DISTRIBUTIONAL AND L^q NORM INEQUALITIES

FOR POLYNOMIALS OVER CONVEX BODIES IN $\mathbf{R}^n$

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§1. Introduction

Let $\mathcal{P}_{d,n}$ be the vector space of all polynomials of degree at most d in $\mathbf{R}^n$. Let K be a convex body of volume 1 in $\mathbf{R}^n$ and let $1 \le q \le \infty$. Since $\mathcal{P}_{d,n}$ is finite dimensional, the norms $\left(\int_K |p|^q\right)^{\frac{1}{q}}$ are all equivalent to each other. Recently there has been considerable interest in the behaviour of the constants in these equivalences as q varies when we consider arbitrary unit-volume convex bodies K. See for example the work of Brudnyi and Ganzburg [BG], Gromov and Milman [GM], Bourgain [Bour], Bobkov [Bobk] and Nazarov, Sodin and Volberg [NSV].

In this paper, we wish to complete the analysis of the constants in these equivalences as well as to extend these results to the vector-valued setting. For a (real or complex) Banach space X with norm $\|\cdot\|$ and a polynomial $p : \mathbf{R}^n \to X$ of degree at most d, we define the functional $p^{\#}(x) = \|p(x)\|^{\frac{1}{d}}$. For a convex body K in $\mathbf{R}^n$ of volume 1, we consider the usual L^q norms of $p^{\#}$ over K; that is, $\|p^{\#}\|_q = \left(\int_K p^{\#}(x)^q dx\right)^{\frac{1}{q}} = \left(\int_K \|p(x)\|^{\frac{q}{d}} dx\right)^{\frac{1}{q}}$. When $q = 0$, we set $\|p^{\#}\|_0 = \exp \int_K \log p^{\#}(x)\, dx$ and $\|p^{\#}\|_\infty$ is the usual L^∞ norm of $p^{\#}$.

Let $0 \le r \le q \le \infty$. Hölder's inequality gives a trivial inequality for the L^q norms with (best possible) constant 1 and for the reverse inequality we have:

Theorem 1 *Let $p : \mathbf{R}^n \to X$ be a polynomial of degree at most d, let K be a convex body in $\mathbf{R}^n$ of volume 1 and let $0 \le r \le q \le \infty$. Then there exists an absolute constant C independent of p, d, K, n, q, r and X such that*

$$\|p^{\#}\|_q \ \le C \ \frac{[nB(n, q+1)]^{\frac{1}{q}}}{[nB(n, r+1)]^{\frac{1}{r}}} \ \|p^{\#}\|_r$$

where B denotes the classical Beta function.

Recall that $nB(n, q+1) = -\int_0^1 u^q d(1-u)^n$; in the limiting cases $q = 0$ and $q = \infty$, the quantity $[nB(n, q+1)]^{\frac{1}{q}}$ is to be understood as $1/n$ and 1 respectively. In particular we note that the estimate in Theorem 1 is independent of the norm $\|\cdot\|$ from X.

By standard estimates for the Beta function we obtain:

Corollary *Let $p : \mathbf{R}^n \to X$ be a polynomial of degree at most d, let K be a convex body in $\mathbf{R}^n$ of volume 1 and let $0 \le r \le q \le \infty$. Then there exists an absolute constant C independent of p, d, K, n, q, r and X such that*

(a) if $n \le r \le q$ then

$$\|p^{\#}\|_q \ \le C \ \|p^{\#}\|_r;$$

(b) if $r \le n \le q$ then

$$\|p^{\#}\|_q \ \le C \ \frac{n}{\max(r, 1)} \ \|p^{\#}\|_r;$$

(c) if $r \le q \le n$ then

$$\|p^{\#}\|_q \ \le C \ \frac{\max(q, 1)}{\max(r, 1)} \ \|p^{\#}\|_r.$$

Up to the numerical constant C, the constant on the right hand side of Theorem 1 is optimal if one seeks an inequality valid for *arbitrary* convex bodies K. One simply takes $p(x) = x_1^d$ and $K = \{(x_1, x') \in \mathbf{R}^n : 0 \leq x_1 \leq 1, |x'| \leq 1 - x_1\}$. The scalar-valued case $q = \infty, r \leq 1$ (in which case the constant on the right hand side is essentially n) is due to Brudnyi and Ganzburg [BG]. For dimensionless bounds, the scalar-valued cases $r = 0$, $q \geq 0$ and $r = d$, $q \geq 2d$ are due to Bobkov [Bobk] (in these cases the dimensionless bound on the right hand side is essentially q). One can then extrapolate these bounds to get sharp dimension free Khinchine-Kahane type inequalities in the exponential class. This refined earlier work of Bourgain [Bour] which in turn extended a result of Gromov and Milman [GM] to the general degree d case from the linear case $d = 1$. Nazarov, Sodin and Volberg [NSV] have also obtained Bobkov's dimensionless bound in the case $r = 0$ and $q \geq 0$ (by different methods), as well as other interesting results. Our Theorem 1 may be viewed as a completion of all these results, giving the precise behaviour in all the parameters, d, n, q and r.

The case $r \leq 1$ and general q has a stronger formulation in terms of distributional inequalities for vector-valued polynomials over convex bodies in $\mathbf{R}^n$ (which may be of independent interest for certain problems in real and harmonic analysis). In fact, we have:

Theorem 2 *Let $p : \mathbf{R}^n \to X$ be a polynomial of degree at most d, and let K be a convex body in $\mathbf{R}^n$ of volume 1. Let $0 \leq q \leq \infty$. Then there exists an absolute constant C independent of p, d, K, n, q and X so that for any $\alpha > 0$,*

$$\|p^\#\|_q \ \alpha^{-1} |\{x \in K : p^\#(x) \leq \alpha\}| \ \leq \ Cn(nB(n, q+1))^{\frac{1}{q}}.$$

In particular, we have:

Corollary *Let $p : \mathbf{R}^n \to X$ be a polynomial of degree at most d, let K be a convex body in $\mathbf{R}^n$ of volume 1 and let $0 \leq q \leq \infty$. Then there exists an absolute constant C independent of p, d, K, n, q and X so that for any $\alpha > 0$,*

(a) if $n \leq q$ then

$$\|p^\#\|_q \ \alpha^{-1} |\{x \in K : p^\#(x) \leq \alpha\}| \ \leq C\, n;$$

(b) if $q \leq n$ then

$$\|p^\#\|_q \ \alpha^{-1} |\{x \in K : p^\#(x) \leq \alpha\}| \ \leq C \max(q, 1).$$

As before, up to the constant C, the inequalities are sharp (to see this we use the same example as for Theorem 1). The scalar-valued case $q = \infty$ is due to Brudnyi and Ganzburg [BG]. Nazarov, Sodin and Volberg [NSV] have obtained Theorem 2 independently by somewhat different methods. In §6, Remark 2 below, we shall show how one can obtain the case $r \leq 1$ and general q in Theorem 1 from Theorem 2.

In common with Bobkov's work [Bobk] (and that of Nazarov, Sodin and Volberg [NSV]) the main tool in this current work is the utilisation of a certain powerful extremal result of Kannan, Lovász and Simonovits which we now state. For $a, b \in \mathbf{R}^n$ and $\lambda \geq 1$ define the measures $\mu_{a,b,\lambda}$ by $\langle \phi, \mu_{a,b,\lambda} \rangle = \int_0^1 \phi(a(1-t)+bt)(\lambda - t)^{n-1}dt$.

Theorem ([KLS]). *Suppose f_1, f_2, f_3, f_4 are continuous nonnegative integrable functions on $\mathbf{R}^n$ and $\alpha, \beta > 0$. Suppose that for every $a, b \in \mathbf{R}^n$ and $\lambda \geq 1$,*

$$\left(\int f_1 d\mu_{a,b,\lambda}\right)^{\alpha}\left(\int f_2 d\mu_{a,b,\lambda}\right)^{\beta} \leq \left(\int f_3 d\mu_{a,b,\lambda}\right)^{\alpha}\left(\int f_4 d\mu_{a,b,\lambda}\right)^{\beta}.$$

Then for every convex open set K in $\mathbf{R}^n$

$$\left(\int_K f_1\right)^{\alpha}\left(\int_K f_2\right)^{\beta} \leq \left(\int_K f_3\right)^{\alpha}\left(\int_K f_4\right)^{\beta}.$$

(Note that the reverse implication is straightforward.)

Finally, C will denote a generic *absolute* constant whose precise value may change from line to line.

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§2. Reduction to weighted inequalities in dimension 1

We shall first prove the results in the scalar-valued setting and then show in §5 how one can extend the arguments to the vector-valued setting. In the scalar-valued setting, by the Kannan, Lovász and Simonovits theorem of the introduction, Theorems 1 and 2 are equivalent (after a limiting argument because $\chi_{\{x \in K \,:\, |p(x)| \leq \alpha\}}$ is not a continuous function) to Theorems 3 and 4 respectively:

Theorem 3 *Let $p : \mathbf{R} \to \mathbf{C}$ be a polynomial of degree at most d, $n \in \mathbf{N}$, $\lambda \geq 1$ and $0 \leq r \leq q \leq \infty$. Then there exists an absolute constant C independent of the above parameters such that*

$$\left(\frac{\int_0^1 |p(t)|^{\frac{q}{d}}(\lambda - t)^{n-1}dt}{\int_0^1 (\lambda - t)^{n-1}dt}\right)^{\frac{1}{q}} \leq C\frac{[nB(n, q+1)]^{\frac{1}{q}}}{[nB(n, r+1)]^{\frac{1}{r}}}\left(\frac{\int_0^1 |p(t)|^{\frac{r}{d}}(\lambda - t)^{n-1}dt}{\int_0^1 (\lambda - t)^{n-1}dt}\right)^{\frac{1}{r}}.$$

Theorem 4 Let *Let $p : \mathbf{R} \to \mathbf{C}$ be a polynomial of degree at most d, $n \in \mathbf{N}$, $\lambda \geq 1$ and $0 \leq q \leq \infty$. Then there exists an absolute constant C independent of the above parameters* *so that for any $\alpha > 0$,*

$$\left(\frac{\int_0^1 |p(t)|^{\frac{q}{d}}(\lambda-t)^{n-1}dt}{\int_0^1 (\lambda-t)^{n-1}dt}\right)^{\frac{1}{q}} \frac{\alpha^{-\frac{1}{d}}\int_0^1 \chi_{\{|p(t)|\le\alpha\}}(\lambda-t)^{n-1}dt}{\int_0^1 (\lambda-t)^{n-1}dt} \;\le\; Cn\,(nB(n,q+1))^{\frac{1}{q}}.$$

Although the forms of the inequalities in Theorems 3 and 4 make sense only for $0 < r \le q < \infty$, it is clear how to extend them when $r = 0, q = 0$ and/or $r = \infty, q = \infty$. For instance, when $q = 0$, the conclusion of Theorem 4 takes the form

$$\exp\frac{1}{d}\left(\frac{\int_0^1 [\log|p(t)|](\lambda-t)^{n-1}dt}{\int_0^1 (\lambda-t)^{n-1}dt}\right) \frac{\alpha^{-\frac{1}{d}}\int_0^1 \chi_{\{|p(t)|\le\alpha\}}(\lambda-t)^{n-1}dt}{\int_0^1 (\lambda-t)^{n-1}dt} \;\le\; C.$$

To prove Theorems 3 and 4, we may of course assume that $r > 0$ and $q < \infty$ and then pass to the limit.

§3. **Proof of Theorem 3**

We begin with the proof of Theorem 3. We first need some preliminary lemmas. The first is a well-known elementary Remez type inequality. It is also a simple consequence of the case $n = 1$ of Theorem 1 or Theorem 3 and as such is already contained in [BG], for instance. We include a simple proof for the convenience of the reader.

Lemma 1 *There is an absolute constant C so that if $p : \mathbf{R} \to \mathbf{C}$ is a polynomial of degree at most d, if $0 \le r \le q \le \infty$, and if $t \ge u$, then*

$$\Big(\frac{1}{t}\int_0^t |p|^{\frac{q}{d}}\Big)^{\frac{1}{q}} \;\le\; C\frac{t}{u}\Big(\frac{1}{u}\int_0^u |p|^{\frac{r}{d}}\Big)^{\frac{1}{r}}.$$

(We have the usual interpretation in the limiting cases $r, q = 0, \infty$.)

Proof of Lemma 1 We may assume that $q = \infty, r = 0$ and $u = 1$. So we want to show

$$\| |p|^{\frac{1}{d}} \|_{L^\infty[0,t]} \;\le\; C\,t\exp\frac{1}{d}\int_0^1 \log|p(s)|ds$$

for $t \ge 1$. Clearly we may also assume that $p(z) = \prod(z-\zeta_j)$ is monic. Now

$$\max_{0\le s\le t}|p(s)|^{\frac{1}{d}} \;=\; \max_{0\le s\le t}\prod|s-\zeta_j|^{\frac{1}{d}} \;=\; \max_{0\le s\le 1}\prod|st-\zeta_j|^{\frac{1}{d}} \;\le t\max_{0\le s\le 1}\prod|s-\zeta_j/t|^{\frac{1}{d}}.$$

Moreover $t \ge 1$ and $|\zeta_j| \ge 2$ implies $|s-\zeta_j/t| \le 2|\zeta_j| \le 4|s-\zeta_j|$ for $0 \le s \le 1$; so that we are left with proving

$$\max_{0\le s\le 1}\prod_{|\zeta_j|\le 2}|s-\zeta_j/t|^{\frac{1}{d}} \;\le\; C\exp\Big\{\frac{1}{d}\int_0^1 \sum_{|\zeta_j|\le 2}\log|s-\zeta_j|ds\Big\}.$$

The term on the left of this inequality is bounded by 3, while the term on the right is bounded below by $\exp\gamma$ where $\gamma = \inf_{|\zeta|\le 2} \int_0^1 \log|s-\zeta|ds$. The lemma is established with $C = 12\mathrm{e}^{-\gamma}$. □

Lemma 2 *There is an absolute constant C so that if $0 < r \le 2m$,*

$$\Big(\int_0^{\frac{r}{2}} (1-t/m)^{m-1} t^{r+1} dt\Big)^{\frac{1}{r}} \ge C\Big(\int_0^m (1-t/m)^{m-1} t^{r+1} dt\Big)^{\frac{1}{r}}.$$

We remark that the term on the right hand side of Lemma 2 is itself bounded below by $m[(m+1)B(m+1,r+1)]^{\frac{1}{r}}$.

Proof of Lemma 2

$$\int_0^{\frac{r}{2}} (1-t/m)^{m-1} t^{r+1} dt \ \ge \int_0^{\frac{r}{4}} (1-t/m)^{m-1} t^{r+1} dt \ge \int_0^{\frac{r}{4}} \mathrm{e}^{-2t} t^{r+1} dt \ge \frac{1}{r+2}\Big(\frac{r}{4}\Big)^{r+2} \mathrm{e}^{-\frac{r}{2}}.$$

But

$$\int_0^m (1-t/m)^{m-1} t^{r+1} dt \le (r+1)\int_0^\infty (1-t/m)^m t^r dt \le (r+1)\int_0^\infty \mathrm{e}^{-t} t^r dt = (r+1)!$$

Taking r'th roots establishes the lemma. □

Lemma 3 *There is an absolute constant C so that if $p : \mathbf{R} \to \mathbf{C}$ is a polynomial of degree at most d, if $0 < r \le q < \infty$ and if $\frac{r}{2} \le t \le x$, then*

$$\int_0^t |p|^{\frac{q}{d}} \le \frac{C^q t^{q+1}}{m^q[(m+1)B(m+1,r+1)]^{\frac{q}{r}}}\left[\int_0^x (1-u/m)^{m-1}\Big(\int_0^u |p|^{\frac{r}{d}}\Big) du\right]^{\frac{q}{r}}.$$

Proof By Lemma 1, we have for $t \ge u$

$$u^{r+1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)^{\frac{r}{q}} \le C^r t^{r+\frac{r}{q}} \int_0^u |p|^{\frac{r}{d}}.$$

Multiplying this inequality by $(1-u/m)^{m-1}$ and integrating with respect to u from 0 to t yields

$$\Big[\int_0^t u^{r+1}(1-u/m)^{m-1} du\Big]\Big(\int_0^t |p|^{\frac{q}{d}}\Big)^{\frac{r}{q}} \le C^r t^{r+\frac{r}{q}} \int_0^t (1-u/m)^{m-1}\Big(\int_0^u |p|^{\frac{r}{d}}\Big) du.$$

Lemma 2 and the remark following its statement now imply that

$$\Big(\int_0^t |p|^{\frac{q}{d}}\Big)^{\frac{r}{q}} \le \frac{C^r t^{r+\frac{r}{q}}}{m^r[(m+1)B(m+1,r+1)]} \int_0^t (1-u/m)^{m-1}\Big(\int_0^u |p|^{\frac{r}{d}}\Big) du.$$

Lemma 3 now follows upon taking $\frac{q}{r}$'th roots. □

Proof of Theorem 3 We may assume that $0 < r \le q < \infty$. For ease of notation we write m for $n-1$, and denote $[(m+1)B(m+1,q+1)]^{\frac{1}{q}}$ by A_q (for m fixed). We assume $m \ge 2$ (otherwise the proof simplifies), and changing variables we see that we have to show, for each $\lambda \ge 1$ and all polynomials p of degree at most d

$$\left(\frac{\int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}}(1-t/m)^m dt}{\int_0^{\frac{m}{\lambda}}(1-t/m)^m dt}\right)^{\frac{1}{q}} \le C\frac{A_q}{A_r}\left(\frac{\int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{r}{d}}(1-t/m)^m dt}{\int_0^{\frac{m}{\lambda}}(1-t/m)^m dt}\right)^{\frac{1}{r}}.$$

<u>Case 1 : m $\le \lambda$</u>.

Notice that if $0 \le t \le 1$, $e^{-2} \le (1-t/m)^m \le 1$ for $m \ge 2$. Moreover,

$$\left(\frac{\lambda}{m}\int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}}dt\right)^{\frac{1}{q}} \le C\left(\frac{\lambda}{m}\int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{r}{d}}dt\right)^{\frac{1}{r}}$$

for $0 < q, r < \infty$ by Lemma 1. Finally, since A_q is an increasing function of q this case is complete.

<u>Case 2 : m $> \lambda$</u>.

Let $x = \frac{m}{\lambda}$; then $1 \le x \le m$ as $\lambda \ge 1$. For $1 \le x \le m$, $\int_0^x (1-t/m)^m dt$ is bounded above and below by absolute constants. So we wish to see that for $1 \le x \le m$ and $0 < r \le q$,

$$\left(\int_0^x |p|^{\frac{q}{d}}(1-t/m)^m dt\right)^{\frac{1}{q}} \le C\frac{A_q}{A_r}\left(\int_0^x |p|^{\frac{r}{d}}(1-t/m)^m dt\right)^{\frac{1}{r}}. \tag{1}$$

Now

$$\int_0^x |p|^{\frac{q}{d}}(1-t/m)^m dt = \int_0^x (1-t/m)^{m-1}\left(\int_0^t |p|^{\frac{q}{d}}\right)dt \ + \ (1-x/m)^m\int_0^x |p|^{\frac{q}{d}}.$$

We shall concentrate on the first term, the arguments for the second being similar but easier. We distinguish two subcases of (1):

<u>Subcase (i) : r/2 $\le x$</u>.

In this subcase, $r \le 2m$ and

$$\int_0^x (1-t/m)^{m-1}\left(\int_0^t |p|^{\frac{q}{d}}\right)dt = \int_{\frac{r}{2}}^x (1-t/m)^{m-1}\left(\int_0^t |p|^{\frac{q}{d}}\right)dt + \int_0^{\frac{r}{2}} (1-t/m)^{m-1}\left(\int_0^t |p|^{\frac{q}{d}}\right)dt.$$

The estimate for the second term here is a special case ($x = r/2$) of subcase (ii) below, so it suffices to deal with the first term. Using Lemma 3 we have

$$\int_{\frac{r}{2}}^x (1-\frac{t}{m})^{m-1}\left(\int_0^t |p|^{\frac{q}{d}}\right)dt \le \frac{C^q}{m^q A_r^q}\left[\int_{\frac{r}{2}}^x (1-\frac{t}{m})^{m-1}t^{q+1}dt\right]\left[\int_0^x (1-\frac{u}{m})^{m-1}\left(\int_0^u |p|^{\frac{r}{d}}\right)du\right]^{\frac{q}{r}}$$

$$\leq \frac{C^q(q+1)m^{q+1}A_q^q}{m^qA_r^q(m+1)}\Big(\int_0^x |p|^{\frac{r}{d}}(1-\frac{t}{m})^m dt\Big)^{\frac{q}{r}}.$$

Taking q'th roots establishes subcase (i).

Subcase (ii) : $1 \leq x \leq$ r/2.

$$\int_0^x (1-t/m)^{m-1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)dt = \int_0^1 (1-t/m)^{m-1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)dt + \int_1^x (1-t/m)^{m-1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)dt. \tag{2}$$

The first term is easy to deal with since by Lemma 1

$$\int_0^1 (1-t/m)^{m-1}\int_0^t |p|^{\frac{q}{d}}dt \leq \int_0^1 |p|^{\frac{q}{d}} \leq C^q\Big(\int_0^1 |p|^{\frac{r}{d}}\Big)^{\frac{q}{r}} \leq C^q\Big(\int_0^1 |p|^{\frac{r}{d}}(1-t/m)^{m-1}dt\Big)^{\frac{q}{r}}.$$

For the second term, Lemma 1 implies that for $t \geq u$

$$u^{r+1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)^{\frac{r}{q}} \leq C^r t^{r+\frac{r}{q}} \int_0^u |p|^{\frac{r}{d}}.$$

Multiplying this inequality by $(1-u/m)^{m-1}$ and integrating with respect to u from 0 to t yields

$$\Big[\int_0^t u^{r+1}(1-u/m)^{m-1}du\Big]\Big[\int_0^t |p|^{\frac{q}{d}}\Big]^{\frac{r}{q}} \leq C^r t^{r+\frac{r}{q}} \int_0^t \Big(\int_0^u |p|^{\frac{r}{d}}\Big)(1-u/m)^{m-1}du$$

$$\leq C^r t^{r+\frac{r}{q}} \int_0^t |p|^{\frac{r}{d}}(1-u/m)^m du \leq C^r t^{r+\frac{r}{q}} \int_0^x |p|^{\frac{r}{d}}(1-u/m)^m du$$

provided $t \leq x$. But

$$\int_0^t u^{r+1}(1-u/m)^{m-1}du \geq \mathrm{e}^{-t}\int_0^{\frac{t}{2}} u^{r+1}du = \frac{\mathrm{e}^{-t}t^{r+2}}{2^{r+2}(r+2)}.$$

Thus for $1 \leq t \leq x \leq \frac{r}{2}$,

$$\int_0^t |p|^{\frac{q}{d}} \leq C^q \mathrm{e}^{\frac{tq}{r}} t^{1-\frac{2q}{r}}\Big[\int_0^x |p|^{\frac{r}{d}}(1-u/m)^m du\Big]^{\frac{q}{r}} \leq C^q\Big[\int_0^x |p|^{\frac{r}{d}}(1-u/m)^m du\Big]^{\frac{q}{r}}.$$

Now multiplying both sides of *this* inequality by $(1-t/m)^{m-1}$ and integrating with respect to t from 1 to x gives

$$\int_1^x (1-t/m)^{m-1}\Big(\int_0^t |p|^{\frac{q}{d}}\Big)dt \leq C^q\Big[\int_1^x (1-t/m)^{m-1}dt\Big]\Big[\int_1^x |p|^{\frac{r}{d}}(1-u/m)^m du\Big]^{\frac{q}{r}}$$

$$\leq C^q \Big[\int_0^x |p|^{\frac{r}{d}} (1-u/m)^m du \Big]^{\frac{q}{r}}$$

(as $(1-t/m)^{m-1} \leq 1$ and $x \leq \frac{r}{2} \leq q$). Taking q'th roots finishes subcase (ii) of (2), and hence (1), proving Theorem 3. □

§4. **Proof of Theorem 4**

The first step in proving Theorem 4 is the special case $n = 1, q = \infty$:

Lemma 4 *There is an absolute constant C so that for all polynomials $p : \mathbf{R} \to \mathbf{C}$ of degree at most d and all intervals I,*

$$\|p\|_{L^\infty(I)}^{\frac{1}{d}} \;\; \alpha^{-\frac{1}{d}} |\{x \in I \,:\, |p(x)| \leq \alpha\}| \;\leq\; C|I|.$$

This lemma is an old result and in fact the best constant C is known to be 4. This is due to Dudley and Randol, [DR]. However this result for *some* absolute constant C is an easy consequence of a classical inequality of H. Cartan [C] which we now state:

Cartan's lemma *Let $w_1, w_2, \ldots, w_d$ be d points in the complex plane $\mathbf{C}$ and let $h > 0$. Then the set of points $z \in \mathbf{C}$ such that the inequality*

$$\prod_{j=1}^{d} |z - w_j| \;\leq\; h^d$$

holds can be covered by at most d circles, the sum of whose radii is $2eh$.

Note, in particular, Cartan's lemma implies the corresponding statement of Lemma 4 for monic (as opposed to L^∞ - normalised) polynomials. We provide a proof of Lemma 4 for completeness.

Proof of Lemma 4

We may assume that $I = [0,1]$ by translating and dilating the polynomial p. Observe that the statement of the lemma is invariant under multiplication of p by any nonzero constant, and (up to changing the value of C) under multiplication of p by a function, whose d'th root is bounded above and below by absolute constants. So if $p(z) = A \prod (z - \zeta_j)$, we may multiply p by $\prod_{|\zeta_j| \geq 2} |\zeta_j| (z - \zeta_j)^{-1}$ and then by $(A \prod_{|\zeta_j| \geq 2} |\zeta_j|)^{-1}$ without changing matters. Thus we may assume that $p(z) = \prod_{|\zeta_j| \leq 2} (z - \zeta_j)$. This modified $p(z)$ is now monic, has degree $k \leq d$ say, and when restricted to the unit interval $[0,1]$ satisfies $\|p\|_\infty^{\frac{1}{d}} \leq 3$. We may therefore assume $\alpha \leq 1$ and Cartan's inequality tells us that

$$|\{x \in [0,1] \,:\, |p(x)| \leq \alpha\}| \;\leq\; C\alpha^{\frac{1}{k}} \;\leq\; C\alpha^{\frac{1}{d}},$$

completing the proof of the lemma. □

Note the case $q = \infty$ of Theorem 4 and thus Theorem 2 is now an immediate consequence: we merely have to observe that for $0 \leq t \leq 1$ and $\lambda \geq 1$, we have $(\lambda - t)^{n-1} \leq n \int_0^1 (\lambda - s)^{n-1} ds$.

Proof of Theorem 4 Again we may assume that $0 < q < \infty$. For ease of notation we again write m for $n-1$ and assume $m \geq 2$ (the cases $m = 0$ and $m = 1$ are easier). Let

$$I^q \;=\; \frac{\int_0^1 |p(t)|^{\frac{q}{d}} (\lambda - t)^m dt}{\int_0^1 (\lambda - t)^m dt} \qquad \text{and} \qquad II \;=\; \frac{\alpha^{-\frac{1}{d}} \int_0^1 \chi_{\{|p| \leq \alpha\}} (\lambda - t)^m dt}{\int_0^1 (\lambda - t)^m dt}.$$

We wish to show that $I \cdot II \leq C(m+1)[(m+1)B(m+1, q+1)]^{\frac{1}{q}}$. We immediately make the change of variables $t \to \frac{\lambda}{m} t$ in all integrals, so that

$$I^q \;=\; \frac{\int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}} (1 - \frac{t}{m})^m dt}{\int_0^{\frac{m}{\lambda}} (1 - \frac{t}{m})^m dt} \qquad \text{and} \qquad II \;=\; \frac{\alpha^{-\frac{1}{d}} \int_0^{\frac{m}{\lambda}} \chi_{\{|p| \leq \alpha\}} (1 - \frac{t}{m})^m dt}{\int_0^{\frac{m}{\lambda}} (1 - \frac{t}{m})^m dt}$$

(for a possibly different polynomial p). Note that if $D := \int_0^{\frac{m}{\lambda}} (1 - \frac{t}{m})^m dt$, then for $m \leq \lambda$, we have $D \;\geq\; \int_0^{\frac{m}{\lambda}} (1 - 1/m)^m dt \;\geq\; \frac{m}{\lambda} \frac{1}{2e}$ while for $m \geq \lambda$, $D \geq \int_0^1 (1 - t/m)^m dt \geq \int_0^1 (1 - 1/m)^m dt \geq \frac{1}{2e}$.

<u>Case 1 : m $\leq \lambda$</u>.

In this case we have

$$I^q \;\leq\; \frac{2e\lambda}{m} \int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}} (1 - t/m)^m dt \;\leq\; \frac{2e\lambda}{m} \int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}} dt,$$

while

$$II \;\leq\; \frac{2e\lambda}{m} \alpha^{-\frac{1}{d}} \int_0^{\frac{m}{\lambda}} \chi_{\{|p| \leq \alpha\}} (1 - t/m)^m dt \;\leq\; \frac{2e\lambda}{m} \alpha^{-\frac{1}{d}} \int_0^{\frac{m}{\lambda}} \chi_{\{|p| \leq \alpha\}} dt$$

so that

$$I \cdot II \leq (2e)^{1+\frac{1}{q}} \Big(\frac{\lambda}{m} \int_0^{\frac{m}{\lambda}} |p(t)|^{\frac{q}{d}} dt \Big)^{\frac{1}{q}} \cdot \alpha^{-\frac{1}{d}} \frac{\lambda}{m} \int_0^{\frac{m}{\lambda}} \chi_{\{|p| \leq \alpha\}} dt \leq C(2e)^{1+\frac{1}{q}}$$

by Lemma 4. Thus $I \cdot II$ is bounded above by an absolute constant in this case.

<u>Case 2 : m $> \lambda$</u>.

In this case, since D is uniformly bounded below and the numerators of I and II are decreasing with λ, we may take $\lambda = 1$ and reduce matters to showing that

$$\widetilde{I} \cdot \widetilde{II} \;\leq\; C(m+1)[(m+1)B(m+1, q+1)]^{\frac{1}{q}}$$

where

$$\widetilde{I}^q \;=\; \int_0^m |p(t)|^{\frac{q}{d}}(1-t/m)^m dt \quad \text{and} \quad \widetilde{II} \;=\; \alpha^{-\frac{1}{d}}\int_0^m \chi_{\{|p|\le\alpha\}}(1-t/m)^m dt.$$

Now

$$\widetilde{I}^q = \int_0^m (1-t/m)^m \frac{d}{dt}\Big\{\int_0^t |p(s)|^{\frac{q}{d}} ds\Big\} dt = \int_0^m (1-t/m)^{m-1}\Big\{\int_0^t |p(s)|^{\frac{q}{d}} ds\Big\} dt$$

which in turn is less than $\int_0^1 H\,dt + C^q H \int_1^m (1-t/m)^{m-1} t^{q+1}\,dt$ where $H = \int_0^1 |p(s)|^{\frac{q}{d}} ds$, by Lemma 1. Hence

$$\widetilde{I}^q \le H\big[1 + C^q \int_1^m (1-t/m)^{m-1} t^{q+1} dt\big] \le H\Big[1 + C^q m^{q+2}\int_0^1 (1-s)^{m-1} s^{q+1} ds\Big]$$

$$= \;H\Big[1 + C^q m^{q+1}(q+1)B(m+1,q+1)\Big].$$

Therefore $\widetilde{I} \le C H^{\frac{1}{q}}(m+1)[(m+1)B(m+1,q+1)]^{\frac{1}{q}}$. On the other hand,

$$\widetilde{II} = \alpha^{-\frac{1}{d}}\int_0^m \chi_{\{|p|\le\alpha\}}(1-t/m)^m dt = \alpha^{-\frac{1}{d}}\int_0^m (1-t/m)^{m-1}\int_0^t \chi_{\{|p(s)|\le\alpha\}} ds\, dt$$

$$\le\; \alpha^{-\frac{1}{d}}\Big[\int_0^1\int_0^1 \chi_{\{|p(s)|\le\alpha\}} ds dt \;+\; \int_1^m (1-t/m)^{m-1}\int_0^t \chi_{\{|p(s)|\le\alpha\}} ds dt\Big]$$

$$\le\; \frac{C}{K} + \frac{C}{K}\int_1^m (1-t/m)^{m-1} t\, dt \le \frac{C}{K}\int_0^\infty \mathrm{e}^{-t/2} t dt \le \frac{C}{K}$$

by Lemma 4, where $K = \||p|^{\frac{1}{d}}\|_{L^\infty[0,1]}$. Thus

$$\widetilde{I}\cdot\widetilde{II} \;\le\; C\left[\frac{\||p|^{\frac{1}{d}}\|_{L^q[0,1]}}{\||p|^{\frac{1}{d}}\|_{L^\infty[0,1]}}\right](m+1)[(m+1)B(m+1,q+1)]^{\frac{1}{q}}$$

which in turn is less than $C(m+1)[(m+1)B(m+1,q+1)]^{\frac{1}{q}}$ as required, completing the proof of Theorem 4. (Note that we have used in passing that $n[nB(n,q+1)]^{\frac{1}{q}}$ is bounded below uniformly in n and q.) □

§5. The vector-valued case

To extend Theorems 1 and 2 to the vector-valued setting, we first observe that our arguments extend to a wider class of functions than polynomials of degree at most d. Following a preliminary version of [NSV], we say that a function $u : \mathbf{R}^n \to \mathbf{R}$ is of class $\mathcal{L}$ if it is the restriction to $\mathbf{R}^n$ of a plurisubharmonic function $\widetilde{u} : \mathbf{C}^n \to \mathbf{R}$ such that $\limsup\limits_{|z|\to\infty} \frac{\widetilde{u(z)}}{\log|z|} \leq 1$. When $n = 1$, $u(x) = \frac{1}{d}\log|p(x)|$ is of class $\mathcal{L}$ if $p : \mathbf{R} \to \mathbf{C}$ is a polynomial of degree d. We can write such a p as $p(x) = A\prod_{j=1}^d (x - \zeta_j)$, so that $\frac{1}{d}\log|p(x)| = \frac{1}{d}\log|A| + \frac{1}{d}\sum_{j=1}^d \log|x-\zeta_j|$, and the distinguishing feature of a function of class $\mathcal{L}$ (when $n = 1$) is that it can be written as $u(x) = \text{constant} + \int \log|x-\zeta|d\mu(\zeta)$ where μ is a positive measure of mass at most one in the plane. This is the well-known Riesz representation for subharmonic functions, see for example Hayman's book [H]. In particular, it is not difficult to see that the key lemmas, Lemma 1 and Lemma 4, remain valid if one replaces $|p(x)|^{\frac{1}{d}}$ with $\exp u(x)$, where u is a general function of class $\mathcal{L}$ in one dimension. With these remarks in mind the reader will have no trouble extending Theorems 1 and 2 to functions of class $\mathcal{L}$ to obtain the following:

Theorem 5 *Let $u : \mathbf{R}^n \to \mathbf{R}$ be a function of class $\mathcal{L}$, $0 \leq r \leq q \leq \infty$ and K be a convex body in $\mathbf{R}^n$ of volume 1. Then there exists an absolute constant C independent of r, q, K, n and u so that*

$$\|\mathrm{e}^u\|_{L^q(K)} \leq C\frac{[nB(n,q+1)]^{\frac{1}{q}}}{[nB(n,r+1)]^{\frac{1}{r}}}\|\mathrm{e}^u\|_{L^r(K)}.$$

Theorem 6 *Let $u : \mathbf{R}^n \to \mathbf{R}$ be a function of class $\mathcal{L}$, $0 \leq q \leq \infty$ and K be a convex body in $\mathbf{R}^n$ of volume 1. Then there exists an absolute constant C independent of q, K, n and u so that*

$$\|\mathrm{e}^u\|_{L^q(K)}\ \|\mathrm{e}^{-u}\|_{L^{1,\infty}(K)} \leq Cn[nB(n,q+1)]^{\frac{1}{q}}.$$

To obtain the vector-valued extension of Theorems 1 and 2, we simply observe that whenever $p : \mathbf{R}^n \to X$ is a polynomial of degree at most d with values in a Banach space X, $u(x) = \frac{1}{d}\log\|p(x)\|$ is a function of class $\mathcal{L}$. In fact, the estimate $\limsup\limits_{|z|\to\infty} \frac{\widetilde{u(z)}}{\log|z|} \leq 1$ is straightforward, and using the fact $\|w\| = \sup\limits_{\ell\in X^*,\|\ell\|\leq 1} |\ell(w)|$ for any $w \in X$, one easily sees that $\widetilde{u}(z)$ is plurisubharmonic.

§6. Further remarks

1. If we let $m \to \infty$ in inequality (1), we have, since $(1-t/m)^m \leq \mathrm{e}^{-t}$ for $m > 0$ and $0 < t \leq m$,

$$\Big(\int_0^x |p(t)|^{\frac{q}{d}}\mathrm{e}^{-t}dt\Big)^{\frac{1}{q}} \leq C\frac{\max(q,1)}{\max(r,1)}\Big(\int_0^x |p(t)|^{\frac{r}{d}}\mathrm{e}^{-t}dt\Big)^{\frac{1}{r}}$$

by the dominated convergence theorem, where C is absolute: combining this with Lemmas 1 and 4 yields the following results.

Proposition 1 *There exists an absolute constant C such that if p is a polynomial of degree at most d, $N > 0$, and $0 < r \leq q < \infty$,*

$$\left(\frac{\int_0^N |p(t)|^{\frac{q}{d}} \mathrm{e}^{-t} dt}{\int_0^N \mathrm{e}^{-t} dt}\right)^{\frac{1}{q}} \leq C \frac{\max(q,1)}{\max(r,1)} \left(\frac{\int_0^N |p(t)|^{\frac{r}{d}} \mathrm{e}^{-t} dt}{\int_0^N \mathrm{e}^{-t} dt}\right)^{\frac{1}{r}}.$$

Proposition 2 *There exists an absolute constant C such that if p is a polynomial of degree at most d, $N > 0$, and $0 < q < \infty$,*

$$\left(\frac{\int_0^N |p(t)|^{\frac{q}{d}} \mathrm{e}^{-t} dt}{\int_0^N \mathrm{e}^{-t} dt}\right)^{\frac{1}{q}} \cdot \frac{\alpha^{-\frac{1}{d}} \int_0^N \chi_{\{|p(t)| \leq \alpha\}} \mathrm{e}^{-t} dt}{\int_0^N \mathrm{e}^{-t} dt} \leq C \max(q,1).$$

Propositions 1 and 2 are also true if one replaces $|p(t)|^{\frac{1}{d}}$ with $\exp u$, where u is any function of class $\mathcal{L}$. Using another theorem of Kannan, Lovász and Simonovits [KLS] (which is similar to their theorem stated in the introduction except that the measures μ are replaced by measures with exponential densities) we then obtain

Theorem 7 *Let X be a Banach space and let $p : \mathbf{R}^n \to X$ be a polynomial of degree at most d. Suppose $0 < r \leq q < \infty$ and μ is a* **log-concave probability measure** *on $\mathbf{R}^n$. Then there is an absolute constant C such that*

$$\left(\int \|p(x)\|^{\frac{q}{d}} d\mu(x)\right)^{\frac{1}{q}} \leq C \frac{\max(q,1)}{\max(r,1)} \left(\int \|p(x)\|^{\frac{r}{d}} d\mu(x)\right)^{\frac{1}{r}},$$

and for the sublevel set estimate:

Theorem 8 *There exists an absolute constant C such that if $p : \mathbf{R}^n \to X$ is a polynomial of degree at most d, $0 < q < \infty$, and μ is a* **log-concave probability measure** *on $\mathbf{R}^n$, then*

$$\left(\int \|p(x)\|^{\frac{q}{d}} d\mu(x)\right)^{\frac{1}{q}} \cdot \alpha^{-\frac{1}{d}} \mu\{x \in \mathbf{R}^n : \|p(x)\| \leq \alpha\} \leq Cq.$$

A measure is said to be *log-concave* if it is supported by an affine subspace L of $\mathbf{R}^n$, and with respect to Lebesgue measure on L has a density of the form $\mathrm{e}^{-g(x)}$ where the set $K = \{x : g(x) < \infty\}$ and $g|_K$ are convex. In addition to characteristic functions of convex bodies, these measures include gaussians $\mathrm{e}^{-|x|^2} dx$. Of course we can let q or $r \to 0$ in Theorems 7 and 8 to obtain estimates in the $exp - log$ class L^0.

2. To see why Theorem 2 implies the case $r \le 1$ and general q of Theorem 1, we first observe that Theorem 2 has a trivial reverse inequality. Considering the sublevel set for $\|p(x)\|$ with $\alpha^{\frac{q}{d}} = 2\int_K \|p\|^{\frac{q}{d}}$, we have $1/4 \le \|\|p\|^{\frac{1}{d}}\|_{L^q(K)} \cdot \sup_{\alpha>0} \alpha^{-\frac{1}{d}} |\{x \in K : \|p(x)\| \le \alpha\}|$ uniformly for $q > 0$. Hence, by Theorem 2

$$1/4 \;\le\; \|\|p\|^{\frac{1}{d}}\|_{L^q(K)} \cdot \sup_{\alpha>0} \alpha^{-\frac{1}{d}} |\{x \in K : \|p(x)\| \le \alpha\}| \;\le\; Cn(nB(n, q+1))^{\frac{1}{q}}.$$

In particular, using these inequalities with $q \le 1$, we see that the "norms" $\|\|p\|^{\frac{1}{d}}\|_{L^q(K)}$ for $q \le 1$ and $[\sup_{\alpha>0} \alpha^{-\frac{1}{d}} |\{x \in K : \|p(x)\| \le \alpha\}|]^{-1} = \|\|p\|^{-1/d}\|^{-1}_{L^{1,\infty}(K)}$ are uniformly equivalent. Therefore for any $q \ge r$, we have

$$\left(\int_K \|p(x)\|^{\frac{q}{d}} dx\right)^{\frac{1}{q}} \;\le\; Cn(nB(n, q+1))^{\frac{1}{q}} \left(\int_K \|p(x)\|^{\frac{r}{d}} dx\right)^{\frac{1}{r}}.$$

3. It is easy to see that the conclusion of Theorem 2 has the following equivalent formulation for general finite-volume convex bodies K:

$$\left(\frac{1}{|K|}\int_K \|p\|^{\frac{q}{d}}\right)^{\frac{1}{q}} \;\le\; Cn(nB(n, q+1))^{\frac{1}{q}} \, \frac{|K|}{|E|} \, \|\|p\|^{\frac{1}{d}}\|_{L^\infty(E)}$$

uniformly over all closed subsets E of K (with the same constant C). Somewhat surprisingly, one can replace the L^∞ norm on the right side with the smaller L^r norm, $\left(\frac{1}{|E|}\int_E \|p\|^{\frac{r}{d}}\right)^{\frac{1}{r}}$, incurring only an extra factor of 2 in the estimate. This was observed in [BG] for the case $q = \infty$ and follows by considering the non-decreasing rearrangement of $\|p\|$ over E, $p_*(\tau)$ (i.e., p_* is the inverse of the measure of the sublevel sets of $\|p\|$ restricted to E). The estimate in Theorem 2 implies a lower bound for p_*, namely

$$\left(\frac{1}{|K|}\int_K \|p\|^{\frac{q}{d}}\right)^{\frac{1}{q}} \frac{\tau}{|K|} \;\le\; Cn(nB(n, q+1))^{\frac{1}{q}} [p_*(\tau)]^{\frac{1}{d}}$$

for $0 \le \tau \le |E|$. Raising this to the r'th power, integrating in τ and then taking the r'th root gives the desired bound.

4. The convexity of the set K is crucial in obtaining the form of the constant in Theorem 1. If instead one asks for the form of the constant B in the inequality

$$\left(\int_F \|p\|^{\frac{q}{d}}\right)^{\frac{1}{q}} \le B\left(\int_F \|p\|^{\frac{r}{d}}\right)^{\frac{1}{r}}$$

where F is now an arbitrary (unit-volume) compact set in $\mathbf{R}^n$ and $0 \le r \le q \le \infty$, one may see that not only must B contain a factor of $|cvxF|^{1-\frac{r}{q}}$ (where $cvxF$ denotes the convex hull of F) but also a factor $n^{1-\frac{r}{q}}$. To see this, consider the example $p(x) = x_1^d$ as before and $F = \{(x_1, x') \in \mathbf{R}^n : x_1 \in (0, 1/n) \cup (1-\epsilon, 1), |x'| \leq 1 - x_1\}$ for suitable ϵ much smaller than $1/n$. The proof of the resulting inequality

$$\Big(\int_F \|p\|^{\frac{q}{d}}\Big)^{\frac{1}{q}} \leq C n^{1-\frac{r}{q}} |cvxF|^{1-\frac{r}{q}} \Big(\int_F \|p\|^{\frac{r}{d}}\Big)^{\frac{1}{r}}$$

is due to Brudnyi and Ganzburg [BG] (at least in the case $q = \infty$). To see this, we first observe

$$\int_F \|p(x)\|^{\frac{q}{d}} \leq \int_F \|p(x)\|^{\frac{r}{d}} \| \|p\|^{\frac{1}{d}} \|_{L^\infty(F)}^{q-r} \leq \int_F \|p(x)\|^{\frac{r}{d}} \| \|p\|^{\frac{1}{d}} \|_{L^\infty(cvxF)}^{q-r}$$

$$\leq \int_F \|p(x)\|^{\frac{r}{d}} \{ \frac{1}{(nB(n, r+1))^{\frac{1}{r}}} \| \|p\|^{\frac{1}{d}} \|_{L^r(cvxF)} \}^{q-r}$$

where the last inequality follows from the case $q = \infty$ and general r of Theorem 1. Next, using the following equivalent formulation of Theorem 2 (which we derived in Remark 3 above)

$$\left(\frac{1}{|K|}\int_K \|p\|^{\frac{q}{d}}\right)^{\frac{1}{q}} \leq Cn(nB(n, q+1))^{\frac{1}{q}} \inf_{E \subset K} \frac{|K|}{|E|} \left(\frac{1}{|E|}\int_E \|p\|^{\frac{r}{d}}\right)^{\frac{1}{r}} \qquad (3)$$

when $q = r$, $E = F$ and $K = cvxF$, we obtain the result. Interestingly, (3) can be thought of as a way to formulate the analogue of Lemma 1 in the higher-dimensional context, and it is natural to enquire as to whether the constant $n(nB(n, q+1))^{\frac{1}{q}}$ can be improved upon if we restrict E to range over *convex* subsets of K. This however is not the case. To see this, take $K = \{(x_1, x') \in \mathbf{R}^n : 0 < x_1 < n-1, |x'| < 1 - \frac{1}{n-1}x_1\}$, $E = \{(x_1, x') \in \mathbf{R}^n : 0 < x_1 < 1, |x'| < 1 - \frac{1}{n-1}x_1\}$, and $p(x) = x_1^d$. (Of course, $X = \mathbf{R}$ here).

5. If $p : \mathbf{R}^n \to \mathbf{C}$ is a polynomial of degree at most d, it is well known that $\omega = |p|$ is an A_q weight when $q > d+1$ with A_q bounds independent of the coefficients of p ; see [RS]. Theorem 2, when $q = d$, can be viewed as a sharp endpoint result of this nature. Recall that a weight ω is in A_q if

$$\frac{1}{|B|}\int_B \omega(x)dx \cdot \left[\frac{1}{|B|}\int_B \omega(x)^{-q'/q}dx\right]^{q/q'} \leq A < \infty$$

for all balls B in $\mathbf{R}^n$. The smallest constant A for which the above holds is called the A_q bound, $A_q(\omega)$, for ω. Using Theorem 2 with $q = d$, we see that there is an absolute constant C such that if $p : \mathbf{R}^n \to X$ is a polynomial of degree at most d with values in a Banach space X and $q > d+1$,

$$A_q(\|p\|) \leq (Cd)^d \left[\frac{q-1}{q-(d+1)}\right]^{q-1}.$$

We remark that this estimate remains valid when we allow the A_q bound to also vary over all convex bodies K in $\mathbf{R}^n$, not just Euclidean balls B. See also [NSV].

6. The theorem of Kannan, Lovász and Simonovits which we used relies heavily on the non-negativity of the functions involved. However there are phenomena, closely related to sublevel set problems for polynomials, which are highly oscillatory in nature; most notably estimates for oscillatory integrals. For example, it follows from Theorem 7.2 of [CCW] that if $Q = [0,1]^n$, $p : Q \to \mathbf{R}$ is a polynomial of degree at most d so that $\int_Q p = 0$ and $\|p\|_{L^\infty(Q)} = 1$, then for λ large and real,

$$\left|\int_Q \mathrm{e}^{i\lambda p(x)}dx\right| \leq \frac{C_{d,n}}{|\lambda|^{\frac{1}{d}}}. \tag{4}$$

Can we expect improvement to this along the lines enjoyed by sublevel sets? In particular if $\int_Q p = 0$ and $\|p\|_{L^1(Q)} = 1$, can we take $C_{d,n}$ in (4) to be $C\min(d,n)$? On average the answer is yes, because a direct consequence Theorem 2 is that for $\|p\|_{L^1(K)} = 1$, K a convex body of volume 1, $\phi \in \mathcal{S}(\mathbf{R})$ with $0 \leq \hat{\phi} \leq \chi_{[-1,1]}$

$$\left|\frac{1}{\mu}\int\Big\{\int_K \mathrm{e}^{i\lambda p(x)}dx\Big\}\phi(\lambda/\mu)d\lambda\right| \leq C\min(d,n)\,\mu^{-\frac{1}{d}}$$

with C absolute. To see this, note that the left side is equal to $\int_K \hat{\phi}(\mu p(x))dx$ which is in turn equivalent to $|\{x \in K : |p(x)| \leq \mu^{-1}\}|$. This well-known argument also demonstrates the fact that oscillatory integral estimates imply sublevel set estimates.

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