

Numerical viscosity and resistivity in MHD simulations using high-order methods

Martin Obergaulinger

Institut für Kernphysik, Technische Universität Darmstadt
& Departament d'Astronomia i Astrofísica, Universitat de València

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In collaboration with

Tomasz Rembiasz, Miguel Ángel Aloy, Pablo Cerdá-Durán (València) and
Ewald Müller (MPA, Garching)

For more details, see

http://www.uv.es/camap/tmweb/Web_tm.html

and Rembiasz et al. (2017)



The ideal Newtonian MHD equations

Gas

$$\begin{aligned}\partial_t \rho + \vec{\nabla} \cdot [\rho \vec{v}] &= 0, \\ \partial_t \rho v^i + \nabla_j [\rho v^i v^j + P_\star \delta^{ij} - b^i b^j] &= 0, \\ \partial_t e_\star + \vec{\nabla} \cdot [(e_\star + P_\star) \vec{v} - \vec{b}(\vec{b} \cdot \vec{v})] &= 0.\end{aligned}$$

$$P_\star = P_{\text{gas}} + \vec{b}^2/2, \quad e_\star = e_{\text{int}} + \rho \vec{v}^2/2 + \vec{b}^2/2.$$

Magnetic field

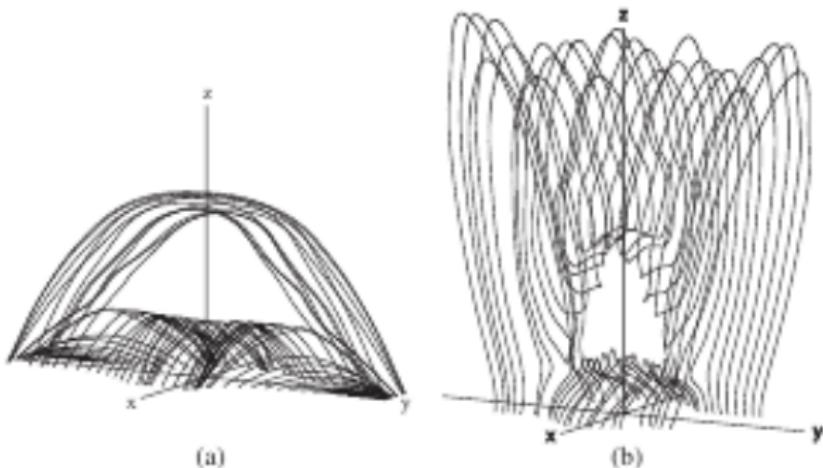
$$\begin{aligned}\partial_t \vec{b} &= \vec{\nabla} \times (\vec{v} \times \vec{b}), \\ \vec{\nabla} \cdot \vec{b} &= 0,\end{aligned}$$

including a non-evolutionary constraint equation. Current density $\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{b}$.



Numerical (M)HD is ubiquitous...

- ▶ from the solar system
- ▶ to the galactic scale and beyond
- ▶ in particular stars from birth to death

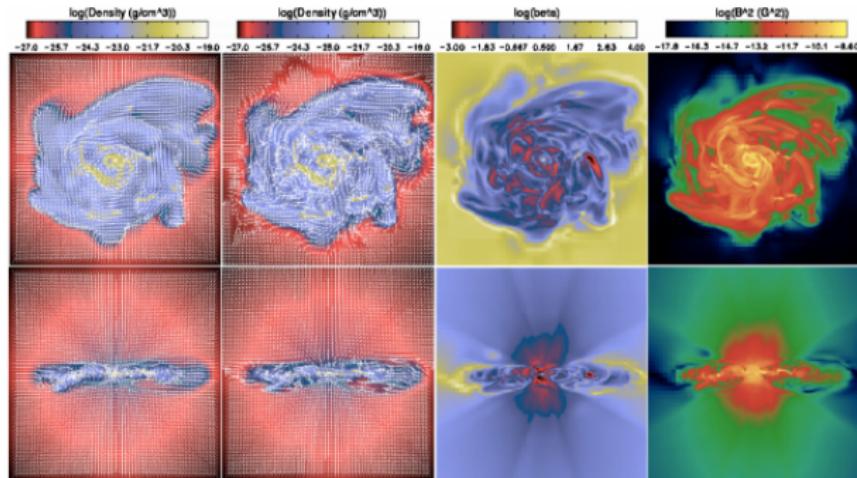


Plasmoid ejection in a solar flare model (Birn et al., 2009)



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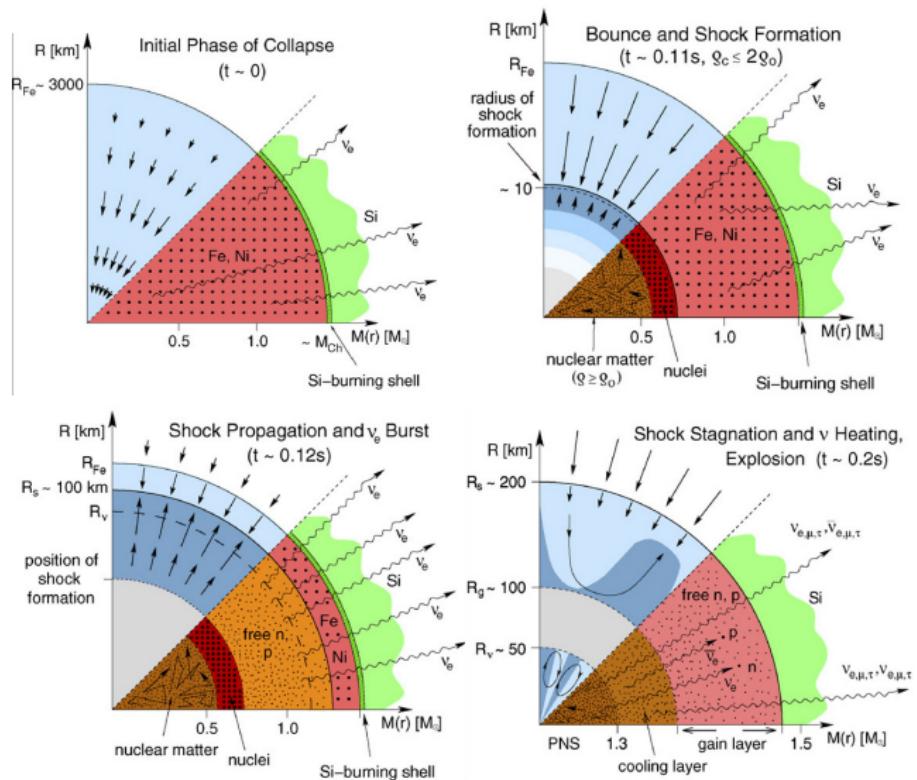


MHD simulations of galaxy formation (Wang & Abel, 2009)



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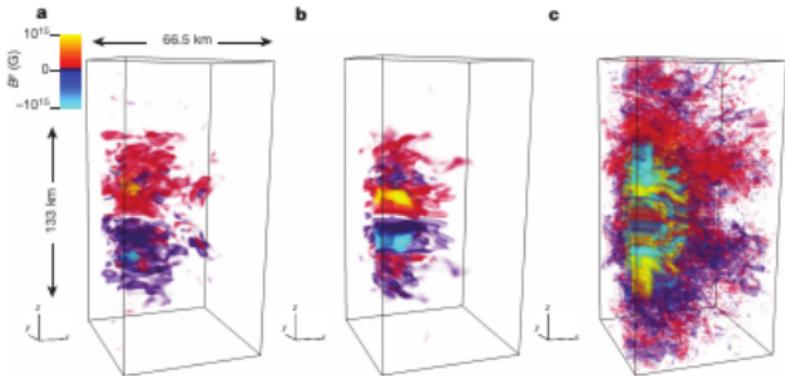


Janka et al. (2007)



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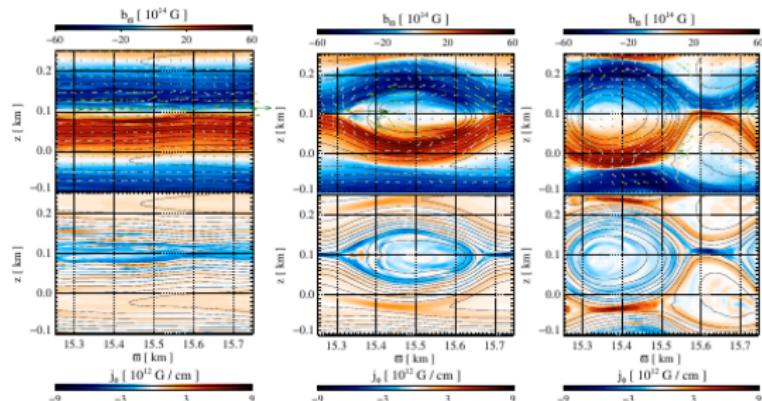


Turbulent dynamo in a rotating stellar core (Mösta et al., 2015)



... and so are numerical errors

- ▶ round-off errors
- ▶ truncation errors
unavoidable due to finite resolution
- ▶ can be classified as numerical diffusion, dispersion, ...
- ▶ may produce phenomena that are not present in the equations



artificial reconnection in an ideal MHD simulation (Obergaulinger et al., 2009)

→ verification

Compare: validation

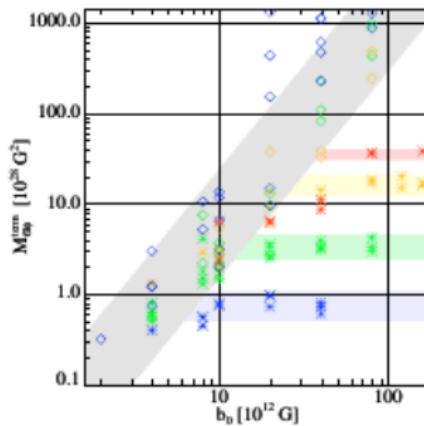
These errors are to be distinguished from the modelling errors due to a lack of knowledge about the underlying equations, material properties, initial and boundary conditions.



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→ **verification**



affecting the key results (scaling of the MRI saturation amplitude)

Compare: validation

These errors are to be distinguished from the modelling errors due to a lack of knowledge about the underlying equations, material properties, initial and boundary conditions.



Diffusive/dissipative effects

Visco-resistive Navier-Stokes equations

$$\begin{aligned}\partial_t \rho + \nabla \cdot (\rho \vec{v}) &= 0, \\ \partial_t \rho v^i + \nabla_j [\rho v^i v^j + P_* \delta^{ij} - b^i b^j + T^{ij}] &= 0, \\ \partial_t e_* + \vec{\nabla} \cdot \left[(e_* + P_*) \vec{v} - \vec{b}(\vec{b} \vec{v}) + \vec{v} \cdot T + \eta \left(\vec{b} \cdot \nabla \vec{b} - \frac{1}{2} \nabla \vec{b}^2 \right) \right] &= 0, \\ \partial_t \vec{b} - \nabla \times [\vec{v} \times \vec{b} + \eta (\nabla \times \vec{b})] &= 0.\end{aligned}$$

Viscous stress tensor

$$\vec{T} = \left[\rho \left(\frac{2}{3} \nu - \xi \right) \vec{\nabla} \cdot \vec{v} \right] \vec{I} - \vec{b} \otimes \vec{b} - \rho \nu \left[\nabla \otimes \vec{v} + (\nabla \otimes \vec{v})^T \right]$$

Viscosities

- ν : kinematic shear viscosity
- ξ : kinematic bulk viscosity

Units: $[\nu, \xi] = \text{cm}^2 \text{s}^{-1}$



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Viscous stress tensor

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- ▶ redistribution of momentum
- ▶ smoothing of shear layers
- ▶ dissipation of kinetic into internal energy
- ▶ timescale: $\tau_{\text{vis}} \sim \frac{L^2}{\{\nu, \xi\}}$



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Resistive electric field

$$\vec{E}_{\text{res}} = \eta \vec{\nabla} \times \vec{b}.$$

- ▶ η : (Ohmic) resistivity
- ▶ $[\eta] = \text{cm}^2 \text{s}^{-1}$
- ▶ diffusion of current sheets
- ▶ dissipation of magnetic into internal energy
- ▶ timescale: $\tau_{\text{vis}} \sim \frac{L^2}{\eta}$



Benefits of numerical dissipation

- ▶ stabilisation of discontinuous solutions
- ▶ preventing unphysical oscillations
- ▶ helps to prevent entropy from decreasing
- maintaining positivity properties
- ▶ an implicit sub-grid model for turbulent dissipation



Space-time discretisation

Conservation laws

- ▶ $\partial_t u + \vec{\nabla} \cdot \vec{f} = S$
- $\Rightarrow \partial_t \int_V dV u + \oint_{\partial_V} d\vec{A} \cdot \vec{f} = \int dV S$
- ▶ for a given volume, the conserved variable u changes only due to fluxes through the surface, \vec{f} , and sources in the volume, S .
- ▶ mass, momentum, energy
- ▶ magnetic flux obeys an analogous law, but is conserved on surfaces rather than volumes: $\partial_t \vec{w} + \vec{\nabla} \times \vec{F} = 0$
- $\Rightarrow \partial_t \int_A dA \vec{w} + \oint_{\partial_A} d\vec{L} \cdot \vec{F} = 0$



Space-time discretisation

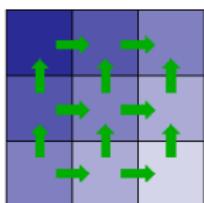
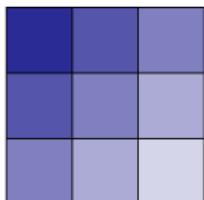
- ▶ divide the simulation domain into a grid of cells of finite volume
- ▶ we use *orthogonal* grids, e.g., Cartesian, cylindrical, or spherical coordinates
- ▶ discretise the physical variables in an integral way,

- ▶ volume average $u_{i,j,k} = \mathcal{V}_{i,j,k}^{-1} \int_{\mathcal{V}_{i,j,k}} d\mathcal{V} u(x, y, z)$
- ▶ surface average

$$w_{i-1/2,j,k}^1 = \mathcal{A}_{i-1/2,j,k}^{-1} \int_{\mathcal{A}_{i-1/2,j,k}} d\mathcal{A} w^1(x, y, z)$$

u, w are unknown \Rightarrow these formulae can be applied, if at all, only for initial data.

- ▶ apply the Gauss and Stokes theorems to these averages; requires numerical approximation of
 - ▶ surface averages of the flux \vec{f}
 - ▶ edge averages of the flux \vec{F}
- \Rightarrow fulfills conservation laws automatically



Space-time discretisation

- ▶ to evolve the conserved variables, discretised in a FV way:
 - compute left and right interface states from cell averages using a *reconstruction* algorithm: preferably using high-order accurate function
 - obtain the final interface flux from the left and right state using a Riemann solver: preferably exact solvers
- ⇒ semi-discrete equations, treated with an ODE solver, e.g., Runge-Kutta
- ▶ all steps affect the accuracy of the method, but focus now mainly on reconstruction

Space-time discretisation

Original values, e.g., cell average values of the conserved variables, $u_{i,j,k}$



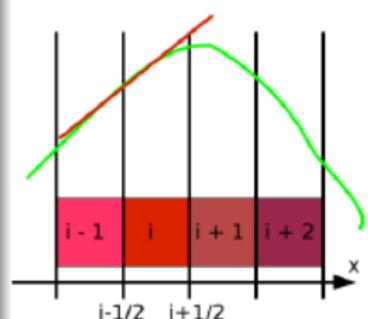
Approximate the functional dependence $(u(\vec{r}))_{i,j,k}$ inside each cell by (simple) functions, e.g.,

- ▶ constants,
- ▶ linear function,
- ▶ higher-order polynomials,

requiring accuracy and stability

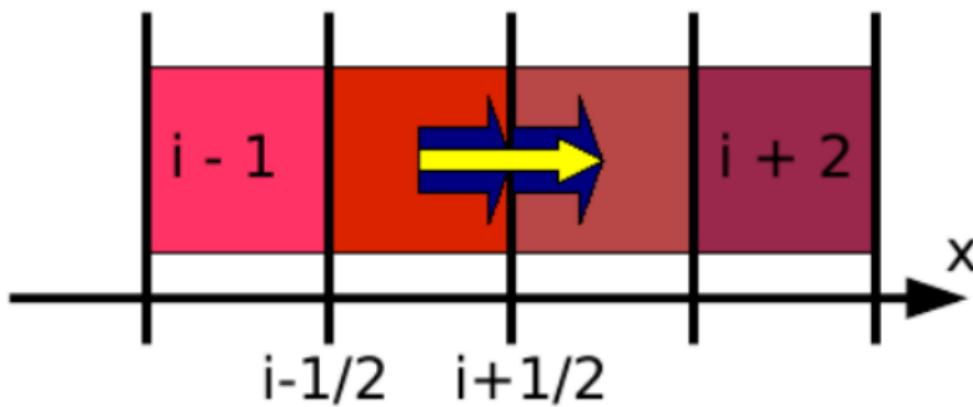


Evaluate the interpolant at a given point \vec{r}_0 , or integrate over a domain, e.g., a cell surface

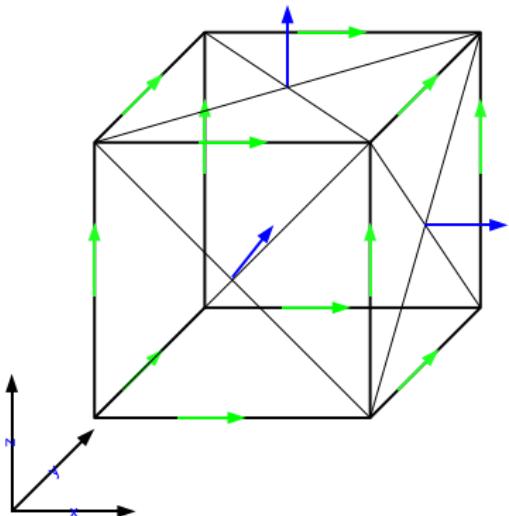


Space-time discretisation

- ▶ reconstruct the volume averages of the conserved hydro variables to surface averages, yielding left and right states
- ▶ solve the Riemann problem at cell interfaces
- ▶ high-order reconstruction of surface states translates into high-order accuracy of the divergence operation



Space-time discretisation



Constrained transport

- ▶ hydro variables discretised in cell volumes
- ▶ magnetic field on cell surface
- ▶ electric field on cell edges
- divergence constraint built-in
- ▶ requires mapping between the staggered grids



The simplest case: linear advection equation

$$\partial_t u + a \partial_x u = 0$$

- ▶ use an upwind solver depending on the sign of a
- ▶ forward discretisation of the differential
- ▶ general formulation for both signs of a
- ▶ introduces a 2nd-order operator with a coefficient $\propto |a|$

$$(-a \partial_x u)_i \rightarrow \begin{cases} -a(u_i - u_{i-1}) & \text{if } a > 0, \\ -a(u_{i+1} - u_i) & \text{if } a < 0, \end{cases}$$



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$$(-a \partial_x u)_i = \frac{-1}{2\Delta x} [a(u_{i+1} - u_{i-1}) + |a|(u_{i+1} - 2u_i + u_{i-1})]$$

Generalisation to hyperbolic systems

$$\partial_t u^k + \vec{\nabla} \cdot \vec{f}^k = 0$$

- ▶ linearise and transform to characteristic variables
- linear advection equations
- ▶ dissipation term $\propto |\lambda|$

$$\Lambda = X^{-1}AX$$

$\Lambda = \text{diag}(\lambda^k)$ is the matrix of eigenvalues, X contains the eigenvectors,
 $A = \partial_u f$ is the Jacobian of the system



Generalisation to hyperbolic systems

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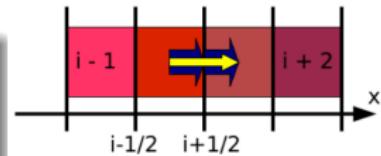
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Finite volumes - upwind flux

$$f_{i+\frac{1}{2}} = \begin{cases} (\lambda w)^L & \text{if } \lambda > 0 \\ (\lambda w)^R & \text{if } \lambda < 0 \end{cases}$$



Generalisation to hyperbolic systems

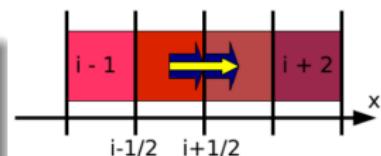
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Finite volumes - upwind flux

$$f_{i+\frac{1}{2}} = \frac{1}{2}\lambda \left[w^L + w^R \right] + \frac{1}{2}|\lambda| \left[w^L - w^R \right]$$

- ▶ diffusion $\propto \lambda$
- ▶ and \propto jump between L and R states → accurate reconstruction minimises diffusion in smooth regions



Main questions

- ▶ Is numerical diffusion (viscosity/resistivity) a good description for the errors in common settings?
- ▶ If so, how do the errors depend on numerical and physical parameters of a simulation?
- ▶ Implications for the choice of numerical methods?

Methodology

- ▶ Perform simulations of systems whose solutions in the case of non-vanishing viscosity and/or resistivity are known, varying the spatial resolution, time step, and numerical methods.
- ▶ Quantify the effective viscosity/resistivity in the simulations by comparing to the analytic/approximate solution
- ▶ Extract the scaling of the diffusion coefficients with space/time resolution.



Our code and models

as used here

- ▶ Newtonian, visco-resistive MHD
- ▶ ideal-gas EOS
- ▶ 1d/2d Cartesian
- ▶ reconstruction: PCM, TVD-PLM, monotonicity-preserving (MP3/5/7/9;
Suresh & Huynh, 1997)
- ▶ Riemann solvers: Lax-Friedrichs, HLL(E)
- ▶ hybrid MPI+OpenMP parallelisation

Our code and models

further elements

- ▶ 3d grid, spherical or cylindrical coordinates
- ▶ general EOS, including Y_e and composition
- ▶ special relativistic MHD
- ▶ post-Newtonian gravity
- ▶ neutrino transport



Dimensional analysis

General assumption

- ▶ numerical diffusion coefficients are the sum of a contribution due to spatial and one due to temporal discretisation (perhaps other contributions as well)
- ▶ they have dimensions velocity times length → a product of a typical length and a typical velocity of the system (to be determined)
- ▶ and they should scale with the grid spacing / time step in a way that depends on the numerical method

- ▶ $\nu = \nu_x + \nu_t$
- ▶ $\nu \propto \mathcal{V} \times \mathcal{L}$
- ▶ $\nu_x \propto (\Delta x)^r, \nu_t \propto (\Delta t)^q$
- ▶ $\nu_x = \mathfrak{R}^{\Delta x} \mathcal{V} \mathcal{L} \left(\frac{\Delta x}{\mathcal{L}} \right)^r$
 $\nu_t = \mathfrak{R}^{\Delta t} \mathcal{V} \mathcal{L} \left(\frac{\mathcal{V} \Delta t}{\mathcal{L}} \right)^q$
- ▶ using the CFL time step with a maximum characteristic speed
 $v_{\max} \nu_t = \mathfrak{R}^{\Delta t} \mathcal{V} \mathcal{L} \left(\frac{C_{\text{CFL}} \Delta x}{\mathcal{L}} \right)^q \left(\frac{\mathcal{V}}{v_{\max}} \right)^q$
- ▶ in multi-D: the spatial contribution is the sum of individual parts with potentially different \mathcal{L}



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Sound waves

Initial conditions

Uniform background:

$\rho_0 = 0, P_0 = 0, \vec{v}_0 = 0.$ $c_s = \sqrt{\Gamma P_0 \rho_0^{-1}}$ is the sound speed.

$$v_1^x = \epsilon \sin(kx),$$

$$\rho_1 = \frac{v_1^x}{c_s} \rho_0,$$

$$P_1 = \frac{v_1^x}{c_s} \Gamma \rho_0.$$

Sine wave is damped exponentially if (numerical) viscosity is present:

$$\begin{aligned} v_1^x &\propto \exp[i(kx - \omega t)] \\ \omega &= \frac{-i(4\nu/3 + \xi)k^2}{2} \\ &\pm k c_s \sqrt{1 - \frac{k^2 \rho_0 (4\nu/3 + \xi)^2}{4\Gamma p_0}}. \end{aligned}$$

Weak damping acts with a combination of bulk and shear viscosity:

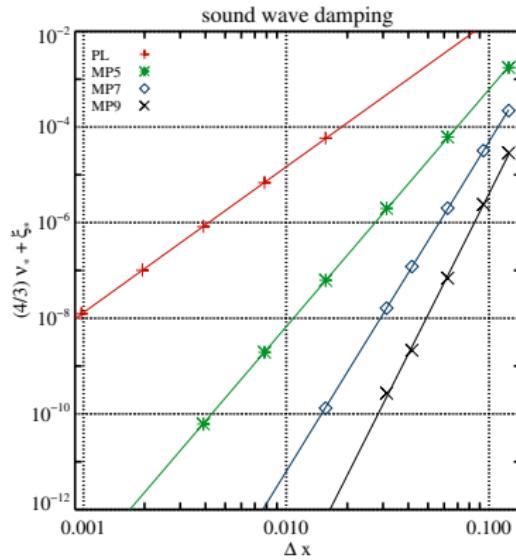
$$\begin{aligned} v^x(x, t) &= \hat{v}_{1x} e^{-\mathfrak{D}_s t} e^{ik(x \mp c_s t)}, \\ \mathfrak{D}_s &= \frac{k^2}{2} \left(\frac{4}{3}\nu + \xi \right). \end{aligned}$$



Sound waves

Several series of simulations changing

- ▶ reconstruction scheme
- ▶ Riemann solver
- ▶ spatial grid spacing (small CFL to limit temporal error)
- ▶ Runge-Kutta integrator
- ▶ CFL factor



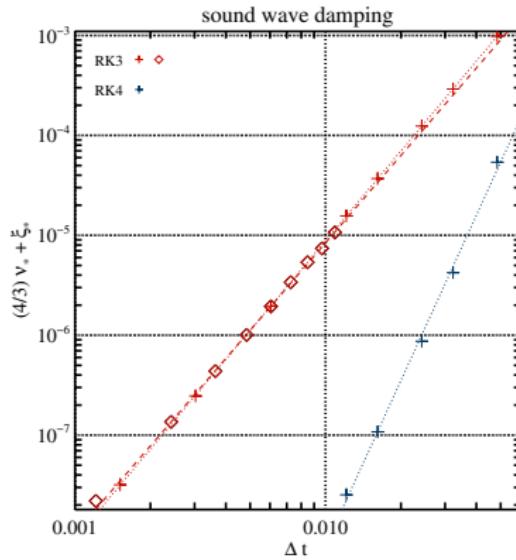
Damping coefficient due to spatial errors (CFL small) for different reconstruction methods.



Sound waves

Several series of simulations changing

- ▶ reconstruction scheme
- ▶ Riemann solver
- ▶ spatial grid spacing (small CFL to limit temporal error)
- ▶ Runge-Kutta integrator
- ▶ CFL factor



Damping coefficient due to temporal errors (MP9 reconstruction) for different time steps (either fixed CFL and varying Δx or varying CFL for fixed grid) and Runge-Kutta integrators.

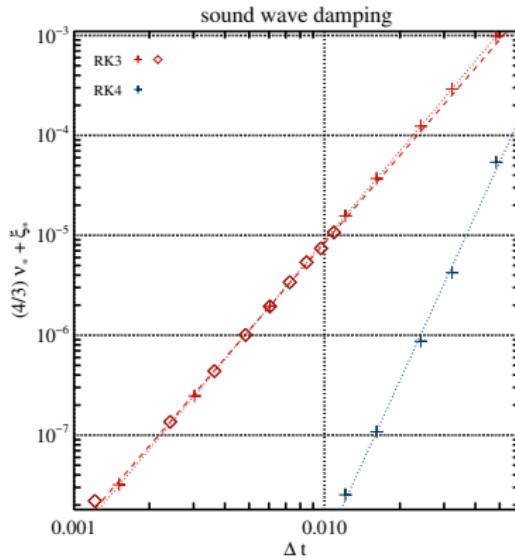


Sound waves

Several series of simulations changing

- ▶ reconstruction scheme
- ▶ Riemann solver
- ▶ spatial grid spacing (small CFL to limit temporal error)
- ▶ Runge-Kutta integrator
- ▶ CFL factor

→ scaling with grid spacing and time step confirmed.



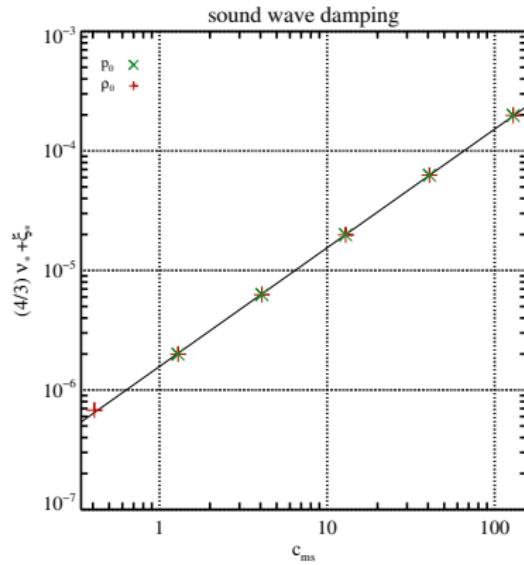
Damping coefficient due to temporal errors (MP9 reconstruction) for different time steps (either fixed CFL and varying Δx or varying CFL for fixed grid) and Runge-Kutta integrators.



Sound waves

¿What about the velocity and length scales?

- ▶ change sound speed
- ▶ and wave length.



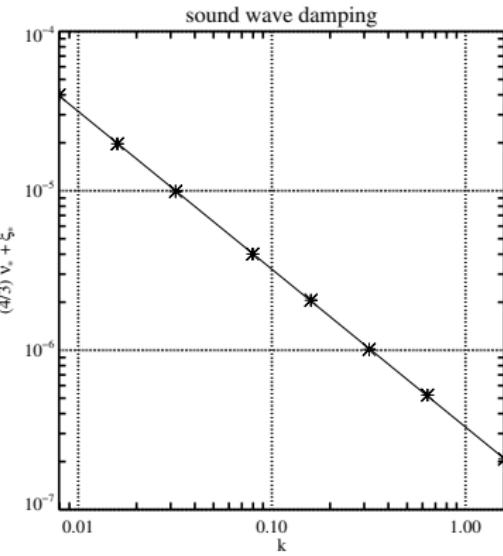
Damping coefficient as a function of sound speed.



Sound waves

¿What about the velocity and length scales?

- ▶ change sound speed
- ▶ and wave length.



Damping coefficient as a function of wavelength.

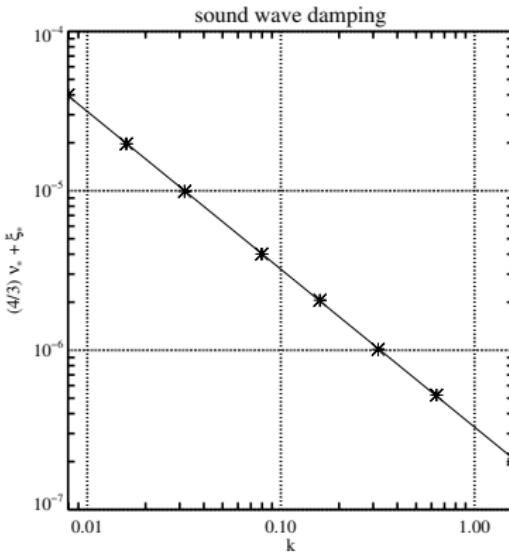


Sound waves

What about the velocity and length scales?

- ▶ change sound speed
- ▶ and wave length.

Consistent with linear scalings of dissipation with c_s and $1/k$
→ identified the typical velocity and length scale.



Damping coefficient as a function of wavelength.



Alfvén waves

Initial conditions

Uniform background:

$\rho_0 = 1, b_0^x = 1, P_0 = 2 \times 10^{-3},$
 $v_0^x = v_0^y = b_0^y = 0.$ Perturbation

$$b_{1y} = \epsilon \sin kx,$$

$$v_{1y} = -\frac{b_{1y}}{\sqrt{\rho_0}}.$$

Solution

$$v_y(x, t) = v_0 e^{-\mathfrak{D}_A t} e^{ik(x \mp c_A t)},$$

with damping rate

$$\mathfrak{D}_A = \frac{k^2}{2}(\eta + \nu).$$

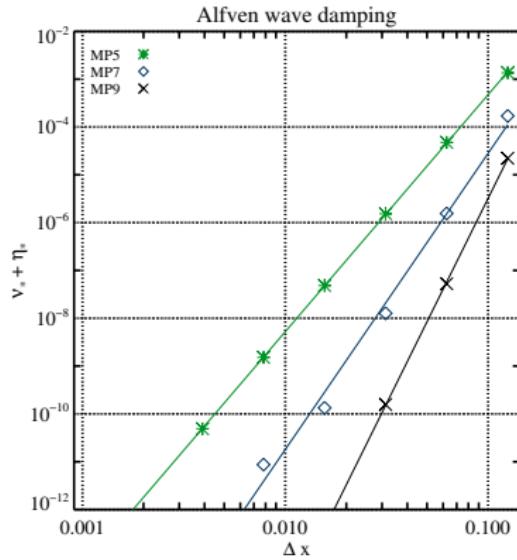


Alfvén waves

Vary

- ▶ resolution
- ▶ time step
- ▶ solvers
- ▶ sound and Alfvén speeds
- ▶ wavelength

to determine the scaling relations for the damping coefficient.



Damping as a function of grid resolution.

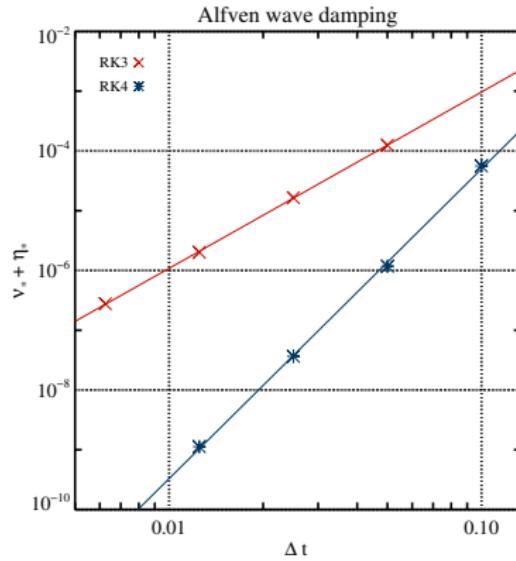


Alfvén waves

Vary

- ▶ resolution
- ▶ time step
- ▶ solvers
- ▶ sound and Alfvén speeds
- ▶ wavelength

to determine the scaling relations for the damping coefficient.



Damping as a function of time step.

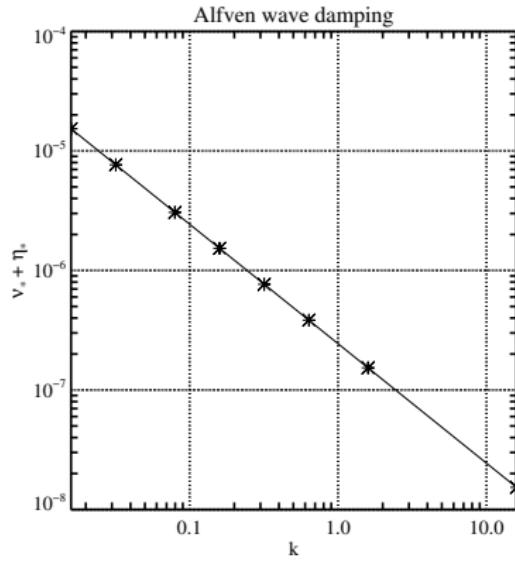


Alfvén waves

Vary

- ▶ resolution
- ▶ time step
- ▶ solvers
- ▶ sound and Alfvén speeds
- ▶ wavelength

to determine the scaling relations for the damping coefficient.



Damping as a function of wave vector.



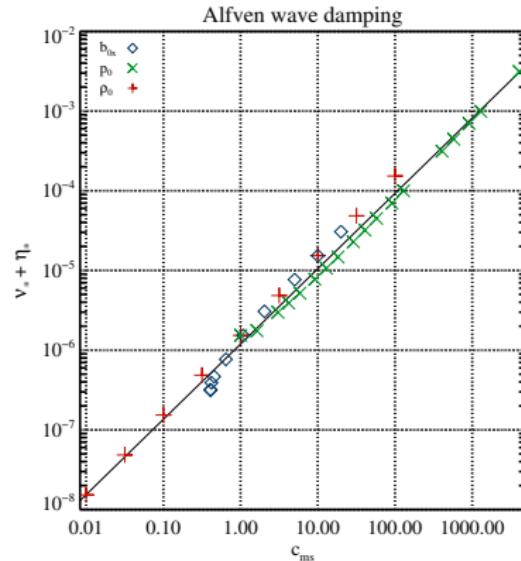
Alfvén waves

Vary

- ▶ resolution
- ▶ time step
- ▶ solvers
- ▶ sound and Alfvén speeds
- ▶ wavelength

to determine the scaling relations for the damping coefficient.

→ confirmed resolution scalings;
typical velocity is now the fast magnetosonic speed.



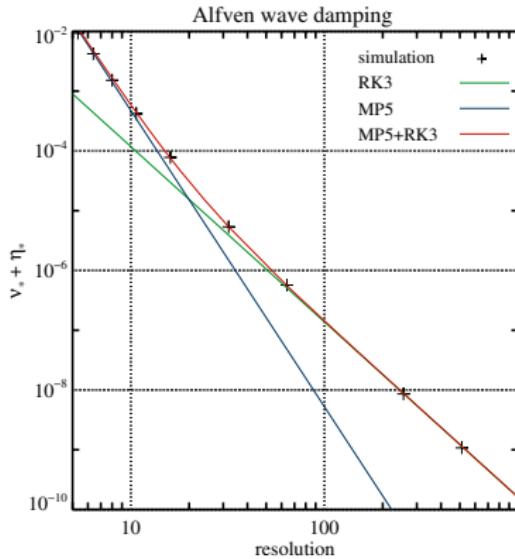
Damping as a function of fast magnetosonic speed

$$c_{\text{ms}} = \left(\frac{1}{2} \left(c_A^2 + c_s^2 + \sqrt{(c_A^2 + c_s^2)^2 - 4c_A^2 c_s^2 \cos^2 \theta} \right) \right)^{1/2},$$

in this case $c_{\text{ms}} = \max(c_s, c_A)$.



Alfvén waves



Damping as a function of grid resolution.

When the spatial and temporal errors are of the same order, we confirm the additive nature.



Magnetosonic waves

Initial conditions

$$\rho_0 = 0, P_0 = 1, b_{0y} = 1, b_{0x} = 0$$

$$v_{x1}(x, 0) = \epsilon \sin(k_x x),$$

$$v_{y1}(x, 0) = v_{x1} \frac{k_x^2 b_{0x} b_{0y}}{b_{0x}^2 k_x^2 - \rho_0 \omega^2},$$

$$b_{y1}(x, 0) = v_{x1} \frac{k_x b_{0y} \omega \rho_0}{\rho_0 \omega^2 - b_{0x}^2 k_x^2},$$

$$\rho_1(x, 0) = v_{x1} \frac{k_x \rho_0}{\omega},$$

$$e_1(x, 0) = v_{x1} \frac{k_x p_0 \Gamma}{\omega(\Gamma - 1)},$$

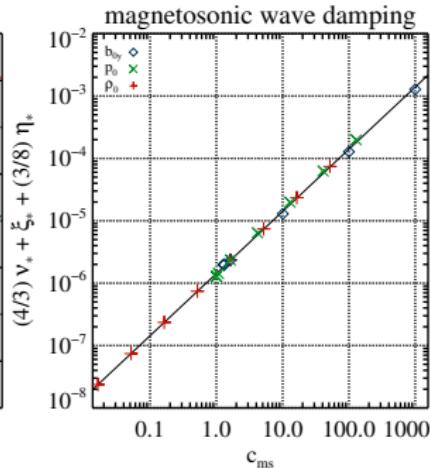
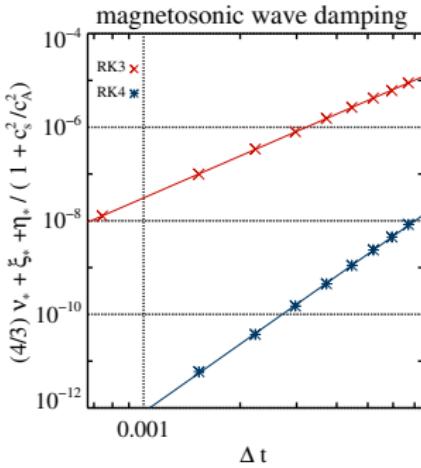
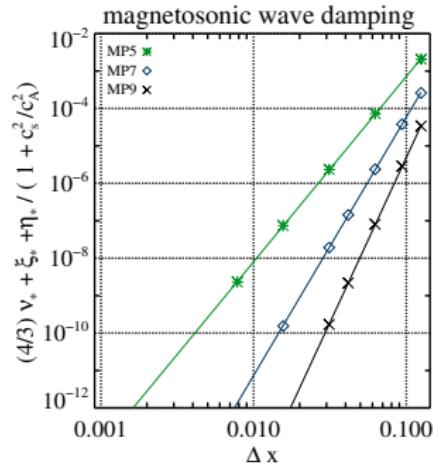
$$v_x(x, t) = \epsilon e^{-\mathfrak{D}_{\text{ms}} t} e^{ik(x \mp c_{\text{ms}} t)}$$

with a damping rate

$$\mathfrak{D}_{\text{ms}} = \frac{k^2}{2} \left(\frac{4}{3} \nu + \xi + \frac{\eta}{1 + c_s^2/c_A^2} \right).$$



Magnetosonic waves



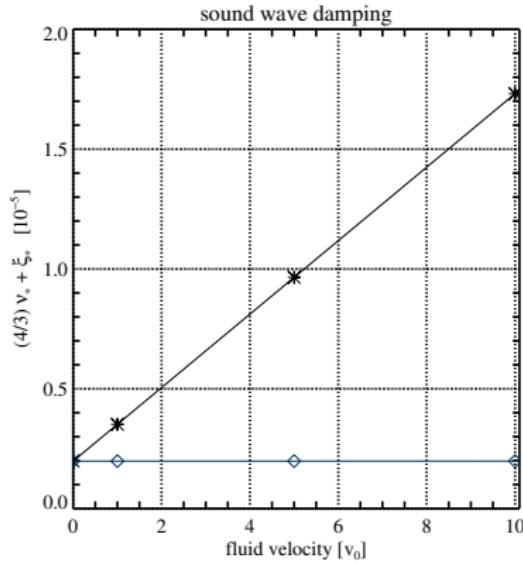
Damping scales with numerical parameters as for the other tests. \mathcal{V} is the fast magnetosonic speed.



Boosted sound waves

To further check \mathcal{V} , we add a uniform velocity to the sound wave test, either parallel or perpendicular to the wave propagation.

→ the relevant \mathcal{V} is the fastest characteristic velocity in the parallel direction.



Damping of sound waves as a function of parallel (black asterisks) or perpendicular (blue diamonds) advection velocity.



Sound waves in 2d

Initial conditions

like the sound waves in 1d, but on a grid of size $L_x \times L_y = 1 \times L_y$ with periodic BCs in both directions (which fixes the wave vector)



Sound waves in 2d

Numerical viscosity in multi-D

We describe the viscosity as a tensor. Damping rate is now

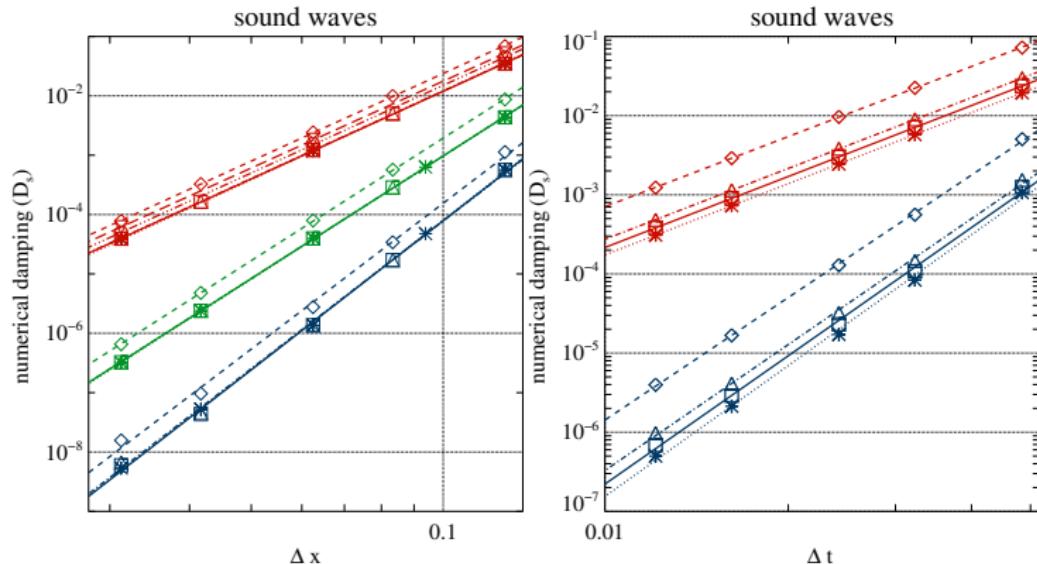
$$\begin{aligned}\mathfrak{D}_s &= \frac{1}{2} \mathbf{k}^T \left(\frac{4}{3} \nu_* + \xi_* \right) \mathbf{k} = \\ &= \frac{1}{2} \left[k_x^2 \left(\frac{4}{3} \nu_{\text{sp}}^{xx} + \xi_{\text{sp}}^{xx} \right) + k_y^2 \left(\frac{4}{3} \nu_{\text{sp}}^{yy} + \xi_{\text{sp}}^{yy} \right) + k^2 \left(\frac{4}{3} \nu_t + \xi_t \right) \right].\end{aligned}$$

In our setup (uniform grid, $L_x = 1$)

$$\begin{aligned}\mathfrak{D}_s &= 2\pi^2 c_s \left\{ \left[(4/3) \mathfrak{N}_\nu^{\Delta x} + \mathfrak{N}_\xi^{\Delta x} \right] [(\Delta x)^r (1 + L_y^{-r-1})] \right. \\ &\quad \left. + \left[(4/3) \mathfrak{N}_\nu^{\Delta t} + \mathfrak{N}_\xi^{\Delta t} \right] \left[(1 + L_y^{-2})^{(1+q)/2} C_{\text{CFL}}^q (\Delta x)^q \right] \right\}.\end{aligned}$$



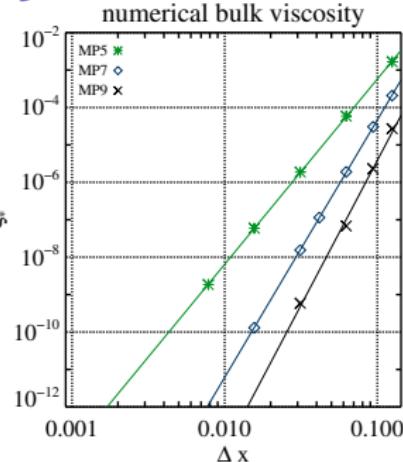
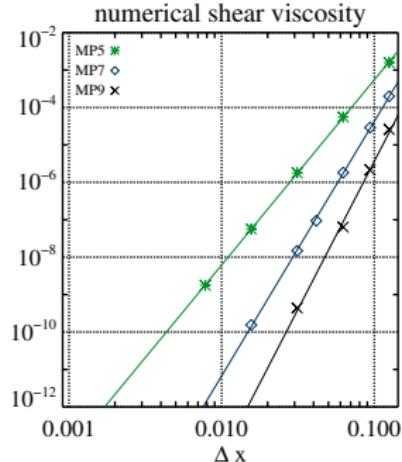
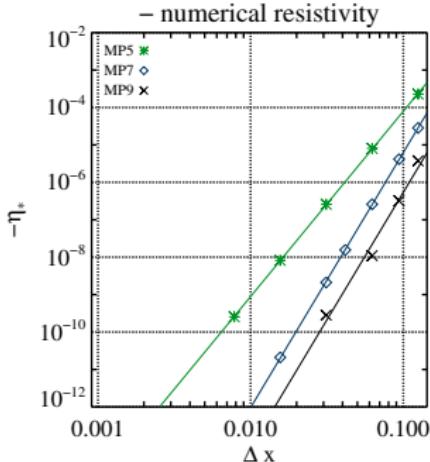
Sound waves in 2d



- ▶ consistent with 1d
- ▶ depending on propagation direction can be less diffusive than 1d
(explicable within theory, though surprising)



Numerical viscosities and resistivity



- ▶ The three sets of tests probe different combinations of ν, ξ, η
- solve for them.
- ▶ surprising result: $\eta < 0$
- ▶ $|\eta| \ll \nu \rightarrow$ most likely unmeasurably small resistivity



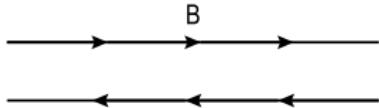
Tearing modes: basics

$$\nabla \cdot \mathbf{B} = 0$$

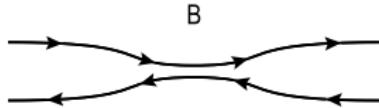
$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

The resistive term spoils: *flux freezing* and *conservation of flux*.

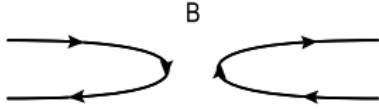
a)



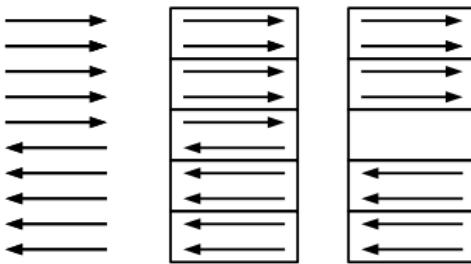
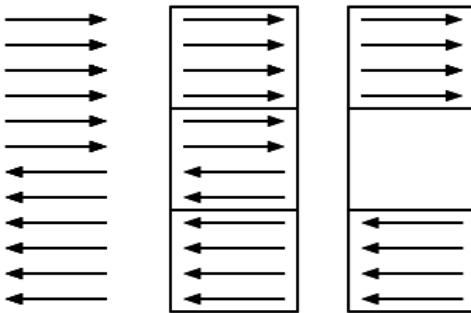
b)



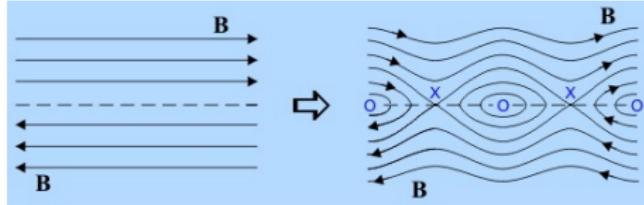
c)



Tearing modes: basics



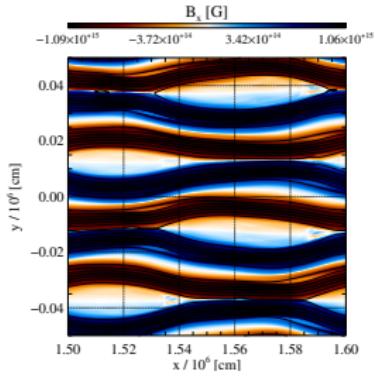
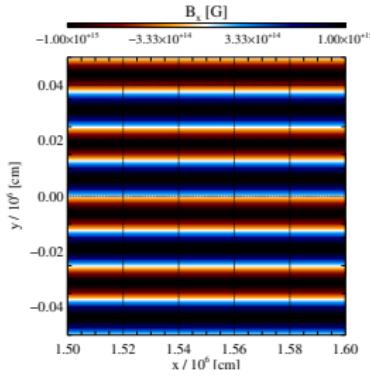
Tearing modes: basics



- ▶ reconnects magnetic field lines
- ▶ its growth-rate

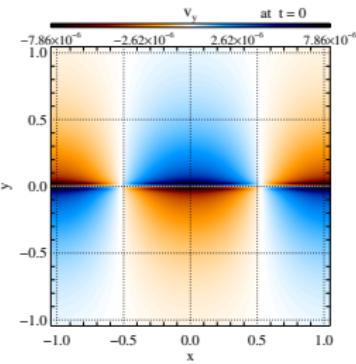
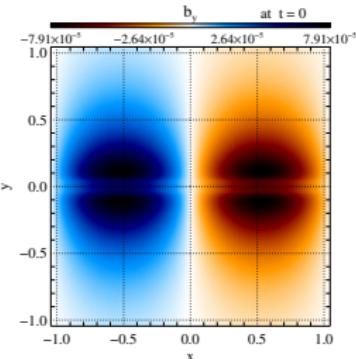
$$\Gamma_{\text{TM}} \propto \eta^{4/5} \nu^{-1/5} B^{2/5}$$

(Furth, Killeen & Rosenbluth, 1963)



Tearing modes: simulations

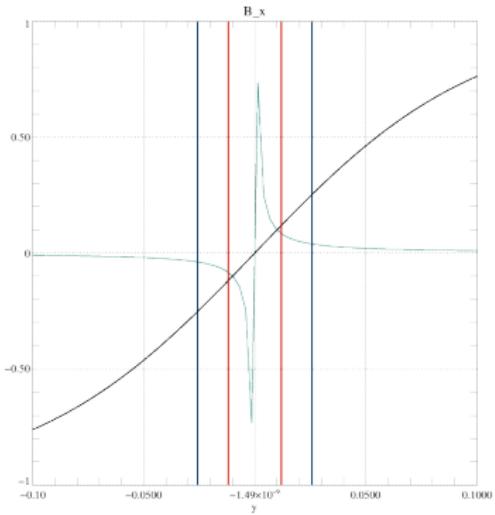
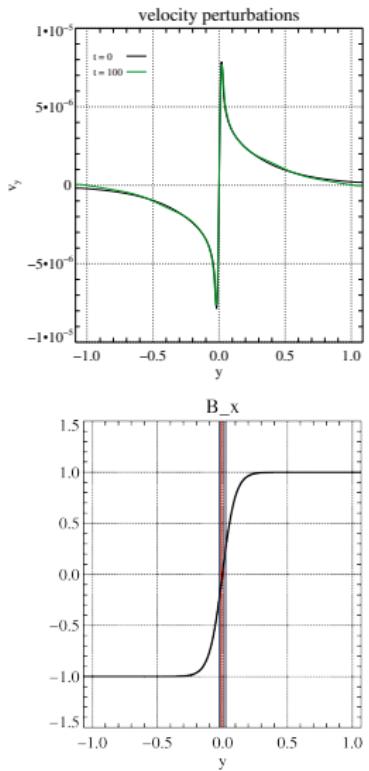
- ▶ uniform density
- ▶ medium at rest
- ▶ $b_{0x} = b_0 \tanh(y/a)$
- ▶ balance the gradient of magnetic pressure by gas pressure or by the z-component of the field
- ▶ perturbation setup as eigenmodes of the system



initial b_y and v_y



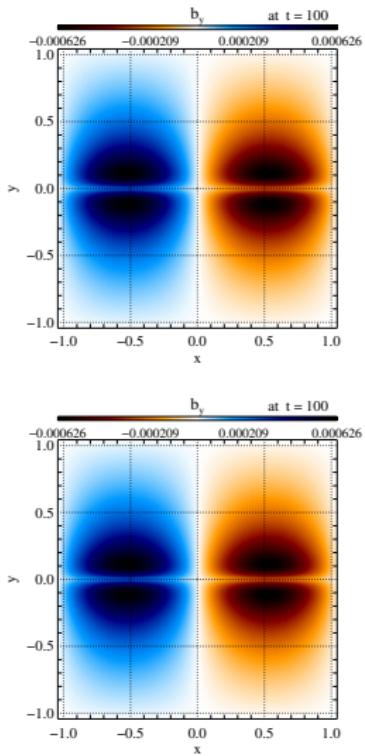
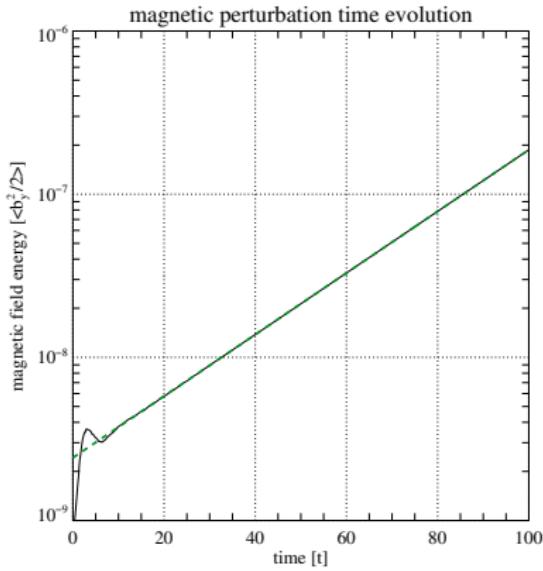
Tearing modes: simulations



Resistive-viscous layer
 $L_{\epsilon_{RV}}(\eta, \nu) \approx 2\epsilon_{RV}$ where
 $\epsilon_{RV} = (\eta\nu)^{1/6} \left(\frac{\sqrt{\rho_0}\delta}{b_0 k} \right)^{1/3}$



Tearing modes: simulations

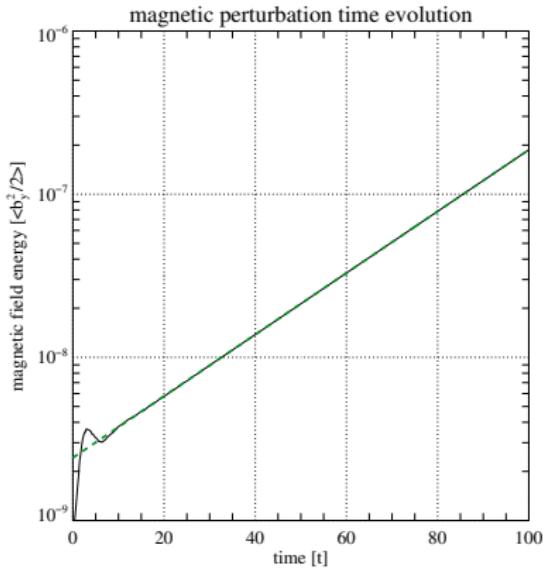


$$b_{1y}(x, y, t) = b_1(y) e^{ikx + \gamma t},$$
$$\gamma \propto \eta^{5/6} \nu^{-1/6}$$

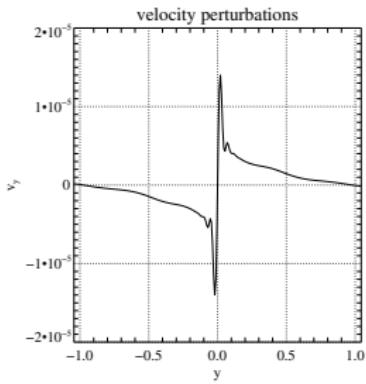
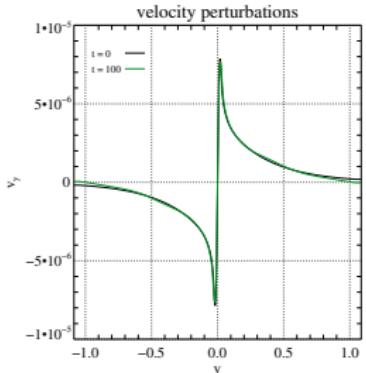
At $t = 100$: same form



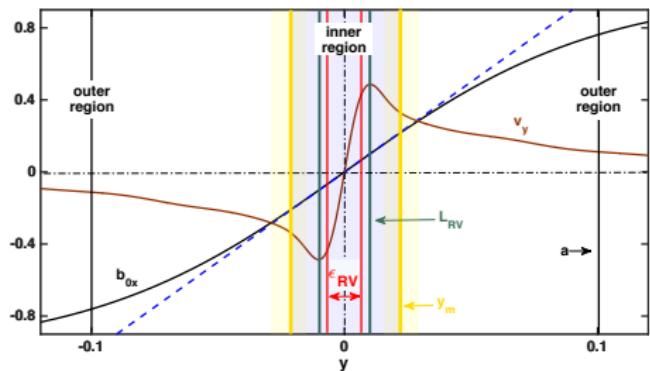
Tearing modes: simulations



$$b_{1y}(x, y, t) = b_1(y) e^{ikx + \gamma t},$$
$$\gamma \propto \eta^{5/6} \nu^{-1/6}$$



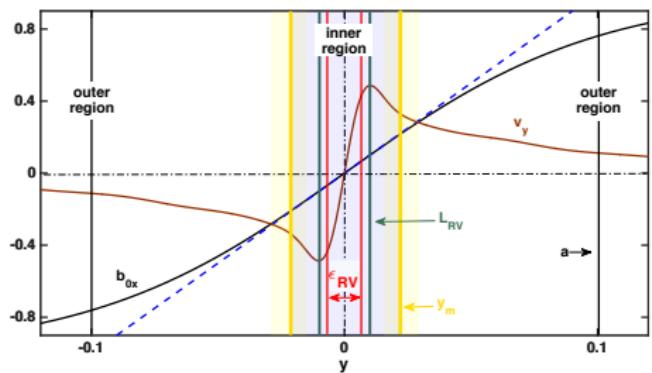
Theoretical analysis



- ▶ WKB ansatz
- ▶ take into account visco-resistive effects only near the current sheet, in a layer of width ϵ_{RV}
- ▶ outside ideal MHD



Theoretical analysis



Requirements

- ▶ diffusion of the current sheet slow compared to TM growth:
 $a^2/\eta \gg \gamma^{-1}$
- ▶ Alfvén crossing through the RV layer short: $a/c_{AA} \ll \gamma^{-1}$
→ $\frac{a^2}{\eta} \gg \gamma^{-1} \gg \frac{a}{c_A}$

Serious restriction of the parameter range in which we can use our simulations.



Theoretical analysis

- ▶ $P_m \gtrsim 1$ (left of blue line)

▶

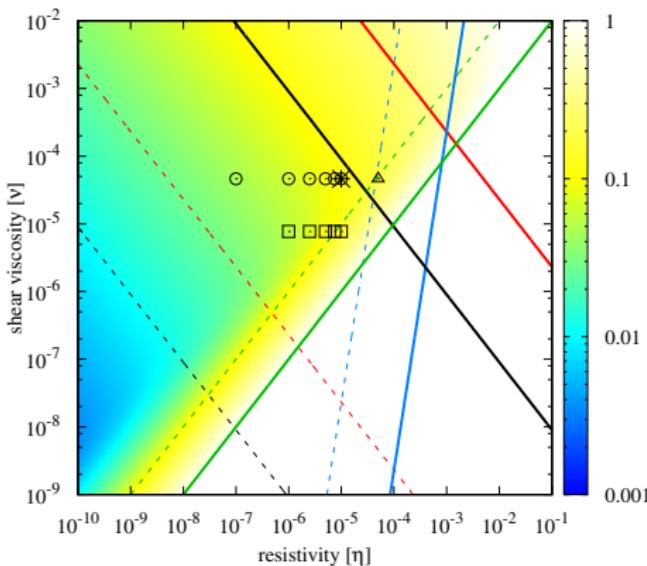
$$\mathcal{C}_1 \equiv \gamma a^2 \eta^{-1} \gg 1,$$

$$\mathcal{C}_2 \equiv a^{-1} c_A \gamma^{-1} \gg 1,$$

$$\mathcal{C}_3 \equiv a L_{RV}^{-1} \gg 1,$$

The colour scale shows a combination of the \mathcal{C}_i and P_m . The smaller, the better.

- ▶ However: resolution limits our models from reaching the best sets of parameters.

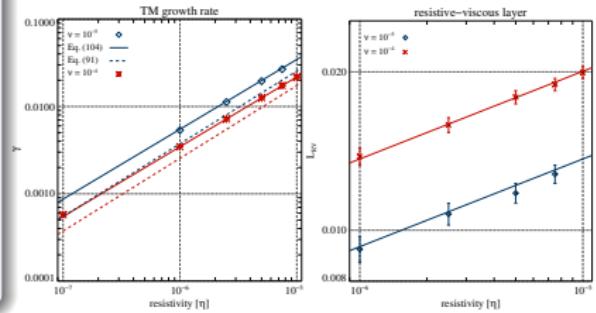


Determining numerical resistivities

Basic idea

- ▶ both growth rate and width of resistive-viscous layer depend on η and ν (checked in simulations with physical viscosity and resistivity)
- ▶ measure γ and ϵ_{RV} as functions of numerical and physical parameters
- ▶ fit the corresponding relations

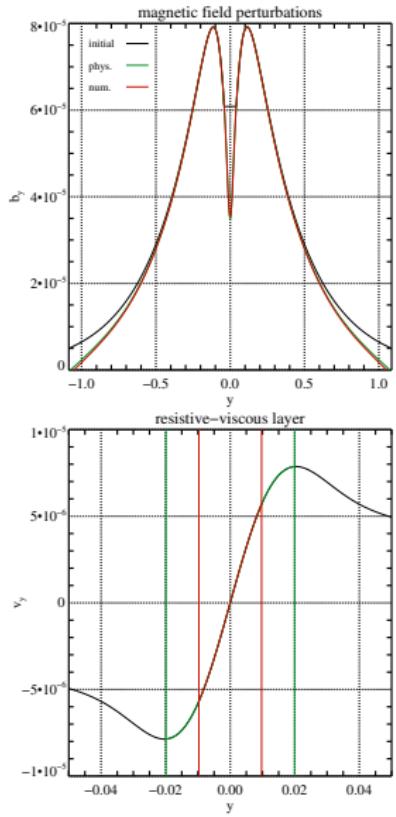
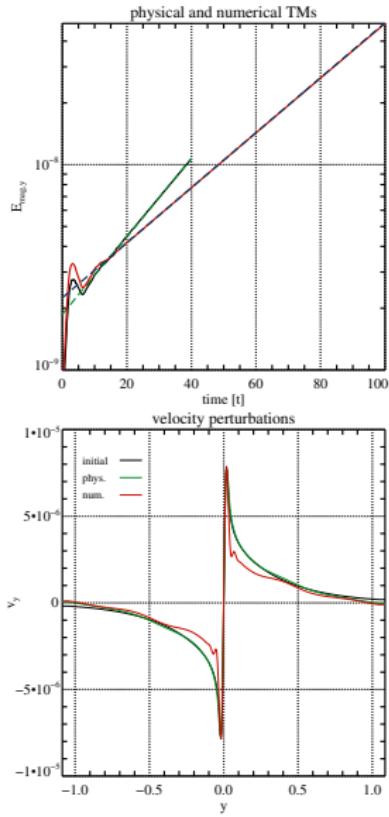
$$\begin{aligned} & \gamma \propto \eta^{5/6} \nu^{-1/6} \\ & L_{\epsilon_{RV}}(\eta, \nu) \approx 2\epsilon_{RV} \text{ with} \\ & \epsilon_{RV} = (\eta\nu)^{1/6} \left(\frac{\sqrt{\rho_0}\delta}{b_0 k} \right)^{1/3} \end{aligned}$$



TM growth rate and width of RV layer

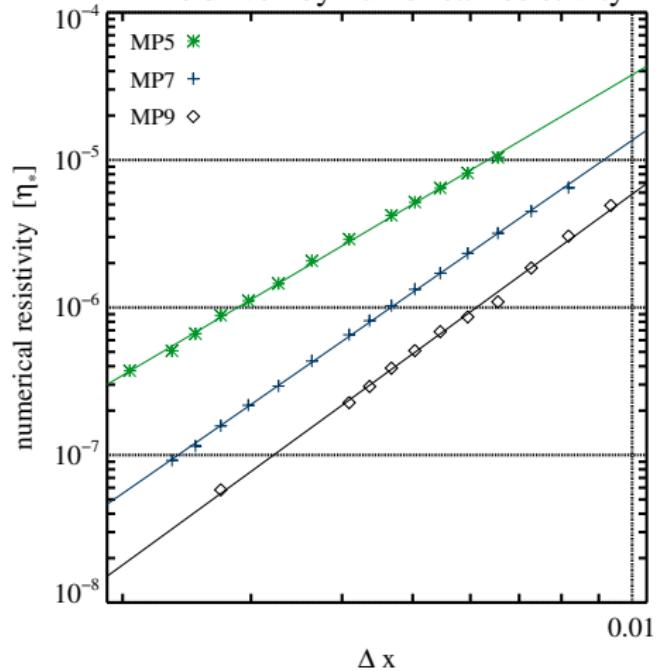


Determining numerical resistivities



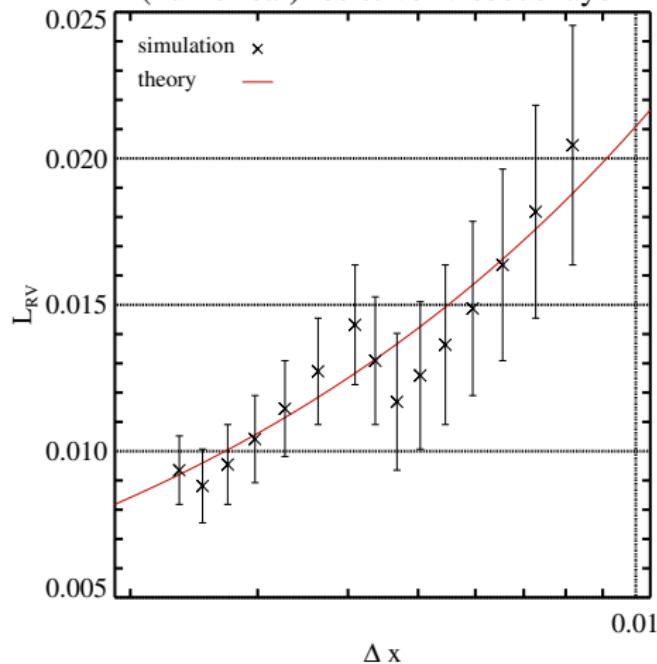
Numerical diffusion coefficients

TMs driven by numerical resistivity



As a function of Δx

(numerical) resistive–viscous layer

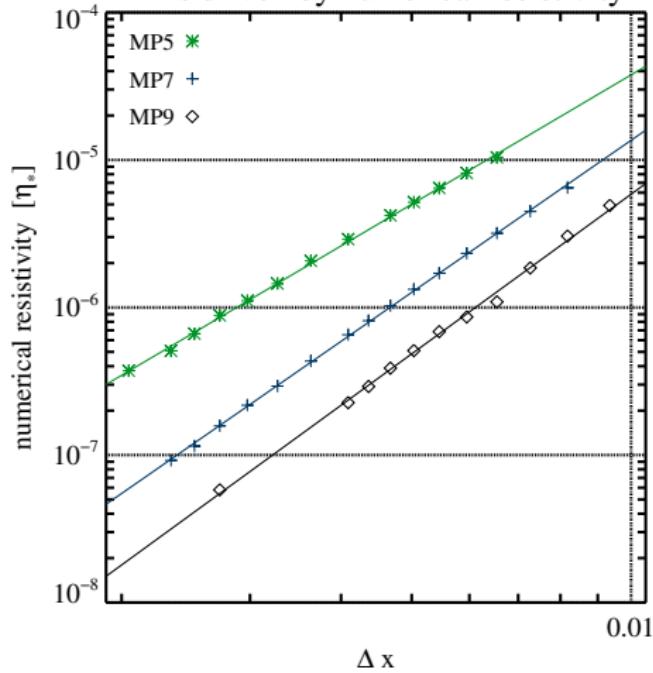


cross-check: RV layer



Numerical diffusion coefficients

TMs driven by numerical resistivity



As a function of Δx

- reconstruction schemes scale with less than their nominal order (exponents less than 3!)
- reason:

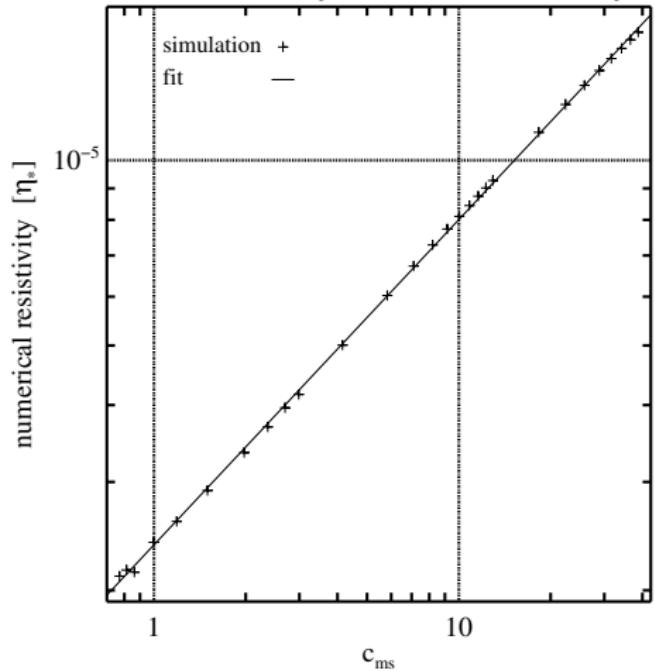
$$\eta_* = \mathfrak{M}_\eta^{\Delta x} \mathcal{V} \mathcal{L} \left(\frac{\Delta x}{\mathcal{L}} \right)^r,$$

- but \mathcal{L} is not constant (such as a)
- instead: $\mathcal{L} \propto \epsilon_{RV}$
- which in turn depends on η (lower $\eta \rightarrow$ more concentrated reconnection)

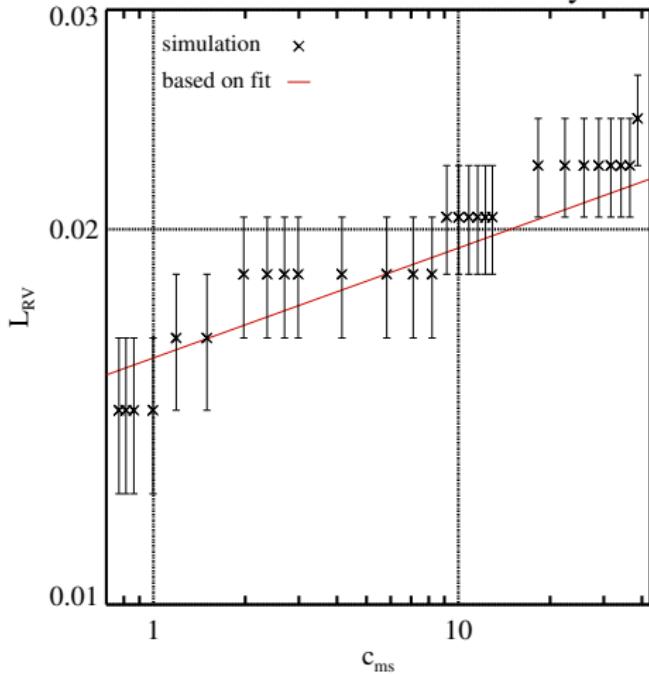


Numerical diffusion coefficients

TMs driven by numerical resistivity



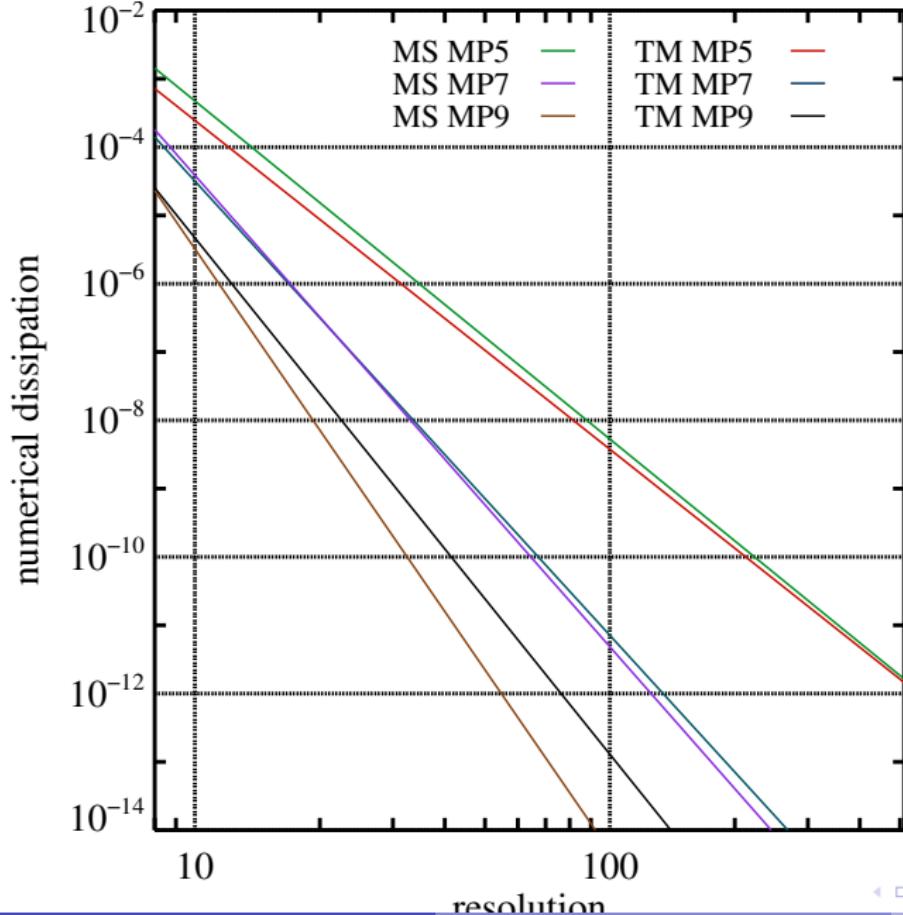
numerical resistive–viscous layer



\mathcal{V} = fast magnetosonic speed



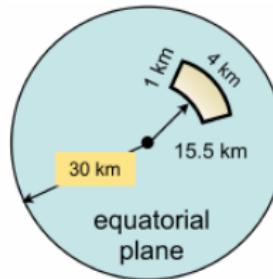
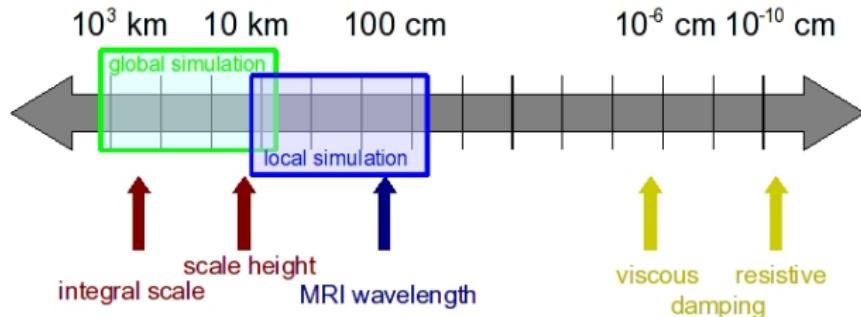
Numerical diffusion from waves and TMs



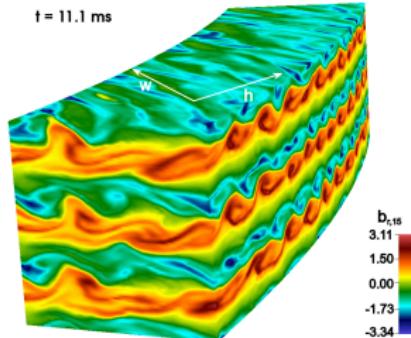
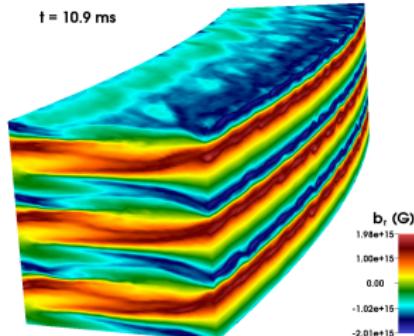
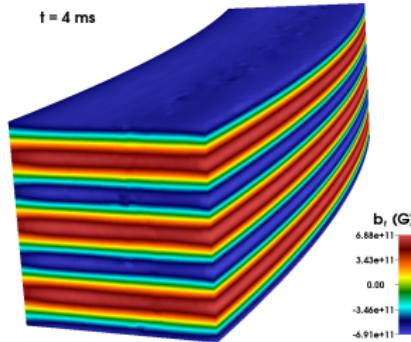
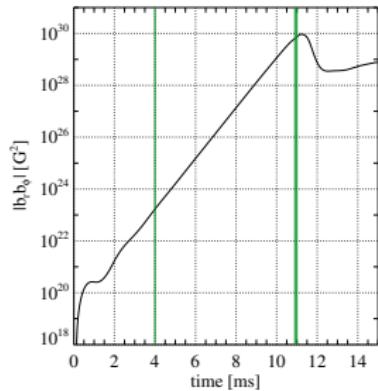
Consistency between
the two types of
problems (accounting
for the scalings with \mathcal{V}
and \mathcal{L})
→ indicating the
applicability of the
ansatz in quite different
settings



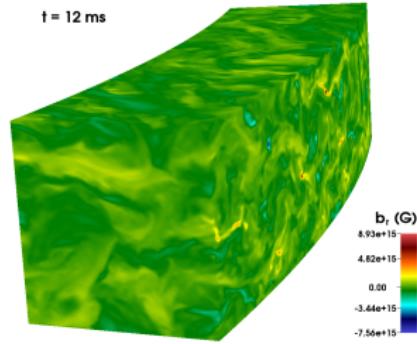
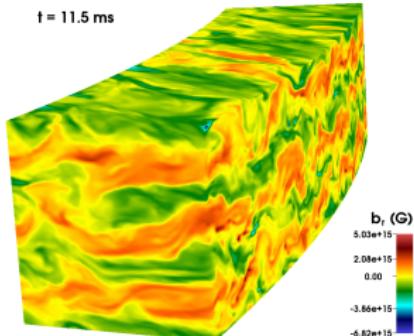
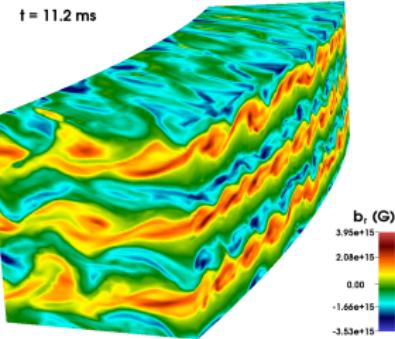
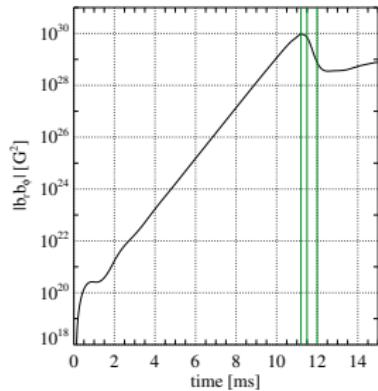
Astrophysical Application: MRI in CCSNe



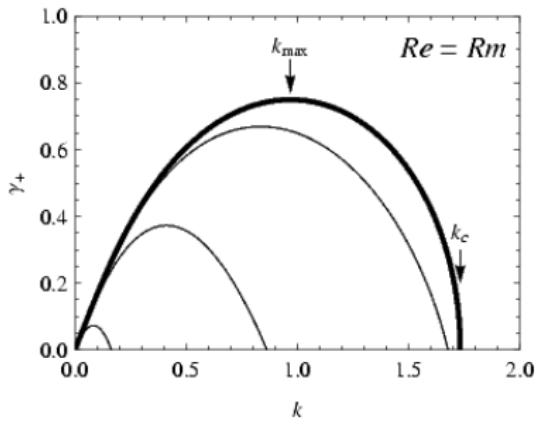
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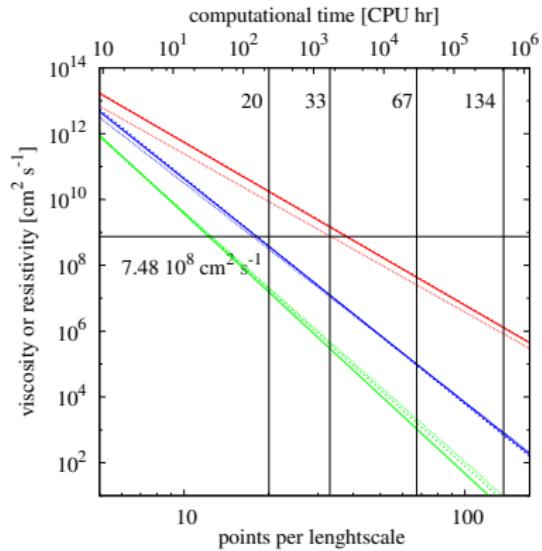


MRI in (numerically) resistive-viscous MHD



$$R_e = R_m = 0.1, 1, 10, \infty$$

From Pessah & Chan (2008)



$$\text{Our goal } R_e^*, R_m^* \leq 100$$



MRI in (numerically) resistive-viscous MHD

name	reco.	reso.	box	$\gamma_{\text{MRI}} [\text{s}^{-1}]$	$\mathcal{M}_{r\phi}^{\text{term}}$ [10^{30} G^2]
PLM-8	PLM	8	s	926	2.2
PLM-10	PLM	10	s	959	2.3
PLM-16	PLM	16	s	1089	1.9
PLM-20	PLM	20	l	1116	1.9
PLM-34	PLM	34	s	1123	1.8
MP5-8	MP5	8	s	1093	1.1
MP5-10	MP5	10	s	1104	1.4
MP5-16	MP5	16	s	1127	1.05
MP5-20	MP5	20	s	1133	0.82
MP5-34	MP5	34	s	1127	1.03
MP9-8	MP9	8	s	1104	0.79
MP9-10	MP9	10	s	1122	1.3
MP9-16	MP9	16	s	1130	1.1
MP9-20	MP9	20	l	1126	1.1
MP9-25	MP9	25	l	1127	1.0
MP9-34	MP9	34	s	1127	0.93
MP9-67	MP9	67	l	1127	0.73
MP9-134	MP9	134	s	1128	0.73

From TR, M. Obergaulinger, P. Cerdá-Durán,
M.A. Aloy & E. Müller (2016JPhCS.719a2009R)



Summary

- ▶ numerical errors behave like resistivity and diffusivity
 - ▶ total error is the sum of temporal and spatial contributions,
 - ▶ each of which scales with $\text{resolution}^{\text{order}} \times \mathcal{V}\mathcal{L}$
 - ▶ i.e., diffusion coefficient is a function of numerical and physical properties
 - ▶ tested the ansatz in simple waves and tearing modes
 - ▶ the former are rather straightforward
 - ▶ the latter reveal possible difficulties in determining the typical velocity and length of a system
- application to other problems possible, but not always an easy task.

