

SOME SPECIAL SOLUTIONS OF THE SCHRÖDINGER EQUATION

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ABSTRACT. We describe a certain “self-similar” family of solutions to the free Schrödinger equation in all dimensions, and derive some consequences of such solutions for two specific problems.

1. INTRODUCTION

In this paper we describe certain “self-similar” solutions to the initial value problem associated to the free Schrödinger equation:

$$(1) \quad \begin{cases} i\partial_t u + \Delta_x u = 0 & (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \\ u(x, 0) = f(x). \end{cases} \quad n \geq 2,$$

A straightforward application of the Fourier transform allows us express the solution u of (1) as

$$u(x, t) = \int_{\mathbb{R}^{n-1}} e^{-\pi it|\xi|^2 + 2\pi ix \cdot \xi} \hat{f}(\xi) d\xi,$$

where \hat{f} is the Fourier transform of f . As usual, we denote this solution by $e^{it\Delta} f(x)$.

In Section 2 we describe in detail the particular solutions that we have in mind, and in Section 3 we give applications to two established problems in harmonic analysis and the theory of Schrödinger equations. These problems concern weighted estimates for solutions to (1) in two variants, and are related to the restriction/extension operators for the base of the paraboloid.

Notation. For non-negative quantities X and Y we use $X \lesssim Y$ ($X \gtrsim Y$) to denote the existence of a positive constant C , depending on at most n , such that $X \leq CY$ ($X \geq CY$). We write $X \sim Y$ if both $X \lesssim Y$ and $X \gtrsim Y$.

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2. SPECIAL SOLUTIONS

Let $0 < \delta \ll 1$ and $0 < \sigma < \frac{1}{2}$. We consider the function of one variable

$$(2) \quad g = \sum_{\ell \in \mathbb{N}, 1 \leq \ell \leq \delta^{-\sigma}} \chi_{(\ell\delta^\sigma - \delta, \ell\delta^\sigma + \delta)}.$$

Note that g is simply the characteristic function of a union of disjoint, equally spaced subintervals of $[0, 1]$ of equal size. We take as initial data in (1) f such that

$$(3) \quad \hat{f}(\xi) = \prod_{j=1}^{n-1} g(\xi_j),$$

where $\xi = (\xi_1, \dots, \xi_{n-1})$. The corresponding solution of the free Schrödinger equation is given by

$$(4) \quad e^{it\Delta} f(x) = \prod_{j=1}^{n-1} \int_0^1 e^{-\pi it\xi_j^2 + 2\pi i x_j \cdot \xi_j} g(\xi_j) d\xi_j.$$

We wish to identify a set $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R}$ upon which $|e^{it\Delta} f(x)|$ is “large”. In order to achieve this we look for points (x, t) for which there is essentially no cancellation in the above integrals; i.e. for which the phases $-t\xi_j^2/2 + x_j \xi_j$ are within a small (say $1/10$) neighbourhood of \mathbb{Z} for all $\xi_j \in \text{supp}(g)$ and $1 \leq j \leq n-1$. By the product structure of (4), it suffices to consider the integral

$$\int_0^1 e^{-\pi it\xi^2 + 2\pi is\xi} g(\xi) d\xi,$$

where $s \in \mathbb{R}$.

If $\xi \in \text{supp}(g)$, then $\xi = \ell\delta^\sigma + \varepsilon$ for some positive integer $\ell \leq \delta^{-\sigma}$ and $|\varepsilon| \leq \delta$. Now consider $s, t \in \mathbb{R}$ of the form $s = p\delta^{-\sigma}$ and $t = 2q\delta^{-2\sigma}$, where $p, q \in \mathbb{N}$. For such s , t and ξ we have

$$s\xi - t\xi^2/2 = p\ell + p\delta^{-\sigma}\varepsilon - (q\ell^2 + 2q\ell\delta^{-\sigma}\varepsilon + q\delta^{-2\sigma}\varepsilon^2).$$

Since $p\ell, q\ell^2 \in \mathbb{N}$, we are interested in the values of p and q for which

$$(5) \quad |p\delta^{-\sigma}\varepsilon|, |q\ell\delta^{-\sigma}\varepsilon|, |q\delta^{-2\sigma}\varepsilon^2| \leq c \text{ for all } 1 \leq \ell \leq \delta^{-\sigma} \text{ and } |\varepsilon| < \delta,$$

where c is a positive constant, such as $1/40$ say. As is easily verified, (5) holds precisely when $|p| \lesssim \delta^{\sigma-1}$ and $|q| \lesssim \delta^{2\sigma-1}$, and hence for such s and t ,

$$(6) \quad \left| \int_0^1 e^{-\pi it\xi^2 + 2\pi is\xi} g(\xi) d\xi \right| \sim \delta^{1-\sigma}.$$

We now define $X = \{p\delta^{-\sigma} : p \in \mathbb{N} \text{ with } p \lesssim \delta^{\sigma-1}\}$, and

$$\Lambda = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : x \in X^{n-1} \text{ and } t = 2q\delta^{-2\sigma} \text{ where } q \in \mathbb{N} \text{ and } q \lesssim \delta^{2\sigma-1}\}.$$

Hence by (6), $|e^{it\Delta} f(x)| \sim \delta^{(n-1)(1-\sigma)}$, for all $(x, t) \in \Lambda$. Now since $e^{it\Delta} f(x)$ may be viewed as the Fourier transform of a certain compactly supported measure¹ on \mathbb{R}^n , this estimate continues to hold for (x, t) belonging to an $O(1)$ neighbourhood of Λ . We denote this union of $O(1)$ -balls by Ω (see Figure 1).

¹See (12) for an explicit expression of this.

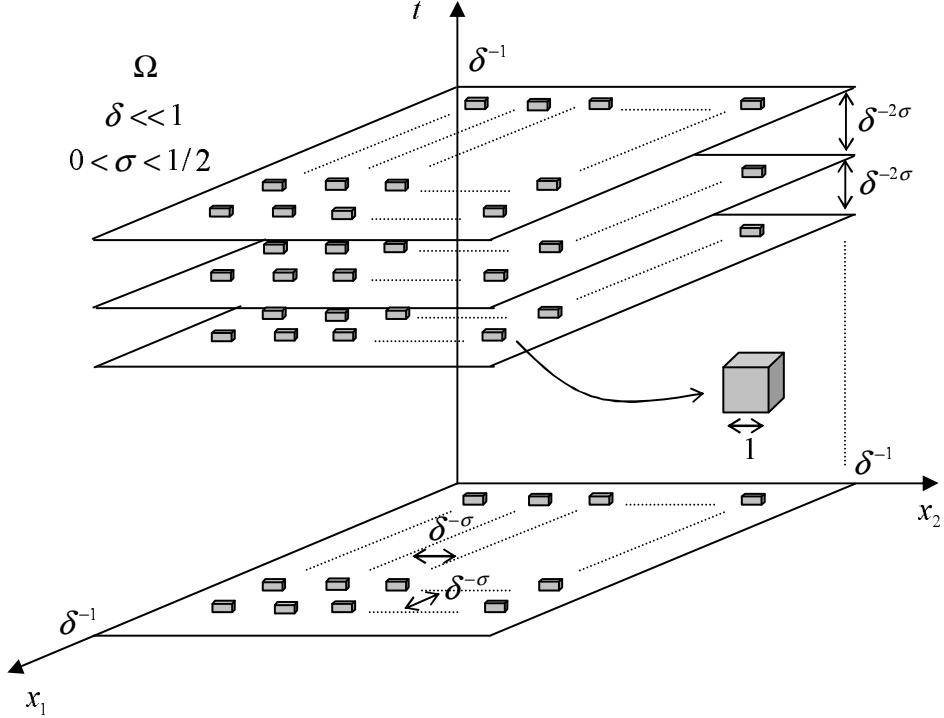


FIGURE 1. The set Ω is the union of $\delta^{2\sigma-1}\delta^{(\sigma-1)(n-1)}$ balls of radius 1 painted in grey.

We notice that for t belonging to the projection of Ω onto the t -axis, $\|e^{it\Delta}f\|_2 \sim \|f\|_2$, so that the mass of $|e^{it\Delta}f(\cdot)|^2$ on the section through Ω at height t accounts for a positive proportion of its total mass at height t .

We now consider a succession of such examples of increasing complexity.

Suppose now that $\delta > 0$ is such that $1/\delta$ is an integer, and let $k \in \mathbb{N}$ be fixed. (We allow the implicit constants in the $\lesssim \gtrsim \sim$ notation to depend on k .) Consider the function of one variable

$$(7) \quad g_k = \sum_{1 \leq \ell_1, \dots, \ell_k \leq \delta^{-\sigma}} \chi_{\{s : |s - (\ell_1 \delta^\sigma + \ell_2 \delta^{\sigma+1} + \dots + \ell_k \delta^{\sigma+k-1})| \leq \delta^k\}}.$$

Note that g_k is “self-similar” in the sense that for $k \geq 2$, g_k may be written as a certain linear combination of rescaled and translated copies of g_{k-1} , where g_1 coincides with the function g defined in (2).

As before we take as initial data in (1) f_k such that

$$\hat{f}_k(\xi) = \prod_{j=1}^{n-1} g_k(\xi_j),$$

where $\xi = (\xi_1, \dots, \xi_{n-1})$. The corresponding solution of the free Schrödinger equation is given by

$$(8) \quad e^{it\Delta} f_k(x) = \prod_{j=1}^{n-1} \int_0^1 e^{-\pi it\xi_j^2 + 2\pi i x_j \cdot \xi_j} g_k(\xi_j) d\xi_j.$$

As before it suffices to consider the integral

$$\int_0^1 e^{-\pi it\xi^2 + 2\pi is\xi} g_k(\xi) d\xi,$$

where $s \in \mathbb{R}$.

If $\xi \in \text{supp}(g)$, then

$$\xi = \sum_{r=1}^k \ell_r \delta^{\sigma+r-1} + \varepsilon$$

for some positive integers $\ell_r \leq \delta^{-\sigma}$ and $|\varepsilon| \leq \delta^k$. Now consider $s, t \in \mathbb{R}$ of the form

$$s = \sum_{m_1=1}^k p_{m_1} \delta^{-\sigma-m_1+1} \quad \text{and} \quad t = 2 \sum_{m_2=1}^k q_{m_2} \delta^{-2\sigma-m_2+1},$$

where $p_{m_1}, q_{m_2} \in \mathbb{N}$. For such s, t and ξ we have

$$\begin{aligned} s\xi - t\xi^2/2 &= \sum_{m_1=1}^k \sum_{r=1}^k p_{m_1} \ell_r \delta^{r-m_1} \\ &\quad + \sum_{m_1=1}^k \varepsilon p_{m_1} \delta^{-\sigma-m_1+1} - \sum_{m_2=1}^k \sum_{r,r'=1}^k q_{m_2} \ell_r \ell_{r'} \delta^{r+r'-m_2-1} \\ &\quad - \sum_{m_2=1}^k \sum_{r=1}^k \varepsilon \ell_r q_{m_2} \delta^{-\sigma+r-m_2} - \sum_{m_2=1}^k \varepsilon^2 q_{m_2} \delta^{-2\sigma-m_2+1}. \end{aligned}$$

In order for the above expression for the phase to belong to a small (say $1/10$) neighbourhood of \mathbb{N} , it suffices for each of the summands to either be integers, or be bounded in absolute value by a sufficiently small constant (depending only on k). As before, this places size restrictions on the integers p_{m_1} and q_{m_2} . It is here where we use the fact that $1/\delta$ is an integer.

We now define

$$X_k = \{p_1 \delta^{-\sigma} + \dots + p_k \delta^{-\sigma-k+1} : p_1, \dots, p_k \in \mathbb{N} \text{ with } p_1, \dots, p_k \lesssim \delta^{\sigma-1}\},$$

and Ω_k to be an $O(1)$ -neighbourhood of

$$\Lambda_k := \{(x, t) : x \in X_k^{n-1}, \quad t = 2(q_1 \delta^{-2\sigma} + \dots + q_k \delta^{-2\sigma-k+1}); \quad q_1, \dots, q_k \lesssim \delta^{2\sigma-1}\}.$$

Arguing as before we find that $|e^{it\Delta} f_k(x)| \sim \delta^{k(n-1)(1-\sigma)}$ whenever $(x, t) \in \Omega_k$.

We note a further ‘‘self-similarity’’ in the family of sets Λ_k . Observe that for $k \geq 2$, Λ_k is a disjoint union of rescaled and translated copies of Λ_{k-1} , where Λ_1 coincides with the set Λ defined previously.

The above examples may be generalised substantially by choosing the family of functions g_k to be self-similar with respect to more general families of affine transformations that in particular may depend on the index k .

3. WEIGHTED ESTIMATES FOR EXTENSION OPERATORS

Let $n \geq 2$ and S be a bounded hypersurface in \mathbb{R}^n with everywhere non-vanishing Gaussian curvature (for instance, S could be the base of a paraboloid or a small portion of the unit sphere \mathbb{S}^{n-1}). If we denote by σ the induced Lebesgue measure on S , then we may define the *extension operator associated to S* to be the mapping $g \mapsto \widehat{gd\sigma}$ where

$$\widehat{gd\sigma}(x) = \int g(\xi) e^{-2\pi i x \cdot \xi} d\sigma(\xi),$$

$g \in L^1(S)$ and $x \in \mathbb{R}^n$.

3.1. Rates of decay. There has recently been considerable interest in studying weighted L^2 inequalities for the extension operator which take the general form

$$(9) \quad \int_{\mathbb{B}} |\widehat{gd\sigma}(Rx)|^2 d\mu(x) \leq \frac{C(\mu)}{R^\gamma} \|g\|_{L^2(S)}^2$$

where μ is a positive measure supported on the unit ball \mathbb{B} of \mathbb{R}^n , $R \geq 1$, γ is a suitable rate-of-decay exponent and $C(\mu)$ is a constant depending only on μ . See for example [15], [19], [4], [1], [5], [21], [6], [2], [9] and [11]. A specific instance of this type of inequality, which is of particular interest in geometric measure theory, concerns the relation between the exponents $\gamma \geq 0$ and $0 \leq \eta \leq n$ such that for each S there exists a constant C (depending on S , γ and η) for which

$$(10) \quad \int_{\mathbb{B}} |\widehat{gd\sigma}(Rx)|^2 d\mu \leq \frac{C}{R^\gamma} \sup_{x \in \mathbb{R}^n, r > 0} \left\{ \frac{\mu(B(x, r))}{r^\eta} \right\} \|g\|_2^2$$

holds for all $g \in L^2(S)$, all $R \geq 1$ and all Borel measures μ supported in \mathbb{B} . In particular, for each $0 \leq \eta \leq n$ it is of interest to determine the exponent $\gamma(\eta)$ which is defined to be the supremum of the numbers γ for which (10) holds for some constant C .

In two dimensions it is known that

$$\gamma(\eta) = \begin{cases} \eta/2, & 1 \leq \eta \leq 2 \quad \text{Wolff [21]} \\ 1/2, & 1/2 \leq \eta < 1 \quad \text{Mattila [14]} \\ \eta, & 0 \leq \eta < 1/2 \quad \text{Mattila [14].} \end{cases}$$

We note that this piecewise linear function was originally computed for $S = \mathbb{S}^{n-1}$. It is however implicit in the arguments given in [14] and [21] that the same is true for general S of the type discussed here.

In higher dimensions it is well known that $\gamma(\eta) = \eta$ for $0 \leq \eta \leq \frac{n-1}{2}$ (Mattila [14]), and that $\gamma(n) = n - 1$ (Sjölin [19]). However, for $\frac{n-1}{2} < \eta < n$ the currently known upper and lower bounds for $\gamma(\eta)$ do not coincide. Lower bounds in this region were obtained by Sjölin [19] and Bourgain [4], and those of [4] have been improved recently by Erdoğan [9]. Examples leading to upper bounds were also observed

by Mattila [14], Sjölin [19], Katz and Tao [12] and more recently by Iosevich and Rudnev [11]. Our purpose here is to improve these upper bounds further by using our special solutions to the Schrödinger equation from Section 2 by showing that the graph of $\gamma(\eta)$ does not lie above the line segment joining the points $(\frac{n-1}{2}, \frac{n-1}{2})$ and $(n, n-1)$.

Proposition 3.1. *If for all bounded hypersurfaces S , (10) holds for all $g \in L^2(S)$, all $R \geq 1$ and all Borel measures μ supported in \mathbb{B} , and if $(n-1)/2 < \eta < n$, then*

$$\gamma \leq (\eta+1) \left(\frac{n-1}{n+1} \right).$$

It is important to point out that we give upper bounds for the general problem (10) by construction of examples on a particular hypersurface, the paraboloid. These examples do not appear to extend in a routine manner to general hypersurfaces, or even to the specific case of the sphere $S = \mathbb{S}^{n-1}$, due to number-theoretic issues. The case of the sphere is the principal interest of [11] as it relates to the classical distance set conjecture of Falconer [10].

Proof. We take S to be the section of the paraboloid

$$(11) \quad \{\xi = (\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \xi_n = |\xi'|^2/2, 0 \leq \xi_1, \dots, \xi_{n-1} \leq 1\},$$

and observe that

$$(12) \quad \widehat{gd\sigma}(x) = \int_{|\xi'| \leq 1} e^{-2\pi i x' \cdot \xi' + \pi i |\xi'|^2 x_n} \hat{f}(\xi') d\xi' = e^{ix_n \Delta} f(x'),$$

where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\hat{f}(\xi') = g(\xi', |\xi'|^2)(1 + |\xi'|^2)^{1/2}$. Now by Plancherel's Theorem, inequality (10) may be written as

$$(13) \quad \int_{\mathbb{B}} |e^{iRx_n \Delta} f(Rx')|^2 d\mu \leq \frac{C}{R^\gamma} \sup_{x \in \mathbb{R}^n, r > 0} \left\{ \frac{\mu(B(x, r))}{r^\eta} \right\} \|f\|_2^2,$$

and so we may test (13) on functions f of the form (3) with $\delta = 1/R$. For such a function $|e^{iRx_n \Delta} f(Rx')| \gtrsim R^{-(n-1)(1-\sigma)}$ for all $x \in \tilde{\Omega}$, where $\tilde{\Omega}$ is an $O(1/R)$ neighbourhood of

$$\{x \in \mathbb{R}^n : x' \in \tilde{X}^{n-1} \text{ and } x_n = 2qR^{2\sigma-1} \text{ where } q \in \mathbb{N} \text{ and } q \lesssim R^{1-2\sigma}\}$$

and

$$\tilde{X} = \{pR^{\sigma-1} : p \in \mathbb{N} \text{ with } p \lesssim R^{1-\sigma}\}.$$

Notice that $R\tilde{\Omega} = \Omega$.

Setting $d\mu(x) = \chi_{\tilde{\Omega}}(x)dx$ we therefore obtain

$$\int_{\mathbb{B}} |e^{iRx_n \Delta} f(Rx')|^2 d\mu \gtrsim R^{-2(n-1)(1-\sigma)} |\tilde{\Omega}| \sim R^{-2(n-1)(1-\sigma)-\sigma(n+1)}.$$

Furthermore,

$$\sup_{x \in \mathbb{R}^n, r > 0} \left\{ \frac{\mu(B(x, r))}{r^\eta} \right\} \sim \begin{cases} R^{-\sigma(n+1)}, & \frac{n-1}{2} \leq \eta \leq n - \sigma(n+1) \\ R^{\eta-n}, & n - \sigma(n+1) \leq \eta \leq n \end{cases}$$

as the reader will easily verify.

Using these calculations and the fact that $\|f\|_2^2 \sim R^{-(n-1)(1-\sigma)}$, we may deduce that a necessary condition for (13) (and thus (10)) to hold for all $R \geq 1$ is that

$$\gamma \leq (\eta + 1) \left(\frac{n-1}{n+1} \right),$$

as claimed. \square

It is conceivable that the upper bound of Proposition 3.1 may be improved by considering the more sophisticated special solutions of the Schrödinger equation (1) described at the end of Section 2, but we do not pursue this point further.

We note that the examples of Section 2 furnish explicit necessary conditions on the functional $C(\mu)$ so that an inequality of the form (9) might hold for some given R . These conditions are not simply upper bounds on how much mass μ can put on various eccentric tubes, but instead on how much mass can be put on various *arrangements* of eccentric tubes dictated by the set Ω and its variants. On the other hand, if we take for example $\gamma = n - 1$, and we demand validity of (9) for *all* $R \geq 1$ then the examples we present here offer no further necessary conditions than do the “simple” examples where the testing function is essentially the characteristic function of a *single* product of intervals.

3.2. Morrey–Campanato weights. The Stein–Tomas restriction theorem (see [20]), in its equivalent dual form, states that for $r \geq 2(n+1)/(n-1)$, there is a constant C for which

$$(14) \quad \|\widehat{gd\sigma}\|_{L^r(\mathbb{R}^n)} \leq C \|g\|_{L^2(S)},$$

for all $g \in L^2(S)$. This estimate may be viewed using duality as the weighted estimate

$$(15) \quad \int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 V(x) dx \leq C \|V\|_{L^p(\mathbb{R}^n)} \int_S |g|^2 d\sigma,$$

whenever $1 \leq p \leq (n+1)/2$ and $V \in L^p$.

In [17] Ruiz and Vega considered (see also [7], [8] and [18] for $\alpha = 2$) extending (15) by replacing the L^p norm of V by certain *Morrey–Campanato* norms. These norms play a role in the theory of unique continuation – see [13] and [22]. The Morrey–Campanato classes, which are denoted by $\mathcal{L}^{\alpha,p}$, for $\alpha > 0$ and $1 \leq p \leq n/\alpha$, are given by

$$\mathcal{L}^{\alpha,p} = \{V \in L_{loc}^p(\mathbb{R}^n) : \|V\|_{\mathcal{L}^{\alpha,p}} < \infty\},$$

where

$$\|V\|_{\mathcal{L}^{\alpha,p}} = \sup_{x \in \mathbb{R}^n, r > 0} r^\alpha \left(r^{-n} \int_{B(x,r)} |V(y)|^p dy \right)^{1/p}.$$

Notice that $\mathcal{L}^{\alpha,n/\alpha} = L^{n/\alpha}(\mathbb{R}^n)$. We also remark that for $p < n/\alpha$ the class $\mathcal{L}^{\alpha,p}$ contains the Lorentz space $L^{n/\alpha,\infty}(\mathbb{R}^n)$. Furthermore, when $p = 1$, it is natural to think of V as a measure μ rather than a locally L^1 function, and then

$$\|\mu\|_{\mathcal{L}^{\alpha,1}} = \sup_{x \in \mathbb{R}^n, r > 0} \left\{ \frac{\mu(B(x,r))}{r^{n-\alpha}} \right\},$$

(similar to what was considered in the previous subsection).

Proposition 3.2. *If $n \geq 2$, $\frac{2n}{n+1} < \alpha \leq n$, $p \geq 1$ and $\frac{\alpha}{n} \leq \frac{1}{p} < \frac{2(\alpha-1)}{n-1}$, there exists a constant C (depending on S , p and α , but independent of g and V) such that*

$$(16) \quad \int_{\mathbb{R}^n} |\widehat{gd\sigma}(x)|^2 V(x) dx \leq C \|V\|_{\mathcal{L}^{\alpha,p}} \int_S |g|^2.$$

The proof given by Ruiz and Vega in [17] was for the case $S = \mathbb{S}^{n-1}$, but all the estimates used go through in the more general case of nonvanishing Gaussian curvature. There are three “endpoint” cases $(\alpha, \frac{1}{p}) = (\frac{2n}{n+1}, \frac{2}{n+1})$, $(\frac{n+1}{2}, 1)$ and $(n, 1)$ respectively. The first of these is the Stein–Tomas restriction theorem (15), the second corresponds to a rescaled version of Mattila’s result ([14]) that $\gamma(\eta) = \eta$ for $\eta = \frac{n-1}{2}$ which was discussed in the previous subsection, and the third is trivial. (The key difficulty to be overcome in [17] was the failure of the Morrey–Campanato spaces to interpolate nicely; see also [3].)

We now consider the sharpness of the condition $\frac{1}{p} < \frac{2(\alpha-1)}{n-1}$ in Proposition 3.2. (See [13] and [22] for related issues concerning Carleman estimates.)

In the first place, a straightforward modification of the standard “Knapp” counterexample provides the following necessary condition for $\alpha < 2$. This condition gives optimal results in dimensions $n = 2, 3$ with the possible exception of the line $\frac{1}{p} = \frac{2(\alpha-1)}{n-1}$.

Lemma 3.3. *Let $n \geq 2$, $\alpha < 2$ and suppose that (16) holds for some S . Then $\frac{1}{p} \leq \frac{2(\alpha-1)}{n-1}$.*

Proof. Let g be the characteristic function of a δ -cell on S ; then $\|g\|_2^2 \sim \delta^{n-1}$, while $|\widehat{gd\sigma}(x)| \geq C\delta^{n-1}$ on a tube of sides $\delta^{-1} \times \delta^{-1} \cdots \times \delta^{-1} \times \delta^{-2}$. Take V to be the characteristic function of this tube. Then

$$\|V\|_{\mathcal{L}^{\alpha,p}} \sim \begin{cases} \delta^{-\alpha} & \text{if } \frac{1}{p} \geq \frac{\alpha}{n-1}, \\ \delta^{-2\alpha + \frac{n-1}{p}} & \text{if } \frac{1}{p} \leq \frac{\alpha}{n-1}. \end{cases}$$

The claim follows from the second of these estimates upon taking δ small. \square

These standard examples only have significance when $\alpha < 2$. As we shall now see, for $\alpha \geq 2$ and $n \geq 4$, the special solutions introduced in Section 2 provide us with further necessary conditions. As in the previous subsection, this will be achieved by taking S to be a bounded subset of the paraboloid. (Again, we point out that these examples do not appear to extend in a routine manner to other curved submanifolds of the type we consider.)

Proposition 3.4. *Suppose that $n \geq 4$ and S is the section of the paraboloid given by (11). If $\alpha \geq 2$ and (16) holds, then $\frac{1}{p} \leq \frac{2\alpha}{n+1}$.*

Proof. As we did in (12), we can write $\widehat{gd\sigma}(x) = e^{ix_n \Delta} f(x')$, where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\widehat{f}(\xi') = g(\xi', |\xi'|^2)(1 + |\xi'|^2)^{1/2}$ and therefore, inequality (16) may be written as

$$(17) \quad \|e^{ix_n \Delta} f(x')\|_{L^2(V)} \leq C \|V\|_{\mathcal{L}^{\alpha,p}}^{1/2} \|\widehat{f}\|_{L^2(\mathbb{R}^{n-1})}.$$

Taking f as in (3) and V as the characteristic function of the set Ω defined in Section (2) (see Figure 1), we have that

$$(18) \quad \|e^{ix_n \Delta} f(x')\|_{L^2(V)} \sim \delta^{(n-1)(1-\sigma)} |\Omega|^{1/2} = \delta^{\frac{(1-\sigma)(n-1)}{2} + \sigma - \frac{1}{2}}.$$

On the other hand,

$$(19) \quad \|\widehat{f}\|_{L^2(\mathbb{R}^{n-1})} \sim \delta^{\frac{(1-\sigma)(n-1)}{2}}.$$

For $p \leq n/\alpha$, $0 < \sigma < 1/2$, and δ small and positive we have

$$(20) \quad \|V\|_{\mathcal{L}^{\alpha,p}} \sim \max\{1, \delta^{\frac{\sigma(n+1)}{p} - \alpha}\}.$$

From (18), (19) and (20), and the fact that $0 < \sigma < 1/2$, we see that a necessary condition for (17) to hold is $\frac{1}{p} \leq \frac{2\alpha}{n+1}$. \square

Figure 2 shows the positive and the negative results discussed here when $\alpha \leq \frac{n}{p}$ and S is the section of the paraboloid (11). We shall address the analogous questions

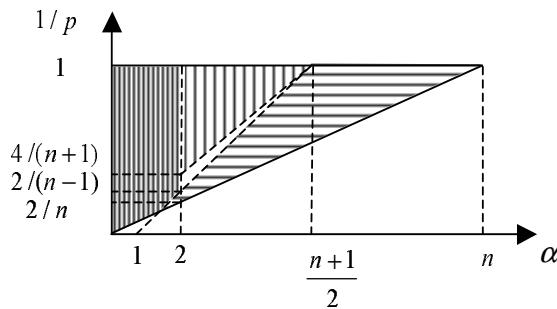


FIGURE 2. Let S be the section of the paraboloid defined in (11). The regions with horizontal lines and vertical lines correspond to the cases where estimate (16) is true and false respectively.

for the full paraboloid – i.e. the initial value problem for the Schrödinger equation (1) where the initial data f is not assumed to have Fourier transform with compact support – in a forthcoming paper. As a consequence of those results we shall obtain some further necessary conditions for (16) to hold in the case of the sphere \mathbb{S}^{n-1} .

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