THE BROUWER FIXED POINT THEOREM AND THE BORSUK–ULAM THEOREM

ANTHONY CARBERY

1. Brouwer fixed point theorem

The Brouwer fixed point theorem states that every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point. When $n = 1$ this is a trivial consequence of the intermediate value theorem.

In higher dimensions, if not, then for some f and all $x \in \mathbb{D}^n$, $f(x) \neq x$. So the map $\tilde{f} : \mathbb{D}^n \to \mathbb{S}^{n-1}$ obtained by sending x to the unique point on $\mathbb{S}^{n-1}$ on the line segment starting at $f(x)$ and passing through x is continuous, and when restricted to the boundary $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ is the identity.

So to prove the Brouwer fixed point theorem it suffices to show there is no map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ which restricted to the boundary $\mathbb{S}^{n-1}$ is the identity. (In fact, this is an equivalent formulation.)

It is enough, by a standard approximation argument, to prove this for C^1 maps g. Consider

$$\int_{\mathbb{D}^n} \det Dg = \int_{\mathbb{D}^n} dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n$$

where Dg is the derivative matrix of g. On the one hand this is zero as Dg has less than full rank at each $x \in \mathbb{D}^n$, and on the other hand it equals, by Stokes' theorem,

$$\int_{\mathbb{S}^{n-1}} g_1 dg_2 \wedge \cdots \wedge dg_n.$$

This quantity manifestly does not depend on the behaviour of g_1 in the interior of $\mathbb{D}^n$, and, by symmetry, likewise depends only on the restrictions of $g_2, \ldots, g_n$ to $\mathbb{S}^{n-1}$. But on $\mathbb{S}^{n-1}$, g is the identity I, so that reversing the argument, this quantity also equals

$$\int_{\mathbb{D}^n} \det DI = |\mathbb{D}^n|.$$

In do Carmo's book *Differential Forms and Applications*, an argument along the above lines is attributed essentially to E. Lima. Is there a similarly simple proof of the Borsuk–Ulam theorem via Stokes' theorem?

Date: Revised May 2010.

2. Borsuk–Ulam theorem

The Borsuk–Ulam theorem states that for every continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ there is some x with $f(x) = f(-x)$. Once again, when $n = 1$ this is a trivial consequence of the intermediate value theorem.

In higher dimensions, we first note that it suffices to prove this for smooth f. Knowing it for smooth functions, uniformly approximate continuous $f : \mathbb{S}^n \to \mathbb{R}^n$ by smooth maps f_m, for each of which there is an x_m with $f_m(x_m) = f_m(-x_m)$; then there is a subsequence (which we'll also call x_m) convergent to some $x \in \mathbb{S}^n$. Then $f_m(x_m) = (f_m(x_m) - f(x_m)) + f(x_m)$; the first term is as small as we please for m sufficiently large, and the second term converges to $f(x)$ by continuity of f. So $f_m(x_m) \to f(x)$ and similarly $f_m(-x_m) \to f(-x)$.

So assume for contradiction that f is smooth and that for all x, $f(x) \neq f(-x)$. Then

$$\tilde{f}(x) := \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

is a smooth map $\tilde{f} : \mathbb{S}^n \to \mathbb{S}^{n-1}$ such that $\tilde{f}(-x) = -\tilde{f}(x)$ for all x, i.e. $\tilde{f}$ is *odd, antipodal* or *equivariant* with respect to the map $x \mapsto -x$.

Thus it suffices to show there is no equivariant smooth map $h : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Equivalently, thinking of the closed upper hemisphere of $\mathbb{S}^n$ as $\mathbb{D}^n$, it suffices to show there is no smooth map $g : \mathbb{D}^n \to \mathbb{S}^{n-1}$ which is equivariant on the boundary. Once again this formulation is equivalent to the Borsuk–Ulam theorem and shows (since the identity map is equivariant) that it generalises the Brouwer fixed point thorem.

2.1. **Case $n = 2$.** As above, it's enough to show that there does not exist a $g : \mathbb{D}^2 \to \mathbb{S}^1$ which is equivariant on the boundary, i.e. such that $g(-x) = -g(x)$ for $x \in \mathbb{S}^1$.

If there did exist such a (smooth) g, consider

$$\int_{\mathbb{D}^2} \det Dg = \int_{\mathbb{D}^2} dg_1 \wedge dg_2$$

On the one hand this is zero as Dg has less than full rank at each x, and on the other hand it equals, by Stokes' theorem,

$$\int_{\mathbb{S}^1} g_1 dg_2 = -\int_{\mathbb{S}^1} g_2 dg_1$$

So it's enough to show that

$$\int_0^1 (g_1(t)g_2'(t) - g_2(t)g_1'(t))dt \neq 0$$

for $g = (g_1, g_2) : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$ satisfying $g(t + 1/2) = -g(t)$ for all $0 \leq t \leq 1$.

Now clearly

$$(g_1(t)g_2'(t) - g_2(t)g_1'(t))dt$$

represents the element of net arclength for the curve $(g_1(t), g_2(t))$ measured in the anticlockwise direction. Indeed, $|g| = 1$ implies $\langle g, g' \rangle = \frac{1}{2}\frac{d}{dt}|g|^2 = 0$, so that

$\det(g, g') = \pm|g||g'| = \pm|g'|$, with the plus sign occuring when g is moving anti-clockwise. And the point of the equivariance condition is that $(g_1(1/2), g_2(1/2)) = -(g_1(0), g_2(0))$, and that

$$\int_0^1 g_1(t)g_2'(t)dt = 2\int_0^{1/2} g_1(t)g_2'(t)dt.$$

In passing from $(g_1(0), g_2(0))$ to $(g_1(1/2), g_2(1/2))$ the total net arclength traversed is clearly an odd multiple of π, and so we're done.

Note that the argument shows that for any smooth equivariant $h : \mathbb{S}^1 \to \mathbb{S}^1$ we have

$$\int_{\mathbb{S}^1} h_1 dh_2 = -\int_{\mathbb{S}^1} h_2 dh_1 \neq 0,$$

and indeed is an odd multiple of π.

In the higher dimensional case $n \geq 3$ we'd be done by the same argument if we could show that

$$\int_{\mathbb{S}^{n-1}} g_1 dg_2 \wedge dg_3 \cdots \wedge dg_n \neq 0$$

whenever $g : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is equivariant.

2.2. **Dimensional reduction.** The material in this subsection is based upon notes by Shchepin available at http://www.mi.ras.ru/~scepin/elem-proof-reduct.pdf with some details added, and is adapted to the smooth case under consideration.

Theorem 1. *Suppose $n \geq 4$ and there exists a smooth equivariant map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$. Then there exists a smooth equivariant map $\tilde{f} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-2}$.*

Once this is proved, only the case $n = 3$ of the Borsuk–Ulam theorem remains outstanding. Note that only the case when f is surjective is interesting here (as if f avoids a pair of points we can simply retract its image into an equator).

Proof. Starting with f, we shall identify suitable equators $E_{n-1} \subseteq \mathbb{S}^n$ and $E_{n-2} \subseteq \mathbb{S}^{n-1}$, and build a smooth equivariant map $\tilde{f} : E_{n-1} \to E_{n-2}$.

We first need to know that there is some pair of antipodal points $\{\pm A\}$ in the target $\mathbb{S}^{n-1}$ whose preimages under f are at most "one-dimensional". This is intuitively clear by dimension counting but for rigour we appeal to Sard's theorem.[1] Indeed, Sard's theorem tells us that the image under f of the set $\{x \in \mathbb{S}^n : \text{ rank } Df(x) < n - 1\}$ is of Lebesgue measure zero: so there are plenty of points $A \in \mathbb{S}^{n-1}$ at all of whose preimages x – if there are any at all – $Df(x)$ has full rank $n - 1$. By the implicit function theorem, for each such x there is a neighbourhood $B(x, r)$ such that $B(x, r) \cap f^{-1}(A)$ is diffeomorphic to the interval $(-1, 1)$. The whole of the compact set $f^{-1}(A)$ is covered by such balls, from which we can extract a finite subcover: so indeed $f^{-1}(A)$ is covered by finitely many diffeomorphic copies of $(-1, 1)$.

[1]Shchepin works instead with polyhedra homeomorphic to spheres and piecewiese linear surjections between them. We may assume that the image of each face in the domain is contained in a face of the target. In this setting linear algebra shows that there is a pair of antipodes whose inverse images are finite unions of line segments.

Now for $x \in f^{-1}(A)$ and $y \in f^{-1}(-A) = -f^{-1}(A)$ with $y \neq -x$, consider the unique geodesic great circle joining x to y. The family of such is clearly indexed by the two-parameter family of points of $f^{-1}(A) \times f^{-1}(-A) \setminus \{(x, -x) : f(x) = A\}$. Their union is therefore a manifold in $\mathbb{S}^n$ of dimension at most three. Since $n \geq 4$ there must be points $\pm B \in \mathbb{S}^n$ *outside* this union(and necessarily outside $f^{-1}(A) \cup f^{-1}(-A)$). Such a point has the property that *no* geodesic great circle passing through it meets points of both $f^{-1}(A)$ and $f^{-1}(-A)$ other than possibly at antipodes. In particular, no *meridian* joining $\pm B$ meets both $f^{-1}(A)$ and $f^{-1}(-A)$.

We now identify E_{n-2} as the equator of $\mathbb{S}^{n-1}$ whose equatorial plane is perpendicular to the axis joining A to $-A$; and we identify E_{n-1} as the equator of $\mathbb{S}^n$ whose equatorial plane is perpendicular to the axis joining B to $-B$. We assume for notational simplicity that B is the north pole $(0, 0, \dots, 0, 1)$.

Lemma 1. *Suppose $B = (0, 0, \dots, 0, 1) \in \mathbb{S}^n$ and that $X \subseteq \mathbb{S}^n$ is a closed subset such that no meridian joining $\pm B$ meets both X and $-X$. Let $\mathbb{S}^n_{\pm}$ denote the open upper and lower hemispheres respectively. Then there is an equivariant diffeomorphism $\psi : \mathbb{S}^n \to \mathbb{S}^n$ such that*

$$X \subseteq \psi(\mathbb{S}^n_+).$$

Proof. Since X is closed and $\pm B \notin X$, there is an $\epsilon > 0$ such that spherical caps centred at $\pm B$ subtending angles of 2ϵ at the origin do not meet $\pm X$. Consider the projections of $\pm X$ along meridians from B and $-B$ on the equator E of $\mathbb{S}^n$ which lies in the plane perpendicular to the axis joining $\pm B$; call these $\pm\Pi(X)$. These are disjoint closed sets and are therefore positively separated. For $x \in E$ let $d_{\pm}(x)$ be the geodesic distance in E to $\pm\Pi(X)$, and let

$$\theta(x) = (\pi/2 - \epsilon)\frac{d_+(x) - d_-(x)}{d_+(x) + d_-(x)}.$$

Then for all $x \in E$, $\theta(-x) = -\theta(x)$ and $|\theta(x)| \leq \pi/2 - \epsilon$; for $x \in \Pi(X)$ we have $\theta(x) = -\pi/2 + \epsilon$ and for $x \in -\Pi(X)$ we have $\theta(x) = \pi/2 - \epsilon$. Mollify θ if necessary to obtain a *smooth* function $\tilde{\theta} : E \to [-\pi/2 + \epsilon/2, \pi/2 - \epsilon/2]$ satisfying $\tilde{\theta}(-x) = -\tilde{\theta}(x)$ and such that for $x \in \Pi(X)$ we have $-\pi/2 + \epsilon/2 \leq \tilde{\theta}(x) \leq -\pi/2 + 3\epsilon/2$.

For $x \in E$ let $\psi(x)$ be the point of $\mathbb{S}^n$ on the meridian through x with latitude $\tilde{\theta}(x)$. Then $\psi : E \to \mathbb{S}^n$ is a smooth equivariant map and clearly we can extend this to be an equivariant diffeomorphism ψ of $\mathbb{S}^n$ to itself which maps $\mathbb{S}^{n-1}_+$ to the region above $\psi(E)$.

Finally, since for $x \in X$ we have $\tilde{\theta}(x) \leq -\pi/2 + 3\epsilon/2$, we have that X is contained in the region above $\psi(E)$, that is, $X \subseteq \psi(\mathbb{S}^n_+)$.

□

Continuing with the proof of the theorem, we apply the lemma with $X = f^{-1}(A)$. Let ϕ be restriction of ψ to $E = E_{n-1}$. Consider the restriction $\widehat{f}$ of f to $\phi(E)$: it has the property that $\widehat{f}(\phi(E))$ does not contain $\pm A$. Let r be the standard retraction of $\mathbb{S}^{n-1} \setminus \{\pm A\}$ onto its equator E_{n-2}; finally let $\tilde{f} = r \circ \widehat{f} \circ \phi$, which is clearly smooth and equivariant.

□

Remark. It is clear from the construction of ψ in Lemma 1 that we may assume that it fixes meridians and acts as the identity on small neighbourhoods of $\pm B$. Moreover it is also clear that we may find a smoothly varying family of diffeomorphisms ψ_t of $\mathbb{S}^n$, $0 \leq t \leq 1$, such that ψ_0 is the identity and $\psi_1 = \psi$.

2.3. **Case $n = 3$.** The treatment in Proposition 1 and Lemma 2 below is also based upon the notes of Shchepin peviously cited, but it incorporates some significant differences of detail, and is adapted to the smooth case under consideration.

Proposition 1. *Suppose that $f : \mathbb{S}^3 \to \mathbb{S}^2$ is a smooth equivariant map. Then there exists a smooth $f^\dagger : \mathbb{D}^3 \to \mathbb{S}^2$ which is equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$, and moreover maps $\mathbb{S}^2_\pm$ to itself.*

Proof. Firstly, by identifying the closed upper hemisphere of $\mathbb{S}^3$ with the closed disc $\mathbb{D}^3$ we obtain a smooth map

$$\widehat{f} : \mathbb{D}^3 \to \mathbb{S}^2$$

which is equivariant on $\partial\mathbb{D}^3 = \mathbb{S}^2$.

Then the restriction of $\widehat{f}$ to $\partial\mathbb{D}^3 = \mathbb{S}^2$ gives a smooth equivariant map $g : \mathbb{S}^2 \to \mathbb{S}^2$. If we could take g to be the identity, we would be finished (by the argument given for the Brouwer fixed point theorem). Further, if g is such that *either* for all x, $g(x) \neq x$, *or* for all x, $g(x) \neq -x$, we can use the map

$$\frac{\theta g(x) \pm (1-\theta)x}{|\theta g(x) \pm (1-\theta)x|}$$

to "graft the identity onto the outside of $\widehat{f}$" to obtain a map to which the Brouwer argument applies. So the "bad" g are those for which there exist x and y with $g(x) = x$ and $g(y) = -y$. For such g we might hope instead to be able to graft an equivariant hemisphere-preserving $g^\dagger$ onto the outside of $\widehat{f}$ – or what is almost as good, to find such a $g^\dagger$ and a smooth family of diffeomorphisms $\psi_t : \mathbb{S}^2 \to \mathbb{S}^2$ such that ψ_0 is the identity and $\psi_1 = \psi$ where $g^\dagger(x) \neq -g \circ \psi(x)$ for all x.

Lemma 2. *If $g : \mathbb{S}^2 \to \mathbb{S}^2$ is a smooth equivariant map, then there exists a smooth equivariant $g^\dagger : \mathbb{S}^2 \to \mathbb{S}^2$ which preserves the upper and lower hemispheres of $\mathbb{S}^2$, and a smooth family of diffeomeorphisms ψ_t such that $\psi_0 = I$, $\psi_1 = \psi$ and such that for all x,*

$$g^\dagger(x) \neq -g \circ \psi(x).$$

Proof. Pick points $\pm A$ in the target $\mathbb{S}^2$ whose inverse images under g are at most finite. This we can do by Sard's theorem as the image under g of the set $\{x \in \mathbb{S}^2 : \text{ rank } Dg(x) < 2\}$ has Lebesgue measure zero in $\mathbb{S}^2$. So there are plenty of points $A \in \mathbb{S}^2$ at all of whose preimages x (if there are any at all), $Dg(x)$ has full rank 2. By the inverse function theorem, for each such x there is a neighbourhood $B(x, r)$ such that $B(x, r) \cap g^{-1}(A) = \{x\}$. The whole of the compact set $g^{-1}(A)$ is covered by such balls, from which we can extract a finite subcover: so indeed $g^{-1}(A)$ consists of (at most) finitely many points. We may assume without loss of generality that A is the north pole.

Pick now $\pm B$ in the domain $\mathbb{S}^2$ such that the merdinial projections of the members of $g^{-1}(\pm A)$ on the equator whose plane is perpendicular to the axis joining $\pm B$ are distinct. Assume without loss of generality that B is the north pole. Now applying Lemma 1 (and the remarks following it) with $X = g^{-1}(A)$ we see there is an equivariant diffeomorphism $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ such that $g^{-1}(A) \subseteq \psi(\mathbb{S}^2_+)$. Thus $g_* := g \circ \psi$ is a smooth equivariant map from $\mathbb{S}^2$ to itself and $g_*^{-1}(A) \subseteq \mathbb{S}^2_+$. Moreover $\psi = \psi_1$ where ψ_t is a smooth family of diffeomorphisms such that $\psi_0 = I$.

Let E be the equator and define $\tilde{g} : E \to E$ as the meridinial projection of $g_*(x)$ on E. It is well-defined since $g_*^{-1}(\pm A) \cap E = \emptyset$. Then $\tilde{g}$ is clearly smooth and equivariant.

We next extend $\tilde{g}$ to a small strip around E. For $x \in \mathbb{S}^2$ let $l(x) \in [-\pi/2, \pi/2]$ denote its latitude with respect to E, and (if $x \neq \pm B$) let $\overline{x}$ denote its meridinial projection on E. For $0 < r < \pi/2$ let $E_r = \{x \in \mathbb{S}^2 : |l(x)| \leq r\}$. Consider only r so small that $E_r \cap g_*^{-1}(\pm A) = \emptyset$. Let $d = \text{dist}\,(\pm A, g_*(E))$. Since g_* is uniformly continuous, there is an $r > 0$ such that for $x \in E_r$ we have $d(g_*(x), g_*(\overline{x})) < d/10$. Now extend $\tilde{g}$ to E_r by defining $\tilde{g}(x)$ to be the point with the same longitude (merdinial projection) as $\tilde{g}(\overline{x})$ and with latitude $\pi l(x)/2r$. This extension is still equivariant and smooth.

Finally extend $\tilde{g}$ to a map from $\mathbb{S}^2$ to itself by defining $\tilde{g}(x) = A$ for $l(x) > r$ and $\tilde{g}(x) = -A$ for $l(x) < -r$. We see that $\tilde{g}$ is continuous, equivariant, preserves the upper and lower hemispheres and – except possibly on the sets $\{x : l(x) = \pm r\}$ – is smooth.

Consider, for $x \in E_r$, the three points $g_*(x)$, $g_*(\overline{x})$ and $\tilde{g}(x)$. Now $g_*(\overline{x})$ and $\tilde{g}(x)$ lie on the same meridian, and $g_*(\overline{x})$ is distant at least d from $\pm A$. On the other hand, $g_*(x)$ is at most $d/10$ from $g_*(\overline{x})$. Thus $g_*(x)$ is at least $9d/10$ from $\pm A$ and lives in a $d/10$-neighbourhood of the common meridian containing $g_*(\overline{x})$ and $\tilde{g}(x)$. So for $x \in E_r$, $g_*(x)$ cannot equal $-\tilde{g}(x)$. For $l(x) > r$ we have $\tilde{g}(x) = A$ and $g_*(x) \neq -A$ because $g_*^{-1}(-A)$ is contained in the lower hemisphere. Similarly for $l(x) < -r$, $\tilde{g}(x) \neq -g_*(x)$. Thus for all $x \in \mathbb{S}^2$ we have $\tilde{g}(x) \neq -g_*(x)$. By continuity, $\tilde{g}(x)$ and $-g_*(x)$ are positively separated on $\mathbb{S}^2$. So for any sufficiently small uniform perturbation $\tilde{\tilde{g}}$ of $\tilde{g}$ we will have the same property $\tilde{\tilde{g}}(x) \neq -g_*(x)$ for all x.

We now mollify $\tilde{g}$ in small neighbourhoods of $\{x : l(x) = \pm r\}$ (and then renormalise to ensure that the target space remains $\mathbb{S}^2$!) to obtain $g^\dagger$ which is smooth, equivariant, preserves the upper and lower hemispheres and, being uniformly very close to $\tilde{g}$, is such that $g^\dagger(x) \neq -g_*(x)$ for all x.

□

Finishing now with the proof of the proposition, for $x \in \mathbb{S}^2$ and $0 \leq t \leq 1$, let $\tilde{f} : \mathbb{D}^3 \to \mathbb{S}^2$ be defined by

$$\tilde{f}(tx) = \frac{(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x)}{|(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x)|} \text{ when } 2/3 \leq t \leq 1,$$

$$\tilde{f}(tx) = g \circ \psi_{3t-1}(x) \text{ when } 1/3 \leq t \leq 2/3$$

and

$$\tilde{f}(tx) = \widehat{f}(3tx) \text{ when } 0 \leq t \leq 1/3.$$

This makes sense because for all $2/3 \leq t \leq 1$ we have $(3t-2)g^\dagger(x) + (3-3t)(g \circ \psi)(x) \neq 0$; then $\tilde{f}$ has all the desired properties of $f^\dagger$ (including continuity) except possibly for smoothness at $t = 1/3$ and $2/3$. To rectify this we mollify $\tilde{f}$ in a small neighbourhood of $\{t = 1/3\}$ and $\{t = 2/3\}$ and renormalise once more to ensure that the target space is indeed still $\mathbb{S}^2$. The resulting $f^\dagger$ now has all the properties we need. □

Let C be the cylinder $\mathbb{D}^2 \times [-1, 1]$ in $\mathbb{R}^3$ with top and bottom faces $D_\pm$ and curved vertical boundary $V = \mathbb{S}^1 \times [-1, 1]$. Let $S_\pm$ be the upper and lower halves of $S = \partial C$. Let E be the equator of S.

Now C, with the all points on each vertical line of V identified, is diffeomorphic to $\mathbb{D}^3$, and S is also diffeomorphic to $\mathbb{S}^2$, so that we immediately get:

Corollary 1. *Under the same hypotheses as Proposition 1, there exists a smooth map from C to S which is equivariant on ∂C, which is constant on vertical lines in V and which maps $D_\pm$ into $S_\pm$.*

The final argument needed to complete the proof of the Borsuk–Ulam theorem is:

Proposition 2. *There is no smooth map $f : C \to S$ which is equivariant on ∂C, which is constant on vertical lines in V and which maps $D_\pm$ into $S_\pm$.*

Proof. If such an f existed, then

$$\int_C \det Df = \int_C df_1 \wedge df_2 \wedge df_3$$

where Df is the derivative matrix of f. On the one hand this is zero as Df has less than full rank at almost every $x \in C$, and on the other hand it equals, by Stokes' theorem,

$$\int_{\partial C} f_3\ df_1 \wedge df_2 = \int_V f_3\ df_1 \wedge df_2 + 2 \int_{D_+} f_3\ df_1 \wedge df_2$$

by equivariance.

Now f maps V into E, so that $f_3 = 0$ on V, and the first term on the right vanishes.

As for the second term,

$$\int_{D_+} f_3\ df_1 \wedge df_2 = \int_{D_+ \cap \{x\,:\,f_3(x)=1\}} f_3\ df_1 \wedge df_2 + \int_{D_+ \cap \{x\,:\,f_3(x)<1\}} f_3\ df_1 \wedge df_2.$$

The region of D_+ on which $f_3(x) < 1$ consists of patches on which $f_1^2(x)+f_2^2(x) = 1$, and so $2f_1\ df_1 + 2f_2\ df_2 = 0$. Taking exterior products with df_1 and df_2 tells us that on such patches we have $f_1\ df_1 \wedge df_2 = f_2\ df_1 \wedge df_2 = 0$. Multiplying by f_1 and f_2 and adding we get that $h\ df_1 \wedge df_2 = 0$ for all h supported on a patch on which $f_3(x) < 1$. So for any h we have

$$\int_{D_+ \cap \{x\,:\,f_3(x)<1\}} h\ df_1 \wedge df_2 = 0.$$

Hence

$$\int_{D_+} f_3\ df_1 \wedge df_2 = \int_{D_+ \cap \{x\,:\,f_3(x)=1\}} df_1 \wedge df_2$$

$$= \int_{D_+ \cap \{x\,:\,f_3(x)=1\}} df_1 \wedge df_2 + \int_{D_+ \cap \{x\,:\,f_3(x)<1\}} df_1 \wedge df_2$$

$$= \int_{D_+} df_1 \wedge df_2.$$

By Stokes' theorem once again we have

$$\int_{D_+} df_1 \wedge df_2 = \int_{\partial D_+} f_1 df_2 = -\int_{\partial D_+} f_2 df_1,$$

and, since f restricted to ∂D_+ is equivariant, this quantity is nonzero (and indeed is an odd multiple of π), by the remarks in the proof of the case $n = 2$ above.

So no such f exists and we are done.

□

Acknowledgement. The author would like to thank Pieter Blue, Marina Iliopoulou and Mark Powell for helpful suggestions and comments.