Definition 1 (Grothendieck, [3, §16.8]). Suppose $k$ a field, $X/k$ a smooth variety. Then the diagonal embedding $X \hookrightarrow X \times X$ is given by an ideal $I \triangleleft O_{X \times X}$, and the sheaf of differential operators of order $n \in \mathbb{N}$ is typically defined as the $O_X$-dual of the quotient $O_{X \times X}/I^{n+1}$:

$$D_{X/k,n} := \text{Mor}_{O_X}(O_{X \times X}/I^{n+1}, O_X).$$

The resulting filtered sheaf is a sheaf of noncommutative rings, which will simply be denoted $D_{X/k}$, and the category of right $D_{X/k}$-modules that are quasicoherent as $O_X$-modules will be denoted $\text{Mod}^r(D_{X/k})$.

Theorem 2 (Kashiwara, [6, Theorem 2.3.1]). Suppose $Z \hookrightarrow X$ a closed immersion of smooth varieties over a field $k$ of characteristic $0$; then the category of right $D_{Z/k}$-modules is naturally equivalent to the category of right $D_{X/k}$-modules set-theoretically supported on $Z$.

Theorem 3 (Hodges, [5]). Suppose $k$ algebraically closed of characteristic $0$, and suppose $X$ smooth over $k$. Then the functor $- \otimes_{O_X} D_{X/k}$ induces an equivalence of $K$-theory spectra

$$K(X) \cong K(D_{X/k}).$$

About the Proof. For affines, this follows from the $K'$-equivalence of a filtered ring and its $0$-th filtered piece [7, Theorem 7]. The general case follows from using Kashiwara’s Theorem to devise a localization sequence for $K(D_{-/k})$, which can be compared to the localization sequence for $K$.

Example 4 (Bernstein-Gelfand-Gelfand, [2]). If $X$ is singular, then $D_{X/k}$ is an unpleasant ring, and neither Kashiwara’s nor Hodges’ Theorem holds for right $D_{X/k}$-modules. To illustrate, suppose that $C$ is the affine cone over the Fermat curve $x^3 + y^3 + z^3 = 0$ (over $\mathbb{C}$, let us say); then $X$ is normal, and has an isolated Gorenstein singularity at the origin.

Nevertheless, the ring $D(C)$ of differential operators is neither left nor right noetherian: if $e$ denotes the Euler operator $x\partial_x + y\partial_y + z\partial_z$, and if $D^{(j)}(C)$ (respectively, $D^{(j)}_o(C)$) is the $R$-module of homogenous differential operators of degree $j$ (resp., and of order $n$), then the two-sided ideals

$$J_k := \sum_{j>1} D^{(j)} + \sum_{n \geq 0} e^n D^{(1)}_k$$

form an ascending chain that does not stabilize.

5. The standard method for rectifying this is defining deviancy down by forcing Kashiwara’s Theorem; namely, for a singular scheme $Z$, one embeds $Z$ (at least locally) into a smooth scheme $X$ and defines the category of right $D_{Z/k}$-modules to be the full subcategory of right $D_{X/k}$-modules set-theoretically supported along

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\[1\text{For simplicity I will use the term “variety” for a separated noetherian scheme of finite type.}\]

\[2\text{I will stick to right $D$-modules here.}\]
immersion; the resulting theory of nilimmersion of (Grothendieck, [4, 4.1]). The infinitesimal site \((X_{\text{int}}/k)\) of \(X/k\) is the category of diagrams \(X \leftarrow S \rightarrow T\) in which the morphism \(S \rightarrow T\) is a closed nilimmersion of \(k\)-schemes, and the morphism \(S \rightarrow X\) is étale.\(^3\) There is a natural forgetful functor \((S, T) \mapsto T\) to the category of \(k\)-schemes; pull back the étale topology along this functor.

7. There is a stack in categories on the infinitesimal site of \(X\):

\[
\begin{align*}
\text{Mod}^{\mathcal{D}}_{X/k, \text{qc}} : (X_{\text{int}}/k)^{\text{op}} & \longrightarrow \text{Cat} \\
(S, T) & \longrightarrow \text{Mod}_{\text{qc}}(\mathcal{O}_T) \\
(f, g) & \longrightarrow H^0 g^*.
\end{align*}
\]

**Definition 8** (Beilinson-Drinfeld, [1, Definition 7.10.3]). A \(\mathcal{D}\)-crystal on \(X\) is a cartesian section of the stack \(\text{Mod}^{\mathcal{D}}_{X/k, \text{qc}}\). More precisely, a \(\mathcal{D}\)-crystal \(M\) assigns to every object \((S, T)\) a quasicoherent \(\mathcal{O}_T\)-module \(M_{(S, T)}\) and to every morphism \((f, g) : (S, T) \rightarrow (S', T')\) an isomorphism

\[M_{(S, T)} \rightarrow H^0 g^* M_{(S', T')}.\]

The category of such will be denoted \(\text{Cris}^!((X/k)\).

**Example 9.** Suppose \(X\) a smooth \(k\)-scheme. Then for any object \((S, T) \in (X_{\text{int}}/k)\), let \(p_T : T \rightarrow \text{Spec } k\) denote the structure morphism of \(T\), and set

\[t \omega_{X/k}(T) := H^n p^*_T \mathcal{O}_{\text{Spec } k}.\]

It follows from the smoothness property of \(X\) that there exists a morphism \(q : T \rightarrow X\) of \(k\)-schemes, so that \(H^n p^*_T \mathcal{O}_{\text{Spec } k} \cong H^0 q^* \omega_{X/k}\), where \(\omega_{X/k}\) is the dualizing sheaf of top-degree differential forms.\(^4\) Thus \(t \omega_{X/k}\) is a \(\mathcal{D}\)-crystal.

**Proposition 10** (Beilinson-Drinfeld, [1, Proposition 7.10.12]). If \(X\) is a smooth \(k\)-scheme, then the category \(\text{Cris}^!((X/k)\) is equivalent to the category \(\text{Mod}^\mathcal{D}'((D_{X/k})\).

**About the Proof.** The question is local, so assume \(X\) affine. If \(p_{r_1}, p_{r_2}\) are the projections from the formal completion of the diagonal, \(\text{Cris}^!((X/k)\) is equivalent to the category of quasicoherent \(\mathcal{O}_X\)-modules \(M\) equipped with isomorphisms \(p_{r_1}^* M \cong p_{r_2}^* M\) satisfying the obvious cocycle condition. There is a natural isomorphism

\[M \otimes_{\mathcal{O}_X} D_X \cong p_{r_2}^* p_{r_1}^* M,\]

and adjunction then converts the isomorphism \(p_{r_1}^* M \cong p_{r_2}^* M\) into the structure of a right \(D_X\)-module; the cocycle condition guarantees associativity. \(\square\)

**Theorem 11** (Beilinson-Drinfeld, [1, Lemma 7.10.11]). Kashiwara’s Theorem holds for \(\mathcal{D}\)-crystals; i.e., for any closed immersion \(Z \rightarrow X\) of schemes (not necessarily smooth), the category of \(\mathcal{D}\)-crystals on \(Z\) is naturally equivalent to the category of \(\mathcal{D}\)-crystals on \(X\) set-theoretically supported on \(Z\).

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\(^3\) I can replace “étale” more generally with “quasi-finite” or less generally with “Zariski open immersion;” the resulting theory of \(\mathcal{D}\)-crystals is the same in each instance.

\(^4\) Observe however that \(\omega_{X/k}(T)\) is only a truncation of the dualizing complex \(\omega_T/k\).
12. The appropriate functorialities of \( \mathcal{D} \)-crystals do not exist in general. It is more natural not to truncate \( g \), and to consider instead the following \((\infty, 1)\)-stack:

\[
\text{HMod}^{[X/k, \text{qc}]} : \left( \text{X}^{\text{inf}}/k \right)^{\text{op}} \to \left( \infty, 1 \right) \text{Cat}
\]

\[
(S, T) \to \text{Cplx}(\text{Mod}_{\text{qc}}(O_T))
\]

\[
(f, g) \to g'.
\]

**Definition 13.** A homotopy \( \mathcal{D} \)-crystal on \( X \) is a homotopy cartesian section of the stack \( \text{HMod}^{[X/k, \text{qc}]} \). The category of such will be denoted \( \text{HCris}^!(X/k) \).

**Example 14.** The assignment \((S, T) \to \omega_{T/k}\) is a homotopy \( \mathcal{D} \)-crystal on \( X \).

**Proposition 15.** If \( X \) is a smooth \( k \)-scheme, then the category \( \text{HCris}^!(X/k) \) is equivalent to the category \( \text{Cplx}(\text{Mod}^r(D_{X/k})) \).

**Theorem 16.** Kashiwara’s Theorem holds for homotopy \( \mathcal{D} \)-crystals; i.e., if \( Z \to X \) is any closed immersion of schemes (not necessarily smooth), there is a natural equivalence between the \((\infty, 1)\)-category of homotopy \( \mathcal{D} \)-crystals on \( Z \) and the full subcategory of the \((\infty, 1)\)-category of homotopy \( \mathcal{D} \)-crystals on \( X \) set-theoretically supported on \( X \).

**Conjecture 17.** For any scheme \( X \), the \( K \)-theory of the \((\infty, 1)\)-category of \( \mathcal{D} \)-crystals on \( X \) is naturally equivalent to \( K'(X) \).

**Strategy.** Again the analogue of Kashiwara’s theorem permits a quick reduction to the affine case. In this case it seems possible to work directly with the definition of \( K \)-theory of \((\infty, 1)\)-categories, but since the definition is necessarily complicated, I have not yet managed to check all the details unless \( X \) is Cohen-Macaulay. □

**References**


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