Numerical PDEs and adaptivity on general meshes

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EPSRC

LONDON MATHEMATICAL SOCIETY EST. 1865

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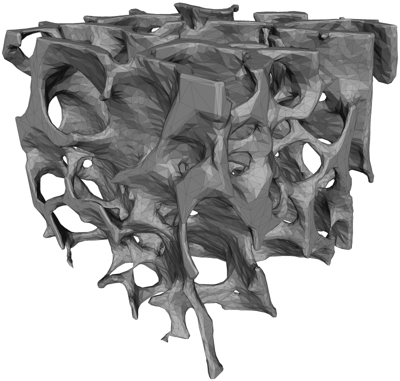
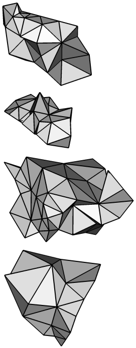
General meshes motivation

Dealing with complicated geometries: linear elastic analysis of trabecular (spongy) bone.



Initial geometry made of 1,179,569 tetrahedral elements.



Sample elements from agglomerated mesh made of 8000 elements.

[C, Dong, Georgoulis, Houston, Springer Briefs, 2017]

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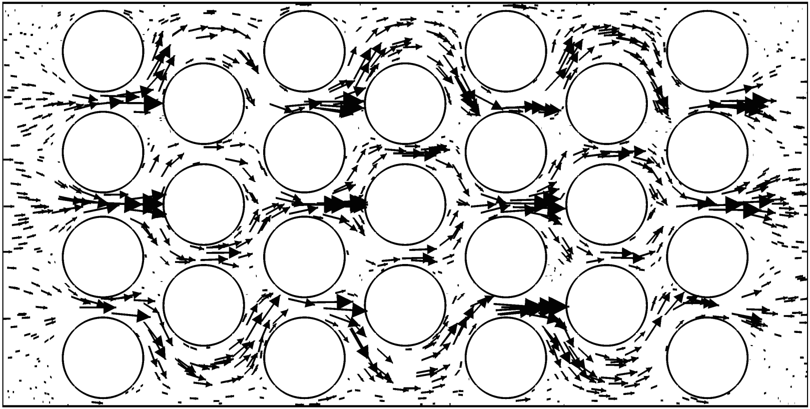
General meshes motivation

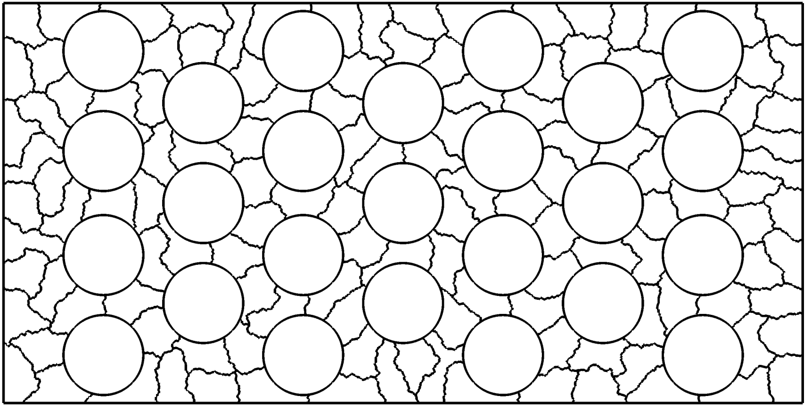
Complicated geometries & adaptivity: interstitial Flow Modelling related to interstitial drug transport to cancer cells



Transport field by incompressible Navier-Stokes.

General meshes motivation

Complicated geometries & adaptivity: interstitial Flow Modelling related to interstitial drug transport to cancer cells



Initial agglomerated mesh consisting of 128 elements

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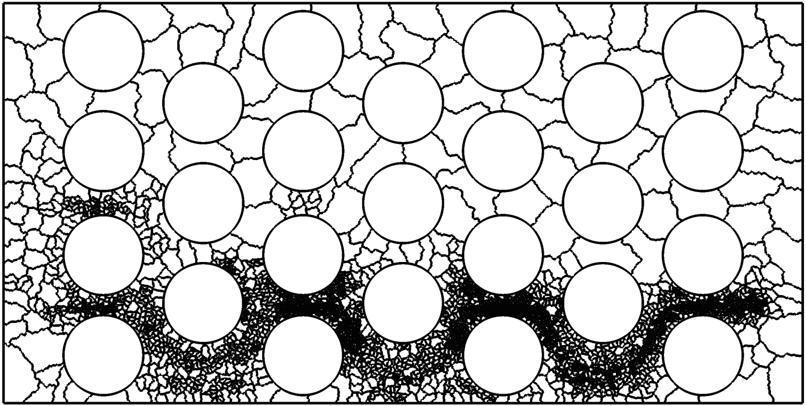
General meshes motivation

Complicated geometries & adaptivity: interstitial Flow Modelling related to interstitial drug transport to cancer cells



Goal oriented adaptivity by P. Houston (Nottingham).

[C, Dong, Georgoulis, Houston, Springer Briefs, 2017]

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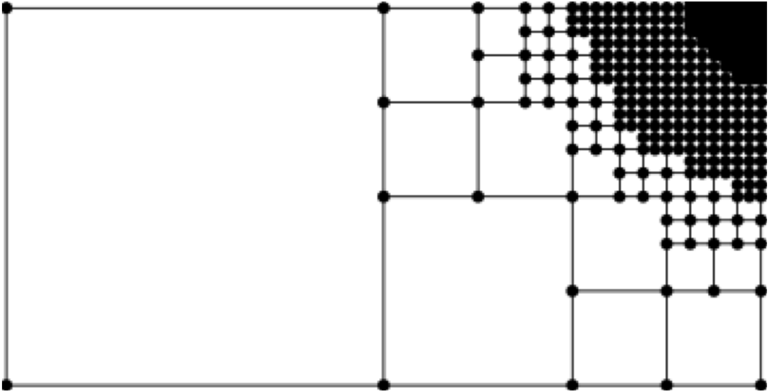
General meshes motivation

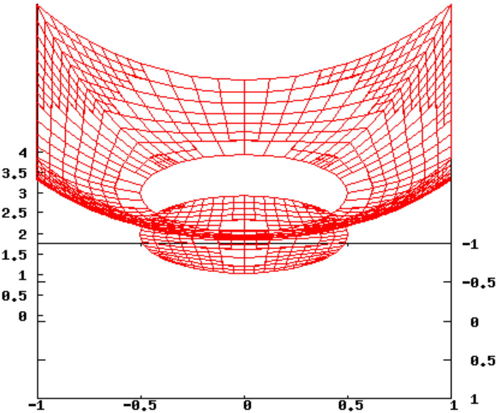
Mesh refinement & coarsening is trivial and fully local



$\rightarrow$ Hanging nodes a thing of the past!

General meshes motivation

Fitted discretisation of curved boundaries: interface diffusion.



Solution adapted mesh by Y. Sabawi (Iraq).

[C, Georgoulis, Sabawi, Math. Comp., 2017.]

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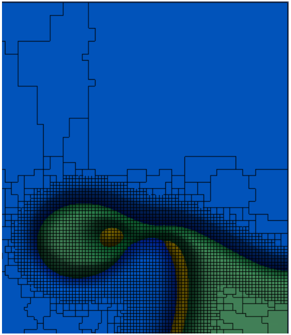
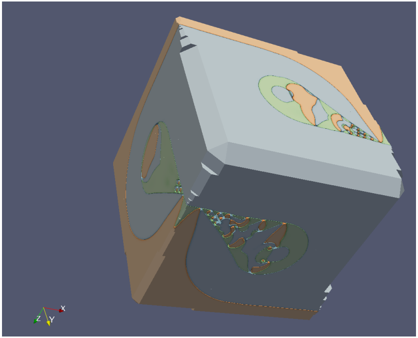
General meshes motivation

Aggressively adapted meshes: truly solution-adapted meshes





Cyclic competition reaction-diffusion system.
[Sutton, PhD Thesis, Leicester 2017]

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C^0-conforming polygonal elements

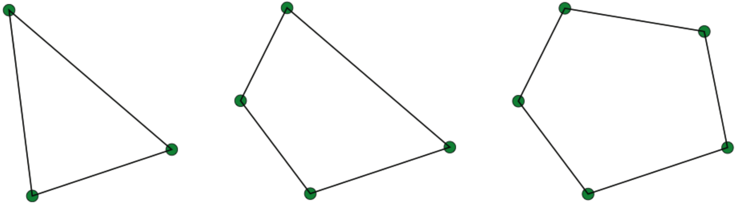
Classical C^0-conforming FEM families ($\mathcal{P}_p$-triangles, $\mathcal{Q}_p$-affine quads) are instances of

Generalised harmonic FE of order p: on an element κ

$$V_h^\kappa := \{v \in H^1(\kappa) : \Delta v \in \mathcal{P}_{p-2}(\kappa); \quad v|_{\partial\kappa} \in C^0(\partial\kappa) \quad \text{and} \quad v|_e \in \mathcal{P}_p(e), \forall e \in \partial\kappa\}$$

where $\mathcal{P}_{-1} := \{0\}$, hence $p = 1$ gives the harmonic FE.

As such, they yield the same element irrespective of the shape:



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where $\mathcal{P}_{-1} := \{0\}$, hence $p = 1$ gives the harmonic FE.

As such, they yield the same element irrespective of the shape:

- 👎 Only known implicitly as solutions of local PDEs
- 👍 Contains $\mathcal{P}_p(\kappa)$, i.e. a space of physical frame polynomials;
- 👍 It is just $\mathcal{P}_p(e)$ on each edge $e \in \partial\kappa$.

Approaches to polygonal FEM

Augmented FE are mentioned as early as Strang & Fix (1973).

Most FEM on polygonal (and polyhedral) meshes play with these ingredients:

1. contains $\mathcal{P}_p(\kappa)$, i.e. a space of physical frame polynomials;
2. is just $\mathcal{P}_p(e)$ on each edge $e \in \partial\kappa$.

Approaches to polygonal FEM

Augmented FE are mentioned as early as Strang & Fix (1973).

Examples of conforming methods[1]

- Composite Finite Elements (CFEs) [Hackbusch & Sauter, Numer. Math., 1997]
- Harmonic FEM & Polygonal FEM [Sukumar &Tabarraei, Int. J. Numer. Methods Eng., 2004]
- BEM-based FEM [Copeland, Langer & Pusch, DDM XVIII, 2009]
- Extended FEM [Fries & Belytschko, Int. J. Numer. Methods Eng., 2010]
- Nodal Mimetic Finite Difference (MFD) [Brezzi, Buffa & Lipnikov, M2AN, 2009]
- Virtual Element Method (VEM) [da Veiga, Brezzi, C, Manzini, Marini & Russo, M3AS, 2013]

[1] Seminal papers: [Babuska & Osborn, SINUM, 1983] and [Babuska & Melenk, CMAME, 1996].

Approaches to polygonal FEM

Augmented FE are mentioned as early as Strang & Fix (1973).

Examples of non-conforming methods

- Mimetic Finite Difference (MFD) [Brezzi, Lipnikov, Shashkov, SINUM, 2005]
- Nonconforming VEM [Ayuso de Dios, Lipnikov & Manzini, M2NA, 2016]
- HDG [Cockburn, Gopalakrishnan & Lazarov, SINUM, 2009]
- Weak Galerkin [Wand & Ye, J. Comput. Appl. Math. 2013]
- Hybrid High-Order (HHO) [Di Pietro & Ern, CMAME, 2015]
- Gradient scheme framework [Droniou, Eymard, Gallouet & Herbin, M3AS, 2013]
- Reconstruction FEM [Georgoulis & Pryer, CMAME, 2018]
- Agglomerated DG [Bassi, L. Botti, A. Colombo, S. Rebay, Comput. Fluids, 2012]
  Composite DG [Antonietti, Giani, Houston, SIAM J. Sci. Comput., 2013]
- *hp*-IPDG [C, Georgoulis & Houston, M3AS, 2014]

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Outline & goals

1. *hp*-version IP-dG methods (*hp*-DGFEM)
   - Extending *hp*-DGFEM to extremely general meshes → including non shape regular elements with degenerating or curved interfaces
   - A posteriori analysis of fitted discretisations on curved domains
2. C^0-conforming Virtual Element Method (VEM)
   - A posteriori error analysis & adaptivity

NOTE:The VEM framework is much more than just polytopic FEM.

Eg. globally C^k, div-free, H(div) and H(curl) conforming, Trefftz.

DRIVING PRINCIPLES

- Computational cost should be comparable to that of standard FEMs
- Allow the use of standard FEM locally.
- Allow flexible mesh adaptation

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Linear PDEs with non-negative characteristic form

Includes elliptic, parabolic, hyperbolic, as well as hypoelliptic and mixed-type PDEs.

On $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, bounded open polyhedral domain[1], consider

$$-\nabla \cdot (A\nabla u) + \mathbf{b} \cdot \nabla u + cu = f$$

with

$$\xi^\top A(x)\xi \geq 0 \quad \forall \xi \in \mathbb{R}^d, \quad \text{a.e.} \quad x \in \bar{\Omega}.$$

Supplemented on $\Gamma = \partial\Omega$ with

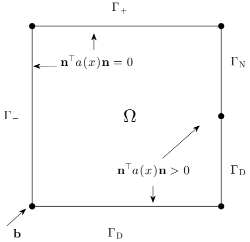
$$u = g_{\mathrm{D}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \cup \Gamma_-,$$

$$\mathbf{n} \cdot (A\nabla u) = g_{\mathrm{N}} \quad \text{on} \quad \Gamma_{\mathrm{N}}.$$

where

$$\Gamma_0 = \left\{ x \in \Gamma : \mathbf{n}(x)^\top a(x)\mathbf{n}(x) > 0 \right\} =: \Gamma_D \cup \Gamma_N$$

$$\Gamma_- = \left\{ x \in \Gamma \backslash \Gamma_0 : \mathbf{b}(x) \cdot \mathbf{n}(x) < 0 \right\}, \quad \Gamma_+ = \Gamma \setminus \Gamma_0 \cup \Gamma_-$$



[1]Later: Curved boundaries/multi-compartment problems.

Physical frame hp-DGFEM(p) on polytopic meshes

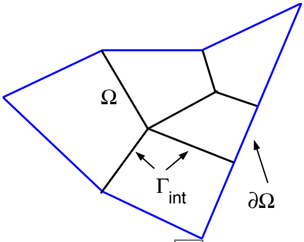
On meshes $\mathcal{T}_h$ made of non-overlapping polygons/polyhedra, set[2]

$$V_h^{\mathbf{p}} := \{v \in L^2(\Omega) : \quad v|_\kappa \in \mathcal{P}_{p_\kappa}(\kappa), \quad \forall \kappa \in \mathcal{T}_h\}$$

- Local space independent of element shape;
- dG space with minimal number of degrees of freedom per element.

(IP-)hp-DGFEM(p) method: Find $u_h \in V_h^{\mathbf{p}}$ such that

$$B(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h^{\mathbf{p}}$$



[2] $\mathcal{P}_p(\kappa)$ space of polynomials of total degree up to p.

Physical frame hp-DGFEM(p) on polytopic meshes

$B(u,v) := B_{\mathrm{d}}(u,v) + B_{\mathrm{ar}}(u,v)$ and

$$B_{\mathrm{d}}(u,v) := \sum_{\kappa\in\mathcal{T}} \int_\kappa A\nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\Gamma_{\mathrm{int}}\cup\Gamma_{\mathrm{D}}} (\{A\nabla u\cdot n\}\cdot[v] + \{A\nabla v\cdot n\}\cdot[u] - \sigma[u][v])\,\mathrm{d}s$$

$$B_{\mathrm{ar}}(u,v) := \sum_{\kappa\in\mathcal{T}} \int_\kappa (\boldsymbol{b}\cdot\nabla u + cu)v\,\mathrm{d}x - \sum_{\kappa\in\mathcal{T}} \int_{\partial_-\kappa\setminus\Gamma_{\mathrm{N}}} (\boldsymbol{b}\cdot\mathbf{n})\lfloor u\rfloor v^+\,\mathrm{d}s$$

where

$$\{u\}|_{\partial\kappa_i\cap\partial\kappa_j} = \frac{u_{\kappa_i}+u_{\kappa_j}}{2} \qquad [u]|_{\partial\kappa_i\cap\partial\kappa_j} = u_{\kappa_i}\mathbf{n}_{\kappa_i} + u_{\kappa_j}\mathbf{n}_{\kappa_j} \qquad \lfloor u\rfloor|_{\partial\kappa} = u^+ - u^-$$

with u^-, u^+ upwind and downwind values[3] and

σ a interior discontinuity-penalization (IP-) parameter.

Finally,

$$\ell(v) := \sum_{\kappa\in\mathcal{T}} \int_\kappa fv\,\mathrm{d}x + (\text{boundary terms}).$$

[3] Here, $u^- := 0$ if $\partial_-\kappa \subset \Gamma$.

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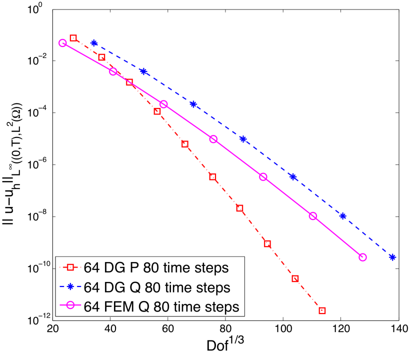
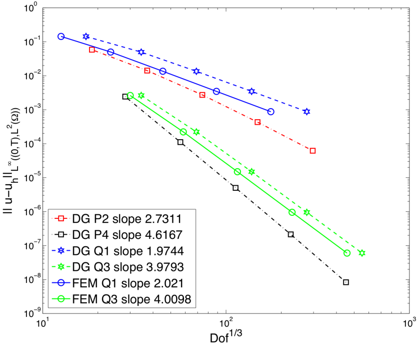
Heat equation in 2D: square mesh comparison

Forcing chosen so that $u(t, x, y) = \sin(20\pi t)e^{(-5(x-0.5)^2-5(y-0.5)^2)}$.

p-convergence



h-convergence ($dt \equiv h/10$)



'DG Q' and 'FEM Q': DG and conforming FEM tensor-product elements in space, with DG time-stepping.

IP-parameter and stability

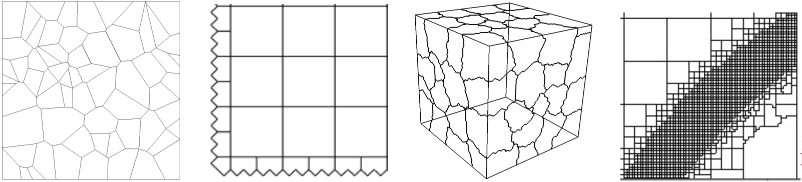
DG-norm: $|||w|||^2 := |||w|||_{\mathrm{ar}}^2 + |||w|||_{\mathrm{d}}^2$ with

$$|||w|||_{\mathrm{d}} := \Big(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla w|^2 \mathrm{d}\mathbf{x} + \int_{\Gamma \setminus \Gamma_{\mathrm{N}}} \sigma |[w]|^2 \mathrm{d}s \Big)^{1/2},$$

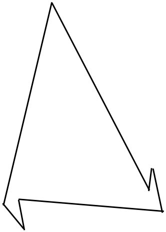
Stability analysis and IP-parameter choice of *hp*-DGFEM depends on the inverse estimates from every face $f \in \partial\kappa$ into κ.

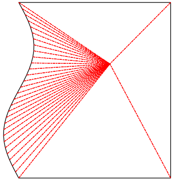
For κ simplex/hexahedron: $$\|v\|_f^2 \leq C \frac{|f| p^2}{|\kappa|} \|v\|_\kappa^2 \qquad \forall v \in \mathcal{P}_p(\kappa).$$

What about this kind of meshes??



Challenges





- Classical *hp*-inverse estimates not sharp in presence of arbitrarily small, degenerating $(d-k)$-dim interfaces, $k = 1, \ldots, d-1$.

  $\rightarrow$ new sharp *hp*-inverse estimates.

- No sharp *hp*-approximation results for L_2-projector (key for first order terms Houston, Schwab & Süli ('02)) over polytopic meshes.

  $\rightarrow$ error analysis via *hp*-inf-sup stability on stronger norms.

  à la Johnson & Pitkäranta ('86), Buffa, Hughes & Sangalli ('06), Ayuso & Marini ('09), C., Chapman, Georgoulis, & Jensen ('13).

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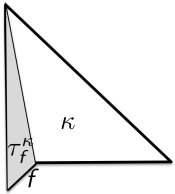
Inverse estimates for arbitrarily small interfaces

Inverse estimates for d-simplexes/d-hexahedra. For each face $f \subset \partial\kappa$

$$\|v\|_f^2 \leq C \frac{|f| p^2}{|\kappa|} \|v\|_\kappa^2 \qquad \forall v \in \mathcal{P}_p(\kappa).$$

For each element $\kappa \in \mathcal{T}$, define the family $\mathcal{F}_f^\kappa$ of all simplices contained in κ and having f as one of their faces.



Inverse estimate for polytopes. For $v \in \mathcal{P}_p(\kappa)$, we have

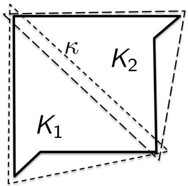
$$\|v\|_f^2 \leq C_{\mathrm{inv}} \frac{p^2 |f|}{|\tau_f^\kappa|} \|v\|_{\tau_f^\kappa}^2 \leq C_{\mathrm{inv}} \frac{p^2 |f|}{|\tau_f^\kappa|} \|v\|_\kappa^2, \quad \forall \tau_f^\kappa \in \mathcal{F}_f^\kappa.$$

The first inequality permits arbitrarily small elemental interfaces!

An inverse estimate

We say κ is p-coverable if it can be covered by at most m_κ shape-regular simplexes K_i, $i = 1, \ldots, m_\kappa$, with $|K_i| \geq c_{as}|\kappa|$ and

$$\mathrm{dist}(\kappa, \partial K_i) \lesssim \mathrm{diam}(K_i)/p^2$$



Lemma (C., Georgoulis & Houston, M3AS, 2014)

If κ is p-coverable and $f \subset \partial\kappa$ is one of its faces, then for each $v \in \mathcal{P}_p(\kappa)$

$$\|v\|_f^2 \lesssim C_{\mathrm{INV}}(p, \kappa, f)\frac{p^2|f|}{|\kappa|}\|v\|_\kappa^2,$$

with $C_{\mathrm{INV}}(p, \kappa, f) := \min\left\{\frac{|\kappa|}{\sup_{\tau_f^\kappa \subset \kappa}|\tau_f^\kappa|}, p^{2(d-1)}\right\}.$

This inverse estimate permits arbitrarily small elemental interfaces!
It leads to new jump penalisation parameter.

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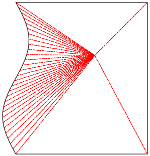
hp-DGFEM(p) a priori analysis

Inverse estimate permits to generalise a priori hp-error analysis [Houston, Schwab & Süli, SINUM,2002] to meshes with

- possibly degenerating interfaces
- no shape regularity assumptions,
- uniformly bounded number of faces

Alternative assumption: allowing for arbitrary number of faces. For all κ, for all $f \in \partial\kappa$ there exists $\tau_f^\kappa \in \mathcal{F}_f^\kappa$:

$$h_\kappa \leq C_s \frac{d|\tau_f^\kappa|}{|f|}.$$



Lemma (C., Dong & Georgoulis, SIAM J. Sci. Comput., 2014)

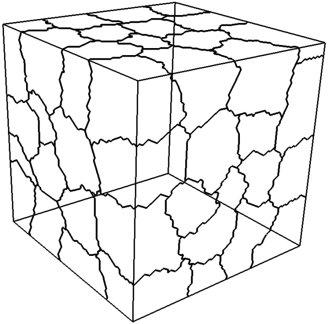
Under the above assumption, for each $v \in \mathcal{P}_p(\kappa)$,

$$\|v\|^2_{\partial\kappa} \leq C_s C_{\mathrm{inv}} d \frac{p^2}{h_\kappa} \|v\|^2_\kappa.$$

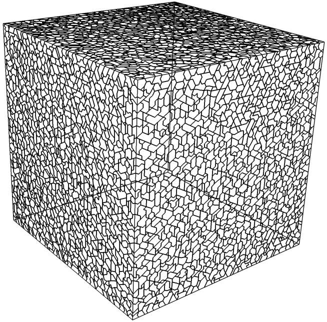
Transport problem in 3D with known solution

$\Omega = (0,1)^3, \quad A \equiv 0, \quad \mathbf{b} = (-y, z, x), \quad c = xy^2 z$

Forcing chosen so that $u(x,y,z) = 1 + \sin(\pi x y^2 z/8)$.



64 agglomerated elements



32768 agglomerated elements

Transport problem in 3D

Agglomerated mesh 'DGFEM' vs square mesh 'DGFEM(P)'

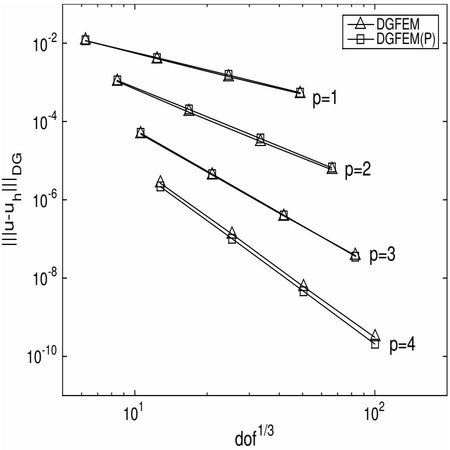


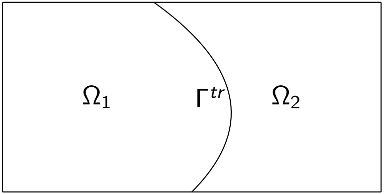
Figure: Convergence under h–refinement for uniform $p = 1, 2, 3, 4$.

Multidomain pbm with flux-balancing interface conditions
Eg. modelling semipermeable membranes

$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^{tr}$,

$\Gamma^{tr} := (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$ Lipschitz

P permeability function



$$
\begin{cases}
u_t - \Delta u = f, & \text{in } (0, T] \times \Omega_1 \cup \Omega_2 \\
u = u_0 & \text{in} \{0\} \times \Omega_1 \cup \Omega_2 \\
u = 0, & \text{on } (0, T] \times \partial\Omega \\
\mathbf{n}^1 \cdot \nabla u_1 = P(u)(u_2 - u_1)|_{\Omega_1} & \text{on } (0, T] \times \bar{\Omega}_1 \cap \Gamma^{tr} \\
\mathbf{n}^2 \cdot \nabla u_2 = P(u)(u_1 - u_2)|_{\Omega_2} & \text{on } (0, T] \times \bar{\Omega}_2 \cap \Gamma^{tr}
\end{cases}
$$

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Curved interface problems

Unfitted mesh approaches

- A number of very successful methods available (unfitted FEM, immersed interface, fictitious domain, composite FE, cut-cell, ...).
- Using PDE stability linking error with the residual is cumbersome $\Rightarrow$ Energy norm a posteriori analysis difficult[4]!

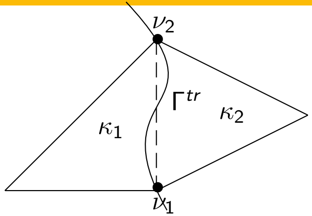
Fitted mesh approach

- Physical frame *hp*-DGFEM with curved elements to fit the interface.
- Natural approach to energy norm a posteriori analysis.
- Applies to problems with non-essential boundary conditions on a single domain with curved boundary.
- Numerical difficulty moved to hard quadrature evaluation.

[4] See [Dörfler & Rumpf, Math Comp 1998, Ainsworth & Rankin, Tech Rep 2012]

Fitted *hp*-DGFEM(p) discretisation

The mesh $\mathcal{T}_h$ is standard, but may contain curved elements to fit the interface.



Elliptic problem Find $u_h \in V_h^{\mathbf{p}}$:

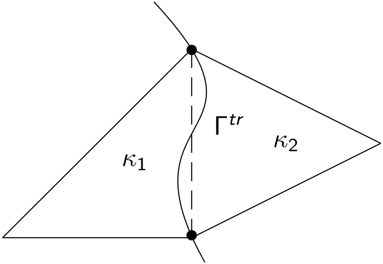
$$B(u_h, v_h) = \langle f, v_h \rangle \qquad \text{for all } v_h \in V_h^{\mathbf{p}}$$

$$B(u_h, v_h) = \sum_{K \in \mathcal{T}} \int_K \nabla u_h \cdot \nabla v_h dx - \int_{\Gamma \setminus \Gamma^{tr}} (\{\nabla u_h\} \cdot [\![v_h]\!] + \{\nabla v_h\} \cdot [\![u_h]\!]) ds$$
$$+ \int_{\Gamma \setminus \Gamma^{tr}} \frac{\sigma}{\mathbf{h}} [\![u_h]\!] \cdot [\![v_h]\!] ds + \int_{\Gamma^{tr}} P(u_h) [\![u_h]\!] \cdot [\![v_h]\!] ds;$$

Parabolic problem By standard timestepping, eg. backward Euler.

Elements with curved faces

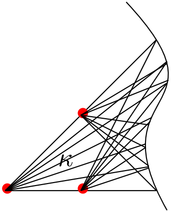
For elements with curved faces, take the process described earlier to the limit!

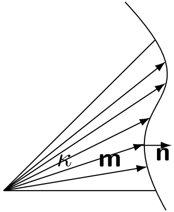


Assumptions:

Star-shaped w.r.t. •

$\frac{\mathbf{m}}{|\mathbf{m}|} \cdot \mathbf{n} \geq c > 0$



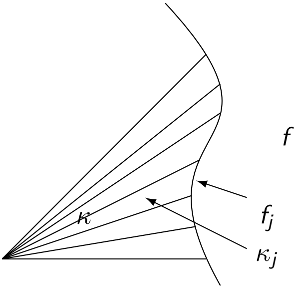


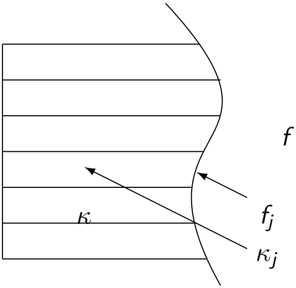
Inverse estimate

Lemma (C., Georgoulis & Sabawi, Math. Comp., 2017)

Let κ be a simplex/hexahedron with a curved face F. For each $v \in \mathcal{P}_p(\kappa)$,

$$\|v\|_f^2 \leq C \frac{p^2}{h_\kappa} \|v\|_\kappa^2.$$





Apply inverse estimate from each f_j to κ_j and sum up.

KP recovery operator

Lemma (C., Georgoulis & Sabawi, Math. Comp., 2017)

Given the above mesh assumptions, there exists a recovery operator $\mathcal{E} : S_h^p \to H^1(\Omega_1 \cup \Omega_2)$, *such that*

$$\sum_{\kappa \in \mathcal{T}} \|\nabla^\alpha (v_h - \mathcal{E}(v_h))\|_\kappa^2 \leq C_\alpha \sum_{E \subset \Gamma \backslash \Gamma^{tr}} \|\sqrt{\theta\eta}\mathbf{h}^{1/2-\alpha} [\![v_h]\!] \|_E^2, \qquad (1)$$

for all $v_h \in S_h^p$, $C_\alpha > 0$, $\alpha = 0, 1$, *independent of* v_h, θ *and* $\mathbf{h}$.

- Here, θ, η measure how far κ is from straight. They must satisfy some mild saturation assumptions of the approximation of the geometry by the mesh.
- If κ is not curved, then $\eta = \theta = 1$ and recovery operator and bound reduces to that of Karakashian & Pascal [Karakashian & Pascal, SINUM,2003].
- Note: reconstruction not continuous across the interface.
  A posteriori analysis for meshes with internal curved interfaces or degenerating interfaces still open!

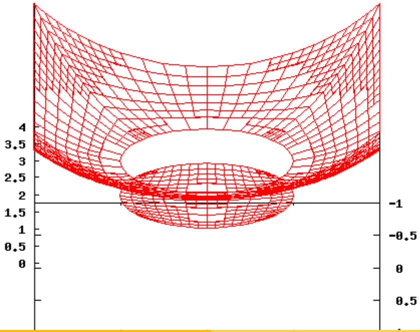
Elliptic problem with curved interface

$\Omega = (-1,1)^2$, $\Omega_1 = \{x^2 + y^2 < 0.5^2\}, \Omega_2 = \Omega \setminus \bar{\Omega}_1, C_{tr} = 0.75$

Forcing & boundary conditions chosen so that

$$u = \begin{cases} (x^2+y^2)^{3/2}, & \text{in } \Omega_1 \\ (x^2+y^2)^{3/2}+1, & \text{in } \Omega_2 \end{cases}$$

$p = 1$. Implementation of interface by 4^{th}-order (polynomial) mapping.



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$p = 1$. Implementation of interface by 4^{th}-order (polynomial) mapping.

Total Dofs	Estimate	Rate	E.norm	Rate	Est./E.norm
768	7.6178	–	0.84377	–	9.03
3072	3.9836	0.9349	0.39946	1.0789	9.97
12288	2.0257	0.9755	0.19146	1.0610	10.58
49152	1.02	0.9897	0.093261	1.0378	10.9
196608	0.51507	0.9857	0.045982	1.0202	11.2

Convection-diffusion problem (straight interface)

[S. Metcalfe, PhD Thesis, Leicester, 2015]

A conforming approach: the C^0-conforming VEM

Model problem: find $u \in V = H_0^1(\Omega)$:

$$A(u, v) := (\nabla u, \nabla v) = (f, v) \qquad \forall v \in V$$

Recall the local generalised harmonic FE of order p:

$$V_h^\kappa := \{v \in H^1(\kappa) : \Delta v \in \mathcal{P}_{p-2}(\kappa); \quad v|_{\partial\kappa} \in C^0(\partial\kappa) \quad \text{and} \quad v|_e \in \mathcal{P}_p(e), \forall e \in \partial\kappa\}$$

from which we may construct a C^0-conforming space as

$$V_h = \{v \in C^0(\Omega) : v|_\kappa \in V_h^\kappa, \ \forall \kappa \in \mathcal{T}_h\} \subset H_0^1(\Omega)$$

yielding the generalised harmonic formulation: find $u_h \in V_h$:

$$A(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V$$

ISSUE: hard to compute!!!

A conforming approach: the C^0-conforming VEM

VEM approach : find $u_h \in V_h$:

$$\sum_{\kappa \in \mathcal{T}_h} A_h^\kappa(u_h, v_h) =: A_h(u_h, v_h) = \langle f, v_h \rangle := \sum_{\kappa \in \mathcal{T}_h} \langle f, v_h \rangle_\kappa \qquad \forall v_h \in V$$

with A_h^κ and $\langle \cdot, \cdot \rangle_\kappa$ local discrete forms computable by accessing only directly available information on the ansatz, ie. the DoF, ...

... and such that:

- Stability. There exists $\alpha_*, \alpha^* > 0$ independent of h and κ such that

$$\alpha_*(\nabla v_h, \nabla v_h)_\kappa \leq A_h^\kappa(v_h, v_h) \leq \alpha^*(\nabla v_h, \nabla v_h)_\kappa \qquad \forall v_h \in V_h^\kappa$$

- Polynomial consistency. For all $p \in \mathcal{P}_p(\kappa)$ and $v_h \in V_h^\kappa$

$$A_h^\kappa(p, v_h) = (\nabla p, \nabla v_h)_\kappa$$
$$\langle f, p \rangle_\kappa = (f, p)_\kappa$$

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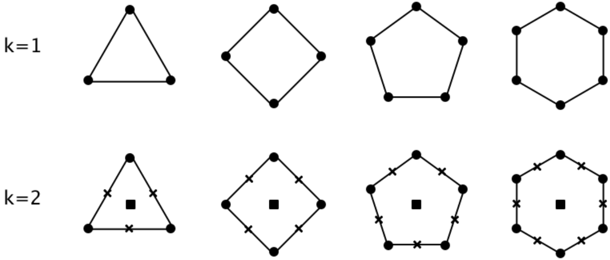
Local DoF for VEM of order p

2D element:

- vertex value
- (if $p > 1$) edge polynomial moments of degree $\leq p - 2$
- (if $p > 1$) internal polynomial moments of degree $\leq p - 2$



3D element: the above on each face + analogous internal moments.

Computability of a local H^1-projector

[Beirao da Veiga, Brezzi, C, Manzini, Marini & Russo, M3AS, 2013]

Recall:

$$V_h^\kappa := \{ v \in H^1(\kappa) : \Delta v \in \mathcal{P}_{p-2}(\kappa); \quad v|_{\partial\kappa} \in C^0(\partial\kappa) \quad \text{and} \quad v|_e \in \mathcal{P}_p(e), \forall e \in \partial\kappa \}$$

CRUCIAL OBSERVATION: The H^1-type projector Π_p^1:

$$\begin{cases} (\nabla \Pi_p^1 v_h, \nabla p)_E = (\nabla v_h, \nabla p)_E & \forall p \in \mathcal{P}_p(\kappa) \\ \frac{1}{|\kappa|} \int_\kappa \Pi_p^1 v_h \, dx = \begin{cases} \frac{1}{\sharp(v)} \sum_v v_h(v) & \text{if } p = 1 \\ \dfrac{1}{|\kappa|} \displaystyle\int_\kappa v_h \, dx & \text{if } p > 1 \end{cases} \end{cases}$$

is computable just by accessing the DoF of v_h.

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CRUCIAL OBSERVATION: The H^1-type projector Π_p^1:

$$\begin{cases} (\nabla \Pi_p^1 v_h, \nabla p)_E = -(v_h, \Delta p)_E + \sum_{e\in\partial\kappa}(v_h, \mathbf{n}\cdot\nabla p)_e \\ \frac{1}{|\kappa|}\int_\kappa \Pi_p^1 v_h \, dx = \begin{cases} \frac{1}{\sharp(v)}\sum_v v_h(v) & \text{if } p = 1 \\ \frac{1}{|\kappa|}\int_\kappa v_h \, dx & \text{if } p > 1 \end{cases} \end{cases}$$

is computable just by accessing the DoF of v_h.

DRAWBACK: L^2-projection is not computable.

An enhanced Virtual Element space

[Ahmad, Alsaedi, Brezzi, Marini & Russo, C&MA, 2013]

$$V_h^\kappa := \{ v \in H^1(\kappa) : \Delta v \in \mathcal{P}_p(\kappa);$$
$$v|_{\partial\kappa} \in C^0(\partial\kappa) \quad \text{and} \quad v|_e \in \mathcal{P}_p(e), \forall e \in \partial\kappa;$$
$$(v - \Pi_p^1 v, p)_\kappa = 0 \quad \forall p \in \mathcal{P}_{p,p-1}(\kappa) \quad \}$$

$L^2(E)$-projection $\Pi_p^0 v_h : V_h^\kappa \to \mathcal{P}_p(\kappa)$ is computable:

- moments up to degree $p-2$ are DoF
- moments of degree p and $p-1$ coincide with those of $\Pi_p^1 v_h$

GLOBAL VE SPACE formed by glueing elementwise spaces:

$$V_h = \{ \chi \in C^0(\Omega) : \chi|_\kappa \in V_h^\kappa, \ \forall \kappa \in \mathcal{T}_h \} \subset H_0^1(\Omega)$$

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VEM computable, stable, and p-consistent forms

We fix the local bilinear form as

$$A_h^\kappa(u_h, v_h) := (\nabla \Pi_p^1 u_h, \nabla \Pi_p^1 v_h)_\kappa + ((I - \Pi_p^0) u_h, (I - \Pi_p^0) v_h)_{S_A}$$

with VEM stabilising term, eg.

$$((I - \Pi_p^1) u_h, (I - \Pi_p^0) v_h))_{S_A} := h_\kappa^{d-2} \overrightarrow{\mathrm{DoF}}_\kappa((I - \Pi_p^1) u_h) \cdot \overrightarrow{\mathrm{DoF}}_\kappa((I - \Pi_p^1) v_h)$$

with $\overrightarrow{\mathrm{DoF}}_\kappa(\cdot)$ the vector of appropriately scaled Degrees of Freedom.

For the local right-hand side:

$$\langle f, v_h \rangle_\kappa := (f, \Pi_p^0 v_h)_\kappa$$

In this setting, optimal a priori error bound in the L^2 and H^1 norms can be proven under appropriate shape-regularity assumptions.

[da Veiga, Brezzi, C., Manzini, Marini, Russo, M3AS, 2013], [Ahmad, Alsaedi, Brezzi, Marini & Russo, C&MA, 2013],

[C., Manzini, Sutton, IMAJNA, 2016]

Energy norm a posteriori error analysis

1 The VEM does NOT satisfy Galerkin orthogonality. For $v_h \in V_h$,

$$A(u - u_h, v_h) = \langle f, v_h \rangle - (f, v_h)_h + [A_h(u_h, v_h) - A(u_h, v_h)]$$

2 The usual residuals:

$$(f + \Delta u_h)|_\kappa$$

and

$$[\nabla u_h]|_f$$

can't be computed!

Energy norm a posteriori error analysis

1. The VEM does NOT satisfy Galerkin orthogonality. For $v_h \in V_h$,

$$A(u - u_h, v_h) = \langle f, v_h \rangle - (f, v_h)_h + [A_h(u_h, v_h) - A(u_h, v_h)]$$

2. We *can* compute the *projected* residuals

$$(\Pi_p^0 f + \Delta \Pi_p^1 u_h)|_\kappa$$

and

$$[\nabla \Pi_p^1 u_h]\big|_f$$

A posteriori error bound

[C, Georgoulis, Pryer, Sutton, Numer. Math., 2017] & [Sutton, PhD, Leicester, 2017]

Theorem (Upper bound)

There exists a constant C, independent of h, u and u_h, such that

$$\|\nabla(u - u_h)\|^2 + \|\nabla(u - \Pi_p^0 u_h)\|^2 \leq C \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 + \Theta_\kappa^2 + \mathcal{S}_\kappa^2$$

where

$$\eta_\kappa^2 := h_\kappa^2 \|\Pi_p^0 f + \Delta \Pi_p^1 u_h\|_\kappa^2 + \sum_{f \subset \partial\kappa} h_f \|[\nabla \Pi_p^0 u_h]\|_f^2 \quad \textit{(residual)}$$

$$\Theta_\kappa^2 := h_\kappa^2 \|f - \Pi_p^0 f\|_\kappa^2 \quad \textit{(data oscillation)}$$

$$\mathcal{S}_\kappa^2 := (u_h - \Pi_p^0 u_h, u_h - \Pi_p^0 u_h)_{S_A} \quad \textit{(projection indicator)}$$

This can be generalised to diffusion-advection-reaction problems with non-constant coefficients: extra virtual inconsistency terms appear.

A posteriori error bound: local lower bound

Theorem (Local lower bound)

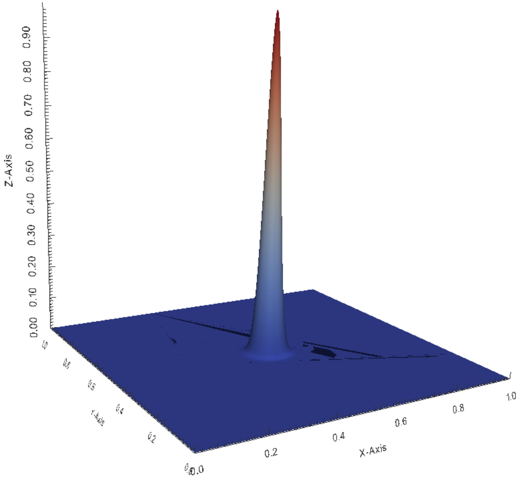
There exists a constant C, independent of h, u and u_h, such that

$$\eta_\kappa^2 \leq C \sum_{\hat{\kappa} \in \omega_\kappa} \left(\|\nabla(u - u_h)\|_{\hat{\kappa}}^2 + \|\nabla(u - \Pi_p^0 u_h)\|_{\hat{\kappa}}^2 + \Theta_{\hat{\kappa}}^2 \right)$$

with ω_κ the patch made of κ and its neighbours.

Proof is based on the classical bubble function techniques.

Numerical example



$$-\Delta u + \boldsymbol{b}\cdot\nabla u + cu = f$$

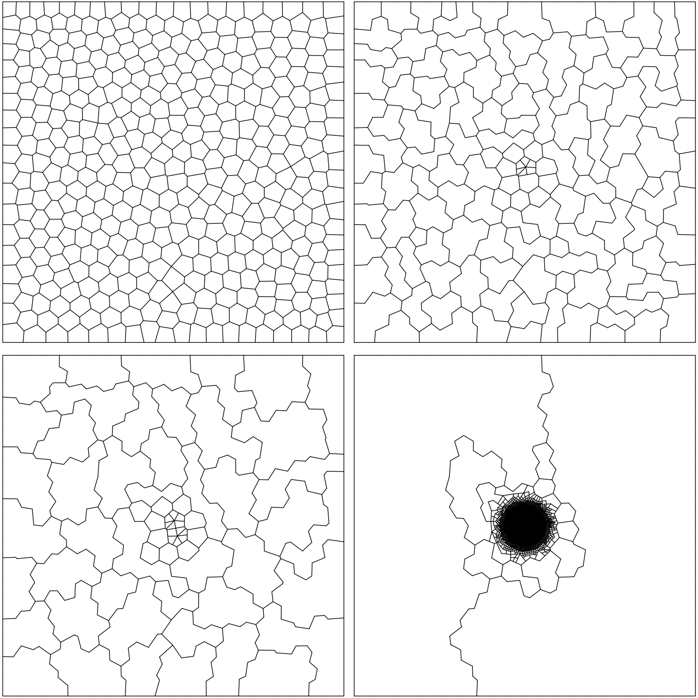
with

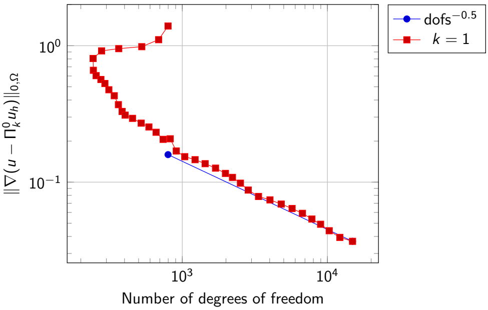
$$\boldsymbol{b}(x,y) = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$$

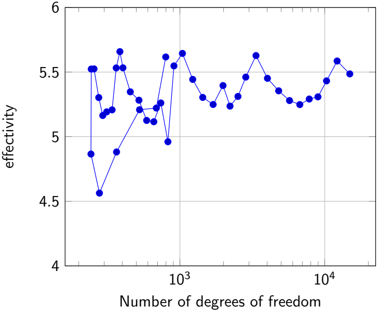
$$c(x,y) = \sin(x)\sin(y)$$

and f such that exact solution is:

$$u(x,y) = \exp(-(1000(x-0.5)^2 + 1000(y-0.5)^2))$$




10^0
10^{-1}
10^3
10^4
$\|\nabla(u - \Pi_k^0 u_h)\|_{0,\Omega}$
Number of degrees of freedom
$\text{dofs}^{-0.5}$
$k = 1$


6
5.5
5
4.5
4
10^3
10^4
effectivity
Number of degrees of freedom

Conclusions

- Possible to *efficiently* generalise classical FE families

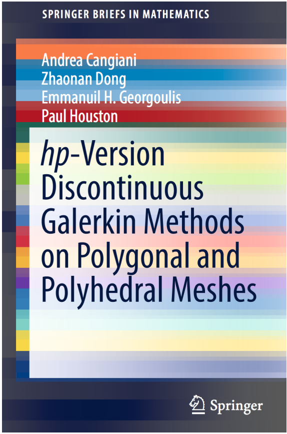
  → Fitted discretisation a posteriori analysis for curved boundary (natural b.c.)

- General meshes can be advantageous in mesh (in facts, *hp*-) adaptivity

Outlook

- A posteriori analysis for degenerating interfaces (and curved boundary with essential b.c)

- More applications & implementations of adaptive algorithms



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