Quantum Groups and Shuffle Algebras

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Vacation Scholarship Project Summer 2020

1 Introduction

This paper will discuss two different types of algebras, quantum groups and shuffle algebras. First we will need to introduce some key ideas from the theory of Lie algebras to be able to understand quantum groups and shuffle algebras more fully. Then, the construction of associative algebras from generators and relations will be discussed. This is again important to be able to understand quantum groups and shuffle algebras more fully. The running example we will use throughout is the Lie algebra $\mathfrak{sl}_2(F)$ to illustrate the different ideas presented. Then, the idea of a loop Lie algebra and the quantum loop algebra will be presented before intuitively defining the affine quantum group of a Lie algebra. Finally, we will look at two different shuffle algebras, mainly that which is presented by B. Enriquez in [1] and the similar version presented by A. Negut in [5]. The main point of this paper is to demonstrate the homomorphism between quantum groups and shuffle algebras endowed with a shuffle product.

2 Background Information / Key Definitions

2.1 Rings

Definition 2.1. A *ring* ([4] , [2]) is a non-empty set R with two binary operations, an addition operation and a multiplication operation, such that the following axioms are satisfied:

1. R is an abelian group under the addition operation. The identity is written as 0_R and the inverse for an element a is written as $-a \ \forall \ a \in R$
2. Closure under multiplication
3. Associativity under multiplication
4. Multiplication is distributive over addition

If the ring has a multiplicative identity, written as 1_R, then the ring is called a *ring with identity.*

A *commutative ring* has $ab = ba \ \forall \ a, b \in R$.

2.2 Fields

Definition 2.2. A *field* is a ring R, as above, with $R \setminus \{0\}$ being an abelian group under the multiplication operation. [7]

An example of a field is the complex numbers, $\mathbb{C}$. This is the field we are going to focus on in this paper.

2.3 Lie Algebras

Definition 2.3. A *Lie algebra* L is a vector space over an arbitrary field F with an operation:

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

called a Lie bracket, such that the following three axioms are satisfied:

1. the operation is a bilinear map:
$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[x, ay + bz] = a[x, y] + b[x, z]$$
2. $[x, x] = 0 \qquad \forall x \in L$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \forall x, y, z \in L$

This third axiom is known as the *Jacobi Identity.*

Definition 2.4. An *associative algebra* is a vector space with a bilinear product:

$$A \times A \longrightarrow A$$
$$(x, y) \mapsto x \cdot y$$

such that:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x, y, z \in A$$

It can be shown that any associative algebra A has an associated Lie bracket defined as:

$$[x, y] = xy - yx$$

for all x, y in A.

Definition 2.5. If L is a Lie Algebra over a field F then a *Lie subalgebra* of L is a vector subspace over F, K of L such that:

$$[x, y] \in K \qquad \forall x, y \in K$$

where the Lie bracket here is the same as the Lie bracket for L.

Definition 2.6. Let L_1 and L_2 be Lie algebras. A *Lie algebra homomorphism* $T : L_1 \longrightarrow L_2$ is a linear map satisfying:

$$T([x, y]) = [T(x), T(y)] \qquad \forall x, y \in L_1$$

Definition 2.7. A *representation* of a Lie algebra L over the field F, consists of a pair (V, φ) where:

1. V is a vector space over F
2. gl $(V) = \{$all linear maps from $V \longrightarrow V\}$ and $\varphi : L \longrightarrow \mathrm{gl}(v)$ is a Lie algebra homomorphism

(V, φ) is a representation $\iff \varphi([x, y])v = \varphi(x)(\varphi(y)v) - \varphi(y)(\varphi(x)v) \qquad \forall x, y \in L$

Definition 2.8. If we have two representations of a Lie algebra L, (V_1, φ_1) and (V_2, φ_2), then the linear map $T : V_1 \longrightarrow V_2$ is a *homomorphism of representations* if:

$$T(\varphi_1(x)v) = \varphi_2(x)T(v)$$

for $v \in V_1$ and $x \in L$

2.4 Generators and Relations

Let S be a set. A *word* on S is a finite sequence of elements of S, $w = s_1, s_2, \dots s_n$ for $s_i \in S$

The free algebra over the field F generated by S is the vector space $F\langle S\rangle$ with basis equal to the set of all words on S i.e. an element of $F\langle S\rangle$ is a formal linear combination:

$$\sum_i a_i w_i$$

for $a_i \in F$ and w_i is a word.

$F\langle S\rangle$ is an associative algebra with a product determined by $w_1 \cdot w_2 = w_1 w_2$ where w_1 and w_2 are words.

Let $R \subseteq F\langle S\rangle$ be a subset. Then,

$$\langle R\rangle = \text{span}\{xry \mid x, y \in F\langle S\rangle, r \in R\}$$

is the two-sided ideal generated by R.

The quotient, $F\langle S\rangle/\langle R\rangle$, is another algebra with product $[x][y] = [xy]$ for $x, y \in F\langle S\rangle$, where $[x]$ denotes the image of $x \in F\langle S\rangle$ in the quotient.

It is convention to write $x \in (F\langle S\rangle/\langle R\rangle)$ instead of $[x]$ for $x \in F\langle S\rangle$.

Definition 2.9. The algebra generated by S with relations $\{r = 0 | r \in R\}$ is $\frac{F\langle S\rangle}{\langle R\rangle}$ such that $\{r = 0 \mid r \in R\}$.

For example, the algebra generated by x, y with relations $xy - yx = 0$ is $\frac{F\langle x,y\rangle}{\langle xy-yx\rangle} = F[x, y]$.

2.5 The Example of $\mathfrak{sl}_2(F)$

First, we need to define $\mathfrak{gl}_n(F)$. This is defined as the set of $n \times n$ matrices with entries in the field, F.

For example, in the case $n = 2$, $\mathfrak{gl}_n(F) = \mathfrak{gl}_2(F)$ and can be written as:

$$\mathfrak{gl}_2(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c \in F \right\}$$

The algebra $\mathfrak{gl}_n(F)$ is an associative algebra under matrix multiplication. Hence, it is also a Lie algebra with Lie bracket:

$$[x, y] = xy - yx \quad \forall x, y \in \mathfrak{gl}_n(F)$$

i.e. the commutator of x and y.

The algebra $\mathfrak{sl}_2(F)$ is a Lie subalgebra of $\mathfrak{gl}_2(F)$. Therefore, it has the same Lie bracket as $\mathfrak{gl}_2(F)$. (Note: this does *not* imply that $\mathfrak{sl}_2(F)$ is an *associative* subalgebra)

We define $\mathfrak{sl}_2(F)$ to be the set of 2×2 matrices with entries in the field F such that $tr(x) = 0$ $\forall x \in \mathfrak{sl}_2(F)$.

This is written succinctly as:

$$\mathfrak{sl}_2(F) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in F \right\}$$

A basis for $\mathfrak{sl}_2(F)$ is:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since these elements e, f, and h are basis elements of $\mathfrak{sl}_2(F)$, they are also generators of $\mathfrak{sl}_2(F)$.

These generators have the following relations:

$$[e, f] = h, \qquad [e, h] = -2e, \qquad [f, h] = 2f$$

For example, the proof of the relation $[e, f] = h$ is as follows:

$$\begin{aligned} [e, f] &= ef - fe \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= h \qquad \square \end{aligned}$$

The other relations are proved in the same manner.

A Representation of $\mathfrak{sl}_2(F)$

For any vector space, V, a representation of $\mathfrak{sl}_2$ on V is $E, F, H : V \longrightarrow V$ satisfying:

$$E = \varphi(e), \quad F = \varphi(f), \quad H = \varphi(h)$$

where φ is a homomorphism.

Then we have a representation $\iff$ the relations for e, f, h also hold for E, F, H.

That is, iff:

$$[E, F] = H, \quad [E, H] = -2E, \quad [F, H] = 2F$$

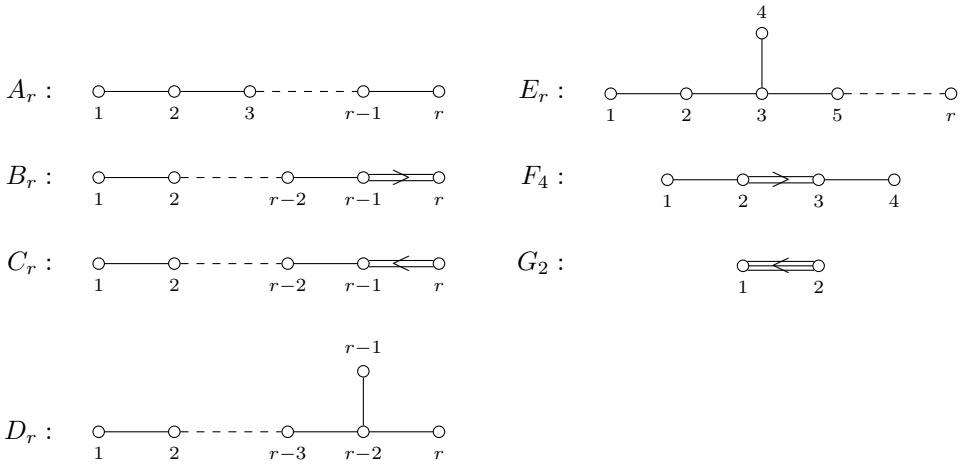
2.6 Simple Lie Algebras

There is a class of finite dimensional Lie algebras called *simple Lie algebras*.

Definition 2.10. A Lie algebra, L, is a *simple Lie algebra* if L is not abelian and the only ideals of L are 0 and L. [8]

These types of Lie algebras can be classified, or represented, by graphs called *finite Dynkin diagrams*. More precisely, there is a bijection between the set of isomorphism classes of simple Lie algebras and the set of finite types of Dynkin diagrams.

Here are all of the finite Dynkin diagrams:



where r is an indication of the number of nodes in the Dynkin diagram. Note, the E_r Dynkin diagrams are only defined for $r = 6, 7, 8$.

The Dynkin diagrams with no double or triple edges are known as *simply laced*. The only simply laced Dynkin diagrams are A_r, D_r and E_r.

Another way of representing simply laced Lie algebras is with a *Cartan matrix*. If we have a Dynkin diagram with r labelled nodes then the a_{ij} entry of the $r \times r$ Cartan matrix is as follows:

$$(a_{ij})_{i,j=1,2,\ldots,n} = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if there is an edge between node } i \text{ and node } j \\ 0, & \text{if there is no edge between node } i \text{ and node } j \end{cases}$$

2.7 Examples of constructing the Cartan Matrix from the Dynkin Diagram

Example 1: The Dynkin diagram for $\mathfrak{sl}_2(F)$ is the diagram A_1: ○

Hence, the Cartan matrix for $\mathfrak{sl}_2(F)$ is: $\begin{pmatrix} 2 \end{pmatrix}$

Example 2: The Dynkin diagram for $\mathfrak{sl}_3(F)$ is the diagram A_2: ○——○

Hence, the Cartan matrix for $\mathfrak{sl}_3(F)$ is: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Example 3: The Dynkin diagram for $\mathfrak{sl}_4(F)$ is the diagram A_3: ○——○——○

Hence, the Cartan matrix for $\mathfrak{sl}_4(F)$ is: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

3 Quantum Groups

3.1 Universal Enveloping Algebra

Definition 3.1. If $\mathfrak{g}$ is a Lie algebra with basis $\{x_1, \ldots x_n\}$, then the *universal enveloping algebra* of $\mathfrak{g}$, $U(\mathfrak{g})$, is the algebra with generators $x_1, \ldots x_n$ and relations $x_i x_j - x_j x_i = [x_i, x_j]$.

If $\mathfrak{g} = \mathfrak{sl}_2(F)$ then $U(\mathfrak{g}) = U(\mathfrak{sl}_2(F))$ which has generators e, f and h and relations:

$$\begin{aligned} ef - fe &= h \\ he - eh &= 2e \\ hf - fh &= -2f \end{aligned}$$

If we restrict our attention to the simply laced cases, there is a method which constructs the *universal enveloping algebra* of the simple Lie algebra from the Dynkin diagram. Here are the steps to do so:

1. Form the Cartan matrix from the Dynkin diagram as shown above in 2.7
2. Then, $U(\mathfrak{g})$ is the algebra with generators:

$$\begin{aligned} &e_1, \ldots, e_r \\ &h_1, \ldots, h_r \\ &f_1, \ldots, f_r \end{aligned}$$

   known as the "Chevally generators"
3. These generators then have relations:

$$\begin{aligned} [h_i, e_j] &= a_{ij} e_j \\ [h_i, f_j] &= -a_{ij} f_j \\ [h_i, h_j] &= 0 \\ [e_i, f_j] &= \delta_{ij} h_i \quad \text{where } \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if} i \neq j \end{cases} \end{aligned}$$

3.2 Loop Lie Algebra

Definition 3.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The *loop Lie algebra* is the vector space:

$$\begin{aligned} \mathcal{L}\mathfrak{g}[t, t^{-1}] &= \{\text{Laurent polynomials where coefficients are elements of } \mathfrak{g}\} \\ &= \mathbb{C} - \text{span}\{\ xt^n \mid x \in \mathfrak{g},\ n \in \mathbb{Z}\ \} \end{aligned}$$

with Lie bracket: $[xt^m, yt^n] := [x, y]_{\mathfrak{g}} t^{m+n}, \quad x, y \in \mathfrak{g}$

where $[\cdot, \cdot]_{\mathfrak{g}}$ denotes the Lie bracket corresponding to the Lie algebra $\mathfrak{g}$.

Notation: $x[n] := xt^n$

Remark: If $\mathfrak{g}$ is a simple Lie algebra then this loop Lie algebra is closely related to the affine Lie algbera $\hat{\mathfrak{g}}$, which is a central extension of $\mathcal{L}\mathfrak{g}$. [3]

For example, let $\mathfrak{g}$ be a simple Lie algebra and let $n_+ \subseteq \mathfrak{g}$ be the subalgebra generated by the generators $e_1, \ldots, e_r$. Then, it can be shown that the universal enveloping algebra of $\mathcal{L}n_+$, $U(\mathcal{L}n_+)$, is generated by:

$$e_i[n] := e_i t^n \ \text{ for } \ i = 1, \ldots, r \text{ and } n \in \mathbb{Z}$$

with:

- Vertex relations:

$$(z - w) e_i(z) e_j(w) = (z - w) e_j(w) e_i(z)$$

  for $i, j = 1, \ldots, r$ and $e_i(z) := \sum_{n \in \mathbb{Z}} e_i[n] z^{-n} \in \mathcal{L}n_+ [\![z, z^{-1}]\!]$, where $[\![z, z^{-1}]\!]$ is a formal power series.

and

- Serre relations: if $i \neq j$:

$$\sum_{k=0}^{N}(-1)^k \binom{N}{k} \mathrm{Sym}_{z_1,\ldots,z_N}(e_i(z_i)\ldots e_i(z_k)e_j(w)e_i(z_{k+1})\ldots e_i(z_N)) = 0$$

  where $N = 1 - a_{ij}$ and $\mathrm{Sym}_{z_1,\ldots,z_N}$ is the sum over all permutations of the variables.

The Serre relations of the simply laced cases occur when $N = 1$ or $N = 2$.

In the case where $N = 1$ we have:

$$e_j(w)e_i(z) - e_i(z)e_j(w) = 0$$

This is the case which has no edge in the Dynkin diagram.

In the case where $N = 2$ we have:

$$e_j(w)e_i(z_1)e_i(z_2) + e_j(w)e_i(z_2)e_i(z_1) - 2e_i(z_1)e_j(w)e_i(z_2) - 2e_i(z_2)e_j(w)e_i(z_1) + e_i(z_1)e_i(z_2)e_j(w) + e_i(z_2)e_i(z_1)e_j(w) = 0$$

This is the case which has a single edge in the Dynkin diagram.

This can be done for all relevant N.

3.3 Quantum Loop Algebra

Definition 3.3. The *quantum loop algebra* of n_+ is the algebra $U_q(\mathcal{L}n_+)$ with generators $e_i[n]$ for $i = 1, \ldots, r$ and $n \in \mathbb{Z}$, and:

- Vertex relations:

$$(q^{a_{ij}}z - w)e_i(z)e_j(w) = (z - q^{a_{ij}}w)e_j(w)e_i(z)$$

  for $i, j = 1, \ldots, r$

- Serre relations: if $i \neq j$:

$$\sum_{k=0}^{N}(-1)^k \begin{bmatrix} N \\ k \end{bmatrix}_q \mathrm{Sym}_{z_1,\ldots,z_N}(e_i(z_1)\ldots e_i(z_k)e_j(w)e_i(z_{k+1})\ldots e_i(z_N)) = 0$$

  where $\begin{bmatrix} N \\ k \end{bmatrix}_q$ is the *quantum binomial coefficient* and is defined to be:

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{[N]!_q}{[k]!_q[N-k]!_q}$$

  where: $[m]!_q = [m]_q \ldots [2]_q[1]_q$ and $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$

The quantum loop algebra $U_q(Ln_+)$ is also known as (the positive part of) the affine quantum group.

4 Shuffle Algebras

Sometimes it can be useful to give variables different colours, where each colour is an integer. For example, if we have the variable x, then we might want two different colours of x denoted as x_1 which has colour 1 and x_2 which has colour 2. If we have multiple variables assigned to the same colour, we index them as follows: $\{x_{1,1}, x_{1,2}, \ldots, x_{1,n}\}$ are the first n variables coloured 1 in some arbitrary case.

A function has colour symmetry if the function is symmetric in each set of coloured variables. For example, if we have the function $F(x_{1,1}, \ldots, x_{1,m_1}, \ldots, x_{n,1}, \ldots, x_{n,m_n})$, it will have colour symmetry if it is a symmetric function in $x_{i,1}, \ldots, z_{i,m_i}$ for all $i \in \{1, \ldots, n\}$. [6], page 23.

When we are working with shuffle algebras, we want to colour the variables.

Definition 4.1. A *shuffle algebra* is an algebra whose product works by permuting all of the variables of two functions in such a way that colour symmetry is maintained.

4.1 Enriquez Shuffle Algebra

Proposition 8 from [1] is an interesting proposition to look at in more detail:

Proposition 4.1. For $\mathfrak{g}$ arbitrary, π is a morphism of algebras from $U_q n_+$ to $\overline{Sh}$, endowed with the shuffle product (as defined in the original paper and below). [1]

Components of this proposition

1. $\mathfrak{g}$ is a simple finite-dimensional complex Lie algebra.
   - $\overline{\mathfrak{g}}$ is its affinization
   - $U_q\overline{\mathfrak{g}}$ is the associated quantum algebra
   - $e_\alpha(z)$ are the positive nilpotent currents of $U_q\overline{\mathfrak{g}}$ where z is a formal variable and α is a simple root of $\mathfrak{g}$. This is defined more fully below.
   - $(a_{\alpha\beta})_{1\leq\alpha,\beta\leq r}$ is the Cartan matrix of $\mathfrak{g}$ where r is the rank of $\mathfrak{g}$ i.e. the number of nodes on the Dynkin diagram
2. π is a morphism i.e. a "structure-preserving map" [9]
3. $U_q n_+$ is the positive nilpotent part of $U_q\overline{\mathfrak{g}}$ with:
   - generators: $e_\alpha[n]$ with α a simple root and n an integer with

$$e_\alpha(z) = \sum_{n\in\mathbb{Z}} e_\alpha[n] z^{-n}$$

   and

   - relations:
     - Vertex Relations:

$$(q_\alpha^{a_{\alpha\beta}} z - w) e_\alpha(z) e_\beta(w) = (z - q_\alpha^{a_{\alpha\beta}} w) e_\beta(w) e_\alpha(z)$$

     - Quantum Serre relations; see [1] equation (7)

   where n_+ is the part of $\overline{g}$ generated by $e_1, \ldots, e_r$. Therefore, it is the set of affine Laurent polynomials.
4. $\overline{Sh} = \oplus_{n\in\mathbb{N}^r} \overline{Sh}_n$

- where n is the r-tuples $(n_1, ..., n_r)$ and r = rank of $\mathfrak{g}$ i.e. the number of nodes in the Dynkin diagram of $\mathfrak{g}$
- $\overline{Sh}_n$ is the set of rational functions in variables:

$$\begin{matrix} z_1^{(1)}, \dots, z_{n_1}^{(1)} \\ \dots \\ z_1^{(r)}, \dots, z_{n_r}^{(r)} \end{matrix}$$

of the form:

$$f(z) = \frac{\prod_{\alpha=1}^{r} \prod_{i,j=1, i<j}^{n} (z_i^{(\alpha)} - z_j^{(\alpha)}))}{\prod_{1 \le \alpha \le \beta \le r} \prod_{i=1}^{n\alpha} \prod_{j=1, i<j \text{ if } \alpha=\beta}^{n\beta} (z_i^{\alpha} - q_{\alpha}^{a_{\alpha\beta}} z_j^{\beta})} A(z)$$

  - where $A(z) \in \mathbb{C}[z_i^{(\alpha)}, z_i^{(\alpha)-1}]$. That is to say, each $A(z)$ is a Laurent polynomial and we want $A(z)$ to be symmetric in $z_1^{\alpha}, ..., z_{n_\alpha}^{(\alpha)}$ for $\alpha = 1, ..., r$

5. The final component of our proposition is the "shuffle product":

$$\text{"Shuffle Product"}: \qquad \overline{Sh}_N \times \overline{Sh}_M \to \overline{Sh}_{N+M}$$

with:

$$(f, g) \mapsto f * g$$

such that:

$$(f * g)(z_1, ..., z_{N+M}) = \Sigma_{\sigma \in M_{N,M}} \varepsilon(\sigma) \prod_{i<j, \sigma^{-1}(i) > \sigma^{-1}(j)} \frac{q^{(\alpha(i), \alpha(j))} z_i - z_j}{z_i - q^{(\alpha(i), \alpha(j))} z_j} \times f(z_{\sigma(1)}, ..., z_{\sigma(N)}) g(z_{\sigma(N+1)}, ..., z_{\sigma(N+M)})$$

where:

$$M_{N,M} = \{\text{permutations} \quad \sigma = \{1, ..., N+M\} \mid \sigma(i) < \sigma(j) \text{ if } i < j \le N \text{ or } N < i < j\}$$

Note: N, M and $N + M$ are integers if there is only one colour of variable, tuples if there are two colours of variables, triples if there are three colours of variables, and so on.

4.2 Examples of the Enriquez Shuffle Product

Let $\mathfrak{g} = \mathfrak{sl}_2(F)$ for examples 1 and 2.

The Dynkin diagram for this simple Lie algebra is A_1: ○

Therefore, it has corresponding Cartan Matrix: $\begin{pmatrix} 2 \end{pmatrix}$

Example 1: $\overline{Sh}_1 \times \overline{Sh}_1 \to \overline{Sh}_2$

In this example, we have $N = 1,\ M = 1,\ N + M = 2$.

$\overline{Sh}_1$ elements have the form $A(z) \in \mathbb{C}[z^{\pm 1}]$

$\overline{Sh}_2$ elements have the form:

$$f(z) = \frac{\prod_{\alpha=1}^{1}\prod_{i,j=1,i<j}^{2}(z_i - z_j)}{\prod_{i,j=1,i<j}^{2}(z_i - q^2 z_j)} A(z_1, z_2)$$
$$= \frac{z_1 - z_2}{z_1 - q^2 z_2} A(z_1, z_2)$$

where $A(z_1, z_2) \in \mathbb{C}[\![z_1^{\pm 1}, z_2^{\pm 1}]\!]$

Hence, $\overline{Sh}_1 \times \overline{Sh}_1 \to \overline{Sh}_2$ is:

$$(f * g)(z_1, z_2) = f(z_1)g(z_2) + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} f(z_2)g(z_1)$$

where the first term is the identity shuffle and the second term is the permutation $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ with the corresponding coefficient. It can be shown that this is a member of $\overline{Sh}_2$.

Example 2: $\overline{Sh}_1 \times \overline{Sh}_2 \to \overline{Sh}_3$

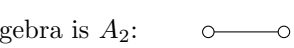
In this example, we have $N = 1,\ M = 2,\ N + M = 3$.

Hence, $\overline{Sh}_1 \times \overline{Sh}_2 \to \overline{Sh}_3$ is:

$$(f * g)(z_1, z_2, z_3) = f(z_1)g(z_2, z_3) + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} f(z_2)g(z_1, z_3) + \frac{q^2 z_1 - z_3}{z_1 - q^3 z_3}\frac{q^2 z_2 - z_3}{z_2 - q^2 z_3} f(z_3)g(z_1, z_2)$$

where the first term is the identity shuffle, the second term is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ with the corresponding coefficient and the third term is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ with the corresponding coefficient. It can be shown that this is a member of $\overline{Sh}_3$.

Example 3: Let $\mathfrak{g} = \mathfrak{sl}_3(F)$. This Lie algebra will give us two different colours of variables.

The Dynkin diagram for this simple Lie algebra is A_2: 

Hence, it has the corresponding Cartan matrix: $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Let's look at the shuffle product:

$$\overline{Sh}_{(0,1)} \times \overline{Sh}_{(1,1)} \to \overline{Sh}_{(1,2)}$$

So, we have $N = (0, 1)$, $M = (1, 1)$ and $N + M = (1, 2)$.

These tuples for N and M are interpreted as follows:

- $N = (0, 1)$ implies that N has 0 variables coloured 1 and 1 variable coloured 2
- $M = (1, 1)$ implies that M has 1 variable coloured 1 and 1 variable coloured 2

The shuffle product is as follows:

$$(f * g)(z_1, z_2, z_3) = f(z_1)g(z_2, z_3) + \frac{q^2 z_1 - z_3}{z_1 - q^2 z_3}\frac{q^{-1} z_1 - z_2}{z_1 - q^{-1} z_2}\frac{q^{-1} z_2 - z_3}{z_2 - q^{-1} z_3} f(z_3)g(z_2, z_1)$$

where z_1 is the variable coloured 2 from $\overline{Sh}_{(0,1)}$, z_2 is the variable coloured 1 from $\overline{Sh}_{(1,1)}$ and z_3 is the variable coloured 2 from $\overline{Sh}_{(1,1)}$.

The first term of this shuffle product is the identity shuffle. The second term of this shuffle product is the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ with the corresponding coefficient.

The permutation which would shuffle the variables z_1 and z_2 is not allowed, as these variables have different colours and as a result colour symmetry would not be preserved. Similarly, z_2 and z_3 cannot be shuffled for the same reason.

4.3 Proof of Enriquez Proposition 8 for $\mathfrak{sl}_2(F)$

We will prove this proposition for our running example of $\mathfrak{sl}_2(F)$.

Proof:

We need to prove that there is a homomorphism of algebras from $U_q(n_+) \longrightarrow \overline{Sh}$. The algebra homomorphism ϕ must be:

$$\phi : U_q(n_+) \longrightarrow \overline{Sh}$$

such that:

1. ϕ is linear
2. $\phi(1_{U_q(n_+)}) = \phi(1_{\overline{Sh}})$
3. $\phi(xy) = \phi(x)\phi(y)$

It can be shown that if we want an algebra homomorphism $\varphi : A \longrightarrow B$ such that $1. - 3.$ from above are satisfied for A and B then:

> If A is generated by a set S with relations R, then for every set function $\varphi_0 : S \longrightarrow B$ such that $\varphi_0(r) = 0 \ \forall r \in R$, there exists a unique homomorphism:
>
> $$\varphi : A \longrightarrow B \ \text{ such that } \ \varphi(s) = \varphi_0(s) \ \ \forall s \in S$$

For $\mathfrak{sl}_2(F)$, n_+ is generated by $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $U_q(n_+)$ is generated by $e_\alpha[n] := e_\alpha t^n$ with the Vertex relations and quantum Serre relations, as defined in 4.1.

So, for the proof in the case of $\mathfrak{sl}_2(F)$, we need to have $\phi : U_q(n_+) \longrightarrow \overline{Sh}$ such that:

1. $\phi(e_\alpha[n]) \in \overline{Sh} \ \ \forall \alpha, n$
2. The Vertex relations and quantum Serre relations are still satisfied after the homomorphism has been applied.

Let us take each of these in order:

1. We can let $\phi(e_\alpha[n]) = \phi(e[n]) = z_p^n \in \overline{Sh}$. We do not need to worry about α since there is only one colour of variable in $\mathfrak{sl}_2$. The variable z_p^n is a formal variable.
2. The Vertex relations for $U_q(n_+)$ are:

$$(q^2x - y)(e_\alpha(x))(e_\beta(y)) = (x - q^2y)(e_\beta(y))(e_\alpha(x))$$

Under the homomorphism ϕ these relations become:

$$(q^2x-y)\phi(e_\alpha(x))\phi(e_\beta(y)) = (x-q^2y)\phi(e_\beta(y))\phi(e_\alpha(x))$$

Since there is only one colour of variable in $\mathfrak{sl}_2(F)$ we have:

$$(q^2x-y)\phi(e(x))\phi(e(y)) = (x-q^2y)\phi(e(y))\phi(e(x))$$

Substituting in

$$e(x) = \sum_n e[n]x^{-n}$$

and

$$e(y) = \sum_m e[m]y^{-m}$$

gives:

$$\begin{aligned}
&\left(q^2x-y\right)\phi\left(\sum_n e[n]x^{-n}\right)\phi\left(\sum_m e[m]y^{-m}\right) = (x-q^2y)\phi\left(\sum_m e[m]y^{-m}\right)\phi\left(\sum_n e[n]x^{-n}\right)\\
\implies &\sum_n x^{-n}\sum_m y^{-m}\left(q^2x-y\right)\phi\left(e[n]\right)\phi\left(e[m]\right) = \sum_n x^{-n}\sum_m y^{-m}\left(x-q^2y\right)\phi\left(e[m]\right)\phi\left(e[n]\right)\\
\implies &\sum_n x^{-n}\sum_m y^{-m}\left(q^2x-y\right)\left(z_1^n\right)*\left(z_2^m\right) = \sum_n x^{-n}\sum_m y^{-m}\left(x-q^2y\right)\left(z_1^m\right)*\left(z_2^n\right)
\end{aligned}$$

where $*$ is the shuffle product.

The left hand side becomes:

$$\begin{aligned}
&\sum_{m,n} x^{-n}y^{-m}\left(q^2x-y\right)\left(z_1^nz_2^m + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^n\right)\\
=&\sum_{m,n}\left(q^2x^{-n+1}y^{-m} - x^{-n}y^{-m+1}\right)\left(z_1^nz_2^m + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^n\right)\\
=&\sum_{m,n} q^2x^{-n+1}y^{-m}\left(z_1^nz_2^m + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^n\right) - \sum_{m,n} x^{-n}y^{-m+1}\left(z_1^nz_2^m + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^n\right)
\end{aligned}$$

Relabelling the indices from n to $n+1$ in the first summation and m to $m+1$ in the second summation gives:

$$\begin{aligned}
&\sum_{m,n} q^2x^{-n}y^{-m}\left(z_1^{n+1}z_2^m + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^{n+1}\right) - \sum_{m,n} x^{-n}y^{-m}\left(z_1^nz_2^{m+1} + \frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^{m+1}z_2^n\right)\\
=&\sum_{m,n} x^{-n}y^{-m}\left(\left(q^2z_1-z_2\right)z_1^nz_2^m + \left(q^2z_2-z_1\right)\left(\frac{q^2z_1-z_2}{z_1-q^2z_2}z_1^mz_2^n\right)\right)\\
=&\sum_{m,n} x^{-n}y^{-m}\left(q^2z_1-z_2\right)\left(z_1^nz_2^m - z_1^mz_2^n\right)
\end{aligned}$$

The right hand side becomes:

$$\begin{aligned}
&\sum_n x^{-n} \sum_m y^{-m} \left(x - q^2 y\right) \left(z_1^m z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^m\right) \\
=&\sum_{m,n} x^{-n} y^{-m} \left(x - q^2 y\right) \left(z_1^m z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^m\right) \\
=&\sum_{m,n} \left(x^{-n+1} y^{-m} - q^2 x^{-n} y^{-m+1}\right) \left(z_1^m z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^m\right) \\
=&\sum_{m,n} x^{-n+1} y^{-m} \left(z_1^m z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^m\right) - \sum_{m,n} q^2 x^{-n} y^{-m+1} \left(z_1^m z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^m\right)
\end{aligned}$$

Relabelling the indices from n to $n+1$ in the first summation and m to $m+1$ in the second summation gives:

$$\begin{aligned}
&\sum_{m,n} x^{-n} y^{-m} \left(z_1^m z_2^{n+1} + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^{n+1} z_2^m\right) - \sum_{m,n} q^2 x^{-n} y^{-m} \left(z_1^{m+1} z_2^n + \frac{q^2 z_1 - z_2}{z_1 - q^2 z_2} z_1^n z_2^{m+1}\right) \\
=&\sum_{m,n} x^{-n} y^{-m} \left(\left(z_2 - q^2 z_1\right) \left(z_1^m z_2^n\right) + \left(z_1 - q^2 z_2\right) \left(\frac{q^2 z_1 - z_2}{z_1 - q^2 z_2}\right) \left(z_1^n z_2^m\right)\right) \\
=&\sum_{m,n} x^{-n} y^{-m} \left(q^2 z_1 - z_2\right) \left(z_1^n z_2^m - z_1^m z_2^n\right)
\end{aligned}$$

Hence, the left hand side is equal to the right hand side. Therefore, the algebra homomorphism ϕ preserves the Vertex relations.
In the case of $\mathfrak{sl}_2(F)$ there are no quantum Serre relations because we only have one colour of variable, so this completes the proof. The quantum Serre relations apply to every pair of different colours $i \neq j$, but we only have one colour in our case.

□

4.4 Negut Shuffle Algebra

In [5], Negut presents a slightly different shuffle algebra. This shuffle algebra also preserves the colour symmetry of the variables.

The product is as follows:

$$\begin{aligned}
R(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots) * R'(\ldots, z_{i1}, \ldots, z_{ik'_i}, \ldots) = \frac{1}{\mathbf{k}! \cdot \mathbf{k}'!} \cdot \\
\operatorname{Sym}\left[R(\ldots, z_{i1}, \ldots, z_{ik_i}, \ldots) R'(\ldots, z_{i,k_i+1}, \ldots, z_{i,k_i+k'_i}, \ldots) \prod_{i,i'=1}^{n} \prod_{j'>k'_{i'}}^{j \leq k_i} \zeta\left(\frac{z_{ij}}{z_{i'j'}}\right)\right]
\end{aligned}$$

where R and R' are rational functions in $\mathbf{k}$ and $\mathbf{k}'$ variables respectively. The "Sym" denotes the symmetrization which preserves the colour of the variables modulo n.

The function ζ is defined to be:

$$\zeta(\frac{z}{w}) = \left(\frac{z q \overline{q}^{2\lceil \frac{i-j}{n} \rceil} - w q^{-1}}{z \overline{q}^{2\lceil \frac{i-j}{n} \rceil} - w}\right)^{\delta^{\frac{\overline{j}}{k}} - \delta^{\frac{\overline{j}}{i+1}}}$$

where q and $\bar{q}$ are nonzero complex parameters, i and j are the colours of the variables z and w respectively, and for any integer k, we write $\bar{k} \in \{0, 1, \ldots, n-1\}$ for the remainder of k divided by n.

4.5 An Example of the Negut Shuffle Product

Let our Lie algbera $\mathfrak{g} = \mathfrak{gl}_2(F)$.

This Lie algebra has two colours of variables associated with it. For our example, we will have two variables coloured 1 and one variable coloured 2. Let the two variables coloured 1 be $z_{1,1}$ and $z_{1,2}$ and the variable coloured 2 be $z_{2,1}$. In this example, we will use f and g to denote our rational functions instead of R and $R^{'}$.

We have:

$$\begin{aligned} f(z_{1,1}, z_{1,2}) * g(z_{2,1}) &= \frac{1}{2!} \cdot \frac{1}{1!} \operatorname{Sym} \left(f(z_{1,1}, z_{1,2}) g(z_{2,3}) \prod_{i,i'=1}^{2} \prod_{j=1}^{2} \prod_{j'=1}^{1} \zeta \left(\frac{z_{ij}}{z_{i'j'}} \right) \right) \\ &= \frac{1}{2} \left[[f(z_{1,1}, z_{1,2}) g(z_{2,3}) + f(z_{1,2}, z_{1,1}) g(z_{2,3})] \left[\zeta \left(\frac{z_{1,1}}{z_{2,1}} \right) \zeta \left(\frac{z_{1,2}}{z_{2,1}} \right) \right] \right] \\ &= [f(z_{1,1}, z_{1,2}) g(z_{2,3})] \left[\zeta \left(\frac{z_{1,1}}{z_{2,1}} \right) \zeta \left(\frac{z_{1,2}}{z_{2,1}} \right) \right] \end{aligned}$$

where:

$$\begin{aligned} \zeta \left(\frac{z_{1,1}}{z_{2,1}} \right) &= \left(\frac{q z_{1,1} - q^{-1} z_{2,1}}{z_{1,1} - z_{2,1}} \right)^{-1} \\ &= \frac{z_{1,1} - z_{2,1}}{q z_{1,1} - q^{-1} z_{2,1}} \end{aligned}$$

and:

$$\begin{aligned} \zeta \left(\frac{z_{1,2}}{z_{2,1}} \right) &= \left(\frac{q z_{1,2} - q^{-1} z_{2,1}}{z_{1,2} - z_{2,1}} \right)^{-1} \\ &= \frac{z_{1,2} - z_{2,1}}{q z_{1,2} - q^{-1} z_{2,1}} \end{aligned}$$

Therefore, we have:

$$f(z_{1,1}, z_{1,2}) * g(z_{2,1}) = [f(z_{1,1}, z_{1,2}) g(z_{2,3})] \left[\left(\frac{z_{1,1} - z_{2,1}}{q z_{1,1} - q^{-1} z_{2,1}} \right) \left(\frac{z_{1,2} - z_{2,1}}{q z_{1,2} - q^{-1} z_{2,1}} \right) \right]$$

In order to preserve colour symmetry, the only variables which could be shuffled were $z_{1,1}$ and $z_{1,2}$.

In [6], Negut presents Theorem 1.5 which is very similar to proposition 8 in [1]. This theorem uses the shuffle product described in 4.4.

References

[1] B. Enriquez, *On Correlation Functions of Drinfield Currents and Shuffle Algebras* Birkhauser Boston, 2000.

[2] K. Erdmann, T. Holm, *Algebras and Representation Theory* Springer, 2010, pages 1-2.

[3] V. Kac, *Infinite dimensional Lie algebras*, Third edition, Chapter 7, Cambridge University Press, 1990.

[4] G. Lee, *Abstract Algebra, an Introductory Course* Springer, 2018, page 135.

[5] A. Negut, *Quantum Toroidal and Shuffle Algebras*, *https://arxiv.org/pdf/1302.6202.pdf*, accessed 5 September 2020.

[6] A. Negut, *A Tale of Two Shuffle Algebras*, *https://arxiv.org/pdf/1908.08395.pdf*, accessed 8 September 2020.

[7] H. A. Priestley, *https://people.maths.ox.ac.uk/flynn/genus2/sheets0405/grfnotes1011.pdf* accessed 25 August 2020.

[8] B. Roberts, *Lie Algebras, University of Idaho*, page 43

[9] https://en.wikipedia.org/wiki/Morphism, accessed 26 May 2020