mathbf{M} , then an object X of the left Bousfield localization $L_H\mathbf{M}$ is *quasifibrant* if some fibrant replacement $r_{\mathbf{M}}X$ of X in \mathbf{M} is fibrant in $L_H\mathbf{M}$.

4.9. Note that this terminology is ambiguous. A better terminology would include a reference to both \mathbf{M} and H — e.g., “ (\mathbf{M}, H) -quasifibrant” — but this seems tiresome.

Lemma 4.10. *If \mathbf{M} is left proper and \mathbf{X} -combinatorial, and H is an \mathbf{X} -small set of homotopy classes of morphisms of \mathbf{M} , then an object X of the left Bousfield localization $L_H\mathbf{M}$ is quasifibrant if and only if it is H -local.*

Proof. H -locality is closed under weak equivalences in \mathbf{M} ; hence if X is quasifibrant it is surely H -local, and the fibrant replacement in \mathbf{M} of an H -local object is H -local. \square

4.11. As a rule, one has essentially no control in a left Bousfield localization over the generating trivial cofibrations. The following proposition (originally — with a different proof — due to M. Hovey) is one of the very few results on the trivial cofibrations of left Bousfield localizations; it is critical for the forthcoming existence theorem 4.46 for *enriched* left Bousfield localizations.

Proposition 4.12 (Hovey, [11, Proposition 4.3]). *Suppose that \mathbf{M} is left proper and \mathbf{X} -combinatorial, and suppose that H is \mathbf{X} -small. Then the left Bousfield localization $L_H\mathbf{M}$ is \mathbf{X} -tractable if \mathbf{M} is.*

Proof. Immediate from 2.7. \square

4.13. Finally, it seems that if \mathbf{M} is only a left model category, then $L_H\mathbf{M}$ exists as a left \mathbf{M} -model category. Such a result has the advantage that it eliminates the dependence of the existence theorem on the left properness of \mathbf{M} . I will not explore this point further here, since it would take me too far afield, and since left properness is frequently satisfied in practice; however, it seems likely that Smith's theorem can be adapted for this purpose.

The failure of right properness

Left Bousfield localizations inherit left properness, but in general they destroy right properness. This is because there is very little control over the fibrations.

Nevertheless, there are often full subcategories that can be regarded as right proper, and this form of right properness is inherited by the quasifibrant objects contained in these subcategories in the left Bousfield localization. In this case, there exist functorial factorizations of morphisms of these quasifibrant objects through quasifibrant objects, so homotopy pullbacks of these quasifibrant objects can be computed effectively. This also provides a nice recognition principle for fibrations of $L_H\mathbf{M}$ with a quasifibrant codomain that lies in such a subcategory.

One can think of this subsection as an enlargement of Reedy's observation that homotopy pullbacks of fibrant objects can be computed by replacing on only one side, or, alternatively, one can think of this subsection as a collection of techniques for coping with the fact that many important combinatorial model categories are simply not right proper.

There are, of course, dual conditions and results to many of those of this section, but they are not discussed here, essentially because left properness is a relatively common condition in practice.

4.14. Suppose \mathbf{X} a universe, \mathbf{M} a model \mathbf{X} -category.

Definition 4.15. (4.15.1) If E is any full subcategory of \mathbf{M} , an *E-placement functor* is a pair (r_E, ϵ_E) consisting of a functor $r_E : \mathbf{M} \rightarrow E$ along with an objectwise weak equivalence ϵ_E from the identity functor to the composite $\iota_E \circ r_E$, where $\iota_E : E \rightarrow \mathbf{M}$ denotes the inclusion.

(4.15.2) A full subcategory E of \mathbf{M} is said to be *stable under trivial fibrations* if for any object X of E and any trivial fibration $Y \rightarrow X$, the object Y is an object of E as well.

(4.15.3) A full subcategory E of \mathbf{M} is said to be *admissibly left exact* if E contains any admissible pullback in E — i.e., if E contains the pullback of any morphism of E along any fibration of E ; more generally, E is said to be *partially left exact* if E contains the pullback of any weak equivalence of E along any fibration of E .

(4.15.4) A partially left exact full subcategory E is said to be *right proper* if in E admissible pullbacks of weak equivalences are weak equivalences.

(4.15.5) Suppose E a partially left exact full subcategory; then a pair (F, η) consisting of a functor $F : E \rightarrow \mathbf{M}$ with an objectwise weak equivalence η from the inclusion functor ι_E to F is said to be *exceptional on E* if F preserves admissible pullback diagrams.

Lemma 4.16. *Suppose E and E' partially (respectively, admissibly) left exact full subcategories of \mathbf{M} and (F, η) an exceptional pair on E . Then the full subcategory $F^{-1}(E')$ of E consisting of those objects X of E such that FX is an object of E' is partially (resp., admissibly) left exact. Moreover, if E' is right proper, then so is $F^{-1}(E')$.*

Proof. Refer to the diagram

$$\begin{array}{ccc}
 F(Y') & \xrightarrow{\quad} & FY \\
 \downarrow & \swarrow & \downarrow \\
 & Y' \xrightarrow{\quad} Y & \\
 & \downarrow & \downarrow \\
 & X' \xrightarrow{\quad} X & \\
 \downarrow & \swarrow & \downarrow \\
 F(X') & \xrightarrow{\quad} & FX
 \end{array}$$

If the interior square is an admissible pullback diagram in E , then the outer square is so in \mathbf{M} . If $X' \rightarrow X$ is a weak equivalence, then so is $F(X') \rightarrow FX$, and if $F(Y') \rightarrow FY$ is a weak equivalence, then so is $Y' \rightarrow Y$. \square

Lemma 4.17. *The full subcategory \mathbf{M}_f of fibrant objects is admissibly left exact and right proper.*

Proof. This is Reedy's observation, [16, Theorem B] or [9, Proposition 13.1.2]. \square

Corollary 4.18. *A partially left exact full subcategory E of \mathbf{M} is right proper if there exists an exceptional fibrant replacement functor on E .*

Corollary 4.19. *The model category $s\mathbf{Set}$ of simplicial sets is right proper.*

Proof. Kan's Ex^∞ is an exceptional fibrant replacement functor. \square

Lemma 4.20. *Suppose E a right proper, partially left exact full subcategory of \mathbf{M} with $\mathbf{M}_f \subset E$. Then there exists a functorial factorization of any morphism in E into a weak equivalence of E followed by a fibration of E . If, in addition, E is stable under trivial fibrations, then there is a functorial factorization of every morphism of E into a trivial cofibration of E followed by a fibration of E .*

Proof. Choose a functorial factorization of every morphism into a trivial cofibration followed by a fibration; this gives in particular a functorial fibrant replacement r . Suppose $X \rightarrow Y$ a morphism of E ; applying the chosen functorial factorization to the vertical morphism on the right in the diagram

$$\begin{array}{ccc}
 X & \longrightarrow & rX \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & rY,
 \end{array}$$

yields $rX \rightarrow Z \rightarrow rY$. Pulling back the resulting fibration $Z \rightarrow rY$ along $Y \rightarrow rY$

produces a diagram

$$\begin{array}{ccc}
 X & \longrightarrow & rX \\
 \downarrow & & \downarrow \\
 Y \times_{rY} Z & \longrightarrow & Z \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & rY,
 \end{array}$$

in which $Y \times_{rY} Z \rightarrow Y$ is a fibration of E , and $Y \times_{rY} Z \rightarrow Z$ — and therefore also $X \rightarrow Y \times_{rY} Z$ — is a weak equivalence of E . If E is stable under trivial fibrations, then applying the chosen factorization to the morphism $X \rightarrow Y \times_{rY} Z$ provides the desired factorization of the second half of the statement. \square

Corollary 4.21. *If E is a right proper, admissibly left exact full subcategory of \mathbf{M} with $\mathbf{M}_f \subset E$, then admissible pullbacks in E are homotopy pullbacks.*

4.22. Suppose for the remainder of this section that the model category \mathbf{M} is left proper and \mathbf{X} -combinatorial, that H is \mathbf{X} -small, and that E is a right proper, admissibly left exact full subcategory of \mathbf{M} with $\mathbf{M}_f \subset E$. Write $\text{loc}_E(H)$ for the full subcategory of E consisting of H -local objects, viewed as a full subcategory of the Bousfield localization $L_H\mathbf{M}$; observe that $(L_H\mathbf{M})_f \subset \text{loc}_E(H)$. The objective is to show that $\text{loc}_E(H)$ is right proper and admissibly left exact, whence the effective computability of homotopy pullbacks of H -local objects of E .

Lemma 4.23. *The subcategory $\text{loc}_E(H)$ is a right proper partially left exact full subcategory of $L_H\mathbf{M}$, and if E is stable under trivial fibrations, then so is $\text{loc}_E(H)$.*

Proof. Weak equivalences in $L_H\mathbf{M}$ between H -local objects are weak equivalences of \mathbf{M} , and the trivial fibrations of $L_H\mathbf{M}$ are exactly those of \mathbf{M} . \square

Lemma 4.24. *If E is stable under trivial fibrations, then $\text{loc}_E(H)$ admits a functorial factorization, within $\text{loc}_E(H)$, of every morphism into a trivial cofibration of \mathbf{M} followed by a fibration of $L_H\mathbf{M}$.*

Proof. Applying 4.20, there is a functorial factorization of every morphism into a trivial cofibration of $\text{loc}_E(H)$ followed by a fibration of $\text{loc}_E(H)$. But a trivial cofibration in $L_H\mathbf{M}$ between H -local objects is a trivial cofibration in \mathbf{M} as well. \square

Corollary 4.25. *If E is stable under trivial fibrations, then a morphism of $\text{loc}_E(H)$ is a fibration in $L_H\mathbf{M}$ if and only if it is a fibration of \mathbf{M} .*

Proof. One implication is obvious; the other follows from the retract argument. \square

Proposition 4.26. *The subcategory $\text{loc}_E(H)$ is a right proper, admissibly left exact full subcategory of $L_H\mathbf{M}$ with $(L_H\mathbf{M})_f \subset \text{loc}_E(H)$.*

Proof. Suppose $Y \rightarrow X$ a fibration of $\text{loc}_E(H)$, and suppose $X' \rightarrow X$ a morphism of $\text{loc}_E(H)$. To show that the pullback

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

exists in $\text{loc}_E(H)$, it suffices by factorization (4.20) to suppose that $X' \rightarrow X$ is a fibration as well. But then the pullback Y' is a homotopy pullback, and since $\mathbf{R}\text{Mor}(A, -)$ commutes with homotopy pullbacks, Y' is also H -local. \square

Corollary 4.27. *Admissible pullbacks of $\text{loc}_E(H)$ are homotopy pullbacks.*

Corollary 4.28. *If \mathbf{M} is right proper, then the quasifibrant objects form a right proper, admissibly left exact full subcategory $\text{loc}(H)$ of $L_H\mathbf{M}$.*

4.29. Lastly, we now turn to a recognition principle for fibrations in $L_H\mathbf{M}$ with codomains in E or $\text{loc}_E(H)$.

Proposition 4.30. *Suppose $p : Y \rightarrow X$ a fibration of \mathbf{M} . For any fibrant replacement $p' : Y' \rightarrow X'$ of p in $L_H\mathbf{M}$ (i.e., a morphism p' between fibrant objects with a weak equivalence $p \rightarrow p'$ in $L_H\mathbf{M}(\mathbf{1})$) consider the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ p \downarrow & & \downarrow p' \\ X & \longrightarrow & X' \end{array} \tag{4.30.1}$$

(4.30.2) *If $X \in E$, then p is a fibration of $L_H\mathbf{M}$ if there exists a fibrant replacement $p' : Y' \rightarrow X'$ of p such that (4.30.1) is a homotopy pullback square in \mathbf{M} .*

(4.30.3) *If $X \in \text{loc}_E(H)$, then p is a fibration of $L_H\mathbf{M}$ if and only if for any fibrant replacement $p' : Y' \rightarrow X'$, the square (4.30.1) is a homotopy pullback square in \mathbf{M} .*

Proof. To prove the first assertion, factor p' as a weak equivalence $Y' \rightarrow Y''$ of $L_H\mathbf{M}$ followed by a fibration $Y'' \rightarrow X'$ of $L_H\mathbf{M}$; the weak equivalence $Y' \rightarrow Y''$ is even a weak equivalence of \mathbf{M} since it is a weak equivalence of local objects. Also factor $X \rightarrow X'$ as a weak equivalence $X \rightarrow X''$ in E followed by a fibration $X'' \rightarrow X$ in E . Pulling back $Y'' \rightarrow X'$, we have the commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & & \longrightarrow & Y' \\ \downarrow & \searrow & & & \downarrow \\ Z & \longrightarrow & Z'' & \longrightarrow & Y'' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X'' & \longrightarrow & X' \end{array}$$

Since X'' , X' , and Y'' are all objects of E , Z'' is an object of E as well, and it follows from the right properness of E that the weak equivalence $X \rightarrow X''$ is pulled back to a weak equivalence $Z \rightarrow Z''$ of \mathbf{M} . The statement that (4.30.1) is a homotopy pullback

is equivalent to the assertion that the morphism $Y \rightarrow Z''$ is a weak equivalence of \mathbf{M} . It now follows that the morphism $Y \rightarrow Z$ is a weak equivalence of \mathbf{M} . Factor this map into a trivial cofibration $Y \rightarrow Z'$ of $L_H\mathbf{M}$ followed by a fibration $Z' \rightarrow Z$ of $L_H\mathbf{M}$; it follows that $Y \rightarrow Z'$ is in fact a trivial cofibration of \mathbf{M} , and the retract argument thus completes proof.

The proof of the second statement begins similarly; factor p' as before, and now factor the morphism $X \rightarrow X'$ as a weak equivalence $X \rightarrow X''$ in $\text{loc}_E(H)$ followed by a fibration $X'' \rightarrow X$ in $\text{loc}_E(H)$ (which is in addition a weak equivalence in $L_H\mathbf{M}$). Again pull back the fibration $Y'' \rightarrow X'$:

$$\begin{array}{ccccc}
 Y & \longrightarrow & & & Y' \\
 \downarrow & \searrow & & & \downarrow \\
 Z & \longrightarrow & Z'' & \longrightarrow & Y'' \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \longrightarrow & X'' & \longrightarrow & X'.
 \end{array}$$

Now it follows from the right properness of $\text{loc}_E(H)$ that the morphism $Z'' \rightarrow Y''$ is a weak equivalence of $L_H\mathbf{M}$, and it follows from the right properness of E that $Z \rightarrow Z''$ is a weak equivalence of \mathbf{M} . It thus follows that the morphism $Y \rightarrow Z$ is a weak equivalence of $L_H\mathbf{M}$, and since $Z \rightarrow X$ and $p : Y \rightarrow X$ are each fibrations of $L_H\mathbf{M}$, it follows [9, Proposition 3.3.5] that $Y \rightarrow Z$ is a weak equivalence of \mathbf{M} ; hence the composite morphism $Y \rightarrow Z''$ is a weak equivalence of \mathbf{M} , so that (4.30.1) is a homotopy pullback. \square

Application I: Presentations of combinatorial model categories

A presentation of a model \mathbf{X} -category is a Quillen equivalence with a left Bousfield localization of a category of simplicial presheaves on an \mathbf{X} -small category. A beautiful result of D. Dugger indicates that presentations exist for all combinatorial model categories. Hence any tractable model category can (up to Quillen equivalence) be given a representation in terms of generators and relations. We recall Dugger's results here.

Definition 4.31. An \mathbf{X} -presentation (K, H, F) of a model \mathbf{X} -category \mathbf{M} consists of an \mathbf{X} -small category K , an \mathbf{X} -small set H of homotopy classes of morphisms of $s\text{Set}_{\mathbf{X}}(K)$, and a left Quillen equivalence $F : L_H s\text{Set}_{\mathbf{X}}(K)_{\text{proj}} \rightarrow \mathbf{M}$.

Theorem 4.32 (Dugger, [4, Theorem 1.1]). *Every \mathbf{X} -combinatorial model \mathbf{X} -category has an \mathbf{X} -presentation.*

Corollary 4.33. *An \mathbf{X} -combinatorial model \mathbf{X} -category has an \mathbf{X} -small set of homotopy generators.*

Proof. By the theorem it is enough to show this for the projective model category of simplicial presheaves on an \mathbf{X} -small category C ; in this case, the images under the Yoneda embedding of the objects of C provide such a set. \square

Application II: Homotopy images

As a quirky demonstration of the usefulness of left Bousfield localizations, we offer the factorization result 4.35.

Definition 4.34. Suppose $f : X \rightarrow Y$ a morphism of a model \mathbf{X} -category \mathbf{M} .

- (4.34.1) The morphism f is said to be a *homotopy monomorphism* if the natural morphism $X \rightarrow X \times_{\mathbb{Y}}^h X$ is an isomorphism of $\mathrm{Ho} \mathbf{M}$.
- (4.34.2) Dually, the morphism f is said to be a *homotopy epimorphism* if the natural morphism $Y \sqcup^{h, X} Y \rightarrow Y$ is an isomorphism of $\mathrm{Ho} \mathbf{M}$.
- (4.34.3) The *homotopy image* of f is a factorization of f into a cofibration $X \rightarrow f(X)$ followed by a homotopy monomorphism $f(X) \rightarrow Y$ such that for any other such factorization $X \rightarrow X' \rightarrow Y$, there exists a unique morphism $f(X) \rightarrow X'$ in $\mathrm{Ho} \mathbf{M}$.

Theorem 4.35. *Suppose \mathbf{M} left proper and \mathbf{X} -combinatorial; if Y is a fibrant object of \mathbf{M} , any morphism $f : X \rightarrow Y$ has a homotopy image.*

Proof. Let G be an \mathbf{X} -small set of cofibrant homotopy generators for \mathbf{M} . Write G/Y for the disjoint union of the sets $\mathrm{Mor}_{\mathrm{Ho} \mathbf{M}}(R, Y)$ over $R \in G$. Now write

$$\nabla_{G/Y} := \{g \sqcup g \rightarrow g \mid g \in G/Y\},$$

an \mathbf{X} -small set. The \mathbf{M} -model category $L_{\nabla_{G/Y}}(\mathbf{M}/Y)$ then exists.

It now suffices to show that the fibrant objects of the model category $L_{\nabla_{G/Y}}(\mathbf{M}/Y)$ are precisely the fibrations $X' \rightarrow Y$ that are also homotopy monomorphisms, for if so, the homotopy image of a morphism $f : X \rightarrow Y$ is simply a fibrant replacement for f in $L_{\nabla_{G/Y}}(\mathbf{M}/Y)$.

Observe that the fibrant objects are precisely those fibrations $X' \rightarrow Y$ such that the natural morphism

$$\mathbf{R} \mathrm{Mor}_{(\mathbf{M}/Y)}(Z, X') \rightarrow \star$$

is a homotopy monomorphism of \mathbf{M} for any cofibrant object Z of \mathbf{M}/Y . Equivalently, a fibration $X' \rightarrow Y$ is fibrant in $L_{\nabla_{G/Y}}(\mathbf{M}/Y)$ if and only if, for any object Z , the natural morphism

$$\mathbf{R} \mathrm{Mor}_{\mathbf{M}}(Z, X') \rightarrow \mathbf{R} \mathrm{Mor}_{\mathbf{M}}(Z, Y)$$

is a homotopy monomorphism, whence the desired characterization of weak equivalences. \square

Application III: Homotopy limits of right Quillen presheaves

The category of right sections of a right Quillen presheaf \mathbf{F} with its projective model structure is to be thought of as the $(\infty, 1)$ -categorical colax limit of \mathbf{F} . The $(\infty, 1)$ -categorical limit — or homotopy limit — of \mathbf{F} is a left Bousfield localization of the category of right sections.

4.36. Suppose \mathbf{X} a universe. Suppose K and \mathbf{X} -small category, and suppose \mathbf{F} an \mathbf{X} -combinatorial right Quillen presheaf on K .

Definition 4.37. A right section (X, ϕ) of \mathbf{F} is said to be *homotopy cartesian* if for any morphism $f : \ell \rightarrow k$ of K , the morphism

$$\phi_f^h : X_\ell \rightarrow \mathbf{R}f^*X_k$$

is an isomorphism of $\mathrm{Ho} \mathbf{F}_\ell$.

Theorem 4.38. *There exists an \mathbf{X} -combinatorial model structure on the category $\mathrm{Sect}^R \mathbf{F}$ — the homotopy limit structure $\mathrm{Sect}_{\mathrm{holim}}^R \mathbf{F}$ — satisfying the following conditions.*

(4.38.1) *The cofibrations are exactly the projective cofibrations.*

(4.38.2) *The fibrant objects are the projective fibrant right sections that are homotopy cartesian.*

(4.38.3) *The weak equivalences between fibrant objects are precisely the objectwise weak equivalences.*

Proof. For every $k \in \mathrm{Obj} K$, let G_k be an \mathbf{X} -small set of cofibrant homotopy generators of \mathbf{F}_k . For each object k of K , there is a Quillen adjunction

$$D_k : \mathbf{F}_k \rightleftarrows \mathrm{Sect}^R \mathbf{F} : E_k$$

where $E_k(X, \phi) = X_k$; one verifies that for any object A of \mathbf{F}_ℓ ,

$$E_k D_\ell A \cong \coprod_{f: \ell \rightarrow k} f_! A.$$

Hence for every $f : \ell \rightarrow k$ and any object A of \mathbf{F}_ℓ , there is a canonical morphism $f_! A \rightarrow E_k D_\ell A$, and, by adjunction, a canonical morphism $r_{f,A} : D_k(f_! A) \rightarrow D_\ell A$.

Now define the \mathbf{X} -small set

$$H := \{r_{f,A} : D_k(f_! A) \rightarrow D_\ell A \mid [f : \ell \rightarrow k] \in K, A \in G_\ell\}.$$

The claim is that $L_H \mathrm{Sect}_{\mathrm{proj}}^R \mathbf{F}$ is the model category of the theorem.

To verify this claim, it suffices to check that the fibrant objects are as described. Indeed, a right section X is fibrant if and only if it is fibrant in $\mathrm{Sect}_{\mathrm{proj}}^R \mathbf{F}$, and, for any morphism $f : \ell \rightarrow k$ of K and any $A \in G_\ell$,

$$\mathbf{R} \mathrm{Mor}_{\mathrm{Sect}_{\mathrm{proj}}^R \mathbf{F}}(\mathbf{L} D_\ell A, X) \rightarrow \mathbf{R} \mathrm{Mor}_{\mathrm{Sect}_{\mathrm{proj}}^R \mathbf{F}}(\mathbf{L} D_k \mathbf{L} f_! A, X),$$

or equivalently

$$\mathbf{R} \mathrm{Mor}_{\mathbf{F}_k}(A, \mathbf{R} E_\ell X) \rightarrow \mathbf{R} \mathrm{Mor}_{\mathbf{F}_k}(A, \mathbf{R} f^* \mathbf{R} E_k X) \quad (4.38.4)$$

is an isomorphism of $\mathrm{Ho} s\mathbf{Set}_{\mathbf{X}}$. Since the elements of G_ℓ generate \mathbf{F}_ℓ by homotopy colimits, it follows immediately that (4.38.4) is an isomorphism of $\mathrm{Ho} s\mathbf{Set}_{\mathbf{X}}$ for any object A of \mathbf{F}_ℓ , whence it follows that X is homotopy cartesian, as desired. \square

Example 4.39. Denote by \mathbf{N} the category whose objects are nonnegative integers, in which there is a unique morphism $m \rightarrow n$ if and only if $m \leq n$. Consider the right

Quillen presheaf

$$\begin{aligned} \underline{\Omega} : \mathbf{N}^{\text{op}} &\longrightarrow \mathbf{Cat}_{\mathbf{Y}} \\ n &\longmapsto (\star/s\mathbf{Set}_{\mathbf{X}}) \\ [n \leq m] &\longmapsto \Omega^{m-n}, \end{aligned}$$

where of course $\Omega^{m-n} := \underline{\text{Mor}}_{(\star/s\mathbf{Set})}((S^1)^{\wedge(m-n)}, -)$. One verifies easily that the model category $\text{Sect}_{\text{holim}}^R \underline{\Omega}$ is simply the usual Bousfield-Friedlander model category of spectra.

4.40. Note that the results of this section say nothing about the homotopy limits of left Quillen presheaves. As an $(\infty, 1)$ -category, such a homotopy limit should be a coreflexive sub- $(\infty, 1)$ -category of the $(\infty, 1)$ -categorical lax limit; hence it is more properly modeled as a right Bousfield localization. This is a somewhat more delicate issue, which is addressed in 5.25.

The enriched left Bousfield localization

Here we define enriched Bousfield localizations, and we prove an existence theorem.

4.41. Suppose \mathbf{X} a universe, \mathbf{V} a symmetric monoidal model \mathbf{X} -category, and \mathbf{C} a model \mathbf{V} -category.

Definition 4.42. Suppose H a set of homotopy classes of morphisms of \mathbf{C} . A *left Bousfield localization of \mathbf{C} with respect to H enriched over \mathbf{V}* is a model \mathbf{V} -category $L_{(H/\mathbf{V})}\mathbf{C}$, equipped with a left Quillen \mathbf{V} -functor $\mathbf{C} \rightarrow L_{(H/\mathbf{V})}\mathbf{C}$ that is initial among left Quillen \mathbf{V} -functors $F : \mathbf{C} \rightarrow \mathbf{D}$ to model \mathbf{V} -categories \mathbf{D} such that for any f representing a class in H , Ff is a weak equivalence in \mathbf{D} .

Lemma 4.43. *When it exists, an enriched left Bousfield localization is unique up to a unique \mathbf{V} -isomorphism of model \mathbf{V} -categories under \mathbf{C} .*

Proof. Initial objects are essentially unique. □

4.44. Suppose, for the remainder of this section, H a set of homotopy classes of morphisms of \mathbf{C} .

Definition 4.45. (4.45.1) An object Z of \mathbf{C} is *(H/\mathbf{V}) -local* if for any morphism $A \rightarrow B$ representing an element of H , the morphism

$$\mathbf{R}\underline{\text{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R}\underline{\text{Mor}}_{\mathbf{C}}(A, Z)$$

is an isomorphism of $\text{Ho } \mathbf{V}$.

(4.45.2) A morphism $A \rightarrow B$ of \mathbf{C} is an *(H/\mathbf{V}) -local equivalence* if for any fibrant (H/\mathbf{V}) -local object Z , the morphism

$$\mathbf{R}\underline{\text{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R}\underline{\text{Mor}}_{\mathbf{C}}(A, Z)$$

is an isomorphism of $\text{Ho } \mathbf{V}$.

Theorem 4.46. *Suppose that following conditions are satisfied.*

(4.46.A) *The model \mathbf{V} -category \mathbf{C} is left proper and \mathbf{X} -tractable.*

(4.46.B) *The set H is \mathbf{X} -small.*

(4.46.C) The model \mathbf{X} -category \mathbf{V} is \mathbf{X} -tractable.

Then the enriched left Bousfield localization $L_{(H/\mathbf{V})}\mathbf{C}$ exists, and satisfies the following conditions.

(4.46.1) The model category $L_{(H/\mathbf{V})}\mathbf{C}$ is left proper and \mathbf{X} -tractable.

(4.46.2) As a \mathbf{V} -category, $L_{(H/\mathbf{V})}\mathbf{C}$ is simply \mathbf{C} .

(4.46.3) The cofibrations of $L_{(H/\mathbf{V})}\mathbf{C}$ are exactly those of \mathbf{C} .

(4.46.4) The fibrant objects of $L_{(H/\mathbf{V})}\mathbf{C}$ are those fibrant (H/\mathbf{V}) -local objects Z of \mathbf{C} .

(4.46.5) The weak equivalences of $L_{(H/\mathbf{V})}\mathbf{C}$ are the (H/\mathbf{V}) -local equivalences.

Proof. Let S be an \mathbf{X} -small set of cofibrations between cofibrant objects representing all and only the homotopy classes of H . Choose a generating set of cofibrations I for \mathbf{V} with cofibrant domains. Set

$$L_{(H/\mathbf{V})}\mathbf{C} := L_{I \square S}\mathbf{C},$$

the left Bousfield localization of \mathbf{C} by $I \square S$ (4.7). By [9, Proposition 17.4.16], an object Z that is fibrant in \mathbf{C} is $(I \square S)$ -local if and only if for any $X \rightarrow Y$ in I and any $A \rightarrow B$ in S , the diagram

$$\begin{array}{ccc} \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(Y, \underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z)) & \longrightarrow & \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(X, \underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z)) \\ \downarrow & & \downarrow \\ \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(Y, \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)) & \longrightarrow & \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(X, \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)) \end{array}$$

is a homotopy pullback. Thus Z is $(I \square S)$ -local if and only if for any morphism $A \rightarrow B$ of S , the induced morphism $\underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)$ is homotopy right orthogonal [9, Definition 17.8.1] to every element of I . By [9, Theorem 17.8.18], this is equivalent to the condition that $\underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)$ is a weak equivalence in \mathbf{V} . Thus the fibrant objects of $L_{(H/\mathbf{V})}\mathbf{C}$ are exactly those fibrant objects Z of \mathbf{C} such that the morphism $\mathbf{R} \underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z) \rightarrow \mathbf{R} \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)$ is a weak equivalence in \mathbf{V} for every $A \rightarrow B$ representing an element of H .

The inheritance of left properness and the \mathbf{X} -combinatoriality follows from the general theory of left Bousfield localizations (4.7).

The cofibrations are unchanged; hence the the unit axiom (1.27.4.42) holds, and the pushout-product $i \square f$ of a cofibration $i : X \rightarrow Y$ of \mathbf{V} with a cofibration $f : A \rightarrow B$ of \mathbf{C} is a cofibration of \mathbf{C} that is trivial if i is. To show that $L_{(H/\mathbf{V})}\mathbf{C}$ is a model \mathbf{V} -category, it thus suffices to show that if f is a trivial cofibration of $L_{(H/\mathbf{V})}\mathbf{C}$, then $i \square f$ is a weak equivalence. By 1.30, it suffices to verify this for f an element of a generating set of trivial cofibrations of $L_{(H/\mathbf{V})}\mathbf{C}$. By 4.12, $L_{(H/\mathbf{V})}\mathbf{C}$ has an \mathbf{X} -small set of generating trivial cofibrations with cofibrant domains; so let J denote such a set, and let $f \in J$. Now by adjunction, one verifies that $i \square f$ is a weak equivalence if, for any fibrant object Z of $L_{(H/\mathbf{V})}\mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(Y, \underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z)) & \longrightarrow & \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(X, \underline{\operatorname{Mor}}_{\mathbf{C}}(B, Z)) \\ \downarrow & & \downarrow \\ \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(Y, \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)) & \longrightarrow & \mathbf{R} \operatorname{Mor}_{\mathbf{V}}(X, \underline{\operatorname{Mor}}_{\mathbf{C}}(A, Z)) \end{array}$$

is a homotopy pullback. Since both of the vertical morphisms are weak equivalences, the desired result follows.

The right Quillen functor $L_{(H/\mathbf{V})}\mathbf{C} \rightarrow \mathbf{C}$ induces a fully faithful $(\mathrm{Ho}\mathbf{V})$ -functor

$$\mathrm{Ho} L_{(H/\mathbf{V})}\mathbf{C} \rightarrow \mathrm{Ho}\mathbf{C}.$$

The characterization of weak equivalences now follows from 1.37.

Now suppose \mathbf{D} a model \mathbf{V} -category and $F : \mathbf{C} \rightarrow \mathbf{D}$ a left Quillen \mathbf{V} -functor such that for any $f \in S$, Ff is a weak equivalence. Then Ff is a trivial cofibration of \mathbf{D} , and for any $i \in I$, the morphism $F(i \square f) = i \square Ff$ is also a trivial cofibration. Hence any such F factors uniquely through $\mathbf{C} \rightarrow L_{(H/\mathbf{V})}\mathbf{C}$ by the universal property enjoyed by ordinary left Bousfield localizations. This completes the proof. \square

Proposition 4.47. *Suppose that \mathbf{C} , \mathbf{V} , and H together satisfy each of the conditions (4.46.A) through (4.46.C), and, in addition, each of the following conditions.*

(4.47.A) *The tuple $(\mathbf{C}, \otimes_{\mathbf{C}}^{\mathbf{C}}, \underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}})$ is a symmetric monoidal model category.*

(4.47.B) *There exists a set (not necessarily \mathbf{X} -small) G of cofibrant homotopy generators of \mathbf{C} , with the property that for any element $A \in G$ and any fibrant (H/\mathbf{V}) -local object B of \mathbf{C} , $\underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}}(A, B)$ is (H/\mathbf{V}) -local.*

Then the symmetric monoidal structure of \mathbf{C} endows the enriched Bousfield localization $L_{(H/\mathbf{V})}\mathbf{C}$ with the structure of a symmetric monoidal \mathbf{X} -tractable model \mathbf{X} -category.

Proof. Since the cofibrations are unchanged, I is an \mathbf{X} -small set of generating cofibrations for $L_{(H/\mathbf{V})}\mathbf{C}$ with cofibrant domains. It suffices to verify that for any trivial cofibration $i : X \rightarrow Y$ of $L_{(H/\mathbf{V})}\mathbf{C}$ and any element $f : A \rightarrow B$ of I , the pushout-product $i \square_{\mathbf{C}}^{\mathbf{C}} f$ is a weak equivalence. By [9, Proposition 17.4.16], this holds in turn if, for any fibrant object Z of $L_{(H/\mathbf{V})}\mathbf{C}$, the diagram

$$\begin{array}{ccc} \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(Y, \underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}}(B, Z)) & \longrightarrow & \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(X, \underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}}(B, Z)) \\ \downarrow & & \downarrow \\ \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(Y, \underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}}(A, Z)) & \longrightarrow & \mathbf{R}\mathrm{Mor}_{\mathbf{C}}(X, \underline{\mathrm{Mor}}_{\mathbf{C}}^{\mathbf{C}}(A, Z)) \end{array}$$

is a homotopy pullback of \mathbf{V} . The horizontal morphisms are weak equivalences if the objects $\underline{\mathrm{Mor}}_{\mathbf{C}}(A, Z)$ and $\underline{\mathrm{Mor}}_{\mathbf{C}}(B, Z)$ are (H/\mathbf{V}) -local. This follows from the observation that A and B are homotopy colimits of objects of G , and that (H/\mathbf{V}) -locality is preserved under homotopy limits. \square

Application IV: Local model structures

As a final application, we describe the local model structures on categories of presheaves valued in a symmetric monoidal model category.

4.48. Suppose \mathbf{X} a universe, suppose (C, τ) an \mathbf{X} -small site, and suppose \mathbf{V} an \mathbf{X} -tractable symmetric monoidal model category with cofibrant unit $\mathbf{1}_{\mathbf{V}}$.

Notation 4.49. Write $y : C \rightarrow \mathbf{Set}_{\mathbf{X}}(C)$ for the usual Yoneda embedding, and write $y_{\mathbf{V}} : C \rightarrow \mathbf{V}(C)$ for the \mathbf{V} -enriched Yoneda embedding, defined by copowers:

$$y_{\mathbf{V}}X := yX \cdot \mathbf{1}_{\mathbf{V}} : Y \mapsto \mathrm{Mor}_C(Y, X) \cdot \mathbf{1}_{\mathbf{V}}$$

Proposition 4.50. *The category $\mathbf{V}(C)$, with either its injective or projective model structure, is a \mathbf{V} -model category.*

Proof. Suppose $i : K \rightarrow L$ a (trivial) cofibration of \mathbf{V} ; then for any objectwise (trivial) cofibration (respectively, any objectwise (trivial) fibration) $f : X \rightarrow Y$ of $\mathbf{V}(C)$, the morphism $i \square f$ (resp., $\text{mor}_{\square}(i, f)$) is an objectwise (trivial) cofibration (resp., an objectwise (trivial) fibration). \square

Proposition 4.51. *The injective model category $\mathbf{V}(C)_{\text{inj}}$ is symmetric monoidal.*

Proof. Since both cofibrations and weak equivalences are defined objectwise, the pushout-product axiom is immediate. \square

Proposition 4.52. *The projective model category $\mathbf{V}(C)_{\text{proj}}$ is symmetric monoidal if and only if there exists a set I of generating cofibrations and a set J of generating trivial cofibrations for \mathbf{V} satisfying the following condition: for any element $[X \rightarrow Y] \in I$ (respectively, any element $[X \rightarrow Y] \in J$), and any objects $K, L \in \text{Obj } C$, the morphism*

$$(\text{Mor}_C(-, K) \times \text{Mor}_C(-, L)) \cdot X \rightarrow (\text{Mor}_C(-, K) \times \text{Mor}_C(-, L)) \cdot Y$$

is a cofibration (resp. trivial cofibration) of $\mathbf{V}(C)$.

Proof. Clearly if the projective model structure is symmetric monoidal, any cofibration (resp., trivial cofibration) must satisfy the condition demanded of I (resp., J).

Conversely, suppose I and J have been chosen to meet this condition. Then set

$$I_{\mathbf{V}(\text{Obj } C)} := \bigcup_{K \in \text{Obj } C} \left(I \times \prod_{L \neq K} \text{id}_{\emptyset} \right);$$

$$J_{\mathbf{V}(\text{Obj } C)} := \bigcup_{K \in \text{Obj } C} \left(J \times \prod_{L \neq K} \text{id}_{\emptyset} \right).$$

Thus $I_{\mathbf{V}(\text{Obj } C)}$ (resp., $J_{\mathbf{V}(\text{Obj } C)}$) is a set of generating cofibrations (resp., trivial cofibrations) for $\mathbf{V}(\text{Obj } C)$, and thus

$$I_{\mathbf{V}(C)} := e_! I_{\mathbf{V}(\text{Obj } C)} \quad (\text{resp., } J_{\mathbf{V}(C)} := e_! J_{\mathbf{V}(\text{Obj } C)})$$

is a set of generating cofibrations (resp., trivial cofibrations) for $\mathbf{V}(C)_{\text{proj}}$. One now easily verifies that the set $I_{\mathbf{V}(C)}$ (resp., $J_{\mathbf{V}(C)}$) is the set

$$I_{\mathbf{V}(C)} = \{ \text{Mor}_C(-, K) \cdot X \rightarrow \text{Mor}_C(-, K) \cdot Y \mid K \in \text{Obj } C, [X \rightarrow Y] \in I \};$$

$$J_{\mathbf{V}(C)} = \{ \text{Mor}_C(-, K) \cdot X \rightarrow \text{Mor}_C(-, K) \cdot Y \mid K \in \text{Obj } C, [X \rightarrow Y] \in J \}.$$

One thus verifies that the condition of the proposition is precisely the statement that for any element $[i : S \rightarrow T] \in I_{\mathbf{V}(C)}$ (resp., any morphism $[S \rightarrow T] \in J_{\mathbf{V}(C)}$) and any object $L \in \text{Obj } C$, the morphism $S \otimes y_{\mathbf{V}}(L) \rightarrow T \otimes y_{\mathbf{V}}(L)$ is a cofibration (resp.,

trivial cofibration), whence it follows from the pushout-product axiom for the \mathbf{V} enrichment of $\mathbf{V}(C)_{\text{proj}}$ that for any fibration $p : V \rightarrow U$, the morphism

$$\underline{\text{Mor}}_{\square}(i, p) : \underline{\text{Mor}}_{\mathbf{V}(C)}(T, V) \rightarrow \underline{\text{Mor}}_{\mathbf{V}(C)}(S, V) \times_{\underline{\text{Mor}}_{\mathbf{V}(C)}(S, U)} \underline{\text{Mor}}_{\mathbf{V}(C)}(T, U)$$

is a fibration (resp., trivial fibration) that is trivial if p is. \square

Corollary 4.53. *If C has all products, then $\mathbf{V}(C)_{\text{proj}}$ is a symmetric monoidal model category.*

Definition 4.54. An \mathbf{V} -valued presheaf $F : C^{\text{op}} \rightarrow \mathbf{V}$ is said to *satisfy τ -descent* if for any τ -covering sieve $[R \rightarrow yX] \in \tau(X)$, the morphism

$$FX \rightarrow \text{holim}_{Y \in (C/R)^{\text{op}}} FY$$

is an isomorphism of $\text{Ho } \mathbf{V}$. In this case, F will be called an *\mathbf{V} -valued sheaf*.

4.55. It should be noted that the \mathbf{V} -valued sheaves do not satisfy hyperdescent in general; the condition above is only the requirement that a \mathbf{V} -valued sheaf satisfy so-called *Čech descent*. Ensuring that the \mathbf{V} -valued sheaves satisfy descent with respect to all τ -hypercoverings usually requires a further localization, which we leave for interested readers to formulate in general. In the case of simplicial presheaves, this further localization is simply the hypercompletion, and in the case of spectral presheaves it is the spectral hypercompletion.

Theorem 4.56. *There exist two \mathbf{X} -tractable \mathbf{V} -model structures on the \mathbf{V} -category $\mathbf{V}(C)$ — the τ -local projective model structure $\mathbf{V}(C, \tau)_{\text{proj}}$ and (respectively) the τ -local injective model structure $\mathbf{V}(C, \tau)_{\text{inj}}$ — satisfying the following conditions.*

(4.56.1) *The cofibrations are exactly the projective (resp., injective) cofibrations.*

(4.56.2) *The fibrant objects are the projective (resp., injective) fibrant sheaves.*

(4.56.3) *The weak equivalences between fibrant objects are precisely the objectwise weak equivalences.*

Proof. Set

$$H := \{ R \cdot \mathbf{1}_{\mathbf{V}} \rightarrow y_{\mathbf{V}} X \mid [R \rightarrow yX] \in \tau(X) \}.$$

Then set

$$\begin{aligned} \mathbf{V}(C, \tau)_{\text{proj}} &:= L_{(H/\mathbf{V})} \mathbf{V}(C)_{\text{proj}}; \\ \mathbf{V}(C, \tau)_{\text{inj}} &:= L_{(H/\mathbf{V})} \mathbf{V}(C)_{\text{inj}}. \end{aligned}$$

It now suffices to show that the fibrant objects are as described, and it suffices to do this for the τ -local projective model category $\mathbf{V}(C, \tau)_{\text{proj}}$. For any τ -covering sieve $[R \rightarrow yX] \in \tau(X)$, write $R \cong \text{colim}_{Y \in (C/R)} yY$; one verifies easily that the corresponding colimit $R \cdot \mathbf{1}_{\mathbf{V}} \cong \text{colim}_{Y \in (C/R)} y_{\mathbf{V}} Y$ is a homotopy colimit. Thus a fibrant object of $\mathbf{V}(C, \tau)_{\text{proj}}$ is an objectwise fibrant \mathbf{V} -valued presheaf F such that

$$FX \simeq \mathbf{R} \underline{\text{Mor}}_{\mathbf{V}(C)_{\text{proj}}}^{\mathbf{V}}(y_{\mathbf{V}} X, F) \rightarrow \mathbf{R} \underline{\text{Mor}}_{\mathbf{V}(C)_{\text{proj}}}^{\mathbf{V}}(R \cdot \mathbf{1}_{\mathbf{V}}, F) \simeq \text{holim}_{Y \in (C/R)^{\text{op}}} FY$$

for any τ -covering sieve $[R \rightarrow yX] \in \tau(X)$. \square

Proposition 4.57. *Suppose $\pi : (C, \tau) \twoheadrightarrow (D, \nu)$ a morphism of sites (hence a functor $\pi^{-1} : D \rightarrow C$). Then there is a Quillen adjunction*

$$\pi^* = (\pi^{-1})_! : \mathbf{V}(D, \nu)_{\text{proj}} \rightleftarrows \mathbf{V}(C, \tau)_{\text{proj}} : (\pi^{-1})^* = \pi_*$$

Theorem 4.58. *The local injective model category $\mathbf{V}(C, \tau)_{\text{inj}}$ is symmetric monoidal, and if the projective model category $\mathbf{V}(C)_{\text{proj}}$ is symmetric monoidal, then so is $\mathbf{V}(C, \tau)_{\text{proj}}$.*

Proof. The proofs of the statements are identical, since by 4.47, it suffices to show that there exists a set G of cofibrant homotopy generators of \mathbf{V} such that for any element $Z \in G$, any object $W \in C$, and any local injective fibrant object $Y \in \mathbf{V}(C)$, the presheaf $\underline{\text{Mor}}_{\mathbf{V}(C)}(Z \otimes y_{\mathbf{V}}W, Y)$ satisfies τ -descent.

To verify this, suppose X an object of C , and suppose $[R \twoheadrightarrow yX] \in \tau(X)$; then one verifies easily that the morphism

$$\text{colim}_{U \in (C/R)^{\text{op}}} \mathbf{1}_{\mathbf{V}} \cdot (yW \times yU) \twoheadrightarrow \mathbf{1}_{\mathbf{V}} \cdot (yW \times yX)$$

is a weak equivalence between cofibrant objects. Hence

$$Z \otimes_{\mathbf{V}(C)}^{\mathbf{V}} \text{colim}_{U \in (C/R)^{\text{op}}} \mathbf{1}_{\mathbf{V}} \cdot (yW \times yU) \twoheadrightarrow Z \otimes_{\mathbf{V}(C)}^{\mathbf{V}} (\mathbf{1}_{\mathbf{V}} \cdot (yW \times yX)),$$

and therefore

$$\begin{array}{c} \underline{\text{Mor}}_{\mathbf{V}(C)}^{\mathbf{V}}((Z \otimes_{\mathbf{V}(C)}^{\mathbf{V}} y_{\mathbf{V}}W) \otimes y_{\mathbf{V}}X, Y) \\ \downarrow \\ \text{holim}_{U \in (C/R)^{\text{op}}} \underline{\text{Mor}}_{\mathbf{V}(C)}^{\mathbf{V}}((Z \otimes_{\mathbf{V}(C)}^{\mathbf{V}} y_{\mathbf{V}}W) \otimes y_{\mathbf{V}}U, Y) \end{array}$$

are weak equivalences of \mathbf{V} , whence the desired descent statement. \square

5. The dreaded right Bousfield localization

The right Bousfield localization of a model category \mathbf{M} relative to a set of objects K is ordinarily defined as a model category $R_K\mathbf{M}$ equipped with a right Quillen functor $\mathbf{M} \rightarrow R_K\mathbf{M}$ satisfying a universal property dual to that of left Bousfield localizations.

Existence theorem

Suppose \mathbf{X} a universe, \mathbf{M} an \mathbf{X} -cofibrantly generated model category, and K an \mathbf{X} -small set of objects of \mathbf{M} . Hirschhorn's existence theorem for right Bousfield localizations [9, Theorem 5.1.1] only works when \mathbf{M} is right proper. The key point is that if $i : A \twoheadrightarrow B$ is a K -colocal cofibration and $p : Y \twoheadrightarrow X$ is a K -colocal trivial fibration, then it is necessary to show that for any diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & X, \end{array}$$

there exists a lift $B \twoheadrightarrow Y$. It turns out that this is easy to verify in case X (and hence also Y) is fibrant in \mathbf{M} . If \mathbf{M} is right proper, this is sufficient: i has the left lifting

property with respect to p if and only if it has the left lifting property with respect to a replacement fibration $p' : Y' \rightarrow X'$ of p with Y' and X' fibrant [9, Propositions 5.2.5 and 13.2.1].

This leads one to the following observation: if one only seeks the left lifting property of K -colocal cofibrations with respect to K -colocal trivial fibrations *with fibrant codomain*, then the right properness of \mathbf{M} is unnecessary here.

Likewise, the small object argument immediately provides factorizations into cofibrations followed by trivial fibrations when the codomain is fibrant. It is the existence of such factorizations for any morphism that requires right properness [9, Proposition 5.3.5].

Upon inspection of the standard proofs of the existence of $R_{\mathbf{M}}$ for \mathbf{M} right proper, one can confirm that these are the only places where right properness is used. Hence $R_K \mathbf{M}$ exists as a right \mathbf{M} -model category, even if \mathbf{M} is not right proper.

Thus, a simple modification of the traditional proof shows that $R_K \mathbf{M}$ exists as a right \mathbf{M} -model category for any \mathbf{X} -cofibrantly generated model category \mathbf{M} (or in fact for any model \mathbf{X} -category \mathbf{M} satisfying Christensen and Isaksen's weaker condition [3, Hypothesis 2.4]), and that, as a right \mathbf{M} -model category, $R_K \mathbf{M}$ is \mathbf{X} -cofibrantly generated as well.⁶ Here we give a complete proof of the existence of $R_K \mathbf{C}$ as an \mathbf{X} -tractable right \mathbf{C} -model category for any \mathbf{X} -tractable right model category \mathbf{C} .

5.1. Suppose \mathbf{C} an \mathbf{X} -tractable right model \mathbf{X} -category, and K a set of isomorphism classes of objects of $\text{Ho } \mathbf{C}$.

Definition 5.2. (5.2.1) If H is a set of homotopy classes of morphisms of \mathbf{C} , a *right Bousfield localization of \mathbf{C} with respect to H* is a right \mathbf{C} -model \mathbf{X} -category $\mathbf{C} \rightarrow R_H \mathbf{C}$ that is initial among right model \mathbf{X} -categories \mathbf{D} equipped with a right Quillen functor $F : \mathbf{C} \rightarrow \mathbf{D}$ with the property that for any f representing a class in H , $\mathbf{R}F(f)$ is an isomorphism of $\text{Ho } \mathbf{D}$.

(5.2.2) A morphism $A \rightarrow B$ is a *K -colocal equivalence* if for any representative X of an element of K , the morphism

$$\mathbf{R} \text{Mor}_{\mathbf{C}}(X, A) \rightarrow \mathbf{R} \text{Mor}_{\mathbf{C}}(X, B)$$

is an isomorphism of $\text{Ho } s\text{Set}_{\mathbf{X}}$.

(5.2.3) An object Z of \mathbf{M} is *K -colocal* if for any K -colocal equivalence $A \rightarrow B$, the morphism

$$\mathbf{R} \text{Mor}_{\mathbf{M}}(Z, A) \rightarrow \mathbf{R} \text{Mor}_{\mathbf{M}}(Z, B)$$

is an isomorphism of $\text{Ho } s\text{Set}_{\mathbf{X}}$.

(5.2.4) A *right Bousfield localization of \mathbf{C} with respect to K* is nothing more than a right Bousfield localization of \mathbf{C} with respect to the set of K -colocal equivalences.

Proposition 5.3. *A right Bousfield localization $R_K \mathbf{C}$ is essentially unique if it exists.*

⁶If $R_K \mathbf{M}$ happens to be a model category, it does not seem to follow that $R_K \mathbf{M}$ will be cofibrantly generated as a model category, unless some very strong conditions on \mathbf{M} are satisfied, e.g., that every object of \mathbf{M} be fibrant.

Proof. Initial objects are essentially unique. \square

5.4. Suppose now that the set K is \mathbf{X} -small.

Notation 5.5. Suppose I and J are generating \mathbf{X} -small sets of cofibrations and trivial cofibrations, respectively, each with cofibrant domains. For every element $A \in K$, choose a cosimplicial resolution $\Lambda_{\mathbf{C}}^{\bullet} A \rightarrow A$, and set

$$I_{R_K \mathbf{C}} := J \cup \{L_p(\Lambda_{\mathbf{C}}^{\bullet} A) \rightarrow \Lambda^p A \mid \mathbf{p} \in \Delta, A \in K\}.$$

Proposition 5.6. *The category \mathbf{C} is a structured homotopical category $R_K \mathbf{C}$ with the following definitions.*

(5.6.1) *A cofibration of $R_K \mathbf{C}$ is defined to be a cofibration $X \rightarrow Y$ of \mathbf{C} such that there exists a weak equivalence $Y \rightarrow Y'$ of \mathbf{C} such that the composite $X \rightarrow Y'$ is a retract of an element of $\text{cell } I_{R_K \mathbf{C}}$.*

(5.6.2) *A fibration of $R_K \mathbf{C}$ is nothing more than a fibration of \mathbf{C} .*

(5.6.3) *A weak equivalence of $R_K \mathbf{C}$ is a K -colocal equivalence.*

Moreover the identity functor $R_K \mathbf{C} \rightarrow \mathbf{C}$ preserves cofibrations, where as the identity functor $\mathbf{C} \rightarrow R_K \mathbf{C}$ preserves weak equivalences.

Proof. The relevant properties of the weak equivalences and fibrations are straightforward. It remains only to show that the given set of cofibrations is closed under compositions, retracts, and pushouts along arbitrary morphisms.

We first claim that in order for a cofibration $X \rightarrow Y$ of \mathbf{C} to be a cofibration of $R_K \mathbf{C}$, it is necessary and sufficient that there exist a trivial cofibration $Y \rightarrow Y''$ of \mathbf{C} such that the composite $X \rightarrow Y''$ is a retract of an element of $\text{cell } I_{R_K \mathbf{C}}$. Sufficiency is clear. To verify necessity, suppose $Y \rightarrow Y'$ a weak equivalence of \mathbf{C} such that the composite $X \rightarrow Y'$ is a retract of an element of $\text{cell } I_{R_K \mathbf{C}}$. Now factor $Y \rightarrow Y'$ in \mathbf{C} as a trivial cofibration followed by a trivial fibration; it now follows from the retract argument that the composite $X \rightarrow Y''$ is a retract of an element of $\text{cell } I_{R_K \mathbf{C}}$.

We first demonstrate that the cofibrations of $R_K \mathbf{C}$ are closed under composition. Indeed, suppose $X \rightarrow Y$ and $Y \rightarrow Z$ cofibrations of \mathbf{C} , and suppose $Y \rightarrow Y'$ and $Z \rightarrow Z'$ trivial cofibrations such that the composites $X \rightarrow Y'$ and $Y \rightarrow Z'$ are retracts of elements of $\text{cell } I_{R_K \mathbf{C}}$. Now form the pushout of $Y \rightarrow Z$ and $Z \rightarrow Z'$ along $Y \rightarrow Y'$:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & Z' \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & Y' & \longrightarrow & W & \longrightarrow & W'. \end{array}$$

The morphism $X \rightarrow Y'$ is a retract of an element of $\text{cell } I_{R_K \mathbf{C}}$, as is the composite $Y' \rightarrow W'$; hence so is their composite.

Now I show that the cofibrations of $R_K \mathbf{C}$ are closed under retracts. For this, suppose that $X' \rightarrow Y'$ is a cofibration of $R_K \mathbf{C}$, and suppose $X \rightarrow Y$ is a retract

thereof:

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y. \end{array}$$

Now suppose $Y' \rightarrow Z'$ a trivial cofibration of \mathbf{C} such that the composite $X' \rightarrow Z'$ is a retract of an element of cell $I_{R_K\mathbf{C}}$. By factoring the morphism $Z' \rightarrow \star$ as an element of cell J followed by an element of $\text{inj } J$ if necessary, one may assume that Z' is fibrant. Hence one may choose an endomorphism $Z' \rightarrow Z'$ such that the diagram

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & Y' \\ \downarrow & & & & \downarrow \\ Z' & \longrightarrow & & \longrightarrow & Z' \end{array}$$

commutes. Now set $Z := Y \sqcup^{Y'} Z'$; the chosen morphism $Z' \rightarrow Z'$ induces a morphism $Z \rightarrow Z'$ such that the following diagram commutes:

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Z' & \longrightarrow & Z. \end{array}$$

Now form the pushout $Z'' := Y' \sqcup^Y Z$; one has a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Z'' & \longrightarrow & Z \\ & & \downarrow & & \parallel \\ & & Z' & \longrightarrow & Z \end{array}$$

in which the bottom three squares are pushouts, whence the composite $Z \rightarrow Z' \rightarrow Z$ is the identity on Z . Thus $X \rightarrow Z$ is a retract of an element of cell $I_{R_K\mathbf{C}}$. Since $Y \rightarrow Z$ is a trivial cofibration of \mathbf{C} , we are done.

Finally, we show that the cofibrations of $R_K\mathbf{C}$ are closed under pushouts. For this, suppose $X \rightarrow Y$ a cofibration of $R_K\mathbf{C}$, and suppose $X \rightarrow X'$ an arbitrary morphism. There exists a trivial cofibration $Y \rightarrow Y'$ of \mathbf{C} such that the composite $X \rightarrow Y'$ is a

retract of an element of cell $I_{R_K\mathbf{C}}$. Now form the pushout

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y'' & \longrightarrow & Y''' \end{array}$$

Now $Y'' \rightarrow Y'''$ is a trivial cofibration of \mathbf{C} , and $X' \rightarrow Y'''$ is a retract of an element of cell $I_{R_K\mathbf{C}}$. \square

5.7. Note the rough similarity between the definition of the cofibrations given here and the a priori stonger description given by Hirschhorn [9, Proposition 5.3.6], namely, than a morphism is a cofibration just in case it is a retract of an element $X \rightarrow Y$ of cell I such that there exists a weak equivalence $Y \rightarrow Z$ such that the composite $X \rightarrow Z$ is an element of cell $I_{R_K\mathbf{C}}$. The two are in fact equivalent when \mathbf{C} is a model category; this is the content of the following lemma.

If \mathbf{C} is not a model category, there seems to be a genuine difference between the two conditions, but, unfortunately, the distinction seems to be fairly subtle, and we do not have an enlightening example that exhibits it. In any case it is certainly the weaker of these that is needed here.

Lemma 5.8. *Suppose \mathbf{C} an \mathbf{X} -tractable model category; then a morphism of \mathbf{C} is a cofibration of $R_K\mathbf{C}$ if and only if it is a retract of an element $X \rightarrow Y$ of cell I such that there exists a weak equivalence $Y \rightarrow Z$ of \mathbf{C} such that the composite $X \rightarrow Z$ is an element of cell $I_{R_K\mathbf{C}}$.*

Proof. That such a retract is a cofibration of $R_K\mathbf{C}$ is obvious. This is of course true regardless of whether \mathbf{C} is a model category.

In the other direction, suppose $f : X \rightarrow Y$ a cofibration for which there is a weak equivalence $e : Y \rightarrow Z$ of \mathbf{C} such that the composite $g : X \rightarrow Z$ is a retract of an element of cell $I_{R_K\mathbf{C}}$. The claim is that f can be written as a retract of an element $X \rightarrow Y'$ of cell I for which there exists a weak equivalence $Y' \rightarrow Z$ such that the composite $e' \circ f' = g$. Indeed, simply factor f , by the small object argument, as an element $f' : X \rightarrow Y'$ of cell I followed by an element $p : Y' \rightarrow Y$ of $\text{inj } I$, and set $e' = e \circ p$. Then since \mathbf{C} is a model category, p is a trivial fibration.⁷ The retract argument thus implies the claim. \square

Lemma 5.9. *A morphism $Y' \rightarrow Y$ whose codomain Y is fibrant is a trivial fibration in $R_K\mathbf{C}$ if and only if it is an element of $\text{inj } I_{R_K\mathbf{C}}$.*

Proof. Of course $Y' \rightarrow Y$ is an element of $\text{inj } J$ if and only if it is a fibration of \mathbf{C} — and, equivalently, of $R_K\mathbf{C}$. In this circumstance, for any element $A \in K$, the morphism $\mathbf{R}\text{Mor}_{\mathbf{C}}(A, Y') \rightarrow \mathbf{R}\text{Mor}_{\mathbf{C}}(A, Y)$ is a fibration of $s\mathbf{Set}_{\mathbf{X}}$, and by [9, Proposition 16.4.5], will be an equivalence if and only if the morphism $Y' \rightarrow Y$ is an element of $\text{inj } I_{R_K\mathbf{C}}$. \square

Lemma 5.10. *There is a functorial factorization of every morphism $X \rightarrow Y$ of \mathbf{C} with fibrant codomain Y into a cofibration $X \rightarrow Y'$ of $R_K\mathbf{C}$ followed by a trivial fibration $Y' \rightarrow Y$ of $R_K\mathbf{C}$.*

⁷If \mathbf{C} were not a model category, this would follow only if Y were \mathbf{E} -fibrant.

Proof. The usual construction via the small object argument provides a factorization of every morphism $X \rightarrow Y$ (irrespective of the fibrancy of Y) into an element $X \rightarrow Y'$ of cell $I_{R_K \mathbf{C}}$ followed by an element $Y' \rightarrow Y$ of $\text{inj } I_{R_K \mathbf{C}}$. Now $X \rightarrow Y'$ is clearly a cofibration, and, by the previous lemma, $Y' \rightarrow Y$ is a trivial fibration of $R_K \mathbf{C}$. \square

Lemma 5.11. *Cofibrations of $R_K \mathbf{C}$ satisfy the left lifting property with respect to every trivial fibration of $R_K \mathbf{C}$ with fibrant codomain.*

Proof. That this is true of any element of $I_{R_K \mathbf{C}}$ is already contained in 5.9. It thus follows for any retract of elements of cell $I_{R_K \mathbf{C}}$.

Now suppose $X \rightarrow Y$ a cofibration of \mathbf{C} and $e : Y \rightarrow Y'$ a weak equivalence of \mathbf{C} such that the composite $X \rightarrow Y'$ is a retract of elements of cell $I_{R_K \mathbf{C}}$. Suppose W a fibrant object, $Z \rightarrow W$ a trivial fibration of $R_K \mathbf{C}$, and the following a commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & W \\ e \downarrow & & \\ Y' & & \end{array}$$

To prove the lemma, it will now suffice to show that there exists a lift $Y \rightarrow Z$. For this, note that in the right model category (X/\mathbf{C}) , W is fibrant and $Y \rightarrow Y'$ is a weak equivalence; hence there is a homotopy lift $Y' \rightarrow W$ in (X/\mathbf{C}) . Since this homotopy lift is chosen in the slice category, it follows that there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & W. \end{array}$$

Hence there is a lift $\ell : Y' \rightarrow Z$, and the composite $\ell \circ e : Y \rightarrow Z$ is thus a homotopy lift of the diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ Y & \longrightarrow & W \end{array}$$

in (X/\mathbf{C}) . Since Y is cofibrant, the homotopy lifting property of the fibration $Z \rightarrow W$ implies that a strict lift of the diagram, homotopic to $\ell \circ e$, exists in (X/\mathbf{C}) . \square

Lemma 5.12. *The trivial cofibrations of $R_K \mathbf{C}$ are exactly those of \mathbf{C} .*

Proof. If $f : K \rightarrow L$ is a trivial cofibration of \mathbf{C} , then it is a weak equivalence of \mathbf{C} , a retract of an element of cell I , and a retract of an element of cell J . It follows that f is a weak equivalence of $R_K \mathbf{C}$ and a retract of an element of cell $I_{R_K \mathbf{C}}$ as well; thus f is a trivial cofibration of $R_K \mathbf{C}$.

Conversely, suppose $f : K \rightarrow L$ a trivial cofibration of $R_K \mathbf{C}$, and let $i : L \rightarrow L'$ be a trivial cofibration in \mathbf{C} with L' fibrant. Factor the composite morphism $f' := i \circ f$ as a trivial cofibration $j : K \rightarrow K'$ of \mathbf{C} followed by a fibration $p : K' \rightarrow L'$ of \mathbf{C} :

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ j \downarrow & \searrow f' & \downarrow i \\ K' & \xrightarrow{p} & L'. \end{array}$$

Since f' is a weak equivalence of $R_K \mathbf{C}$, so is p . Now f' is a trivial cofibration of $R_K \mathbf{C}$ and by the previous result has the left lifting property with respect to p . The retract argument now implies that f' is a retract of p and is thus a trivial cofibration of \mathbf{C} . Since $f' = i \circ f$, and both f' and i are weak equivalences of \mathbf{C} , it follows that f is a weak equivalence of \mathbf{C} as well. Since f was a fortiori a cofibration of \mathbf{C} , the converse is verified. \square

Proposition 5.13. *The structured homotopical category $R_K \mathbf{C}$ is an \mathbf{X} -tractable right \mathbf{C} -model category. If in addition \mathbf{C} is a right proper model \mathbf{X} -category, then $R_K \mathbf{C}$ is a right proper model category (not necessarily \mathbf{X} -tractable).*

Proof. The factorization axioms are 5.10 and the corresponding factorization in \mathbf{C} , coupled with 5.12. Likewise, the lifting properties are 5.11 and the corresponding property in \mathbf{C} , coupled with 5.12.

The structured homotopical structure is by 5.8 the one provided by Hirschhorn in case \mathbf{C} is right proper. \square

Proposition 5.14. *The left Quillen identity functor $U : R_K \mathbf{C} \rightarrow \mathbf{C}$ induces a coreflexive fully faithful functor of \mathbf{X} -categories — and thus also of $(\text{Ho sSet}_{\mathbf{X}})$ -categories — $\mathbf{L}U : \text{Ho } R_K \mathbf{C} \rightarrow \text{Ho } \mathbf{C}$. The derived right adjoint $\mathbf{R}F : \mathbf{C} \rightarrow \text{Ho } R_K \mathbf{C}$ is essentially surjective.*

Proof. Write $F : \mathbf{C} \rightarrow R_K \mathbf{C}$ for the right adjoint of U . It suffices to show that the unit $X \rightarrow (\mathbf{R}F)(\mathbf{L}U)X$ of the derived adjunction is an isomorphism of $\text{Ho } R_K \mathbf{C}$. But this is clear, as the fibrant replacement in \mathbf{C} of any object of $R_K \mathbf{C}$ is a fibrant replacement in $R_K \mathbf{C}$. \square

Corollary 5.15. *A morphism $A \rightarrow B$ of \mathbf{C} is a K -colocal weak equivalence if and only if the induced morphism $(\mathbf{R}F)A \rightarrow (\mathbf{R}F)B$ is an isomorphism of $\text{Ho } R_K \mathbf{C}$.*

Proposition 5.16. *A cofibrant-fibrant object X of \mathbf{C} is cofibrant as an object of $R_K \mathbf{C}$ if and only if it is K -colocal.*

Proof. Since a K -colocal weak equivalence $A \rightarrow B$ is weak equivalence of $R_K \mathbf{C}$, it follows that if X is cofibrant in $R_K \mathbf{C}$, the morphism

$$\mathbf{R} \text{Mor}_{\mathbf{C}}(X, A) \simeq \mathbf{R} \text{Mor}_{R_K \mathbf{C}}(X, A) \rightarrow \mathbf{R} \text{Mor}_{\mathbf{C}}(X, B) \simeq \mathbf{R} \text{Mor}_{R_K \mathbf{C}}(X, B)$$

is an isomorphism of $\text{Ho sSet}_{\mathbf{X}}$.

On the other hand, if X is K -colocal, it suffices, since by assumption X is fibrant in $R_K \mathbf{C}$, to show that the morphism $\emptyset \rightarrow X$ has the left lifting property with respect to

all trivial fibrations $A \rightarrow B$ of $R_K \mathbf{C}$ with fibrant codomain B . Since X is K -colocal, the map

$$\mathrm{Mor}_{\mathrm{Ho} \mathbf{C}}(X, A) \rightarrow \mathrm{Mor}_{\mathrm{Ho} \mathbf{C}}(X, B)$$

is a bijection, whence the desired lifting property. \square

Corollary 5.17. *An object X of \mathbf{C} is K -colocal if and only if the counit of the derived adjunction $(\mathbf{L}U)(\mathbf{R}F)X \rightarrow X$ is an isomorphism of $\mathrm{Ho} \mathbf{C}$.*

Corollary 5.18. *A weak equivalence of $R_K \mathbf{C}$ between K -colocal objects is a weak equivalence of \mathbf{C} .*

Theorem 5.19. *The right \mathbf{C} -model category $R_K \mathbf{C}$ is a right Bousfield localization of \mathbf{C} with respect to K .*

Proof. The proof that the right \mathbf{C} -model structure described here has the universal property required of a right Bousfield localization is well known, and is [9, Proposition 3.3.18]. \square

Corollary 5.20. *A cofibrant-fibrant object of \mathbf{C} is K -colocal if and only if it can be written as a homotopy colimit (in \mathbf{C}) of a diagram of representatives of elements of K .*

Proof. The proof is in fact identical to the one given by Hirschhorn [9, §5.5]. For any object A of \mathbf{C} , the functor

$$\mathbf{R} \mathrm{Mor}_{\mathbf{C}}(-, A) : \mathrm{Ho} \mathbf{C}^{\mathrm{op}} \rightarrow \mathrm{Ho}(s\mathrm{Set}_{\mathbf{X}})$$

turns homotopy colimits into homotopy limits; hence any homotopy colimit of K -colocal objects is again K -colocal. On the other hand, one verifies easily that any $I_{R_K \mathbf{C}}$ -cell complex can be written as a homotopy colimit of objects of K . \square

Example 5.21. Consider the model category $(\star/s\mathrm{Set}_{\mathbf{X}})$ of pointed \mathbf{X} -small simplicial sets. Let $S^1 := \Delta^1/\partial\Delta^1$, the usual simplicial circle, pointed at its unique vertex; let $S^n := (S^1)^{\wedge n}$ be the n -sphere, again pointed at its unique vertex.

Then $R_{S^n}(\star/s\mathrm{Set}_{\mathbf{X}})$ is a model \mathbf{X} -category (since $(\star/s\mathrm{Set}_{\mathbf{X}})$ is in fact right proper), in which a morphism $X \rightarrow Y$ is a weak equivalence if and only if the induced map on n -fold loop spaces $\mathbf{R}\Omega^n X \rightarrow \mathbf{R}\Omega^n Y$ is an isomorphism in $\mathrm{Ho} s\mathrm{Set}_{\mathbf{X}}$. Equivalently, a morphism $X \rightarrow Y$ of pointed simplicial sets is a weak equivalence of $R_{S^n}(\star/s\mathrm{Set}_{\mathbf{X}})$ if and only if the induced homomorphism $\pi_k X \rightarrow \pi_k Y$ is an isomorphism for any $k > n$. This model category can be regarded as complementary to the n -truncated model structure 5.28.

As observed by Hirschhorn [9, 5.2.7], one can easily construct a fibration $f : S \rightarrow \partial\Delta^{n+1}$, in which S is in effect a subdivision of $\partial\Delta^{n+1}$ (hence of the same homotopy type), and the homomorphism $\pi_n f$ is multiplication by 2. Hence f is not a weak equivalence in $R_{S^n}(\star/s\mathrm{Set}_{\mathbf{X}})$, but it nevertheless has the right lifting property with respect to the sets $J_{(\star/s\mathrm{Set}_{\mathbf{X}})}$ and $\{A \wedge \partial\Delta_+^p \rightarrow A \wedge \Delta_+^p \mid \mathbf{p} \in \Delta, A \in K\}$.

Thus the set $I_{R_{S^n}(\star/s\mathrm{Set}_{\mathbf{X}})}$ is not a set of generating cofibrations for the model category $R_{S^n}(\star/s\mathrm{Set}_{\mathbf{X}})$, though it nevertheless is a set of generating cofibrations for the underlying right model category.

Using a related example, one may show that right Bousfield localizations of right proper tractable model categories need not be combinatorial model categories, even though they are, quite naturally, tractable right model categories.

Theorem 5.22. *Suppose now \mathbf{C} an \mathbf{X} -tractable model category, and suppose $D \subset \mathbf{C}$ any full subcategory of \mathbf{C} satisfying the following properties.*

(5.22.1) *The subcategory D is accessible and accessibly embedded.*

(5.22.2) *The subcategory D is closed under weak equivalences.*

(5.22.3) *The subcategory D is closed under homotopy colimits in \mathbf{C} .*

Then there exists an \mathbf{X} -tractable right Bousfield localization $R_K \mathbf{C}$ of \mathbf{C} such that the K -colocal objects are precisely the objects of D .

Proof. For some regular, \mathbf{X} -small cardinal λ , one has the following: (1) \mathbf{C} is λ -accessible; (2) λ -filtered colimits in \mathbf{C} are homotopy colimits; and (3) D is a λ -accessibly embedded λ -accessible subcategory of $\text{Sect}^L(\mathbf{F})$.

Now let K be the \mathbf{X} -small set of homotopy cartesian, λ -presentable objects of D . Any homotopy cartesian left section of D can be written as a λ -filtered colimit of objects of K , and this λ -filtered colimit is a homotopy colimit, so the proof is complete. \square

Application I: The homotopy limit of left Quillen presheaves

In 4.38 we constructed a model category that plays the role of the homotopy limit of a right Quillen presheaf, which we constructed by taking a left Bousfield localization of a projective model structure. It is perhaps unfortunate that a similar model structure does not exist for *left* Quillen presheaves; however, one can define a *right model structure* on the category of left sections of a left Quillen presheaf that plays the role of the homotopy limit as a right Bousfield localization of an injective model structure.

5.23. Suppose \mathbf{X} a universe. Suppose K and \mathbf{X} -small category, and suppose \mathbf{F} an \mathbf{X} -tractable left Quillen presheaf on K .

Definition 5.24. A left section (X, ϕ) of \mathbf{F} is said to be *homotopy cartesian* if for any morphism $f : \ell \rightarrow k$ of K , the morphism

$$\phi_f^h : \mathbf{L}f^* X_k \rightarrow X_\ell$$

is an isomorphism of $\text{Ho } \mathbf{F}_\ell$.

Theorem 5.25. *There exists an \mathbf{X} -tractable right model structure on the category $\text{Sect}^L \mathbf{F}$ — the homotopy limit structure $\text{Sect}_{\text{holim}}^L \mathbf{F}$ — satisfying the following conditions.*

(5.25.1) *The fibrations are exactly the injective fibrations.*

(5.25.2) *The cofibrant objects are the injective cofibrant left sections that are homotopy cartesian.*

(5.25.3) *The weak equivalences between cofibrant objects are precisely the objectwise weak equivalences.*

Proof. ⁸ We wish to apply 5.22. For this, it suffices to show that the full subcategory of homotopy cartesian left sections of \mathbf{F} is accessible and accessibly embedded.

So suppose q an accessible fibrant replacement functor, and observe that the functor

$$F : \text{Sect}^L(\mathbf{F}) \longrightarrow \prod_{[f:\ell \rightarrow k] \in \nu_1 K} \mathbf{F}_\ell(\mathbf{1})$$

$$(X, \phi) \longmapsto (f^* X_k \rightarrow X_\ell)_f$$

is accessible. Since

$$\prod_{[f:\ell \rightarrow k] \in \nu_1 K} w\mathbf{F}_\ell \subset \prod_{[f:\ell \rightarrow k] \in \nu_1 K} \mathbf{F}_\ell(\mathbf{1})$$

is an accessibly embedded accessible subcategory, its inverse image under $F \circ q$ is as well. But this is exactly the full subcategory of homotopy cartesian left sections of \mathbf{F} , so the proof is complete. \square

Application II: Postnikov towers for simplicial model categories

Any fibrant simplicial set X has a Postnikov tower

$$\cdots \longrightarrow X\langle n \rangle \longrightarrow X\langle n-1 \rangle \longrightarrow \cdots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle,$$

in which each $X\langle n \rangle$ is an n -type, and the morphism $f : X\langle n \rangle \rightarrow X\langle n-1 \rangle$ induces an isomorphism

$$\pi_j(X\langle n \rangle, x) \xrightarrow{\cong} \pi_j(X\langle n-1 \rangle, f(x))$$

for any point x of X . An analogous construction can be made in any tractable, left proper, simplicial model category. For a fully ∞ -categorical variant of this construction, see [14, §5.5.6].

5.26. Suppose \mathbf{X} a universe, \mathbf{M} an \mathbf{X} -tractable, left proper, simplicial model \mathbf{X} -category.

Definition 5.27. For any integer $n \geq -1$, an object X of \mathbf{M} is *n -truncated* if for any object Z of \mathbf{M} , the simplicial set $\mathbf{R}\text{Mor}_{\mathbf{M}}(Z, X)$ is an n -type; the object X is *truncated* if and only if it is n -truncated for some n .

Proposition 5.28. For any integer $n \geq -1$, there exists an \mathbf{X} -tractable, left proper, simplicial model structure on the category \mathbf{M} — the n -truncated model structure $\mathbf{M}_{\leq n}$ — satisfying the following conditions.

- (5.28.1) The cofibrations of $\mathbf{M}_{\leq n}$ are precisely the cofibrations of \mathbf{M} .
- (5.28.2) The fibrant objects of $\mathbf{M}_{\leq n}$ are precisely the fibrant, n -truncated objects of \mathbf{M} .
- (5.28.3) The weak equivalences between the fibrant objects are precisely the weak equivalences of \mathbf{M} .

⁸I am indebted to the referee for leading me to find this proof.

Proof. Let G be an \mathbf{X} -small set of cofibrant homotopy generators of \mathbf{M} . Then one verifies easily that the n -truncated model structure is the left Bousfield localization $L_{H(n)}\mathbf{M}$ with respect to the set

$$H(n) := \{S^j \otimes X \rightarrow X \mid X \in G, n < j\}. \quad \square$$

Corollary 5.29. *For any integers $m \geq n \geq -1$, the identity functor on \mathbf{M} induces a left Bousfield localization $\mathbf{M}_{\leq m} \rightarrow \mathbf{M}_{\leq n}$.*

5.30. For any integers $m \geq n \geq -1$, the full subcategory of $\text{Ho } \mathbf{M}$ comprised of n -truncated objects is a reflexive subcategory. The left adjoint to the inclusion is the left derived functor of the identity $\mathbf{M}_{\leq m} \rightarrow \mathbf{M}_{\leq n}$, which will be denoted

$$\tau_{\leq n} : \text{Ho } \mathbf{M}_{\leq m} \rightarrow \text{Ho } \mathbf{M}_{\leq n}.$$

This functor is the n -truncation functor.

Proposition 5.31. *There exists an \mathbf{X} -tractable right model structure on the presheaf category $\mathbf{M}(\mathbf{N})$ — the Postnikov model structure $\mathbf{M}(\mathbf{N})_{\text{Post}}$ — satisfying the following conditions.*

(5.31.1) *The fibrations of $\mathbf{M}(\mathbf{N})_{\text{Post}}$ are those morphisms $Y \rightarrow X$ such that the induced morphism $Y(0) \rightarrow X(0)$ is a fibration, and for any integers $m \geq n \geq 0$, the morphism*

$$X(m) \rightarrow X(n) \times_{Y(n)} Y(m)$$

is a fibration of the model category $\mathbf{M}_{\leq m}$.

(5.31.2) *In particular, the fibrant objects are those sequences*

$$\cdots \rightarrow Y(n) \rightarrow Y(n-1) \rightarrow \cdots \rightarrow Y(1) \rightarrow Y(0)$$

such that the following conditions hold.

(5.31.2.1) *The object $Y(0)$ is fibrant in \mathbf{M} .*

(5.31.2.2) *For any $n \geq 0$, the object $Y(n)$ is n -truncated.*

(5.31.2.3) *For any integers $m \geq n \geq 0$, the morphism $Y(m) \rightarrow Y(n)$ is a fibration of \mathbf{M} .*

(5.31.3) *The cofibrant objects are those sequences*

$$\cdots \rightarrow X(n) \rightarrow X(n-1) \rightarrow \cdots \rightarrow X(1) \rightarrow X(0)$$

satisfying the following conditions.

(5.31.3.1) *For any integer $n \geq 0$, the object $X(n)$ is cofibrant.*

(5.31.3.2) *For any pair of integers m, n such that $m \geq n \geq 0$, the morphism $X(m) \rightarrow X(n)$ exhibits $X(n)$ as the n -truncation of $X(m)$; that is, the natural morphism*

$$\tau_{\leq n} X(m) \rightarrow X(n)$$

of $\text{Ho } \mathbf{M}_{\leq n}$ is an isomorphism.

(5.31.4) *The weak equivalences between the cofibrant and fibrant objects are precisely the objectwise weak equivalences.*

Proof. The desired right model structure is the holim right model structure of the category of left sections of the left Quillen presheaf

$$\begin{aligned} \mathbf{M}_{\leq \bullet} : \mathbf{N}^{\text{op}} &\longrightarrow \mathbf{Cat}_{\mathbf{Y}}. \\ n &\longmapsto \mathbf{M}_{\leq n} \end{aligned}$$

The properties listed above are formal consequences of this definition, save the characterization of the fibrant objects. By definition, the fibrant objects of $\text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet})$ are those sequences

$$\cdots \longrightarrow Y(n) \longrightarrow Y(n-1) \longrightarrow \cdots \longrightarrow Y(1) \longrightarrow Y(0)$$

such that $Y(0)$ is fibrant in $\mathbf{M}_{\leq 0}$, and for any integers $m \geq n \geq 0$, the morphism $Y(m) \rightarrow Y(n)$ is a fibration of $\mathbf{M}_{\leq n}$. Thus the condition given above is necessary for Y to be fibrant; since a fibration of \mathbf{M} whose domain and codomain are fibrant in $\mathbf{M}_{\leq n}$ is a fibration in $\mathbf{M}_{\leq n}$, it follows that the condition given above suffices as well. \square

5.32. The towers constructed here do indeed coincide with Postnikov towers of spaces. Indeed, suppose $\mathbf{M} = s\mathbf{Set}_{\mathbf{X}}$. Then an object X of \mathbf{M} is an n -type if and only if it is n -truncated in the sense above. For any integers $m \geq n \geq 0$, the n -truncation of a space X is the essentially unique n -type $\tau_{\leq n} X$ for which

$$[\tau_{\leq n} X, Y] \cong [X, Y]$$

for any n -type Y . A sequence

$$\cdots \longrightarrow X(n) \longrightarrow X(n-1) \longrightarrow \cdots \longrightarrow X(1) \longrightarrow X(0)$$

is cofibrant and fibrant if and only if the following conditions hold.

(5.32.1) The space $X(0)$ is a Kan complex.

(5.32.2) Each space $X(n)$ is n -truncated.

(5.32.3) For any integers $m \geq n \geq 0$, the morphism $X(m) \rightarrow X(n)$ is a Kan fibration.

(5.32.4) For any integers $m \geq n \geq 0$, the morphism $X(m) \rightarrow X(n)$ exhibits $X(n)$ as the n -truncation of $X(m)$.

In the category of spaces, every object is the homotopy limit of its Postnikov tower; the analogous question can now be asked in any \mathbf{X} -tractable, left proper, simplicial model category.

5.33. Observe that the Quillen adjunctions

$$\text{id} : \mathbf{M} \rightleftarrows \mathbf{M}_{\leq n} : \text{id}$$

fit together to give rise to a single Quillen functor

$$\text{id} : \mathbf{M}(\mathbf{N})_{\text{inj}} \rightleftarrows \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet}) : \text{id}.$$

This can now be composed with the adjunction $(\text{const}, \text{lim})$ to give a Quillen pair

$$\text{const} : \mathbf{M} \rightleftarrows \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet}) : \text{lim}.$$

Lemma 5.34. *The left Quillen functor*

$$\text{const} : \mathbf{M} \rightarrow \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet})$$

factors (uniquely) through a left Quillen functor

$$P : \mathbf{M} \rightarrow \mathbf{M}(\mathbf{N})_{\text{Post}}.$$

Proof. It suffices to observe that for any cofibrant object X of \mathbf{M} , the object

$$\text{const } X := [\cdots \rightrightarrows X \rightrightarrows X \rightrightarrows \cdots \rightrightarrows X \rightrightarrows X]$$

is cofibrant in $\mathbf{M}(\mathbf{N})_{\text{Post}}$. \square

Definition 5.35. (5.35.1) The *Postnikov tower* of an object X of \mathbf{M} is a cofibrant and fibrant model for

$$P(X) := [\cdots \rightrightarrows X \rightrightarrows X \rightrightarrows \cdots \rightrightarrows X \rightrightarrows X]$$

in $\mathbf{M}(\mathbf{N})_{\text{Post}}$; it will be denoted

$$\cdots \longrightarrow X\langle n \rangle \longrightarrow X\langle n-1 \rangle \longrightarrow \cdots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle$$

(5.35.2) One says that a morphism $X \rightarrow Y$ of \mathbf{M} is ∞ -*connective* if it induces an equivalence $P(X) \rightarrow P(Y)$ in $\mathbf{M}(\mathbf{N})_{\text{Post}}$.

(5.35.3) One says that \mathbf{M} is *hypercomplete* if every object is the homotopy limit of its Postnikov tower, i.e., if the composite

$$\text{Ho } \mathbf{M} \xrightarrow{\mathbf{L}P} \text{Ho } \mathbf{M}(\mathbf{N})_{\text{Post}} \xrightarrow{\text{holim}} \text{Ho } \mathbf{M}$$

is isomorphic (via the unit) to the identity functor on $\text{Ho } \mathbf{M}$.

Proposition 5.36. *There exists a hypercompletion $\mathbf{M} \rightarrow \mathbf{M}^\wedge$ of \mathbf{M} , which is the initial object in the category of left Quillen functors $\mathbf{M} \rightarrow \mathbf{P}$ under which ∞ -connective morphisms are sent to weak equivalences.*

Proof. Simply define \mathbf{M}^\wedge as the left Bousfield localization of \mathbf{M} with respect to the set $\{\lim RP(X) \rightarrow X \mid X \in G\}$, where G is an \mathbf{X} -small set of cofibrant and fibrant homotopy generators of \mathbf{M} , and R is a fibrant replacement functor. \square

Proposition 5.37. *The following conditions are equivalent.*

(5.37.1) *The model category \mathbf{M} is hypercomplete.*

(5.37.2) *The hypercompletion functor $\mathbf{M} \rightarrow \mathbf{M}^\wedge$ is a Quillen equivalence.*

(5.37.3) *Every ∞ -connective morphism of \mathbf{M} is a weak equivalence.*

Proof. It is readily apparent that (5.37.1) and (5.37.2) are equivalent conditions.

If \mathbf{M} is hypercomplete, then every ∞ -connected morphism of \mathbf{M} is a homotopy limit of weak equivalences; hence (5.37.1) implies (5.37.3). Conversely, one sees that the morphism $X \rightarrow \lim_{n \in \mathbf{N}} X\langle n \rangle$ is automatically ∞ -connected, so in fact (5.37.1) and (5.37.3) are equivalent conditions.

This completes the proof. \square

5.38. It is now an elementary exercise to show that the model category $\mathcal{S}_{\mathbf{X}}$ is itself hypercomplete. This is false, however, for the local projective (or the local injective) model category of presheaves of spaces on a site constructed in 4.56.

Application III: Postnikov and coPostnikov towers for spectral model categories

Spectral model categories are model categories enriched in the symmetric monoidal model category of symmetric spectra. Just as it is possible to construct Postnikov towers on certain simplicial model categories, it is likewise possible to construct both Postnikov and coPostnikov towers on certain spectral model categories; however, these towers are constructed *relative to* a choice of t -structure on the homotopy category.

Notation 5.39. Suppose \mathbf{X} a universe. Write $\mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ for the stable symmetric monoidal model category of symmetric spectra in pointed \mathbf{X} -small simplicial sets with the flat stable model structure [12, 17]. For any integer j and any symmetric spectrum E , write $\pi_j E$ for the j -th stable homotopy group of a fibrant replacement of E in $\mathbf{Sp}_{\mathbf{X}}^{\Sigma}$.

5.40. Suppose \mathbf{M} an \mathbf{X} -tractable, proper $\mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ -model category. Suppose also that \mathbf{M} has a zero object 0 in the $\mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ -enriched sense. That is, for any object X of \mathbf{M} ,

$$\underline{\mathrm{Mor}}_{\mathbf{M}}(0, X) \cong 0 \cong \underline{\mathrm{Mor}}_{\mathbf{M}}(X, 0).$$

Suppose further that \mathbf{M} is *stable* in the sense that the suspension functor

$$\begin{array}{ccc} \Sigma : \mathrm{Ho} \mathbf{M} & \longrightarrow & \mathrm{Ho} \mathbf{M} \\ X & \longmapsto & 0 \sqcup^{h, X} 0 \end{array}$$

is an equivalence of categories, with quasi-inverse

$$\begin{array}{ccc} \Omega : \mathrm{Ho} \mathbf{M} & \longrightarrow & \mathrm{Ho} \mathbf{M} \\ X & \longmapsto & 0 \times_X^h 0 \end{array}$$

Finally, suppose $\tau = (\mathrm{Ho}_{\leq n} \mathbf{M}, \mathrm{Ho}_{\geq n} \mathbf{M}, \tau_{\leq n}, \tau_{\geq n})$ a t -structure on the homotopy category $\mathrm{Ho} \mathbf{M}$. (Observe that here we employ *homological* indexing.)

Proposition 5.41. *For any integer n , there exists an \mathbf{X} -tractable, left proper, $\mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ -enriched model structure on the category \mathbf{M} — the n -truncated model structure $\mathbf{M}_{\leq n}$ — satisfying the following conditions.*

- (5.41.1) *The cofibrations of $\mathbf{M}_{\leq n}$ are precisely the cofibrations of \mathbf{M} .*
- (5.41.2) *The fibrant objects of $\mathbf{M}_{\leq n}$ are precisely the fibrant objects of \mathbf{M} that model objects of $\mathrm{Ho}_{\leq n} \mathbf{M}$.*
- (5.41.3) *The weak equivalences between the fibrant objects are precisely the weak equivalences of \mathbf{M} .*
- (5.41.4) *The natural functor $\mathrm{Ho}_{\leq n} \mathbf{M} \rightarrow \mathrm{Ho} \mathbf{M}_{\leq n}$ is an equivalence of full subcategories of $\mathrm{Ho} \mathbf{M}$.*

Proof. Let G be an \mathbf{X} -small set of cofibrant homotopy generators of \mathbf{M} , and let $\tau_{\geq 0} G$ be a set of cofibrant models for the truncations of elements of G to $\mathrm{Ho}_{\geq 0} \mathbf{M}$. One verifies that the n -truncated model structure is the enriched left Bousfield localization $L_{(H(n)/\mathbf{Sp}_{\mathbf{X}}^{\Sigma})} \mathbf{M}$ with respect to the set

$$H(n) := \{S^j \otimes X \rightarrow X \mid X \in \tau_{\geq 0} G, n < j\}. \quad \square$$

Corollary 5.42. *For any integers $m \geq n$, the identity functor on \mathbf{M} induces a left Bousfield localization $\mathbf{M}_{\leq m} \rightarrow \mathbf{M}_{\leq n}$.*

5.43. Observe that since $\mathrm{Ho}_{\leq n} \mathbf{M} \rightarrow \mathrm{Ho} \mathbf{M}_{\leq n}$ is an equivalence of full subcategories of $\mathrm{Ho} \mathbf{M}$, it follows that the left derived functor of the identity functor $\mathbf{M} \rightarrow \mathbf{M}_{\leq n}$ is equivalent to the truncation functor $\tau_{\leq n}$ given by the t -structure.

Proposition 5.44. *For any integer n , there exists an \mathbf{X} -tractable, right model structure on the category \mathbf{M} — the n -cotruncated model structure $\mathbf{M}_{\geq n}$ — satisfying the following conditions.*

- (5.44.1) *The fibrations of $\mathbf{M}_{\geq n}$ are precisely the fibrations of \mathbf{M} .*
- (5.44.2) *The cofibrant objects of $\mathbf{M}_{\geq n}$ are precisely the cofibrant objects of \mathbf{M} that model objects of $\mathrm{Ho}_{\geq n} \mathbf{M}$.*
- (5.44.3) *The weak equivalences between the cofibrant objects are precisely the weak equivalences of \mathbf{M} .*
- (5.44.4) *The natural functor $\mathrm{Ho}_{\geq n} \mathbf{M} \rightarrow \mathrm{Ho} \mathbf{M}_{\geq n}$ is an equivalence of full subcategories of $\mathrm{Ho} \mathbf{M}$.*

Proof. Let G be an \mathbf{X} -small set of cofibrant homotopy generators of \mathbf{M} , and let $\tau_{\geq n} G$ be a set of cofibrant models for the truncations of elements of G to $\mathrm{Ho}_{\geq n} \mathbf{M}$. The n -cotruncated model structure is the right Bousfield localization $R_{\tau_{\geq n} G} \mathbf{M}$ with respect to the set $\tau_{\geq n} G$. \square

Corollary 5.45. *For any integers $m \leq n$, the identity functor on \mathbf{M} induces a right Bousfield localization $\mathbf{M}_{\geq m} \rightarrow \mathbf{M}_{\geq n}$.*

5.46. As above, observe that since $\mathrm{Ho}_{\geq n} \mathbf{M} \rightarrow \mathrm{Ho} \mathbf{M}_{\geq n}$ is an equivalence of full subcategories of $\mathrm{Ho} \mathbf{M}$, it follows that the right derived functor of the identity functor $\mathbf{M} \rightarrow \mathbf{M}_{\geq n}$ corresponds via this equivalence to the cotruncation functor $\tau_{\geq n}$ given by the t -structure.

5.47. Suppose $\mathbf{M} = \mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ itself with its standard t -structure, in which $\mathrm{Ho}_{\leq 0} \mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ is the full subcategory of $\mathrm{Ho} \mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ comprised of those homotopy type of spectra E such that for any integer $j > 0$, $\pi_j E = 0$, and $\mathrm{Ho}_{\geq 0} \mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ is the full subcategory of $\mathrm{Ho} \mathbf{Sp}_{\mathbf{X}}^{\Sigma}$ comprised of those homotopy type of spectra E such that for any integer $j < 0$, $\pi_j E = 0$. Then $\mathbf{Sp}_{\mathbf{X}, \geq 0}^{\Sigma}$ is a right model category of connective spectra, and the right derived functor

$$\tau_{\geq 0} : \mathrm{Ho} \mathbf{Sp}_{\mathbf{X}}^{\Sigma} \rightarrow \mathrm{Ho} \mathbf{Sp}_{\mathbf{X}, \geq 0}^{\Sigma}$$

is the formation of the connective cover.

Notation 5.48. Denote by \mathbf{Z} the category whose objects are integers, in which there is a unique morphism $m \rightarrow n$ if and only if $m \leq n$.

Proposition 5.49. *There exists an \mathbf{X} -tractable, right model structure on the presheaf category $\mathbf{M}(\mathbf{Z})$ — the stable Postnikov model structure $\mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \mathrm{Post}}$ — satisfying the following conditions.*

- (5.49.1) *The fibrations of $\mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \mathrm{Post}}$ are those morphisms $Y \rightarrow X$ such that for any integers $m \geq n$, the morphism*

$$X(m) \rightarrow X(n) \times_{Y(n)} Y(m)$$

is a fibration of the model category $\mathbf{M}_{\leq m}$.

(5.49.2) In particular, the fibrant objects are those sequences

$$\cdots \longrightarrow Y(n) \longrightarrow \cdots \longrightarrow Y(1) \longrightarrow Y(0) \longrightarrow Y(-1) \longrightarrow \cdots$$

such that the following conditions hold.

(5.49.2.1) For any integer n , the object $Y(n)$ is a fibrant representative of an isomorphism class in $\mathbf{Ho}_{\leq n} \mathbf{M}$.

(5.49.2.2) For any integers $m \geq n$, the morphism $Y(m) \rightarrow Y(n)$ is a fibration of \mathbf{M} .

(5.49.3) The cofibrant objects are those sequences

$$\cdots \longrightarrow X(n) \longrightarrow \cdots \longrightarrow X(1) \longrightarrow X(0) \longrightarrow X(-1) \longrightarrow \cdots$$

satisfying the following conditions.

(5.49.3.1) For any integer n , the object $X(n)$ is cofibrant.

(5.49.3.2) For any integers $m \geq n$, the morphism $X(m) \rightarrow X(n)$ exhibits $X(n)$ as the n -truncation of $X(m)$; i.e., the natural morphism

$$\tau_{\leq n} X(m) \rightarrow X(n)$$

of $\mathbf{Ho} \mathbf{M}_{\leq n}$ is an isomorphism.

(5.49.4) The weak equivalences between the cofibrant and fibrant objects are precisely the objectwise weak equivalences.

Proof. As in 5.31, the stable Postnikov right model structure is the holim model structure on the category of left sections of the left Quillen presheaf

$$\begin{array}{ccc} \mathbf{M}_{\leq \bullet} : \mathbf{Z}^{\text{op}} & \longrightarrow & \mathbf{Cat}_{\mathbf{Y}} \\ n & \longmapsto & \mathbf{M}_{\leq n} \end{array} \quad \square$$

Definition 5.50. We will hereafter refer to the model category \mathbf{M} as *right viable* if, for every integer n , the cotruncated right model category $\mathbf{M}_{\geq n}$ is a left proper, \mathbf{X} -tractable model category.

Proposition 5.51. *Suppose that \mathbf{M} is right viable. Then there exists a left proper, \mathbf{X} -tractable model structure on the presheaf category $\mathbf{M}(\mathbf{Z})$ — the stable coPostnikov model structure $\mathbf{M}(\mathbf{Z})_{\Omega^\infty \text{ Post}}$ — satisfying the following conditions.*

(5.51.1) The cofibrations of $\mathbf{M}(\mathbf{Z})_{\Omega^\infty \text{ Post}}$ are those morphisms $Y \rightarrow X$ such that for any integers $m \leq n$, the morphism

$$X(m) \sqcup^{X(n)} Y(n) \rightarrow Y(m)$$

is a cofibration of the model category $\mathbf{M}_{\geq m}$.

(5.51.2) In particular, the cofibrant objects are those sequences

$$\cdots \longrightarrow X(n) \longrightarrow \cdots \longrightarrow X(1) \longrightarrow X(0) \longrightarrow X(-1) \longrightarrow \cdots$$

such that the following conditions hold.

(5.51.2.1) For any integer n , the object $X(n)$ is a cofibrant representative of an isomorphism class in $\mathbf{Ho}_{\geq n} \mathbf{M}$.

(5.51.2.2) For any integers $m \leq n$, the morphism $X(n) \rightarrow X(m)$ is a cofibration of \mathbf{M} .

(5.51.3) The fibrant objects are those sequences

$$\cdots \longrightarrow Y(n) \longrightarrow \cdots \longrightarrow Y(1) \longrightarrow Y(0) \longrightarrow Y(-1) \longrightarrow \cdots$$

satisfying the following conditions.

(5.51.3.1) For any integer n , the object $Y(n)$ is cofibrant.

(5.51.3.2) For any integers $m \leq n$, the morphism $Y(n) \rightarrow Y(m)$ exhibits $X(n)$ as the n -cotruncation of $Y(m)$; i.e., the natural morphism

$$Y(n) \rightarrow \tau_{\geq n} Y(m)$$

of $\text{Ho } \mathbf{M}_{\geq n}$ is an isomorphism.

(5.51.4) The weak equivalences between the cofibrant and fibrant objects are precisely the objectwise weak equivalences.

Proof. The stable coPostnikov structure is the holim model structure on the category of right sections of the right Quillen presheaf

$$\begin{array}{ccc} \mathbf{M}_{\geq \bullet} : \mathbf{Z} & \longrightarrow & \mathbf{Cat}_{\mathbf{Y}} \\ n & \longmapsto & \mathbf{M}_{\geq n} \end{array} \quad \square$$

5.52. Suppose that \mathbf{M} is right viable. Observe that the Quillen adjunctions

$$\text{id} : \mathbf{M} \rightleftarrows \mathbf{M}_{\leq n} : \text{id} \quad \text{and} \quad \text{id} : \mathbf{M}_{\geq n} \rightleftarrows \mathbf{M} : \text{id}$$

fit together to give rise to Quillen functors

$$\text{id} : \mathbf{M}(\mathbf{Z})_{\text{inj}} \rightleftarrows \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet}) : \text{id} \quad \text{and} \quad \text{id} : \text{Sect}_{\text{proj}}^R(\mathbf{M}_{\geq \bullet}) \rightleftarrows \mathbf{M}(\mathbf{Z})_{\text{proj}} : \text{id}.$$

These adjunctions can now be composed with the adjunction $(\text{const}, \text{lim})$ and the adjunction $(\text{colim}, \text{const})$ to give a Quillen pair

$$\text{const} : \mathbf{M} \rightleftarrows \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet}) : \text{lim} \quad \text{and} \quad \text{colim} : \text{Sect}_{\text{proj}}^R(\mathbf{M}_{\geq \bullet}) \rightleftarrows \mathbf{M} : \text{const}.$$

Lemma 5.53. *The left (respectively, right) Quillen functor*

$$\text{const} : \mathbf{M} \rightarrow \text{Sect}_{\text{inj}}^L(\mathbf{M}_{\leq \bullet}) \quad (\text{resp.,} \quad \text{const} : \mathbf{M} \rightarrow \text{Sect}_{\text{proj}}^R(\mathbf{M}_{\geq \bullet}))$$

factors (uniquely) through a left (resp., right) Quillen functor

$$P : \mathbf{M} \rightarrow \mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \text{Post}} \quad (\text{resp.,} \quad Q : \mathbf{M} \rightarrow \mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}}).$$

(Here we assume for the parenthetical case that \mathbf{M} is right viable.)

Proof. For any cofibrant object X of \mathbf{M} , the object

$$\text{const } X := [\cdots = X = X = X = \cdots]$$

is cofibrant in $\mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \text{Post}}$, and for any fibrant object X of \mathbf{M} , the object $\text{const } X$ is fibrant in $\mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}}$. \square

Definition 5.54. (5.54.1) The *stable Postnikov tower* of an object X in \mathbf{M} is a cofibrant and fibrant model for $\text{const } X$ in $\mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \text{Post}}$; it will be denoted

$$\cdots \longrightarrow X\langle n \rangle \longrightarrow \cdots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle \longrightarrow X\langle -1 \rangle \longrightarrow \cdots .$$

(5.54.2) Dually, suppose that \mathbf{M} is right viable. Then the *stable coPostnikov tower* of an object X in \mathbf{M} is a cofibrant and fibrant model for the object $\text{const } X$ in the model category $\mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}}$; it will be denoted

$$\cdots \longrightarrow X\langle n \rangle \longrightarrow \cdots \longrightarrow X\langle 1 \rangle \longrightarrow X\langle 0 \rangle \longrightarrow X\langle -1 \rangle \longrightarrow \cdots .$$

(5.54.3) One says that a morphism $X \rightarrow Y$ of \mathbf{M} is ∞ -*connective* if it induces an equivalence $\text{const } X \rightarrow \text{const } Y$ in $\mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \text{Post}}$.

(5.54.4) If \mathbf{M} is right viable, one says that a morphism $X \rightarrow Y$ of \mathbf{M} is $-\infty$ -*connective* if it induces an equivalence $\text{const } X \rightarrow \text{const } Y$ in $\mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}}$.

(5.54.5) One says that \mathbf{M} is *stably hypercomplete* if every object is the homotopy limit of its stable Postnikov tower, i.e., if the composite

$$\text{Ho } \mathbf{M} \xrightarrow{\text{LP}} \text{Ho } \mathbf{M}(\mathbf{Z})_{\Omega^{-\infty} \text{Post}} \xrightarrow{\text{holim}} \text{Ho } \mathbf{M}$$

is isomorphic (via the unit) to the identity functor on $\text{Ho } \mathbf{M}$.

(5.54.6) Dually, one says that \mathbf{M} is *stably cohypercomplete* if it is right viable, and every object is the homotopy colimit of its stable coPostnikov tower, i.e., if the composite

$$\text{Ho } \mathbf{M} \xrightarrow{\text{RQ}} \text{Ho } \mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}} \xrightarrow{\text{hocolim}} \text{Ho } \mathbf{M}$$

is isomorphic (via the unit) to the identity functor on $\text{Ho } \mathbf{M}$.

Proposition 5.55. *There exists a stable hypercompletion $\mathbf{M} \rightarrow \mathbf{M}^{\wedge}$ of \mathbf{M} , which is the initial object in the category of left Quillen $(\mathbf{Sp}_{\mathbf{X}}^{\Sigma})$ -functors $\mathbf{M} \rightarrow \mathbf{N}$ under which ∞ -connective morphisms are sent to weak equivalences.*

Proof. Define \mathbf{M}^{\wedge} as the enriched left Bousfield localization of \mathbf{M} with respect to the set $\{\lim RP(X) \rightarrow X \mid X \in G\}$, where G is an \mathbf{X} -small set of cofibrant homotopy generators of \mathbf{M} , and R is a fibrant replacement functor. \square

Proposition 5.56. *Dually, if \mathbf{M} is right viable, then there exists a stable cohypercompletion $\mathbf{M} \rightarrow \mathbf{M}^{\vee}$ of \mathbf{M} , which is the initial object in the category of right Quillen $(\mathbf{Sp}_{\mathbf{X}}^{\Sigma})$ -functors $\mathbf{M} \rightarrow \mathbf{N}$ under which $-\infty$ -connective morphisms are sent to weak equivalences.*

Proof. Define \mathbf{M}^{\vee} as the right Bousfield localization of \mathbf{M} with respect to the set $\{\text{colim } Z \mid Z \in G'\}$, where G' is an \mathbf{X} -small set of cofibrant homotopy generators of $\mathbf{M}(\mathbf{Z})_{\Omega^{\infty} \text{Post}}$. \square

5.57. As in the previous section, one has the following pair of results. The proofs are left to the reader.

Proposition 5.58. *The following conditions are equivalent.*

- (5.58.1) The model category \mathbf{M} is stably hypercomplete.
 (5.58.2) The stable hypercompletion functor $\mathbf{M} \rightarrow \mathbf{M}^\wedge$ is a Quillen equivalence.
 (5.58.3) Every ∞ -connective morphism of \mathbf{M} is a weak equivalence.

Proposition 5.59. *Suppose \mathbf{M} is right viable. Then the following conditions are equivalent.*

- (5.59.1) The model category \mathbf{M} is stably cohypercomplete.
 (5.59.2) The stable cohypercompletion functor $\mathbf{M} \rightarrow \mathbf{M}^\vee$ is a Quillen equivalence.
 (5.59.3) Every $-\infty$ -connective morphism of \mathbf{M} is a weak equivalence.

5.60. It is now an elementary exercise to show that the model category \mathbf{Sp}_X^Σ is both stably hypercomplete and stably cohypercomplete. This is false, however, for the local projective model category of presheaves of spectra on a site constructed in 4.56.

5.61. The underlying ∞ -category of the stable hypercompletion \mathbf{M}^\wedge can be described as the homotopy limit of the diagram of ∞ -categories

$$\cdots \xrightarrow{\tau_{\leq n}} \mathbf{M}_{\leq n} \xrightarrow{\tau_{\leq n-1}} \cdots \xrightarrow{\tau_{\leq 1}} \mathbf{M}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathbf{M}_{\leq 0} \xrightarrow{\tau_{\leq -1}} \mathbf{M}_{\leq -1} \xrightarrow{\tau_{\leq -2}} \cdots$$

Similarly, the underlying ∞ -category of the stable cohypercompletion \mathbf{M}^\vee can be described as the homotopy limit of the diagram of ∞ -categories

$$\cdots \xleftarrow{\tau_{\geq n+1}} \mathbf{M}_{\geq n} \xleftarrow{\tau_{\geq n}} \cdots \xleftarrow{\tau_{\geq 2}} \mathbf{M}_{\geq 1} \xleftarrow{\tau_{\geq 1}} \mathbf{M}_{\geq 0} \xleftarrow{\tau_{\geq 0}} \mathbf{M}_{\geq -1} \xleftarrow{\tau_{\geq -1}} \cdots$$

This point of view is explored in [13, §7], where stable hypercompleteness is called “left completeness” and stable cohypercompleteness is called “right completeness.”

References

- [1] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.
- [2] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.
- [3] J. Daniel Christensen and Daniel C. Isaksen. Duality and pro-spectra. *Algebr. Geom. Topol.*, 4:781–812 (electronic), 2004.
- [4] Daniel Dugger. Combinatorial model categories have presentations. *Adv. Math.*, 164(1):177–201, 2001.
- [5] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [6] W. G. Dwyer and D. M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.
- [7] W. G. Dwyer and D. M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(3):267–284, 1980.

- [8] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*, volume 113 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.
- [9] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.
- [10] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [11] Mark Hovey. Homotopy theory of comodules over a Hopf algebroid. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 261–304. Amer. Math. Soc., Providence, RI, 2004.
- [12] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.
- [13] J. Lurie. Derived algebraic geometry I. Stable infinity categories. Available from <http://www.math.harvard.edu/~lurie>.
- [14] J. Lurie. *Higher topos theory*. Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009. Also available electronically from <http://www.math.harvard.edu/~lurie>.
- [15] Michael Makkai and Robert Paré. *Accessible categories: the foundations of categorical model theory*, volume 104 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.
- [16] C. Reedy. Homotopy theory of model categories. Unpublished manuscript, available electronically from <http://www-math.mit.edu/~psh/#Reedy>.
- [17] Brooke Shipley. A convenient model category for commutative ring spectra. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, pages 473–483. Amer. Math. Soc., Providence, RI, 2004.
- [18] J. Smith. Combinatorial model categories. To exist.
- [19] M. Spitzweck. *Operads, algebras and modules in general model categories and motives*. PhD thesis, Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Willhelms Universität Bonn, 2001.

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