ON THE REFLECTOR SHAPE DESIGN

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Abstract

In this paper we study the problem of recovering the reflecting surface in a reflector system which consists of a point light source, a reflecting surface, and an object to be illuminated. This problem involves a fully nonlinear partial differential equation of Monge-Ampère type, subject to a nonlinear second boundary condition. A weak solution can be obtained by approximation by piecewise ellipsoidal surfaces. The regularity is a very complicated issue but we found precise conditions for it.

1. Introduction

In this paper we study a reflector system which consists of a light source at the origin O, a reflecting surface Γ, and a bounded, smooth object Σ to be illuminated. Assume that Γ is a radial graph over a domain U in the unit sphere. Let $f \in L^1(U)$ be the distribution of light from O, and let $g \in L^1(\Sigma)$ be a nonnegative function on Σ. We are concerned with the existence and regularity of reflector Γ such that the light from O is reflected off to the object Σ and the density of reflected light on Σ is equal to g.

Due to its practical importance in electromagnetics and optics, this problem has been extensively studied (see Remark 2.4 below). The law of reflection, namely, the angle of reflection is equal to that of incidence, is simple. However, mathematically it is a difficult open problem (see [Y], problem 21). Even an explicit workable equation, which can be obtained by calculating the Jacobian determinant of the reflection, has never been worked out, as it requires very tricky computation (see Remark 2.3).

In this paper we will first derive the equation. It is a fully nonlinear partial differential equation of Monge-Ampère type, subject to a non-linear second boundary condition (see (1.2). (1.4) below). In particular when the receiver Σ is a domain in a plane passing through the origin, it becomes the standard Monge-Ampère equation. By approximation by piecewise ellipsoidal surfaces, we prove that for any point p in the cone $\mathcal{C}_U = \{tX \mid t > 0, X \in U\}$, there is a weak solution such that the

reflector Γ passes through the point p (Theorem A). However, the regularity is an extremely complicated issue, we find the following interesting phenomena.

- The regularity depends on the position of the reflector. That is, there is a region $\mathcal{D}$ in the cone $\mathcal{C}_U$, depending only on the domain U and the receiver Σ, such that the part of the reflector Γ in $\mathcal{D}$ is smooth, and the part of Γ outside $\mathcal{D}$ may not be smooth (for smooth and positive distributions f, g).
- The domain $\mathcal{D}$ depends on the position and geometry of the object Σ. That is, $\mathcal{D}$ varies if one translates, rotates, or bends the surface Σ.
- The domain $\mathcal{D}$ also depends on the geometry of the boundary $\partial\Sigma$. Namely, $\mathcal{D}$ varies if one deforms the boundary $\partial\Sigma$ smoothly.

These phenomena are special for the reflector problem. Recall that for the second boundary value problem of the Monge-Ampère equation (see (1.5), (1.6) below), or for the reflector problem in the far field case (see (1.7), (1.8) below), the solution is unique up to a constant, and if one solution is smooth, so are all the other solutions. The above phenomena show that the regularity of the reflector problem is a very complicated issue. However in this paper we give a complete resolution for the regularity. In particular we give precise conditions for a point to be in the region $\mathcal{D}$ (see Theorem C below).

In applications it is natural to study the reflector problem in the Euclidean 3-space $\mathbb{R}^3$. But in this paper we will work directly in $\mathbb{R}^{n+1}$, $n \geq 2$, as it does not impose any substantial new difficulty in our treatment. Represent the reflector Γ in the polar coordinate system as a radial graph of a positive function ρ,

$$(1.1) \qquad \Gamma = \{X\rho(X) \mid X \in U\}.$$

Identify a ray from the origin O with a unit vector $X \in S^n$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$. Suppose the ray X is reflected off at a point $X\rho(X) \in \Gamma$ to the point $Z \in \Sigma$. We get a mapping $T : X \to Z$. Computing the Jacobian determinant of T in a local orthonormal frame, we obtain our equation

$$(1.2) \qquad \det\left\{ -D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho - \frac{\cos\theta}{\sin\theta}|D\rho|I \right\} = h \ \ \text{in} \ \ U,$$

where I is the unit matrix, D is the covariant derivative, θ is the angle between the vectors OX and OZ, and h is a positive function depending on $f, g, \rho, D\rho, Z$, and Σ (see (2.34)). We assume that Σ is given implicitly by

$$(1.3) \qquad \Sigma = \{p \in \mathbb{R}^{n+1} \mid \psi(p) = 0\}.$$

In (1.2) we denote by $a \otimes b$ the tensor product, namely, for any two vectors $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_m)$, $a \otimes b = \{a_i b_j\}$ is an

$n \times m$-matrix. The boundary condition for equation (1.2) is the natural one

$$(1.4) \qquad\qquad T(U) = \Sigma.$$

A special case is when Σ is a domain in the plane $\{x_{n+1} = 0\}$. In this case equation (1.2) can be reduced to the standard Monge-Ampère equation

$$(1.5) \qquad\qquad \det D^2 u = h \quad \text{in} \ \ \Omega,$$

where $u = \frac{1}{\rho}$, $h = h(x, u, Du)$, Ω is the projection of U in the plane $\{x_{n+1} = 0\}$, and $x = (x_1, \cdots, x_n) \in \Omega$. Equation (1.5) is an important fully nonlinear equation with various applications in geometry and applied sciences, and has been studied in the last century by many authors. The existence and interior regularity of solutions to the Dirichlet problem were proved in [CY, P2]. The regularity near the boundary was established in [CNS, K]. The boundary condition (1.4) is related, but not equivalent, to the second boundary problem of the Monge-Ampère equation, that is, prescribing the image of the gradient mapping,

$$(1.6) \qquad\qquad Du(\Omega) = \Omega^*,$$

where Ω^* is a domain in $\mathbb{R}^n$. The existence of a weak solution of (1.5) (1.6), for appropriate h, can be found in [B, P1], and the interior regularity was proved by Caffarelli [C2], under the necessary condition that Ω^* is convex. Caffarelli [C3] also proved the global $C^{2,\alpha}$ regularity if both domains Ω and Ω^* are uniformly convex and $C^{2,\alpha}$ smooth, provided $h \in C^\alpha(\overline{\Omega})$, where $\alpha \in (0, 1)$. The global regularity was also obtained by Delanoe [D] in dimension two and Urbas [U] in higher dimensions for $h \in C^{1,1}(\overline{\Omega})$. A crucial ingredient in these papers is a duality, which is not available for the general reflector problem.

Another special case of the reflector problem is the so-called far field case. It can be regarded as the limit of the above problem with $\Sigma = \{dX \mid X \in V\}$, $d \to \infty$, where V is a domain in S^n. The far field case is related to the reflector antenna and has been extensively studied. Suppose a ray X is reflected off by Γ to a direction Y. By computing the Jacobian determinant of the mapping $T: \ X \to Y$, we obtain the equation

$$(1.7) \qquad \det\Big\{-D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho - \frac{|D\rho|^2 - \rho^2}{2\rho}I\Big\} = \frac{f(X)}{g(Y)}h.$$

where $h = (\frac{|D\rho|^2 + \rho^2}{2\rho})^n$ (see [ON, W1, GW]). In the far field case, the boundary condition (1.4) is replaced by

$$(1.8) \qquad\qquad T(U) = V.$$

The existence and regularity of weak solutions to (1.7) and (1.8) were first established in [W1] in dimension $n = 2$, which can be extended

to high dimensions by the a priori estimates in [GW]. The regularity near the boundary was obtained in a recent paper [TW2]. By a duality, namely, a Legendre type transform, in [W1, GW], the far field case is an optimal transportation problem, as stated in Theorem 4.1; [W2] (see also [W3, GO]). Mathematically one may also consider the case when the reflector is a closed surface without boundary. In this case, the existence of weak solutions was proved in [CO], and the regularity was proved in [GW] if $f, g \in C^\infty$ and f, g are pinched by two positive constants. In [CGH] the reflector was proved to be C^1 smooth, assuming only that f, g are pinched by two positive constants. The C^1 regularity was also obtained in [L, TW3] by different proofs.

We always assume that the reflection system is ideal, namely, there is no loss of energy in reflection. Then an obvious compatibility condition for the reflector problem is the energy conservation

$$(1.9) \qquad \int_U f = \int_\Sigma g.$$

We also assume that U is a domain in the unit sphere S^n with Lipschitz boundary, and Σ is a C^2 smooth surface with Lipschitz boundary. If Σ is a radial graph given in (1.12), we also assume that V is a domain with Lipschitz boundary.

Equation (1.2) is a fully nonlinear equation. It is elliptic when the matrix

$$(1.10) \qquad \mathcal{W} = -D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho - \frac{\cos\theta}{\sin\theta}|D\rho|I$$

is positive (or negative) definite. In Section 4.1 we will introduce the R-convexity of functions so that the matrix $\mathcal{W}$ is positive definite. With the R-convexity we can also introduce the notion of weak solutions. Our main existence results are as follows.

Theorem A. *Consider the reflector problem with distributions f and g satisfying the balance condition (1.9).*

(a): *For any point p in $\mathcal{C}_U = \{p \in \mathbb{R}^{n+1} \mid \frac{p}{|p|} \in U\}$ with $|p| > 2\sup_{q\in\Sigma}|q|$, there is a weak solution ρ_p to the reflector problem such that the reflector Γ_{ρ_p}, the radial graph of ρ_p, passes through the point p.*

(b): *Suppose that Σ is contained in the cone $\mathcal{C}_V = \{tX \mid t > 0, X \in V\}$ for a domain $V \subset S^n$ and*

$$(1.11) \qquad \overline{U} \cap \overline{V} = \emptyset,$$

where $\overline{U}$ and $\overline{V}$ denote the closures of U and V. Then for any point $p \in \mathcal{C}_U$, there is a weak solution ρ_p such that the reflector Γ_{ρ_p} passes through the point p.

Part (a) was essentially proved in [KO]. In [KO] the authors proved the existence of a weak solution Γ_ρ satisfying $\inf \rho \geq 2\sup_{q\in\Sigma}|q|$. Part

(b) can be established in a similar way, using the Harnack inequality and the gradient estimate in §4.2 below. Alternatively one can prove Theorem A by a Perron method, as in [W1]. See also Remark 4.4 below. We point out that the proof of Theorem A does not use the explicit equation (1.2). See Remark 4.3 for more details.

The regularity is a much more complicated issue. In addition to the R-convexity of solutions, we also need a convexity condition on the boundary $\partial\Sigma$, namely, R-convexity of $\partial\Sigma$, which will be introduced in §4.4. Recall that to obtain the regularity of solutions to (1.5), (1.6), one must assume that Ω^* is convex [MTW].

Theorem B. *Suppose the distributions f and g are smooth, positive, and satisfy (1.9). Suppose $\Sigma \subset C_V$ for some $V \subset S^n$ and $\overline{U} \cap \overline{V} = \emptyset$. Then we have the following results.*

> **(a):** *If Σ is a smooth radial graph over V, namely,*

$$(1.12) \qquad\qquad \Sigma = \{X\varphi(X) \mid X \in V\}$$

> *for some smooth, positive function φ, and if $\partial\Sigma$ is R-convex, then there is a small, smooth reflector.*
> **(b):** *If furthermore Σ is convex and*

$$(1.13) \qquad\qquad |(q - p) \cdot \nu| > 0$$

> *for any point $p \in C_U$, $q \in \Sigma$, then all weak solutions in Theorem A are smooth, where ν is the normal of Σ at q, and $\cdot$ denotes inner product in $\mathbb{R}^{n+1}$.*
> **(c):** *If Σ is not convex, there exist smooth, positive distributions f and g and a large solution ρ which is not C^1 smooth.*
> **(d):** *If $\partial\Sigma$ is not R-convex with respect to $p \in C_U$, then there exist smooth, positive distributions f and g such that the weak solution ρ_p is not C^1 smooth near p.*

We say that a reflector Γ given by (1.1) is small if $\sup \rho$ is small (compare with the distance from the origin to the object Σ), and Γ is large if $\inf \rho$ large. Assumption (1.13) means that a tangent plane of Σ does not intersect with the cone C_U. It also implies that the right-hand side h of equation (1.2) is a positive function, which is crucial for the regularity of Monge-Ampère type equation. Note that we assume (1.13) holds for all $p \in C_U$. Hence it implies (1.11) and (1.12). But if one is concerned with the regularity of a particular solution Γ, it suffices to assume that (1.13) holds for all points $p \in \Gamma$ and $q \in \Sigma$, and (1.13) is independent of (1.11) and (1.12).

A more precise statement of our regularity is the following.

Theorem C. *Assume that the distributions f and g are smooth, positive, and satisfy the balance condition (1.9). Let ρ_p be the weak solution in Theorem A.*

(a): *There exists a region $\mathcal{D} \subset \mathcal{C}_U$, depending on U and Σ but independent of f, g, such that if $p \in \mathcal{D}$, then Γ_{ρ_p} is smooth near p.*

(b): *A point $p \in \mathcal{D}$ if and only if the following conditions are satisfied:*

(i) $|(q - p) \cdot \nu| > 0 \quad \forall \ \ q \in \Sigma$;

(ii) $\partial\Sigma$ is R-convex with respect to points near p;

(iii) for any $q \in \Sigma$, the following inequality holds

$$(1.14) \qquad II + \frac{\cos\beta}{2\cos^2\alpha} \frac{|q|}{|p| \, |p - q|} I > 0,$$

where I is the unit matrix, II is the second fundamental form (in an orthonormal frame) of Σ along the direction $p - q$, α is the angle of reflection, and β is the angle between Oq and the normal of Σ at q. See (3.30).

(c): *If one of the above three conditions (i)–(iii) is violated, there exist smooth, positive distributions f, g such that the weak solution ρ_p is not C^1 smooth near p.*

Some remarks are in order. We first note that Theorem B is contained in Theorem C. Indeed, from part (b) of Theorem C, we see that if $\partial\Sigma$ is R-convex, and if Σ is a radial graph given in (1.12) with $\overline{U} \cap \overline{V} = \emptyset$, then $B_r(0) \cap \mathcal{C}_U \subset \mathcal{D}$ for $r > 0$ small. If furthermore Σ is convex and (1.13) holds, then $\mathcal{D} = \mathcal{C}_U$. Therefore parts (a) and (b) of Theorem B are contained in Theorem C. Obviously parts (c) and (d) of Theorem B are contained in Theorem C(c).

The regularity in Theorem C is a local result. That is, if Γ is a weak solution obtained in Theorem A, then the part $\Gamma \cap \mathcal{D}$ is smooth, whereas the remaining part may not be smooth. The set $\mathcal{D}$ can be empty, such as when $\partial\Sigma$ is not R-convex for all $p \in \mathcal{C}_U$. It is worth to point out that in Theorem C we do not assume $\overline{U} \cap \overline{V} = \emptyset$. Hence Theorem C applies to weak solutions obtained in Theorem A(a).

Inequality (1.14) has two equivalent forms. Let $\tau = \frac{\cos\theta}{\sin\theta}|D\rho|$ denote the last term in the matrix in (1.2), in which θ also depends on $D\rho$. Then (1.14) is equivalent to the property that τ is a convex function of $\xi =: D\rho$, namely,

$$(1.15) \qquad \{\tau_{\xi_k \xi_l}\} > \delta I,$$

where δ is a positive constant. (1.15) is similar to the condition (A3) in optimal transportation [MTW], which is also a necessary condition for the regularity of optimal mappings. A geometric interpretation of (1.14) is as follows. Let $E_i = \{X e_i(X) \mid X \in S^n\}$ $(i = 0, 1, 2)$ be ellipsoids of revolution with two foci, one at the origin and the other one on Σ. Suppose all E_0, E_1, and E_2 pass the point p and $\gamma_2 = \frac{t\gamma_0 + (1-t)\gamma_1}{|t\gamma_0 + (1-t)\gamma_1|}$ for some $t \in (0, 1)$, where γ_i is the normal of E_i at p. Then (1.14) is

equivalent to

(1.16) $E_0^s \cap E_1^s \subset E_2^s$ and $\{E_0^s \cap E_1^s\} \cap E_2^s = \{p\},$

where E^s denotes the solid body enclosed by E.

In part (c), we say (1.14) is violated if the least eigenvalue of the matrix in (1.14) is negative. It is the borderline when the matrix is semi-definite, such as in the case when $\Sigma \subset \{x_{n+1} = 0\}$. In this case we have the standard Monge-Ampère equation (1.5), for which the interior second derivative estimate holds only for strictly convex solutions [C1, P2]. In §6 we show that if no open subset of Σ is contained in a plane passing through the origin, then for any point $p \in \mathcal{C}_U - \overline{\mathcal{D}}$, at least one condition of (i)–(iii) is violated at p. Hence there exist smooth, positive distributions f, g such that the weak solution ρ_p is not C^1 smooth near p.

Finally we remark that the reflector problem studied in this paper is not an optimal transportation problem. Indeed, if Σ is a domain in the plane $\{x_{n+1} = 0\}$, we get the standard Monge-Ampère equation (1.5). The equation of optimal transportation has the form

(1.17) $\det[D^2 u - c_{xx}(x, y)] = h,$

so the cost function c must be linear in x, namely, $c(x, y) = x \cdot \psi(y)$. Recall that the optimal mapping $T_u : x \to y$ is determined by $D_x c(x, y) = Du(x)$. We conclude that $\psi(y) = Du$, which is in contradiction with (2.8) below. If Σ is a radial graph given in (1.12), we assume that Σ satisfies (1.13) and $\partial\Sigma$ is R-convex, but Σ is not convex. Then $B_r(0) \cap \mathcal{C}_U \subset \mathcal{D}$ for small r but $\mathcal{D} \neq \mathcal{C}_U$. Hence by Theorem C, there exists smooth, positive densities f, g, such that the reflector problem has both smooth and nonsmooth solutions. Recall that in optimal transportation, if one solution is smooth, then all other solutions are smooth too.

This paper is arranged as following. In Section 2 we derive the equation (1.2). We first consider the case when Σ is a domain in the plane $\{x_{n+1} = 0\}$, then consider the general case with Σ given in (1.3). In the section we also discuss briefly properties of ellipsoids of revolution, which play a crucial role in the reflector problem.

In Section 3 we establish the a priori estimates for R-convex solutions, and prove that (1.15) is equivalent to (1.14).

In Section 4 we prove Theorem A. That is, for any point $p \in \mathcal{C}_U$, there is a weak R-convex solution ρ_p such that Γ_{ρ_p} passes through the point p.

In Section 5 we first show that (1.15) is equivalent to the geometric characterization (1.16), which is crucial for the proof of both parts (b) and (c) of Theorem C. Then we prove the regularity result in Theorem C. Using the a priori estimates in §3, we prove that if p is a point in $\mathcal{D}$ and if f, g are positive and smooth, then Γ_{ρ_p} is smooth near p.

In Section 6 we prove that if any condition in Theorem C(b) is violated at a point $p \in \mathcal{C}_U$, then there exists smooth, positive distributions f, g such that the reflector Γ_{ρ_p} is not smooth near p.

Finally, in Section 7, we discuss very briefly R-concave solutions.

We would like to point out that some arguments in §4–§6 are inspired by previous works on the topic [W1, KO] as well as recent work on the optimal transportation [MTW].

2. Derivation of equations

In the far field case, the equation was derived in the polar coordinate system by computing directly on the sphere. But the computation becomes too complicated in the general case. So we project U to a domain Ω in the plane $x_{n+1} = 0$ and regard ρ as a function in Ω. By restricting to a subset we may assume that U is in the north hemisphere. In this section we show that ρ satisfies the equation

$$(2.1) \qquad \det\Big\{ - D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho + \frac{az_{n+1}}{2\rho(\rho x_{n+1} - z_{n+1})}\mathcal{N} \Big\} = h$$

in $\Omega \subset \{x_{n+1} = 0\}$, where a is given in (2.11),

$$(2.2) \qquad\qquad\qquad \mathcal{N} = I + \frac{x \otimes x}{1 - |x|^2},$$

and h is the right-hand side of (2.34) below. h is a positive function of $f, g, \rho, \nabla\rho$ and Σ.

We will first consider the case when the receiver Σ lies in the hyperplane $\{x_{n+1} = 0\}$, and then consider the general case when Σ is given implicitly by (1.3). From (2.1) we then deduce the equation on the sphere. By taking limit we also obtain the equation for the reflector problem in the far field case.

2.1. The case $\Sigma \subset \{x_{n+1} = 0\}$**.** Represent the reflector Γ as a radial graph of ρ as in (1.1). Suppose the ray $X = (x_1, \cdots, x_{n+1})$ is reflected off at a point $X\rho(X) \in \Gamma$ in direction $Y = (y_1, \cdots, y_{n+1})$, $Y \in S^n$, and reaches a point $Z = (z_1, \cdots, z_{n+1}) \in \Sigma$. Denote by γ the unit normal of Γ and by $T : X \to Z$ the reflection mapping. Then by the reflection law,
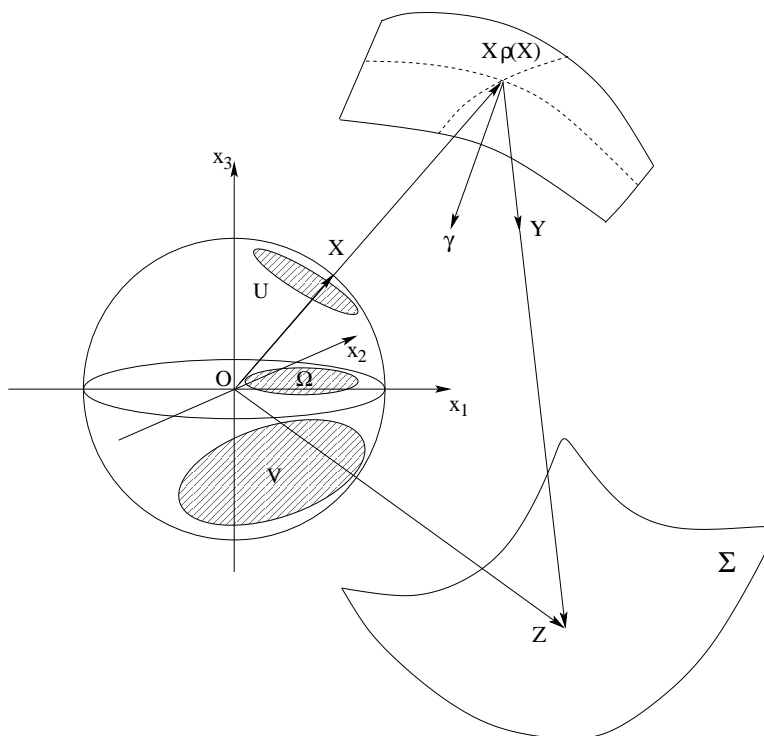
$$(2.3) \qquad\qquad\qquad Y = X - 2(X \cdot \gamma)\gamma.$$

Hence

$$(2.4) \qquad\qquad\qquad Z = X\rho + Yd,$$

where $d = |Z - X\rho|$ is the distance from $X\rho$ to Z, and $X \cdot \gamma$ denotes the inner product in $\mathbb{R}^{n+1}$.

Let Ω be the projection of U on $\{x_{n+1} = 0\}$, so that

$$x = (x_1, \cdots, x_n) \in \Omega$$

if and only if

$$X = (x, x_{n+1}) \in U, \quad x_{n+1} = \sqrt{1 - |x|^2}.$$

In the following we regard ρ as a function on Ω, and T as a mapping on Ω. Therefore we may also write $\rho = \rho(x)$ and $T = T(x)$.

Let dS denote the surface area element. Denote $\partial_i = \frac{\partial}{\partial x_i}$, $z = (z_1, \cdots, z_n)$, $z_{i,j} = \partial_j z_i$, $i, j = 1, \cdots, n$. Then the Jacobian determinant of the mapping T is given by

$$(2.5) \qquad J = \frac{dS_\Sigma}{dS_\Omega} = \begin{vmatrix} z_{1,1}, & \cdots, & z_{1,n}, & \nu_1 \\ \vdots & \ddots & \ddots & \vdots \\ z_{n,1}, & \cdots, & z_{n,n}, & \nu_n \\ z_{n+1,1}, & \cdots, & z_{n+1,n}, & \nu_{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} z_{1,1}, & \cdots, & z_{1,n} \\ \vdots & \ddots & \vdots \\ z_{n,1}, & \cdots, & z_{n,n} \end{vmatrix} = \det Dz,$$

where $\nu = (0, \cdots, 0, 1)$ is the normal of Σ.

Let f and g be the energy distributions on U and Σ, respectively. Note that $dS_\Omega = \omega dS_U$, where

$$\omega(x) = \sqrt{1 - |x|^2}.$$

Hence we have the equation

$$(2.6) \qquad \det Dz = \frac{f}{\omega g}.$$

The explicit formulae for Y, Z, the Jacobian matrix Dz, and the equation (2.6) are given in the following proposition.

Proposition 2.1. *We have*

$$(2.7) \qquad Y = \frac{a}{b} X + \frac{2\rho}{b} \hat{D}\rho,$$

$$(2.8) \qquad Z = -\frac{2\rho^2}{a} \hat{D}\rho,$$

$$(2.9) \qquad Dz = \frac{2\rho^2}{a}(I - q)\left\{ - D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho \right\},$$

$$(2.10) \qquad det\left\{ - D^2\rho + 2\rho^{-1}D\rho \otimes D\rho \right\} = \frac{-a^{n+1}}{2^n \rho^{2n} b} \frac{f}{\omega g},$$

where I is the unit matrix, $D\rho = (\partial_1\rho, \cdots, \partial_n\rho)$ is the gradient of ρ, $\hat{D}\rho = (D\rho, 0)$, $D^2\rho = (\partial_i\partial_j\rho)$ is the Hessian matrix of ρ,

$$(2.11) \qquad a = |D\rho|^2 - (\rho + D\rho \cdot x)^2,$$

$$(2.12) \qquad b = |D\rho|^2 + \rho^2 - (D\rho \cdot x)^2,$$

$$q = \frac{2}{a} D\rho \otimes [D\rho - (\rho + D\rho \cdot x)x].$$

Remark 2.1. Let $u = \frac{1}{\rho}$. Then from (2.10), u satisfies the standard Monge-Ampère equation

$$(2.13) \qquad \det D^2 u = h(x, u, Du)$$

for some h. Noting that U lies in the north hemisphere, we have $x_{n+1} > 0$, $y_{n+1} < 0$. Hence by (2.16) below,

$$(2.14) \qquad a = \frac{y_{n+1}}{x_{n+1}} b < 0.$$

Hence if ρ is an R-convex solution (see §4.1 for definition), the matrix $\{-D^2\rho + 2\rho^{-1}D\rho \otimes D\rho\}$ is positive definite, and Dz is negative definite. Therefore when dimension n is odd, we should replace the left-hand side of (2.6) by $|\det Dz|$, and accordingly the right-hand side of (2.10) should take absolute value.

Proof. First we show that the unit normal is given by

$$(2.15) \qquad \gamma = \frac{\hat{D}\rho - X(\rho + D\rho \cdot x)}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2}}.$$

Indeed, since $\partial_{x_k}(X\rho)$ is tangential to Γ, we have

$$\partial_{x_k}(X\rho) \cdot \gamma = 0, \quad k = 1, \cdots, n.$$

Write $\gamma = (\gamma', \gamma_{n+1})$. Then the above formula becomes

$$(\rho I + D\rho \otimes x)\gamma' = (\frac{x}{\omega} - \omega D\rho)\gamma_{n+1}.$$

Multiplying both sides by

$$(\rho I + D\rho \otimes x)^{-1} = \rho^{-1}\Big(I - \frac{D\rho \otimes x}{\rho + D\rho \cdot x}\Big),$$

we obtain (2.15). One easily verifies that $|\gamma| = 1$.
   From (2.15),

$$X \cdot \gamma = -\frac{\rho}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2}}.$$

By (2.3),

$$
\begin{aligned}
(2.16) \qquad y_{n+1} &= x_{n+1} + \frac{2\rho\gamma_{n+1}}{\sqrt{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2}} \\
&= x_{n+1}\Big(1 - \frac{2\rho(\rho + D\rho \cdot x)}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2}\Big) \\
&= x_{n+1}\frac{|D\rho|^2 - (\rho + D\rho \cdot x)^2}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2} \\
&= x_{n+1}\frac{a}{b}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
Y &= X - 2(X \cdot \gamma)\gamma \\
&= X + \frac{2\rho(\hat{D}\rho - X(\rho + D\rho \cdot x))}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2} \\
&= X\Big(1 - \frac{2\rho(\rho + D\rho \cdot x)}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2}\Big) + \frac{2\rho\hat{D}\rho}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2} \\
&= X\frac{|D\rho|^2 - (\rho + D\rho \cdot x)^2}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2} + \frac{2\rho\hat{D}\rho}{\rho^2 + |D\rho|^2 - (D\rho \cdot x)^2} \\
&= X\frac{a}{b} + \frac{2\rho\hat{D}\rho}{b}.
\end{aligned}
$$

We obtain (2.7). Formula (2.8) follows readily from (2.7). Indeed, in the case $\Sigma \subset \{x_{n+1} = 0\}$, we have

$$d = -\rho\frac{x_{n+1}}{y_{n+1}}.$$

Hence by (2.4),

$$
\begin{aligned}
Z &= X\rho - Y\rho \frac{x_{n+1}}{y_{n+1}} \\
&= X\rho - \big(X\frac{a}{b} + \frac{2\rho\hat{D}\rho}{b}\big)\rho \frac{x_{n+1}}{y_{n+1}}.
\end{aligned}
$$

By (2.16) we then have

$$
Z = -\frac{2\rho^2 \hat{D}\rho}{b} \frac{x_{n+1}}{y_{n+1}} = -\frac{2\rho^2}{a}\hat{D}\rho.
$$

Next we derive (2.9). Differentiating (2.8), we get

$$
z_{i,j} = -\rho_{ij}\frac{2\rho^2}{a} - 4\rho_i\rho_j\frac{\rho}{a} + 2\rho_i\frac{\rho^2 a_j}{a^2},
$$

where $i, j = 1, \cdots, n$,

$$
\begin{aligned}
a_j &= 2\sum_k \rho_k\rho_{kj} - 2\big(\rho + D\rho \cdot x\big)\big(\rho_j + \sum_k (\rho_{kj}x_k + \rho_k\delta_{kj})\big) \\
&= -4(\rho + D\rho \cdot x)\rho_j + 2\sum_k \rho_{kj}\big(\rho_k - (\rho + D\rho \cdot x)x_k\big),
\end{aligned}
$$

and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Therefore

$$
\begin{aligned}
z_{i,j} &= \frac{2\rho^2}{a}\Big[-\rho_{ij} + \frac{2\rho_i}{a}\sum_k \rho_{kj}\big(\rho_k - (\rho + D\rho \cdot x)x_k\big)\Big] \\
&\quad -\frac{4\rho}{a}\rho_i\rho_j\Big[1 + \frac{2\rho}{a}(\rho + D\rho \cdot x)\Big].
\end{aligned}
$$

By our definition of a and b,

$$
1 + \frac{2\rho}{a}(\rho + D\rho \cdot x) = \frac{b}{a}.
$$

Hence

(2.17)

$$
\begin{aligned}
Dz &= \Big\{\frac{2\rho^2}{a}\Big[-\big(\delta_{ik} - \frac{2\rho_i}{a}(\rho_k - (\rho + D\rho \cdot x)x_k)\big)\rho_{kj}\Big] - \frac{4\rho b}{a^2}\rho_i\rho_j\Big\} \\
&= -\frac{2\rho^2}{a}\Big\{(I - q)D^2\rho + \frac{2b}{\rho a}D\rho \otimes D\rho\Big\}, \\
&= -\frac{2\rho^2}{a}(I - q)\Big\{D^2\rho + \frac{2b}{\rho a}(I - q)^{-1}D\rho \otimes D\rho\Big\},
\end{aligned}
$$

where

(2.18)

$$
\begin{aligned}
q &= \frac{2}{a}[D\rho \otimes D\rho - (\rho + D\rho \cdot x)D\rho \otimes x] \\
&= \frac{2}{a}D\rho \otimes [D\rho - (\rho + D\rho \cdot x)x].
\end{aligned}
$$

Lemma 2.1. *We have*

$$(2.19) \qquad \frac{b}{a}(I - q)^{-1} D\rho \otimes D\rho = -D\rho \otimes D\rho.$$

Suppose Lemma 2.1 for a moment. By Lemma 2.1 and (2.17) we obtain (2.9). Formula (2.10) follows directly from (2.6) and (2.9), as we have

$$\frac{f}{\omega g} = \det Dz = \frac{2^n \rho^{2n}}{a^n} \det[I - q] \det\left[- D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho\right].$$

This completes the proof of Proposition 2.1. *q.e.d.*

Proof of Lemma 2.1. To compute the inverse matrix of $I - q$, we observe that if a matrix $\mathcal{M} = I + \xi \otimes \eta$ for any vectors ξ and η, then

$$(2.20) \qquad \det\mathcal{M} \quad = \quad 1 + \xi \cdot \eta,$$

$$(2.21) \qquad \mathcal{M}^{-1} \quad = \quad I - \frac{\xi \otimes \eta}{1 + \xi \cdot \eta}.$$

Hence

$$(2.22) \qquad \det[I - q] \quad = \quad -\frac{b}{a},$$

$$(2.23) \qquad (I - q)^{-1} \quad = \quad I - \frac{a}{b}q.$$

Recall that $-\frac{b}{a} > 0$, as noted after Proposition 2.1.

Next observe that

$$(2.24) \qquad (\alpha \otimes \beta)(\xi \otimes \eta) = (\beta \cdot \xi)\,\alpha \otimes \eta$$

for any vectors α, β, ξ, η. Hence

$$\begin{aligned}
\frac{b}{a}(I - q)^{-1} D\rho \otimes D\rho \quad &= \quad \frac{b}{a}(I - \frac{a}{b}q)D\rho \otimes D\rho \\
&= \quad \frac{b}{a} D\rho \otimes D\rho - \frac{2}{a} D\rho \otimes [D\rho - (\rho + D\rho \cdot x)x]\, D\rho \otimes D\rho \\
&= \quad \frac{1}{a}[b - 2D\rho \cdot [D\rho - (\rho + D\rho \cdot x)x]D\rho \otimes D\rho \\
&= \quad -D\rho \otimes D\rho.
\end{aligned}$$

Lemma 2.1 is proved. q.e.d.

2.2. The general case. Assume that the receiver Σ is given implicitly by (1.3). We assume that

$$\nabla\psi \cdot (Z - X\rho) > 0.$$

If the above inequality does not hold, it suffices to replace ψ by $-\psi$. Denote

$$(2.25) \qquad Z_0 = -\frac{2\rho^2}{a}\hat{D}\rho,$$

that is, Z_0 is the intersection of the output ray with the plane $\{x_{n+1} = 0\}$. Here we allow that $y_{n+1} > 0$; see Remark 2.2 (iv) and the picture below.

Denote $z_0 = -\frac{2\rho^2}{a}D\rho$, so that $Z_0 = (z_0, 0)$. The Jacobian determinant of the mapping $T : X \to Z$ is given by

$$J = \frac{dS_\Sigma}{dS_\Omega} = \frac{-1}{|\nabla\psi|} \begin{vmatrix} z_{1,1}, & \cdots, & z_{1,n}, & \psi_1 \\ \vdots & \ddots & \ddots & \vdots \\ z_{n,1}, & \cdots, & z_{n,n}, & \psi_n \\ z_{n+1,1}, & \cdots, & z_{n+1,n}, & \psi_{n+1} \end{vmatrix},$$

where $\psi_k = \partial_{x_k}\psi$ and $\nabla\psi = (\psi_1, \cdots, \psi_{n+1})$. Differentiating $\psi(Z) = 0$ gives

$$z_{n+1,i} = -\frac{1}{\psi_{n+1}} \sum_{k=1}^{n} \psi_k z_{k,i}.$$

Hence

$$J = \frac{-1}{|\nabla\psi|\psi_{n+1}} \begin{vmatrix} z_{1,1}, & \cdots, & z_{1,n}, & \psi_1 \\ \vdots & \ddots & \ddots & \vdots \\ z_{n,1}, & \cdots, & z_{n,n}, & \psi_n \\ 0, & \cdots, & 0, & |\nabla\psi|^2 \end{vmatrix} = -\frac{|\nabla\psi|}{\psi_{n+1}}\det Dz.$$

We obtain the equation

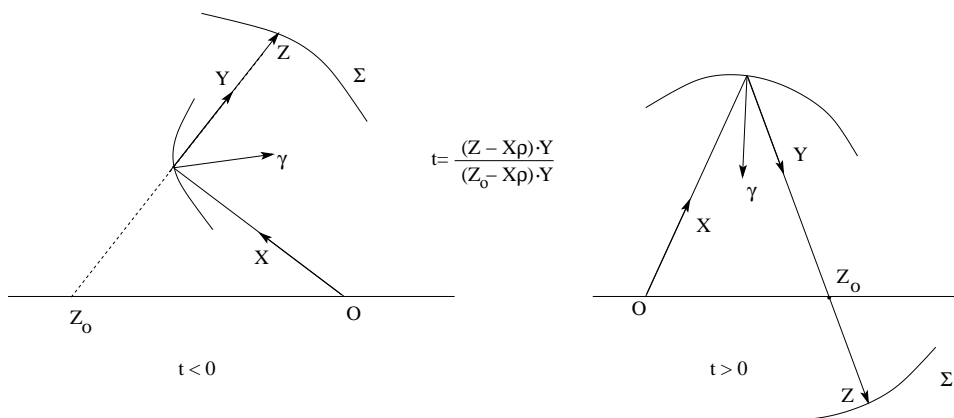(2.26) $$\det Dz = -\frac{f\psi_{n+1}}{\omega g |\nabla\psi|}.$$

Now we compute the matrix Dz. We write the mapping T in terms of the stretch function t in the form

(2.27) $$Z = X\rho + t(Z_0 - X\rho),$$

where Z_0 is the mapping given in (2.25). Then

$$t = \frac{(Z - X\rho)\cdot e_{n+1}}{(Z_0 - X\rho)\cdot e_{n+1}} = \frac{\rho x_{n+1} - z_{n+1}}{\rho x_{n+1}}.$$

Alternatively, we can express the mapping T in the form (2.4). But we found that the computations below are simpler if one uses (2.27) rather than (2.4), as we can use some previous formulas.

From (2.27),

$$(2.28) \qquad Dz = (1-t)D(x\rho) + (z_0 - x\rho) \otimes Dt + tDz_0.$$

Differentiating the identity $\psi((1-t)X\rho + tZ_0) \equiv 0$ gives

$$
\begin{aligned}
Dt &= -\beta \nabla\psi \cdot [(1-t)D(X\rho) + tDZ_0] \\
&= -\beta \partial\psi \cdot [(1-t)D(x\rho) + tDz_0] - \beta(1-t)\psi_{n+1}D(x_{n+1}\rho),
\end{aligned}
$$

where

$$\beta = \frac{1}{(Z_0 - X\rho) \cdot \nabla\psi}$$

and $\partial\psi = (\psi_1, \cdots, \psi_n)$. We have used the fact that the $(n+1)^{th}$-component of Z_0 vanishes. Inserting the above formula to (2.28) and observing that

$$(2.29) \qquad\qquad \xi \otimes (\eta A) = (\xi \otimes \eta)A$$

for any matrix A and vectors ξ, η, which can be verified directly, we obtain,

$$
\begin{aligned}
Dz &= \mathcal{B}\{tDz_0 + (1-t)D(x\rho)\} \\
&\qquad -\beta(1-t)\psi_{n+1}(z_0 - x\rho) \otimes D(x_{n+1}\rho),
\end{aligned}
$$

where

$$\mathcal{B} = I - \beta(z_0 - x\rho) \otimes \partial\psi.$$

By (2.20), (2.21),

$$
\begin{aligned}
(2.30) \qquad \det\mathcal{B} &= 1 - \beta(z_0 - x\rho) \cdot \partial\psi \\
&= 1 - \beta[(Z_0 - X\rho) \cdot \nabla\psi + \rho x_{n+1}\psi_{n+1}] \\
&= -\beta\rho x_{n+1}\psi_{n+1}, \\
\mathcal{B}^{-1} &= I - \frac{(z_0 - x\rho) \otimes \partial\psi}{x_{n+1}\rho\psi_{n+1}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
Dz &= \mathcal{B}\{tDz_0 + (1-t)D(x\rho) \\
&\quad - \beta(1-t)\psi_{n+1}\big(I - \frac{(z_0 - x\rho) \otimes \partial\psi}{x_{n+1}\rho\psi_{n+1}}\big)(z_0 - x\rho) \otimes D(x_{n+1}\rho)\} \\
&= \mathcal{B}\{tDz_0 + (1-t)D(x\rho) + \frac{1-t}{x_{n+1}\rho}(z_0 - x\rho) \otimes D(x_{n+1}\rho)\}.
\end{aligned}
$$

From Proposition 2.1,

$$
Dz_0 = \frac{2\rho^2}{a}[I - q](-D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho).
$$

Hence

(2.31) $$\qquad Dz = \frac{2t\rho^2}{a}\mathcal{B}[I - q]\big\{ - D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho + \frac{a(1-t)}{2t\rho}\mathcal{A}\big\},$$

where

$$
\mathcal{A} = \frac{1}{\rho}(I - q)^{-1}\big[D(x\rho) + \frac{1}{x_{n+1}\rho}(z_0 - x\rho) \otimes D(x_{n+1}\rho)\big].
$$

The matrix $\mathcal{A}$ looks extremely complicated, but it is surprisingly simple.

Lemma 2.2. *We have*
$$
\mathcal{A} = \mathcal{N},
$$
where $\mathcal{N}$ is given in (2.2).

Proof. By direct computation,
$$
D(x\rho) = x \otimes D\rho + \rho I,
$$
and

$$
\begin{aligned}
&(z_0 - x\rho) \otimes D(x_{n+1}\rho) \\
&= -\big(\frac{2\rho^2}{a}D\rho + x\rho\big) \otimes \big(x_{n+1}D\rho - \frac{\rho x_{n+1}}{1 - |x|^2}x\big) \\
&= -\rho x_{n+1}\big(\frac{2\rho D\rho \otimes D\rho}{a} + x \otimes D\rho - \frac{2\rho^2 D\rho \otimes x}{a(1 - |x|^2)} - \rho\frac{x \otimes x}{1 - |x|^2}\big).
\end{aligned}
$$

Hence

(2.32) $$\qquad \mathcal{A} = (1 - q)^{-1}\big\{\mathcal{N} - \frac{2D\rho \otimes D\rho}{a} + \frac{2\rho D\rho \otimes x}{a(1 - |x|^2)}\big\}.$$

By (2.23), (2.18), and (2.24), we have

$$
\begin{aligned}
[I - q]^{-1}D\rho \otimes x &= \big(I - \frac{a}{b}q\big)D\rho \otimes x \\
&= D\rho \otimes x - \frac{2}{b}\big(|D\rho|^2 - (\rho + D\rho \cdot x)D\rho \cdot x\big)D\rho \otimes x \\
&= -\frac{a}{b}D\rho \otimes x.
\end{aligned}
$$

Next by (2.19),

$$[I - q]^{-1} D\rho \otimes D\rho = -\frac{a}{b} D\rho \otimes D\rho.$$

Finally,

$$
\begin{aligned}
(I - q)^{-1} \mathcal{N} &= \left(I - \frac{a}{b}q\right)\left[I + \frac{x \otimes x}{1 - |x|^2}\right] \\
&= \left(I - \frac{2}{b} D\rho \otimes (D\rho - (\rho + D\rho \cdot x)x)\right)\left[I + \frac{x \otimes x}{1 - |x|^2}\right] \\
&= I + \frac{x \otimes x}{1 - |x|^2} - \frac{2}{b} D\rho \otimes D\rho \\
&\quad - \frac{2}{b}\left[\frac{D\rho \cdot x - (\rho + D\rho \cdot x)|x|^2}{1 - |x|^2} - (\rho + D\rho \cdot x)\right] D\rho \otimes x \\
&= I + \frac{x \otimes x}{1 - |x|^2} - \frac{2}{b} D\rho \otimes D\rho + \frac{2\rho}{b(1 - |x|^2)} D\rho \otimes x.
\end{aligned}
$$

Applying the above three formulas to (2.32), we conclude that $\mathcal{A} = \mathcal{N}$.

q.e.d.

Proposition 2.2. *We have*

(2.33)

$$Dz = \frac{2t\rho^2}{a}[I - q]\mathcal{B}\left\{ - D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho + \frac{a(1 - t)}{2t\rho}\mathcal{N}\right\}$$

(2.34)

$$det\left\{ - D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho + \frac{a(1 - t)}{2t\rho}\mathcal{N}\right\} = -\frac{a^{n+1}}{2^n t^n \rho^{2n+1} b} \frac{f}{\omega^2 \beta g |\nabla \psi|}.$$

Proof. Formula (2.33) follows from (2.31), as we have $\mathcal{A} = \mathcal{N}$. From (2.33) and (2.26), the left-hand side of (2.34) is equal to

$$h := -\frac{a^n}{2^n t^n \rho^{2n}} \frac{1}{\det(I - q)\det\mathcal{B}} \frac{f\psi_{n+1}}{\omega g |\nabla \psi|}.$$

By (2.22), $\det(I - q) = -\frac{b}{a}$. From (2.30), $\det\mathcal{B} = -\beta\rho x_{n+1}\psi_{n+1}$. Hence we obtain (2.34). q.e.d.

Remark 2.2.
(i) A significance of equation (2.34) is that the matrix on the left-hand side does not involves the geometry of Σ (its normal and curvature), but depends only on the position of the point $Z = T(X)$. This property enables us to derive an equation for the reflector on the sphere (see §2.3 below). Our a priori estimates in §3 will also be built upon the equation.
(ii) Equation (2.1) is equivalent to (2.34) since

$$\frac{1 - t}{t} = \frac{z_{n+1}}{\rho x_{n+1} - z_{n+1}}.$$

Note that in our derivation of (2.34) we assumed that $t > 0$, namely, $\rho x_{n+1} > z_{n+1}$. But by the equation (1.2) on sphere, one sees that (2.1) is still correct when $\rho x_{n+1} \leq z_{n+1}$.

(iii) In the following we will concentrate on solutions such that the matrix

$$\mathcal{W} = \left\{ -D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho + \frac{a(1-t)}{2t\rho}\mathcal{N} \right\}$$

is positive definite, and discuss briefly solutions with negative definite $\mathcal{W}$ in §7. As noted in Remark 2.1, if the right-hand side of (2.34) is negative, we should replace it by its absolute value.

(iv) In §2.1 (see Remark 2.1), we assumed that $y_{n+1} < 0$. But we don't need to assume it in the general case in §2.2. Indeed, when $y_{n+1} > 0$, we can still define Z_0 and t as in (2.25) and (2.27). Then Z_0 is the intersection of the output ray (in the opposite direction) and the plane $\{x_{n+1} = 0\}$, and t becomes negative. From the above calculation we obtain the same equation.

2.3. Equation on the sphere. Let $(y_1, \cdots, y_n, y_{n+1})$ be another co-ordinate system in $\mathbb{R}^{n+1}$ such that

$$(2.35) \qquad \begin{cases} x_1 = \sin\theta y_1 + \cos\theta y_{n+1}, \\ x_k = y_k, \qquad\qquad k = 2, \cdots, n, \\ x_{n+1} = -\cos\theta y_1 + \sin\theta y_{n+1}, \end{cases}$$

where θ is the angle between the y_{n+1} axis and the plane $\{x_{n+1} = 0\}$, $\theta \in (0, \pi)$. Let $y = (y_1, \cdots, y_n), |y| < 1$ be a point on the plane $\{y_{n+1} = 0\}$ and $Y = (y_1, \cdots, y_n, \sqrt{1 - |y|^2})$ be a point on the unit sphere. Let x be the projection of Y on the plane $\{x_{n+1} = 0\}$. By (2.35) we have

$$(2.36) \qquad \begin{cases} x_1 = \sin\theta y_1 + \cos\theta\sqrt{1 - |y|^2}, \\ x_k = y_k, \qquad\qquad k = 2, \cdots, n. \end{cases}$$

To derive an equation on the sphere, we make use of equation (2.34). Let $\bar{X} \in S^n$ be a point arbitrarily given. Choose a coordinate system $(y_1, \cdots, y_{n+1})$ such that $\bar{X}$ is the north pole in the new coordinates. Suppose the ray $\bar{X}$ is reflected off by Γ and reaches the point $Z \in \Sigma$. Let θ be the angle between $O\bar{X}$ and OZ. By a rotation of the axes $y_1, \cdots, y_n$, we assume that Z lies in the 2-plane spanned by the y_1 and y_{n+1} axes. Now let $x_1, \cdots, x_{n+1}$ be given by (2.35), so that Z is located in $\{x_{n+1} = 0\} \cap \{x_1 \geq 0\}$, and $\theta \in (0, \pi)$. Then

$$|D\rho| = \rho_{y_1} \geq 0 \quad \text{at} \quad \bar{X}$$

and $u_{y_1} \leq 0$ at $y = 0$. By (2.36), one easily verifies that

$$(2.37) \qquad \det u_{x_i x_j} = (\sin\theta)^{-2}\det\left(u_{y_i y_j} + \frac{\cos\theta}{\sin\theta}u_{y_1}\delta_{ij}\right) \quad \text{at} \quad y = 0.$$

Since the point $Z \subset \{x_{n+1} = 0\}$, equation (2.34) becomes

$$\det\left(-D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho\right) = h \quad \text{at} \quad \bar{X}.$$

Let $u = \frac{1}{\rho}$. The above becomes

$$\det D^2 u = u^{2n} h \quad \text{at} \quad \bar{X}.$$

We choose an orthonormal coordinate system on S^n near the north pole. Then the covariant derivatives coincide with the local derivatives as the Christofell symbol Γ_{ij}^k vanishes at $\bar{X}$. Hence by (2.37) we obtain the equation on the sphere. That is:

Proposition 2.3. *The reflector Γ satisfies the equation*

(2.38) $$det(D^2 u - \frac{\cos\theta}{\sin\theta}|Du|I) = u^{2n} h \sin^2\theta,$$

where $u = \rho^{-1}$, θ is the angle between OX and OZ, and h is the right-hand side of (2.34).

Now equation (1.2) follows from (2.38) by letting $\rho = 1/u$ (but the right-hand side h in (1.2) should be equal to $h\sin^2\theta$ in (2.38)).

Remark 2.3. Attempts to derive the equation were previously made in [S, ONP]. In [S] the author considered the case of planer receiver and obtained an equation which is nothing more than the Jacobian, and is of little help to study the regularity of solutions. In [ONP] the authors obtained the equation (in a local orthonormal frame)

$$\det[M \cdot M + D\ell \otimes D\ell] = h \quad \text{on } S^n,$$

where ℓ is the distance traveled by the ray from O to $X\rho(X)$ and then to $T(X) \in \Sigma$, h is a function depending on $f, g, \rho, D\rho$ and Σ,

$$M = 2(\ell - \rho)\frac{\rho D^2\rho - \rho^2 I - 2D\rho \otimes D\rho}{\rho^2 + |D\rho|^2} + \ell I,$$

and I is the unit matrix. Our equation (2.38) is much simpler and is an equation of Monge-Ampère type. In particular, when the receiver Σ is a planer domain, it becomes the standard Monge-Ampère equation (2.13). As the reader can see, the derivation of (2.38) requires a great effort. We discovered all the formulas in Section 2 by direct computation.

Remark 2.4. Due to its importance in applications, the reflector problem has been extensively studied [E, HK, JM, RP, We, Wo] (to cite a few). For example, in a Google Scholar search, one gets nearly six hundred thousand results by inputting the keyword *reflector*, which is surprisingly more than that for the keyword *equation*.

2.4. Equation in the far field case. The equation in the far field case can be obtained from (2.34) by approximation (but we should note that in the far field case, the equation can be obtained by direct computation, and the computation is much simpler than those in §2.1 and 2.2 above; see [ON, W1, GW]). Assume the receiver Σ is the sphere of radius r;

then we can define $\psi(X) = r^2 - |X|^2$. Let g_r be the light distribution on Σ_r under the same reflector Γ. Then when r is sufficiently large,

$$r^n g_r(Z) \to g(Y),$$

$$\frac{r}{t} \to |Z_0 - X\rho|,$$

$$\beta|\nabla\psi| = \frac{|\nabla\psi|}{(Z_0 - X\rho) \cdot \nabla\psi} \to \frac{-1}{|Z_0 - X\rho|}.$$

Note that

$$\frac{\rho}{|Z_0 - X\rho|} = \frac{|y_{n+1}|}{x_{n+1}} = -\frac{a}{b}.$$

We have

$$|Z_0 - X\rho| = -\frac{b}{a}\rho.$$

Sending $r \to \infty$, from (2.34) we obtain the equation for the far field case

$$(2.39) \qquad \det\left\{ -D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho - \frac{a}{2\rho}\mathcal{N} \right\} = \frac{b^n}{2^n \rho^n} \frac{f}{\omega^2 g}.$$

From equation (2.39) we can get the equation on sphere with $\cos\theta = X \cdot Y$, which coincides with (1.7).

2.5. Ellipsoid of revolution. In the study of the reflector problem, ellipsoid of revolution plays a crucial role (in the far field case it is paraboloid of revolution). Suppose E is an ellipsoid obtained by rotating an ellipse along its major axis. Then E has two foci F_1, F_2. A special reflection property of such an ellipsoid is that a ray from one focus will always be reflected to the second focus. In this paper when we say an ellipsoid we always mean it is an ellipsoid of rotation, with two foci F_1, F_2, and one focus (say, F_1) is located at the origin.

In the polar coordinate system, such an ellipsoid E can be represented as the radial graph of a function of the form

$$(2.40) \qquad\qquad e(X) = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon X \cdot \ell}$$
$$= \frac{a^2 - c^2}{a - cX \cdot \ell},$$

where a is the major axis, which equals half of the diameter of E, $c = \frac{1}{2}|F_2|$ is the distance from the center of E to its foci, $\varepsilon = \frac{c}{a}$ is the eccentricity, and $\ell = F_2/|F_2|$.

From the above formula we see that an ellipsoid can be uniquely determined by its major axis a and its focus F_2 (remember the focus F_1 is located at the origin). When F_2 is fixed, then for any $X \in S^n$, $e(X)$ is increasing in a. In other words, E expands or shrinks if we increase or decrease a.

It follows that for any two points Z and p in $\mathbb{R}^{n+1}$, if p does not lie on the segment OZ, there is a unique ellipsoid $E = E_{p,Z}$ with foci O and Z and passing through p.

If the reflector Γ is an ellipsoid with one focus at the origin, then T is an identity map, namely, for all $x \in \Omega$, $T(x) = F_2$ is the second focus. Therefore the Jacobian matrix $\frac{DT}{Dx}$ vanishes completely, namely, the matrix $\mathcal{W}$ must vanish identically. This can also be verified directly. In the special case when F_2 lies on the plane $\{x_{n+1} = 0\}$, by the expression (2.40), $u = \frac{1}{e}$ is a linear function and so $\mathcal{W} = u^{-2} D^2 u \equiv 0$. In the general case, we also let $u = \frac{1}{\rho}$. Then the matrix

$$\mathcal{W} = \frac{1}{u^2}\Big(D^2 u - \frac{c_0}{2}\frac{|Du|^2 - (u - Du \cdot x)^2}{c_0 u + \sqrt{1 - |x|^2}}\mathcal{N}\Big),$$

where $c_0 = F_2 \cdot (-e_{n+1})$, namely, $F_2 \subset \{x_{n+1} = -c_0\}$. By the expression (2.40), one can also verify $\mathcal{W} \equiv 0$ by direct computation.

3. A priori estimates

In this section we establish the a priori estimates for the second derivatives of solutions, assuming that the matrix $\mathcal{W}$ is positive definite.

In the case when $\Sigma \subset \{x_{n+1} = 0\}$, equation (2.1) is the standard Monge-Ampère equation, and the a priori estimates can be found in [P2, GT, G]. Here we consider the general case.

3.1. The a priori estimate. Write equation (2.34) in the form

$$(3.1) \qquad\qquad \log \det\Big\{D^2 u - \frac{\hat{a}(t-1)}{2ut}\mathcal{N}\Big\} = h$$

for a different $h = h(x, u, Du)$, defined on $\Omega \times \mathbb{R}^1 \times \mathbb{R}^n$, where

$$(3.2) \qquad\qquad \hat{a} = |Du|^2 - (u - Du \cdot x)^2.$$

Denote

$$\tau = \frac{(t-1)\hat{a}}{2tu}.$$

Then τ is a function of x, u, and $p := Du$.

Lemma 3.1. *Let $u \in C^4(B_r(x_0))$ be an elliptic solution of (2.34). Suppose τ satisfies*

$$(3.3) \qquad\qquad \{\tau_{p_i p_j}\} \geq \delta I$$

at the solution u, for some positive constant δ. Then we have the estimate

$$(3.4) \qquad\qquad |D^2 u| \leq C \quad in \ \ B_{r/2}(x_0),$$

where C depends on n, r, δ, $\sup_{B_r(x_0)}(|u| + |Du|)$, and h up to its second derivatives.

Proof. Consider u in a ball $B_r(x_0) \subset \Omega$. Denote

$$\mathcal{W} = \{w_{ij}\} = \left\{D^2 u - \frac{\hat{a}(t-1)}{2ut}\mathcal{N}\right\}$$

and $\{w^{ij}\}$ the inverse of $\{w_{ij}\}$. Differentiating equation (3.1) gives

(3.5) $$w^{ij} w_{ij,k} = h_k,$$
$$w^{ij} w_{ij,kk} = h_{kk} + w^{is} w^{jt} w_{ij,k} w_{st,k},$$

where $w_{ij,k} = \partial_{x_k} w_{ij}$ and we have used the formula

$$\frac{\partial}{\partial w_{st}} w^{ij} = -w^{is} w^{jt}.$$

By assumption, $\{w_{ij}\}$ is positive definite. Hence

(3.6) $$w^{ij} w_{ij,kk} \geq h_{kk}.$$

Introduce the auxiliary function

$$H = \eta(x) \sum_{k=1}^{n} w_{kk}(x),$$

where η is a cut-off function, namely, η is nonnegative, smooth, positive in B_r and vanishes outside B_r. Assume that H attains its maximum at some point y_0. Then at y_0,

(3.7) $$(\log H)_i = \frac{\eta_i}{\eta} + \frac{\Sigma w_{kk,i}}{\Sigma w_{kk}} = 0,$$

(3.8) $$(\log H)_{ij} = \frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2} + \frac{\Sigma w_{kk,ij}}{\Sigma w_{kk}} - \frac{\Sigma w_{kk,i} \Sigma w_{kk,j}}{(\Sigma w_{kk})^2}$$

$$= \frac{\eta_{ij}}{\eta} - 2\frac{\eta_i \eta_j}{\eta^2} + \frac{\Sigma w_{kk,ij}}{\Sigma w_{kk}} \leq 0$$

as a matrix. It follows that at y_0,

$$0 \geq \sum_{i,j,k} (w_{kk}) w^{ij} (\log H)_{ij}$$

$$= \sum_{i,j,k} w^{ij} \left(w_{kk,ij} + \left(\frac{\eta_{ij}}{\eta} - 2\frac{\eta_i \eta_j}{\eta^2}\right) w_{kk}\right).$$

By replacing η by η^2, we may assume that $|\frac{\eta_i \eta_j}{\eta^2}| \leq \frac{C}{\eta}$. Denote by K any constant satisfying

(3.9) $$K \leq C\left(1 + \sum_i w_{ii}(y_0) + \frac{1}{\eta}\sum_{i,k} w_{ii} w^{kk}(y_0)\right)$$

for some constant C under control. Then we obtain

$$
\begin{aligned}
0 \;\geq\; & \sum_{i,j,k} w^{ij} w_{kk,ij} - K \\
\geq\; & \sum_{i,j,k} w^{ij}\big[u_{kkij} - (\tau\mathcal{N}_{kk})_{ij}\big] - K.
\end{aligned}
$$

To cancel out the forth derivatives, we use (3.6) to get

$$
0 \geq \sum_k h_{kk} - \sum_{i,j,k} w^{ij}(u_{ijkk} - (\tau\mathcal{N}_{ij})_{kk}).
$$

Noting that $\{u_{ij} - w_{ij}\} = \tau\mathcal{N}$, we have

$$
|\sum_k (w_{kk,i} - u_{kki})| \leq C(1 + \sum w_{kk}).
$$

Hence by (3.7),

$$
\begin{aligned}
|h_{kk}| \;\leq\; & C\big(1 + \sum w_{kk} + \sum h_{p_i} u_{kki}\big) \\
\leq\; & C\big(1 + \frac{1}{\eta} \sum w_{kk}\big) \leq K.
\end{aligned}
$$

Combining the above inequalities, we get

$$
(3.10) \qquad \sum_{i,j,k} w^{ij}\big[(\tau\mathcal{N}_{ij})_{kk} - (\tau\mathcal{N}_{kk})_{ij}\big] \leq K.
$$

One easily verifies that

$$
(\tau\mathcal{N}_{kk})_{ij} = \big[\tau_{p_s p_t} u_{si} u_{tj} + \tau_{p_s} u_{ijs}\big]\mathcal{N}_{kk} + O(1 + u_{kk}).
$$

From (3.5),

$$
w^{ij} u_{ijs} = w^{ij} w_{ij,s} + w^{ij}(\tau\mathcal{N}_{ij})_s = h_s + O(w^{ii} w_{kk}).
$$

By a rotation of coordinates we may assume that $\{w_{ij}(y_0)\}$ is diagonal and

$$
w_{11} \geq w_{22} \geq \cdots \geq w_{nn},
$$

so that $w^{11} \leq \cdots \leq w^{nn}$. Then

$$
w^{ij} u_{si} u_{tj} = w^{ii}(w_{ii} - \tau\mathcal{N}_{ii})^2 = O(w_{ii} + w^{ii}).
$$

We obtain

$$
\sum_{i,j,k} w^{ij}(\tau\mathcal{N}_{kk})_{ij} \leq K.
$$

Similar computation gives

$$
\sum_{i,j,k} w^{ij}(\tau\mathcal{N}_{ij})_{kk} \geq \sum_{i,j,k,s,t} \big(w^{ij}\mathcal{N}_{ij}\big)\{\tau_{p_s p_t} u_{sk} u_{tk} + h_{p_s} u_{kks}\} - K.
$$

As above, we have from (3.7), $|\sum_k u_{kks}| \leq K$. Noting that $\mathcal{N}$ is positive, we obtain

$$
\sum_{i,j,k} w^{ij}(\tau\mathcal{N}_{ij})_{kk} \geq C\Big(\sum_k w^{kk}\Big)\Big(\sum_{k,s,t} \tau_{p_s p_t} u_{sk} u_{tk}\Big) - K.
$$

From (3.10) it follows that

$$(3.11) \qquad \Big(\sum_k w^{kk}\Big)\Big(\sum_{k,s,t} \tau_{p_s p_t} u_{sk} u_{tk}\Big) \le K.$$

Observe that

$$\Big(\sum_{k,s,t} \tau_{p_s p_t} u_{sk} u_{tk}\Big) \ge \frac{\delta}{n}\Big(\sum_k w_{kk}^2\Big).$$

Hence at y_0,

$$\sum w^{ii} \sum w_{kk}^2 \le C\Big(1 + \sum w_{kk} + \frac{1}{\eta}\sum w^{ii} \sum w_{kk}\Big).$$

If $\sum w_{kk}(y_0)$ is sufficiently large, we obtain

$$\eta(y_0) \sum w_{kk}(y_0) \le C.$$

Recall that H attains its maximum at y_0. We obtain $\sum w_{kk} \le C$ in $B_{r/2}(x_0)$. Lemma 3.1 is proved. q.e.d.

Note that we don't need to assume (3.3) for all $p \in \mathcal{C}_U$ and $Z \in \Sigma$. It suffices to assume (3.3) at solution u, namely, at points $p = X\rho(X) \in \mathcal{C}_U$ and $Z = T_\rho(X)$.

The proof of Lemma 3.1 is not new; similar argument has been used in [W1, GW]. We include the proof here for completeness. This argument also applies to equation (1.2) on sphere (see [GW]). In particular the condition (3.3) is independent of the choice of the coordinate system. In the following we show that (3.3) is equivalent to (1.14).

3.2. Verification of (3.3). Denote

$$(3.12) \qquad \hat{\tau} = \frac{t-1}{t}\hat{a}.$$

Then $\tau = \hat{\tau}/2u$ and $\tau_{p_s p_t} = \frac{1}{2u}\hat{\tau}_{p_s p_t}$. Hence τ satisfies (3.3) if and only if $\hat{\tau}$ does. We compute

$$\hat{\tau}_{p_k} = \frac{t_{p_k}}{t^2}\hat{a} + \Big(1 - \frac{1}{t}\Big)\hat{a}_{p_k},$$

$$\hat{\tau}_{p_k p_l} = \Big(\frac{t_{p_k p_l}}{t^2} - \frac{2t_{p_k} t_{p_l}}{t^3}\Big)\hat{a} + \frac{t_{p_k}}{t^2}\hat{a}_{p_l} + \frac{t_{p_l}}{t^2}\hat{a}_{p_k} + \Big(1 - \frac{1}{t}\Big)\hat{a}_{p_l p_k}.$$

Denote

$$(3.13) \qquad \xi = Z_0 - X\rho.$$

Then as in (2.27),

$$(3.14) \qquad Z = X\rho + t\xi.$$

Differentiating $\psi(Z) = 0$ with respect to p_k, we get

$$(3.15) \qquad \frac{t_{p_k}}{t} = -\frac{\nabla\psi \cdot \xi_{p_k}}{\nabla\psi \cdot \xi}.$$

Alternatively, by the implicit function theorem one can solve the equation $\psi(X\rho + t\xi) = 0$ to get

$$t = F(X\rho, \xi).$$

Then $\psi(X\rho + \xi F(X\rho, \xi)) = 0$. By differentiating in ξ, one has

$$\nabla F = -\frac{F}{\nabla\psi \cdot \xi}\nabla\psi.$$

Hence

$$\frac{t_{p_k}}{t} = \frac{\nabla F \cdot \xi_{p_k}}{F} = -\frac{\nabla\psi \cdot \xi_{p_k}}{\nabla\psi \cdot \xi},$$

which is exactly (3.15).

Next we differentiate (3.15) in p_l to get

$$
\begin{aligned}
\left(\frac{t_{p_k}}{t}\right)_{p_l} &= -\frac{\nabla\psi \cdot \xi_{p_k p_l} + \xi'_{p_k}\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l})}{\nabla\psi \cdot \xi} \\
&\quad + \frac{\nabla\psi \cdot \xi_{p_k}}{(\nabla\psi \cdot \xi)^2}\left(\xi'\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l}) + \nabla\psi \cdot \xi_{p_l}\right) \\
&= -\frac{\xi'_{p_k}\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l})}{\nabla\psi \cdot \xi} + \frac{\nabla\psi \cdot \xi_{p_k}}{(\nabla\psi \cdot \xi)^2}\xi'\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l}) \\
&\quad - \frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi} + \frac{\nabla\psi \cdot \xi_{p_k}\nabla\psi \cdot \xi_{p_l}}{(\nabla\psi \cdot \xi)^2},
\end{aligned}
$$

where ξ' is the transpose of ξ. By (3.15),

$$
\begin{aligned}
\left(\frac{t_{p_k}}{t}\right)_{p_l} &= -\frac{1}{\nabla\psi \cdot \xi}\left[\xi'_{p_k}\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l}) + \frac{t_{p_k}}{t}\xi'\nabla^2\psi(\xi_{p_l}t + \xi t_{p_l})\right] \\
&\quad - \frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi} + \frac{t_{p_k}t_{p_l}}{t^2} \\
&= -\frac{1}{t\nabla\psi \cdot \xi}(t_{p_k}\xi' + t\xi'_{p_k})\nabla^2\psi(t_{p_l}\xi + t\xi_{p_l}) - \frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi} + \frac{t_{p_k}t_{p_l}}{t^2}.
\end{aligned}
$$

Recall that $Z = X\rho + t\xi$. We have

(3.16)
$$Z_{p_k} = t_{p_k}\xi + t\xi_{p_k}.$$

Hence

$$
\begin{aligned}
\frac{t_{p_k p_l}}{t} - 2\frac{t_{p_k}t_{p_l}}{t^2} &= \left(\frac{t_{p_k}}{t}\right)_{p_l} - \frac{t_{p_k}t_{p_l}}{t^2} \\
&= -\frac{Z'_{p_k}\nabla^2\psi Z_{p_l}}{t\nabla\psi \cdot \xi} - \frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi}.
\end{aligned}
$$

It follows that

$$\hat{\tau}_{p_k p_l} = -\frac{\hat{a}}{t}\left[\frac{Z'_{p_k}\nabla^2\psi Z_{p_l}}{t\nabla\psi \cdot \xi} + \frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi}\right] + \frac{t_{p_k}}{t^2}\hat{a}_{p_l} + \frac{t_{p_l}}{t^2}\hat{a}_{p_k} + \left(1 - \frac{1}{t}\right)\hat{a}_{p_l p_k}.$$

The above formula can be furthermore simplified as follows. Recall that $\xi = Z_0 - X\rho$, and by (2.25),

$$(3.17) \qquad Z_0 = \frac{2Du}{|Du|^2 - (u - Du \cdot x)^2} = \frac{2}{\hat{a}} Du.$$

We have

$$\xi_{p_k} = \frac{2}{\hat{a}}\left[e_k - \frac{\hat{a}_{p_k}}{\hat{a}} p\right],$$

$$\xi_{p_k p_l} = 2\left[-\frac{\hat{a}_{p_l}}{\hat{a}^2} e_k - \frac{\hat{a}_{p_k}}{\hat{a}^2} e_l - \frac{\hat{a}_{p_k p_l}}{\hat{a}^2} p + 2\frac{\hat{a}_{p_k}\hat{a}_{p_l}}{\hat{a}^3} p\right],$$

where e_k denotes the unit vector in the x_i-axis. Hence

$$-\frac{\hat{a}}{t}\frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi} = \frac{2}{\hat{a}t\nabla\psi \cdot \xi}\left[\psi_k\hat{a}_{p_l} + \psi_l\hat{a}_{p_k} + (\nabla\psi \cdot p)\hat{a}_{p_k p_l} - 2\frac{\nabla\psi \cdot p}{\hat{a}}\hat{a}_{p_k}\hat{a}_{p_l}\right].$$

On the other hand, by (3.15),

$$t_{p_k}\hat{a}_{p_l} = -t\frac{\nabla\psi \cdot \xi_{p_k}}{\nabla\psi \cdot \xi}\hat{a}_{p_l}$$

$$= \frac{2t}{\hat{a}(\nabla\psi \cdot \xi)}\left(\frac{\nabla\psi \cdot p}{\hat{a}}\hat{a}_{p_k}\hat{a}_{p_l} - \psi_k\hat{a}_{p_l}\right).$$

We obtain

$$-\frac{\hat{a}}{t}\frac{\nabla\psi \cdot \xi_{p_k p_l}}{\nabla\psi \cdot \xi} + \frac{t_{p_k}}{t^2}\hat{a}_{p_l} + \frac{t_{p_l}}{t^2}\hat{a}_{p_k} = \frac{2(\nabla\psi \cdot p)}{\hat{a}t\nabla\psi \cdot \xi}\hat{a}_{p_k p_l}.$$

Therefore

$$\hat{\tau}_{p_k p_l} = -\frac{\hat{a}}{t^2(\nabla\psi \cdot \xi)}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + \left[\frac{2(\nabla\psi \cdot p)}{\hat{a}t\nabla\psi \cdot \xi} + \left(1 - \frac{1}{t}\right)\right]\hat{a}_{p_l p_k}.$$

From (3.17),

$$p = \frac{\hat{a}}{2}Z_0 = \frac{\hat{a}}{2}(Z + (1-t)\xi).$$

Note that $a_{p_k p_l} = 2(\delta_{kl} - x_k x_l)$. We obtain

$$(3.18) \qquad \hat{\tau}_{p_k p_l} = -\frac{\hat{a}}{t^2(\nabla\psi \cdot \xi)}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + \frac{2}{t}\frac{\nabla\psi \cdot Z}{\nabla\psi \cdot \xi}(\delta_{kl} - x_k x_l).$$

In the special case when Σ is a convex radial graph given by (1.12), the matrix $\{Z'_{p_k}\nabla^2\psi Z_{p_l}\}$ is nonnegative, and $\hat{a} < 0$, $\nabla\psi \cdot Z > 0$, $\nabla\psi \cdot \xi > 0$. Therefore we proved the following:

Lemma 3.2. *Suppose Σ is a convex radial graph given by (1.12). Then condition (3.3) is satisfied.*

Next we make formula (3.18) more precise. Denote

$$(3.19) \qquad \tau^*_{kl} = -\frac{\hat{a}}{2t}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + \nabla\psi \cdot Z(\delta_{kl} - x_k x_l).$$

Recall that $\nabla\psi \cdot (Z - X\rho) > 0$. We have $t(\nabla\psi \cdot \xi) > 0$ in both cases when $t > 0$ or $t < 0$. Hence the matrix $\{\hat{\tau}_{p_k p_l}\}$ is positive definite if and only if the matrix $\{\tau^*_{kl}\}$ is.

Given a point $\bar{X} \in U$, to evaluate the formula (3.19) at $\bar{X}$, we choose a coordinate system $(y_1, \cdots, y_{n+1})$ such that $\bar{X}$ is the north pole in the new coordinates. From §2.1,

$$\{Z_0 - p\} \cdot Y = -\frac{b}{a}\rho = -\frac{b}{a}|p|,$$

where $p = \bar{X}\rho(\bar{X})$. Hence by (2.27),

$$t = \frac{(Z - p) \cdot Y}{(Z_0 - p) \cdot Y} = -\frac{ad}{b|p|} = -\frac{\hat{a}d}{\hat{b}|p|},$$

where $\hat{b} = |Du|^2 + u^2$, $d = \{Z - p\} \cdot Y$ is the distance between p and Z. We obtain

$$-\frac{\hat{a}}{t} = \frac{\hat{b}|p|}{d}.$$

The formula holds when $t < 0$. Hence at $\bar{X}$ (which corresponds to $y = 0$),

$$(3.20) \qquad \tau^*_{kl} = \frac{\hat{b}|p|}{2d}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + (\nabla\psi \cdot Z)\,\delta_{kl}.$$

Near the point Z we choose a local coordinate $(\hat{x}_1, \cdots, \hat{x}_{n+1})$ such that Y is the negative $\hat{x}_{n+1}$-axis. Suppose in this coordinate system, Σ is given by

$$(3.21) \qquad \hat{x}_{n+1} = \varphi(\hat{x}), \quad \hat{x} = (\hat{x}_1, \cdots, \hat{x}_n).$$

Let $\psi(Z) = \varphi(\hat{x}) - \hat{x}_{n+1}$. Then $|\nabla\psi| = (1 + |\nabla\varphi|^2)^{1/2}$ and

$$\nabla\psi \cdot Z = |Z|(1 + |\nabla\varphi|^2)^{1/2}\cos\beta,$$

where β is the angle between OZ and ν, and $\nu = \nabla\psi/|\nabla\psi|$ is the normal of Σ at Z. Hence

$$(3.22) \qquad \tau^*_{kl} = \frac{\hat{b}|p|}{2d}(Z'_{p_k}\nabla^2\psi Z_{p_l}) + |Z|(1 + |\nabla\varphi|^2)^{\frac{1}{2}}\cos\beta\,\delta_{kl}.$$

Next we compute the matrix $Z'_{p_k}\nabla^2\psi Z_{p_l}$. Recall that

$$Z = X\rho + dY.$$

We compute

$$(3.23) \qquad Z_{p_k} = Y_{p_k}d + Y d_{p_k}.$$

In the coordinates y, by (2.15), the normal of Γ is given by

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{\rho^2 + |D\rho|^2}}(\rho_1, \cdots, \rho_n, -\rho) \\
&= \frac{-1}{\sqrt{u^2 + |Du|^2}}(u_1, \cdots, u_n, u).
\end{aligned}$$

By (2.7),

$$(3.24) \qquad Y = \frac{|Du|^2 - u^2}{|Du|^2 + u^2} e_{n+1} + \frac{-2u}{|Du|^2 + u^2}(u_1, \cdots, u_n, 0)$$

$$= \frac{-1}{|Du|^2 + u^2}(2uu_1, \cdots, 2uu_n, u^2 - |Du|^2).$$

Note that $Du = (u_1, 0, \cdots, 0)$. Hence

$$Y_{p_1} = \frac{2u_1}{(u^2 + u_1^2)^2}(2uu_1, 0, \cdots, 0, u^2 - u_1^2) + \frac{-1}{u^2 + u_1^2}(2u, 0, \cdots, 0, -2u_1)$$

$$= \frac{2u}{(u^2 + u_1^2)^2}(u_1^2 - u^2, 0, \cdots, 0, 2uu_1),$$

$$Y_{p_k} = \frac{-2u}{u^2 + u_1^2} e_k, \quad 1 < k \leq n,$$

where e_k is the unit vector on the y_k-axis. Hence $Y_{p_1}, \cdots, Y_{p_n}, Y$ are orthogonal and

$$(3.25) \qquad |Y_{p_k}| = \frac{2u}{u^2 + u_1^2} \quad \text{for all} \ \ k = 1, 2, \cdots, n.$$

Therefore we may assume that $Y_{p_1}, \cdots, Y_{p_n}$ are in the $\hat{x}_1, \cdots, \hat{x}_n$ axes. Hence we have

$$(3.26) \qquad Z'_{p_k} \nabla^2 \psi Z_{p_l} = (Y_{p_k} d + Y d_{p_k})' \nabla^2 \psi (Y_{p_l} d + Y d_{p_l})$$

$$= d^2 Y'_{p_k} \nabla^2 \varphi Y_{p_l}$$

$$= \frac{4 d^2 u^2}{(u^2 + u_1^2)^2} \partial_k \partial_l \varphi$$

$$= \frac{4 d^2 u^2}{(u^2 + u_1^2)^2}(1 + |D\varphi|^2)^{\frac{1}{2}} II,$$

where

$$(3.27) \qquad II = \frac{1}{\sqrt{1 + |D\varphi|^2}} \partial_k \partial_l \varphi$$

is the second fundamental form of Σ along the direction $-Y$. Note that by our choice of coordinates in (3.21), $\nabla^2 \psi \cdot Y = 0$.

From (3.22) and noting that $u = \frac{1}{\rho} = \frac{1}{|p|}$, we therefore obtain

$$(3.28) \qquad \frac{\tau^*_{kl}}{\sqrt{1 + |D\varphi|^2}} = \frac{\hat{b}|p|}{2d} \frac{4 d^2 u^2}{(u^2 + u_1^2)^2} II + |Z| \cos \beta \, \delta_{kl}$$

$$= \frac{2du}{u^2 + u_1^2} II + |Z| \cos \beta \, \delta_{kl}.$$

Let 2α denote the angle between $-\bar{X}$ and Y, so that α is the angle of reflection. From the expression of Y in (3.24),

$$\cos 2\alpha = \frac{u^2 - u_1^2}{u^2 + u_1^2}.$$

We obtain

$$|u_1| = \left(\frac{1 - \cos 2\alpha}{1 + \cos 2\alpha}\right)^{1/2} u = \frac{\sin \alpha}{\cos \alpha} u,$$

$$u^2 + u_1^2 = u^2 + \frac{\sin^2 \alpha}{\cos^2 \alpha} u^2 = \frac{u^2}{\cos^2 \alpha}.$$

Therefore the matrix

$$(3.29) \qquad \frac{\tau_{kl}^*}{\sqrt{1 + |D\varphi|^2}} = 2|p|d \cos^2 \alpha\, II + |Z| \cos \beta\, I;$$

and condition (3.3) is equivalent to

$$(3.30) \qquad II + \frac{|Z|}{2|p|d} \frac{\cos \beta}{\cos^2 \alpha} I > 0.$$

Note that $|p|, d$, and $|Z|$ are the length of the three sides of the triangle OpZ. Hence $\cos^2 \alpha$ is also determined by $|p|, d$, and $|Z|$.

Suppose Σ is a radial given in (1.12). Then $|Z| \geq \inf_{X \in V} \varphi(X)$ and $\cos \beta \geq c_0 > 0$. By the assumption $\overline{U} \cap \overline{V} = \emptyset$, we have $\alpha < \frac{\pi}{2}$ and $\cos \alpha \geq c_0 > 0$. When ρ is small, $|Z| \approx d$. Hence (3.30) holds and we have therefore proved the following lemma.

Lemma 3.3. *Suppose Σ can be represented by (1.12). If the solution ρ is small, then condition (3.3) is satisfied.*

Lemmas 3.2 and 3.3 show that (1.14) is satisfied under the conditions in parts (a) and (b) of Theorem B.

4. Weak solutions

This section is divided into a few subsections. In §4.1 we introduce two types of weak solutions to the reflector problem. These two weak solutions are indeed equivalent (Lemma 4.6). In §4.2 we establish a Harnack inequality and a gradient estimate for weak solutions. In §4.3 we prove the existence of weak solutions. In §4.4 we discuss the boundary condition. At the end we also show that the reflection cone is a convex cone.

4.1. Definition of weak solutions. First we introduce some terminologies. Let $\Gamma = \Gamma_\rho$ be a reflecting surface, given by (1.1).

- **Supporting ellipsoid**. An ellipsoid $E = \{Xe(X) \mid X \in S^n\}$ is a supporting ellipsoid of $\Gamma = \Gamma_\rho$ at $\bar{X}\rho(\bar{X})$ if one of its foci is at the

origin and the other one on Σ, and E satisfies

(4.1)
$$\rho(\bar{X}) = e(\bar{X}),$$
$$\rho(X') \leq e(X') \quad \forall \ X' \in U.$$

- **R-convexity of reflector**. We say ρ, or $\Gamma = \Gamma_\rho$, is R-convex (with respect to Σ) if for any point $\bar{X} \in U$, there is a supporting ellipsoid at $\bar{X}\rho(\bar{X})$.

Recall that when the reflector is an ellipsoid, the matrix $\mathcal{W}$ vanishes (see §2.5). Hence when Γ is R-convex, the matrix $\mathcal{W}$ is positive semi-definite. If Γ is R-convex, it is obviously convex in usual sense. Hence it is twice differentiable almost everywhere. In particular, Γ has a unique supporting ellipsoid almost everywhere.

Next we define two multiple valued maps, $T : U \to \Sigma$ and $V : \Sigma \to U$. For any $X \in U$,

$$T(X) = \{Z \in \Sigma \quad | \quad Z \text{ is the focus of}$$
$$\text{a supporting ellipsoid of } \Gamma \text{ at } X\rho(X)\},$$
$$V(Z) = \{X \in U \quad | \quad \exists \text{ a supporting ellipsoid of}$$
$$\Gamma \text{ at } X\rho(X) \text{ with } Z \text{ as its focus}\}.$$

For any subset $\omega \subset U$, we denote $T(\omega) = \cup_{X \in \omega} T(X)$. Similarly, we denote $V(\omega) = \cup_{Z \in \omega} V(Z)$ for any subset ω of Σ.

Note that at any differentiable point of ρ, T is single valued and is exactly the reflection mapping. If Γ is smooth and T is one-to-one, then V is the inverse of T.

Remark 4.1. For some points $X \in U$, the above defined set $T(X)$ may be empty. The situation may occur if a ray X misses the object Σ after reflection. To avoid the situation, we extend Σ to

$$\Sigma^* = \Sigma \cup \{\partial B_R(0)\},$$

where R is chosen large such that Γ and Σ are contained in $B_R(0)$. Let $g = 0$ on $\partial B_R(0)$. For any point $X \in U$ and any tangent plane P of Γ at $p = X\rho(X)$, the ray X either hits the object Σ at some point Z, or it misses Σ but hits $\partial B_R(0)$ at a different point Z'. We can define $T(X)$ as the set of Z and Z' for all possible tangent planes P of Γ at p. Here we say $P = \{X\psi(X) \mid X \in S^n\}$ (actually ψ is defined in a semi-sphere) is a tangent plane of Γ at $p = \bar{X}\rho(\bar{X})$ if

$$\rho(\bar{X}) = \psi(\bar{X}),$$
$$\rho(X') \leq \psi(X') + \eta(|X' - \bar{X}|) \ \text{ for } X' \text{ near } \bar{X},$$

for some function η satisfying $\eta(r) \to 0$ as $r \to 0$. The only purpose for this extension is for the convenience of statements below (Lemmas 4.4 and 4.5, Theorems 6.1 and 6.2). For the definition of weak solutions below, we can allow that $T(X)$ or $V(Z)$ is empty.

We say Γ is an *R-polyhedron* if it is R-convex and of piecewise ellipsoids, and for each ellipsoid, one of its foci is at the origin and the other one on Σ. If Γ is an R-polyhedron, there exists finite many ellipsoids $E_1, \cdots, E_k$ such that

$$(4.2) \qquad \Gamma = \cup_{i=1}^k (E_i \cap \Gamma).$$

Any R-convex reflector Γ can be approximated by R-polyhedra. The approximation can be obtained by choosing finitely many points $p_1, \cdots, p_m \in \Gamma$, and shrinking the supporting ellipsoids of Γ at these points slightly.

Let Γ be an R-polyhedron given by (4.2), and let $Z_1, \cdots, Z_k \in \Sigma$ be the foci of the ellipsoids $E_1, \cdots, E_k$. Then for any $Z = Z_k$, $V(Z) = \Gamma \cap E_k$ and $V(\Sigma')$ has measure zero, where $\Sigma' = \Sigma - \{Z_1, \cdots, Z_k\}$. By approximation one sees that if Γ is a general R-convex surface, then for any Borel set $\omega \subset \Sigma$, $V(\omega)$ is also Borel. Therefore we may define

$$(4.3) \qquad \mu_b(\omega) = \mu_{b,\Gamma}(\omega) = \int_{V(\omega)} f \quad \forall\, \omega \in \Sigma.$$

If $V(\omega)$ is empty, we let $\mu_b(\omega) = 0$. For any two Borel sets $\omega_1, \omega_2 \subset \Sigma$ with $\omega_1 \cap \omega_2 = \emptyset$, the set $V(\omega_1) \cap V(\omega_2)$ has measure zero, as ρ is not differentiable at any point in the set. Hence μ_b is countably additive and so it is a measure.

Similarly for any Borel set $\omega \subset U$, one can show that $T(\omega)$ is Borel and define

$$(4.4) \qquad \mu_a(\omega) = \mu_{a,\Gamma}(\omega) = \int_{T(\omega)} g \quad \forall\, \omega \in U.$$

Weak solutions.

- If for any Borel set $\omega \subset \Sigma$,

$$(4.5) \qquad \mu_b(\omega) = \int_\omega g,$$

  we say that Γ_ρ, or equivalently ρ, is a *weak solution of type B* to the reflector problem.

- If for any Borel set $\omega \subset U$,

$$(4.6) \qquad \mu_a(\omega) = \int_\omega f,$$

  then we say Γ_ρ, or equivalently ρ, is a *weak solution of type A* to the reflector problem.

In the definition of type B weak solution, we allow that f is a measurable function and g a measure. In the definition of type A weak solution, we allow that g is a measurable function and f a measure.

Weak solutions of type B have been studied in detail in [KO]. If both f, g are positive measurable functions, type A and type B weak solutions are equivalent (see Lemma 4.6 below). Therefore in the following we will consider Type B weak solutions.

Remark 4.2. In the theory of convex bodies, one can introduce respectively curvature measure and area measure [Sc]. For the standard Monge-Ampère equation, one can introduce a weak solution of Aleksandrov and a weak solution of Brenier. A weak solution of type A introduced above corresponds to that of Aleksandrov, or the curvature measure; and a weak solution of type B corresponds to that of Brenier, or the area measure. In the far field case, a weak solution of Type A was introduced in [W1] and a weak solution of type B was introduced in [CO].

To see how the measure μ_b is related to the area measure, let $\hat{f}(p) = \rho^{-n} f(X)$ (where $X = p/|p|$) be a function on Γ, and let $\hat{V} : \Sigma \to \Gamma$ be a mapping given by

$$(4.7) \qquad \hat{V}(Z) = \{X\rho(X) \in \Gamma \mid X \in V(Z)\}.$$

Then by (4.3),

$$(4.8) \qquad \mu_b(\omega) = \int_{\hat{V}(\omega)} \hat{f}.$$

If $\hat{f} \equiv 1$, then $\mu_b(\omega)$ is the area of $\hat{V}(\omega) \subset \Gamma$.

A basic property of the measure μ_b (and also μ_a) is its weak continuity.

Lemma 4.1. *If Γ_{ρ_k} is a sequence of R-convex polyhedra which converges to Γ_{ρ_0} locally uniformly, then the measures $\mu_{b,\Gamma_{\rho_k}}$ converge to a measure μ_0 weakly. Furthermore, μ_0 is independent of the choice of the sequence Γ_{ρ_k} and coincides with $\mu_{b,\Gamma_{\rho_0}}$.*

For a proof of Lemma 4.1, we refer the reader to [KO]. The weak continuity is a fundamental property of Monge-Ampère type equations, and has also been proved for many other elliptic equations, such as the k-Hessian equations, the p-Laplace equation, and even quasilinear subelliptic equations [TW1]. In particular, the proof of Lemma 3.8 in [TW1] applies to the measure $\mu_{a,\Gamma}$. Note that the weak continuity also implies that that $\mu_{a,\Gamma}$ is a measure.

4.2. Uniform and gradient estimates. Assume that the reflector Γ is a radial graph given by (1.1) and Σ is contained in the cone $\mathcal{C}_V = \{tX \mid t > 0, X \in V\}$, where V is a domain on the sphere S^n. We first establish the a priori estimates under the assumption

$$(4.9) \qquad \overline{U} \cap \overline{V} = \emptyset.$$

Lemma 4.2. *Assume that Γ_ρ is a weak solution to the reflector problem. Then we have the Harnack inequality*

$$(4.10) \qquad \sup_{X \in U} \rho(X) \leq \frac{2}{1-\beta} \inf_{X \in U} \rho(X),$$

where $\beta = \sup\{X \cdot Y \mid X \in U, Y \in V\}$.

Proof. First we claim that for any ellipsoid E with one focus at O and the other one $F_2 \in \mathcal{C}_V$, given as a radial graph by

$$(4.11) \qquad e(X) = \frac{a(1 - \varepsilon^2)}{1 - \varepsilon X \cdot \ell} = \frac{a^2 - c^2}{a - cX \cdot \ell},$$

where $\varepsilon = \frac{c}{a}$, $c = \frac{1}{2}|F_2|$, and $\ell = F_2/|F_2|$ (ref. (2.40)), one has

$$(4.12) \qquad \sup_{X \in U} e(X) \leq \frac{2}{1 - \beta} \inf_{X \in U} e(X).$$

Indeed, (4.12) follows from

$$\inf_{X \in U} e(X) = e(-\ell) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon}$$

and

$$\sup_{X \in U} e(X) \leq \frac{a(1 - \varepsilon^2)}{1 - \varepsilon\beta}.$$

Now let ρ be the weak solution. Choose $X_0 \in U$ such that $\rho(X_0) = \inf \rho$. Let E be a supporting ellipsoid of Γ at X_0 (more precisely at $X_0\rho(X_0)$), given by (4.11). By definition of R-convexity, Γ is contained in E. Hence by (4.12),

$$
\begin{aligned}
\sup_U \rho \quad &\leq \quad \sup_U e \leq \frac{2}{1 - \beta} \inf_U e \\
&\leq \quad \frac{2}{1 - \beta} e(X_0) = \frac{2}{1 - \beta} \inf_U \rho.
\end{aligned}
$$

Lemma 4.2 is proved.   q.e.d.

Next we consider the gradient estimate.

Lemma 4.3. *Assume that Γ is a weak solution to the reflector problem. Then we have the gradient estimate*

$$(4.13) \qquad \sup_{X \in U} |D\rho|(X) \leq C,$$

where C depends only on $\sup_U \rho$, β, and

$$(4.14) \qquad d_0 = \sup\{|Z| : \ Z \in \Sigma\}.$$

Proof. For any point $X_0 \in U$, let E be a supporting ellipsoid of Γ at X_0, given by (4.11), with a focus $Z \in \Sigma$. Then $D\rho(X_0) = De(X_0)$. Note that $\ell = Z/|Z|$ and a, ε are completely determined by X_0 and $\rho(X_0)$ (when Z is fixed). Hence $|De(X_0)|$ is bounded from above by a constant depending on $\sup_U \rho$, β, and d_0.   q.e.d.

Estimate (4.13) holds for nonsmooth solutions. If assumption (4.9) is not satisfied, we consider large reflector, namely, solution ρ satisfying

$$(4.15) \qquad \inf_U \rho(X) > d_0.$$

We claim that (4.15) holds if

$$\text{(4.16)} \qquad \sup_{U} \rho(X) > 2d_0.$$

Indeed, let E be a supporting ellipsoid, given by (4.11). Then one has

$$\text{(4.17)} \qquad \sup_{X} e(X) = a + c > 2d_0.$$

But note that $c = \frac{1}{2}|Z| \leq \frac{1}{2}d_0$. Hence $a \geq \frac{3}{2}d_0$, so $\inf e(X) = a - c > d_0$. In particular, let E be the supporting ellipsoid at $\bar{X}\rho(\bar{X})$ at which $\rho(\bar{X}) < \inf_U \rho + \varepsilon$ for sufficiently small $\varepsilon > 0$, we obtain

$$\begin{aligned} \sup_{U} \rho &\leq \sup_{U} e \leq a + c \\ &\leq \frac{a + c}{a - c} \inf e \leq 2 \inf e. \end{aligned}$$

Hence whenever $\sup_U \rho \geq 2d_0$, we have the Harnack inequality

$$\text{(4.18)} \qquad \sup_{X} \rho(X) < 2 \inf_{X} \rho(X).$$

From (4.18) we obtain accordingly the gradient estimate as in Lemma 4.3.

4.3. Existence of weak solutions. Here we prove Theorem A. It can be proved by either the Perron method [W1] or by approximation by R-convex polyhedra [KO]. Here we adapt the proof in [KO].

Proof of Theorem A. In [KO] the authors proved the existence of large, type B weak solutions which satisfy

$$\text{(4.19)} \qquad \sup \rho \geq 4d_0.$$

But by the discussions in §4.2, (4.19) can be replaced by (4.16). Their proof is as follows.

First one considers the case when $g = \sum_{i=1}^{k} t_i \delta_{Z_i}$ is a discrete measure, where t_i are positive constants and δ_Z is the Dirac measure at Z. Denote $\ell_i = Z_i/|Z_i|$, $c_i = \frac{1}{2}|Z_i|$. Let a_i be a constant greater than c_i and let

$$\text{(4.20)} \qquad e_i(X) = \frac{a_i(1 - \varepsilon_i^2)}{1 - \varepsilon_i X \cdot \ell_i} = \frac{a_i^2 - c_i^2}{a_i - c_i X \cdot \ell_i}$$

be an ellipsoid with a focus at Z_i. Let $\rho(X) = \min\{e_i(X) \mid 1 \leq i \leq k\}$. Then ρ is R-convex and the associated measure $\mu_{b,\Gamma} = \mu_{b,\Gamma}[a_1, \cdots, a_k]$ is concentrated at the points $Z_1, \cdots, Z_k$, where Γ is the radial graph of ρ. Moreover, $\mu_{b,\Gamma}$ is continuous in $a_1, \cdots, a_k$ and $\mu_{b,\Gamma}(Z_i)$ increases if a_i decreases but all other $a_j, j \neq i$ fixed.

Note that in (4.20), $c_i = \frac{1}{2}|Z_i|$ is fixed but one can vary a_i for all i. So fix $a_1 = a$. At the initial stage we choose $a > 2d_0$ suitably large. Then we choose all a_i ($i \geq 2$) sufficiently large, such that $\rho = e_1$, where

$\rho = \min\{e_i \mid 1 \le i \le k\}$ as above. Hence $\mu_{b,\Gamma} = c\delta_{Z_1}$, where $c = \sum_i c_i$. Hence the set

$$\mathcal{R} = \{(a_1, \cdots, a_k) \mid \mu_b(Z_1) \ge t_1 \text{ and } \mu_b(Z_i) \le t_i \text{ for all } i \ge 2\}$$

is not empty. Let $\hat{a}_1 = a$ and for $i \ge 2$,

(4.21) $$\hat{a}_i = \inf\{a_i \mid (a_1, \cdots, a_k) \in \mathcal{R}\}.$$

Let $\hat{e}_i$ be the function in (4.20) with a_i replaced by $\hat{a}_i$ and

(4.22) $$\hat{\rho}(X) = \min\{\hat{e}_i(X) \mid 1 \le i \le k\}.$$

By the monotonicity of $\mu_{b,\Gamma}$ in $(a_1, \cdots, a_k)$, it is easy to prove that $\hat{\rho}$ is a weak solution to the reflector problem with distributions f and $g = \sum_{i=1}^{k} t_i \delta_{Z_i}$. See [KO] for details.

In order to prove that for any point $p \in \mathcal{C}_U$ with $|p| > 2d_0$ in the case of Theorem A(a), or any point $p \in \mathcal{C}_U$ in the case of Theorem A(b), there exists a weak solution ρ such that Γ_ρ passes through p, we first choose a large so that the weak solution $\inf_U \hat{\rho} > |p|$. Observe that for each $i \ge 2$, the infimum $\hat{a}_i$ in (4.21) depends continuously on a and $\hat{a}_i$ is monotone increasing in a. It follows that ρ is continuous and monotone in a. Therefore by the Harnack inequalities (4.18) or (4.10), we can first choose a suitably large so that $\inf_U |\hat{\rho}| > |p|$. Then by the gradient estimates, we can decrease a until at some point $a = a_p$, the point p lies in the graph $\Gamma_{\hat{\rho}}$.

Now we choose a sequence of discrete measures g_k on Σ whose densities converge weakly to g. Let $\hat{\rho}_k$ be the corresponding weak solution obtained as above, such that p is contained in the graph $\Gamma_{\hat{\rho}_k}$. Then by the weak continuity of the measure μ_b, and the estimates in §4.2,

(4.23) $$\hat{\rho} = \lim \hat{\rho}_k$$

is a weak solution of the reflector problem and $\Gamma_{\hat{\rho}}$ passes through the point p.   q.e.d.

Remark 4.3. The definition and existence of weak solutions do not need the explicit equation; it uses only the associated mapping. This is a special feature of Monge-Ampère type equations. In the proof above, we don't need the assumption (1.13). But if (1.13) is not satisfied, there may be infinitely many solutions.

Remark 4.4. As noted before, the weak solution can also be obtained by the Perron method; see [W1] for the far field case. Consider as example the second boundary value problem of the standard Monge-Ampère equation

(4.24) $$\begin{aligned} \det D^2 u &= f(x)/g(Du) \text{ in } \Omega, \\ Du(\Omega) &= \Omega^*, \end{aligned}$$

where f, g satisfy the obvious balance condition $\int_\Omega f = \int_{\Omega^*} g$. The idea is as follows. First suppose Ω^* is convex. We say w is a sub-solution

to (4.24) if $\det D^2 w \geq f(x)/g(Dw)$ in Ω and $Dw(\Omega) \subset \Omega^*$. The set of all sub-solutions is not empty if we replace f/g in (4.24) by $e^{\varepsilon(u-c_0)}f/g$ for some $\varepsilon > 0$ small, where c_0 is any positive constant. The sup of all sub-solutions is bounded when $\varepsilon > 0$, and is a weak solution of (4.24) after sending $\varepsilon \to 0$. If Ω^* is not convex, we may replace Ω^* by a ball $B \supset \Omega^*$, and extend g to B by letting $g = \delta$ and sending $\delta \to 0$ in $B - \Omega^*$.

4.4. The boundary condition. In this subsection we are concerned with the following property: If P is a tangent plane of $\Gamma_{\hat{\rho}}$ at some point $p \in \Gamma_{\hat{\rho}}$, then the ray $X = \frac{p}{|p|}$ will hit the object Σ after being reflected by P, i.e., $T(X) \subset \overline{\Sigma}$ (see the notation T in Remark 4.1).

Lemma 4.4. *Let $\hat{\rho}$ be a weak solution obtained above. Then at any point $X_0 \in U$ where $\hat{\rho}$ is differentiable, one has $T(X_0) \subset \overline{\Sigma}$.*

Proof. The assertion follows from the approximation (4.23). Indeed, choose a regular point X_k of $\hat{\rho}_k$ so that $X_k \to X_0$. Let E_k be the supporting ellipsoid of $\hat{\rho}_k$ at X_k, with a focus $\hat{Z}_k \in \Sigma$. Then E_k converges to an ellipsoid E with a focus $\hat{Z} = \lim \hat{Z}_k \in \overline{\Sigma}$ and E is a supporting ellipsoid of $\hat{\rho}$ at X_0. Since X_0 is a differentiable point, E is the unique supporting ellipsoid at X_0. Hence $T_{\hat{\rho}}(X_0) = \hat{Z}$ and Lemma 4.4 holds. q.e.d.

If X_0 is a singular point of $\hat{\rho}$, there are more than one tangent plane of $\Gamma_{\hat{\rho}}$ at the point $X_0\hat{\rho}(X_0)$, and for some tangent planes the reflected rays may miss the object Σ. For the regularity of $\hat{\rho}$, it is crucial that all reflected rays hit the object Σ, namely, $T(X_0) \subset \overline{\Sigma}$. For this purpose we need to introduce the R-convexity of $\partial\Sigma$. First we introduce the following:

- **Reflection cone**. Let γ_1 and γ_2 be two unit vectors ($\gamma_1 \neq \gamma_2$). Let $p \neq 0$ be a point in $\mathcal{C}_U$. The reflection cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is the set of points $q \in \mathbb{R}^{n+1}$ which satisfy

$$(4.25) \qquad \frac{p-q}{|p-q|} = 2\frac{c_1\gamma_1 + c_2\gamma_2}{|c_1\gamma_1 + c_2\gamma_2|} - \frac{p}{|p|}$$

  for all possible constants c_1, c_2.
- **R-convexity of $\partial\Sigma$**. We say $\partial\Sigma$ is R-convex with respect to a point $p \in \mathcal{C}_U$ if for any unit vectors γ_1, γ_2, the intersection $\mathcal{C}_{p,\gamma_1,\gamma_2} \cap \Sigma$ is connected. We say $\partial\Sigma$ is R-convex with respect to $\mathcal{C}_U$, or simply R-convex, if it is R-convex with respect to all points $p \in \mathcal{C}_U$.

Remark 4.5. The geometric meaning of the reflection cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is as follows. Let Γ_{ρ_1} and Γ_{ρ_2} be two surfaces passing through the point p. Let γ_1 and γ_2 be the normal of Γ_{ρ_1} and Γ_{ρ_2} at p. Let $\rho_t = t\rho_1 + (1-t)\rho_2$, where $-\infty < t < \infty$. Then Γ_{ρ_t} also passes through the point p. By (4.25), a ray from the origin, reflected by Γ_{ρ_t} at p, will fall in the cone $\mathcal{C}_{p,\gamma_1,\gamma_2}$.

Geometrically the R-convexity of $\partial\Sigma$ can be explained as follows. Let Γ_{e_1} and Γ_{e_2} be two ellipsoids with foci $Z_1, Z_2 \in \Sigma$ (the other one is the origin). Let p be a point on $\Gamma_{e_1} \cap \Gamma_{e_2}$ and let γ_1, γ_2 be the normal of E_1, E_2 at p. Let $\rho = te_1 + (1-t)e_2$, where $t \in (0,1)$. Suppose a ray X from O is reflected off by Γ_ρ at p. Then the R-convexity of $\partial\Sigma$ means that reflected ray will hit the object Σ.

In $\mathbb{R}^3$, we have a simple but equivalent definition of R-convexity. That is, $\partial\Sigma$ is R-convex with respect to $p \in \mathcal{C}_U$ if the intersection $\mathcal{C} \cap \Sigma$ is connected for any round convex cone $\mathcal{C}$ with vertex at p, given by $\mathcal{C} = \{X \in \mathbb{R}^{n+1} : \frac{X-p}{|X-p|} \cdot \gamma = \frac{p}{|p|} \cdot \gamma\}$, where γ is the axial direction of $\mathcal{C}$.

From the above remark, we have:

Lemma 4.5. *Let $\hat\rho$ be a weak solution as above. Suppose $p \in \mathcal{C}_U$ and $\partial\Sigma$ is R-convex with respect to points near p. Then the ray $X = \frac{p}{|p|}$ will hit the object Σ after reflection by any tangent plane of $\Gamma\hat\rho$ at p, namely,*

(4.26)
$$T_{\hat\rho}(X_0) \subset \overline{\Sigma}.$$

The weak solution $\hat\rho$ obtained in (4.23) is by definition a type B weak solution. By Lemma 4.4 we have:

Lemma 4.6. *Suppose f, g are positive. Then the solution $\hat\rho$ is also a type A weak solution.*

Lemma 4.6 follows from Lemma 4.4 immediately, as the mapping T is one-to-one almost everywhere and V is the inverse of T.

4.5. Reflection cone. We show that $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is a convex cone, and the cone becomes a plane if and only if the vectors γ_1, γ_2, and $\vec{Op}$ lie in a 2-plane.

Lemma 4.7. *The set $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is a convex cone.*

Proof. Choose an appropriate coordinate system such that p is the origin, the normals γ_1, γ_2 lie in the 2-plane $\{x_1 = \cdots = x_{n-1} = 0\}$, and the ray X in the 2-plane $\{x_2 = \cdots = x_n = 0\}$. Then one can express the unit vector
$$\gamma = \frac{c_1\gamma_1 + c_2\gamma_2}{|c_1\gamma_1 + c_2\gamma_2|}$$
as $\gamma = (0, \cdots, 0, \cos\theta, \sin\theta)$ for some $\theta \in [0, \pi]$ and the ray $X = (\cos\varphi, 0, \cdots, 0 \sin\varphi)$. By the law of reflection, the direction of reflected ray is

$$
\begin{aligned}
Y &= X - 2(X \cdot \gamma)\gamma \\
&= (\cos\varphi, 0, \cdots, 0, -2\sin\theta\cos\theta\sin\varphi, (1 - 2\sin^2\theta)\sin\varphi).
\end{aligned}
$$

Observe that $|Y| = 1$ and $Y \cdot e_1 = \cos\varphi$. Hence Y lies on the convex cone with vertex p, axis $e_1 = (1, 0, \cdots, 0)$, and aperture φ. When $\varphi = \frac{\pi}{2}$, the cone becomes a plane.   q.e.d.

From Lemma 4.7 we see that when Σ is a convex domain in the plane $\{x_{n+1} = 0\}$ containing the origin, then $\partial\Sigma$ is R-convex.

5. Regularity of solutions

In this section we prove the regularity result in Theorem C. We show that if the densities f, g are positive and smooth, then the part of the reflector $\Gamma_{\hat{\rho}}$ located in $\mathcal{D}$ is smooth, where $\hat{\rho}$ is the weak solution obtained in (4.23). We first prove that a local supporting ellipsoid is a global one, then prove a comparison principle, and finally the regularity of solutions.

5.1. A geometric interpretation of (3.3). Let E_i $(i = 0, 1)$ be two ellipsoids with one focus at the origin and the other one Z_i on the receiver Σ, given in polar coordinates by

$$(5.1) \qquad\qquad e_i(X) = \frac{a_i(1 - \varepsilon_i^2)}{1 - \varepsilon_i(X \cdot \ell_i)}, \quad i = 0, 1,$$

where $\ell_i = \frac{Z_i}{|Z_i|}$, $\varepsilon_i = \frac{|Z_i|}{2a_i}$ (see (2.40)). Denote $\Lambda = \{X \in \mathbb{R}^{n+1} \mid e_0(X) = e_1(X)\}$ the intersection. From the above expression,

$$\Lambda = \Big\{ X \in \mathbb{R}^{n+1} \mid 1 - \varepsilon_1(X \cdot \ell_1) = \frac{a_1(1 - \varepsilon_1^2)}{a_0(1 - \varepsilon_0^2)}\big[1 - \varepsilon_0(X \cdot \ell_0)\big] \Big\}$$

is contained in a plane. Let $p \in \Lambda$ be a given point and let γ_0 and γ_1 be the normal of E_0 and E_1 at p. Let $\mathcal{C}_{p, \gamma_1, \gamma_2}$ be the convex cone introduced in §4.4. Let $\mathcal{K} = \Sigma \cap \mathcal{C}_{p, \gamma_1, \gamma_2}$ be the intersection of $\mathcal{C}_{p, \gamma_1, \gamma_2}$ and Σ. Then for any point $Z \in \mathcal{K}$ between Z_0 and Z_1, there is a unique ellipsoid $E = E_{p,Z}$ with foci O and Z, passing through the point p. By the reflection property of ellipsoid, E is tangent to Λ at p.

Let $E = \{X \, e(X) \mid X \in S^n\}$. Denote $X_p = \frac{p}{|p|} \in S^n$ and x_p the projection of X_p on $\{x_{n+1} = 0\}$. Let $w(x) = 1/e(x)$, $w_i(x) = 1/e_i(x)$, $i = 0, 1$. Since E is tangent to Λ at p, we have

$$(5.2) \qquad\qquad Dw(x_p) = \theta Dw_1(x_p) + (1 - \theta) Dw_0(x_p)$$

for some $\theta \in (0, 1)$. The mapping $Z \mapsto \theta$ is one-to-one, so now we can consider w as function of θ. Choose a proper coordinate system such that p is on the positive x_{n+1}-axis and Λ is tangent to the plane $\{x_n = 0\}$. Then $x_p = 0$ and at the origin we have

$$(5.3) \qquad\qquad D(w_1 - w_0) = (0, \cdots, 0, \alpha)$$

for some $\alpha \neq 0$. Recall that the matrix $\mathcal{W} \equiv 0$ at any ellipsoid, namely,

$$(5.4) \qquad\qquad D^2 w = \tau(Dw)\mathcal{N},$$

where τ is the function in Lemma 3.1. Suppose that (3.3) holds. Differentiating (5.4) in θ, we get

$$(5.5) \qquad\qquad \frac{d^2}{d\theta^2} D^2 w = \tau_{p_n p_n} |D(w_1 - w_0)|^2 \mathcal{N} > 0.$$

The above inequality implies that near $x = 0$,

$$(5.6) \qquad\qquad w < \theta w_1 + (1 - \theta) w_0$$

on the plane $x_n = 0$. It implies in particular that

(5.7) $w(x) < \max(w_1(x), w_0(x))$ for x near $0, \neq 0$.

Remark 5.1. Condition (3.3) is closely related to the assumption (A3) in [MTW], and inequality (5.7) corresponds to the geometric property of (A3), discovered by Loeper [L]. In optimal transportation, we have a Monge-Ampère type equation of the form

$$\det(D^2 u - A(x, Du)) = h$$

with $A(x, Du) = D^2_x c(x, T(x, Du))$, where $c(\cdot, \cdot)$ is the cost function and T is the mapping determined by the potential u. By remark 4.1 in [MTW], assumption (A3) is equivalent to

$$A_{ij,p_k p_l} \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2$$

for any vectors $\xi, \eta \in \mathbb{R}^n$, $\xi \perp \eta$, where c_0 is a positive constant, $A = \{A_{ij}\}$ and $A_{ij,p_k p_l} = \frac{\partial^2}{\partial p_k \partial p_l} A_{ij}$. Let $\varphi_t = c(\cdot, y_t) + a_t$, where a_t is chosen such that $\varphi_t(x_0, y_t) = 0$ at x_0, where $\{y_t : t \in [0,1]\}$ is a c-segment (with respect to x_0) connecting y_1 and y_2. Then since the matrix $\{D^2 \varphi_t - A(x, D\varphi_t)\} \equiv 0$, differentiating the matrix in t we get

$$\frac{d^2}{dt^2} D^2 \varphi_t = A_{p_k p_l} \partial_k (\varphi_2 - \varphi_1) \partial_l (\varphi_2 - \varphi_1).$$

It follows that for any vector $\xi \perp D(\varphi_2 - \varphi_1)$,

$$\begin{aligned} \frac{d^2}{dt^2} D^2_\xi \varphi_t &= A_{ij,p_k p_l} \xi_i \xi_j \partial_k (\varphi_2 - \varphi_1) \partial_l (\varphi_2 - \varphi_1) \\ &\geq c_0 |\xi|^2 |D(\varphi_2 - \varphi_1)|^2, \end{aligned}$$

namely, $D^2_\xi \varphi_t$ is convex in t, from which we get Loeper's geometric interpretation of (A3).

Lemma 5.1. *Let w_0, w_1, and w be as above. Suppose inequality (5.7) holds near 0. Then it holds for all $x \in \Omega$.*

Proof. From (5.1), we have the expressions

$$w = c_0 + \sum_{k=1}^n c_k x_k + c_* \sqrt{1 - |x|^2},$$

$$w_i = c_0^i + \sum_{k=1}^n c_k^i x_k + c_*^i \sqrt{1 - |x|^2}, \quad i = 0, 1,$$

where c_k, c_* are constants. Since E is tangent to $\mathcal{N}$, we have $\partial_k w = \partial_k w_0 = \partial_k w_1$ for $k = 1, \cdots, n-1$ at $x = 0$, namely,

(5.8) $c_k = c_k^0 = c_k^1, \quad k = 1, \cdots, n-1.$

By (5.2)

$$c_n = \theta c_n^1 + (1 - \theta) c_n^0.$$

Since all ellipsoids E_0, E_1, E pass through the point p,

$$c_0 + c_* = c_0^0 + c_*^0 = c_0^1 + c_*^1.$$

By (5.6) we also have

$$c_* > \theta c_*^1 + (1 - \theta)c_*^0.$$

Therefore

(5.9) $w < w_\theta := \theta w_1 + (1 - \theta)w_0$

for all $x \in \Omega, x \neq 0$.   q.e.d.

Let E^s denote the solid ellipsoid enclosed by E. Then (5.9) implies that

(5.10) $E_0^s \cap E_1^s \subset E^s.$

In optimal transportation, in order that a local c-support function is a global one, one assumes that the condition (A3) holds everywhere. In our reflector problem, it suffices to assume that (3.3), which corresponds to (A3), holds in a neighborhood of the point x_p.

Remark 5.2. If the least eigenvalue of the matrix in (1.14) is negative, and the corresponding eigenvector $\xi \perp e_n$, then by (5.5), $\frac{d^2}{d\theta^2}D_\xi^2 w < 0$, which implies that (5.10) is not true any more.

If the matrix in (1.14) vanishes identically, such as when Σ lies in a plane passing through the origin, then τ is linear in Du, and (5.5) reduces to $\frac{d^2}{d\theta^2}D^2 w \equiv 0$. Therefore $w \equiv w_\theta$.

5.2. Regularity of solutions. Write equation (2.1) in the form

(5.11) $L[\rho] = \dfrac{f}{g \circ T}$ in $\Omega,$

where by (2.34),

$$L[\rho] = \sigma \det\{-D^2\rho + \frac{2}{\rho}D\rho \otimes D\rho - \frac{a(t-1)}{2t\rho}\mathcal{N}\},$$
$$\sigma = 2^n t^n \rho^{2n+1}\omega^2\beta|a|^{-n-1}b|\nabla\psi|.$$

In the following we assume the conditions (i)–(iii) in Theorem C(b). By (i), σ is a smooth function of $x, \rho, D\rho$. We also assume that f, g are smooth, positive functions.

Lemma 5.2. *Let ρ_1, ρ_2 be weak solutions of (5.11) in Ω' with $f = f_1, f_2$, respectively, where Ω' is a subdomain of Ω. Suppose $f_1 < f_2$ in Ω' and $\rho_1 \leq \rho_2$ on $\partial\Omega'$. Suppose Γ_{ρ_1} (the radial graph of ρ_1 in Ω') lies in the region $\mathcal{D}$. Then $\rho_1 \leq \rho_2$ in Ω'.*

Proof. If the lemma is not true, denote $\omega = \{x \in \Omega' \mid \rho_1(x) > \rho_2(x)\}$. For any $Z \in T_{\rho_2}(x_0)$, where $x_0 \in \omega$, we show that $Z \in T_{\rho_1}(\omega)$. Indeed, let E be the supporting ellipsoid of Γ_2 at $X_0\rho_2(X_0)$, where $X_0 =$

$(x_0, \sqrt{1 - |x_0|^2})$, given by (2.40) with $c = \frac{1}{2}|Z|$ and $\ell = Z/|Z|$. Then since $\rho_1 > \rho_2$ on ω, the surface Γ_1 restricted to ω is not contained in E. We increase a (but fix c and ℓ) until a moment a_1 such that Γ_1 is contained in the ellipsoid E_1 (given by (2.40) with $a = a_1$) and $E_1 \cap \Gamma_1 \neq \emptyset$. Let $x_1 \in \omega$ such that $X_1 \rho_1(X_1) \in E_1 \cap \Gamma_1$, where $X_1 = (x_1, \sqrt{1 - |x_1|^2})$. Then E_1 is a local supporting ellipsoid of Γ_1 at $X_1 \rho_1(X_1)$. Since Γ_1 is in the region $\mathcal{D}$, (5.7) holds, and so by Lemma 5.1, E_1 is a global supporting ellipsoid of Γ_1. It follows that $Z \in T_{\rho_1}(\omega)$, and so

$$(5.12) \qquad\qquad T_{\rho_2}(\omega) \subset T_{\rho_1}(\omega).$$

Hence we obtain

$$(5.13) \qquad\qquad \int_\omega f_2 > \int_\omega f_1 = \int_{T_{\rho_1}(\omega)} g \geq \int_{T_{\rho_2}(\omega)} g,$$

which is in contradiction with the definition of weak solutions (4.6). q.e.d.

The assumption that Γ_1 is contained in the region $\mathcal{D}$ is such that a local supporting ellipsoid is also a global one. If ρ_1 is C^1 smooth, then there is a unique supporting ellipsoid at every point and a local supporting ellipsoid is automatically a global one. Hence Lemma 5.2 holds if ρ_1 is C^1 smooth.

Lemma 5.3. *Let $\hat{\rho}$ be the weak solution obtained in (4.23). Suppose the radial graph of $\hat{\rho}$ in $B_r(x_0)$ is contained in $\mathcal{D}$. Then when r is small, there is a solution $\rho \in C^{3,\alpha}(B_r) \cap C^{0,1}(\overline{B}_r)$ to equation (5.11) such that $\rho = \hat{\rho}$ on $\partial B_r(x_0)$ and $\rho \leq \hat{\rho}$ in B_r.*

Proof. Consider the Dirichlet problem

$$(5.14) \qquad\qquad L[\rho] \;=\; \frac{f - \delta}{g \circ T} \;\; \text{in} \;\; B_r(x_0),$$
$$\rho \;=\; \hat{\rho}_\varepsilon \;\; \text{on} \;\; \partial B_r(x_0),$$

where $\delta > 0$ is a small constant, $B_r(x_0)$ is a ball contained in Ω, $\hat{\rho}_\varepsilon$ is a mollification of $\hat{\rho}$. Namely,

$$\hat{\rho}_\varepsilon(x) = \int_{B_\varepsilon(x)} \varepsilon^{-n} \eta\left(\frac{x - y}{\varepsilon}\right)\hat{\rho}(y),$$

where η is a nonnegative, smooth function in $\mathbb{R}^n$, with compact support in $B_1(0)$ and satisfying $\int \eta = 1$.

Let $w = \hat{\rho}_\varepsilon + C(r^2 - |x|^2)$. We first choose C large then r sufficiently small, so that w is R-convex and satisfies

$$(5.15) \qquad\qquad L[w] \geq \frac{f - \delta}{g \circ T}.$$

Hence w is a subbarrier to (5.14), and we obtain the global gradient estimate for the solution ρ to (5.14).

When r is sufficiently small, one can establish the global a priori estimates for the second derivatives. Indeed, the interior estimate was established in §3. The boundary estimate can be obtained similarly as in [W1]; see [MTW], page 168, for discussion on global regularity of solutions to Monge-Ampère type equations in small ball. With the a priori estimates, the equation becomes uniformly elliptic, and regularity theory by Evans and Krylov [GT] implies that $\rho \in C^{3,\alpha}(\overline{B}_r)$. Therefore one obtains a unique smooth solution $\rho = \rho_{\varepsilon,\delta}$ to the Dirichlet problem (5.14) by the continuity method. By Lemma 5.2, $\rho \le \hat{\rho}$.

Sending $\varepsilon \to 0$ (but δ fixed) and by the interior a priori estimate, we see that $\rho_{\varepsilon,\delta}$ converges to a solution $\rho = \rho_\delta \in C^{3,\alpha}(B_r) \cap C^{0,1}(\overline{B}_r)$ to equation (5.14) which satisfies the boundary condition $\rho_\delta = \hat{\rho}$ on $\partial B_r(x_0)$. By Lemma 5.2, we have $\rho_\delta \le \hat{\rho}$ in B_r. Let $\rho^- = \lim_{\delta \to 0} \rho_\delta$. Then $\rho^- \in C^{3,\alpha}(B_r) \cap C^{0,1}(\overline{B}_r)$ is a solution to equation (5.11) which satisfies $\rho^- = \hat{\rho}$ on $\partial B_r(x_0)$ and $\rho^- \le \hat{\rho}$ in B_r.   q.e.d.

Replace (5.14) by

$$(5.16) \qquad \begin{aligned} L[\rho] &= \frac{f + \delta}{g \circ T} \quad \text{in} \ \ B_r(x_0), \\ \rho &= \hat{\rho}_\varepsilon \quad \text{on} \ \ \partial B_r(x_0). \end{aligned}$$

By the same argument we conclude that there is a solution $\rho^+ \in C^{3,\alpha}(B_r) \cap C^{0,1}(\overline{B}_r)$ to equation (5.11) such that $\rho^+ = \hat{\rho}$ on $\partial B_r(x_0)$ and $\rho^+ \ge \hat{\rho}$ in B_r. As both solutions ρ^+ and ρ^- are smooth, we conclude that $\rho^+ = \rho^-$. Hence we have proved the following:

Theorem 5.1. *Let $\hat{\rho}_p$ be the weak solution obtained in Theorem A. Suppose that f, g are positive and smooth. Then if p is a point in $\mathcal{D}$, $\hat{\rho}_p$ is smooth near p.*

Remark 5.3. Condition (i) in Theorem C(b) guarantees the right-hand side of (3.1) is a smooth function. Condition (iii), which is equivalent to (3.3), was used in the a priori estimate (3.4). Condition (ii) (the R-convexity of $\partial\Sigma$) has also been used in the above proof. Indeed, when considering the Dirichlet problems (5.14) and (5.16), we assumed implicitly that the right-hand sides are smooth and positive. Therefore when using the comparison principle (Lemma 5.2), we need Lemma 4.5.

Remark 5.4. We can also prove the $C^{1,\alpha}$ regularity of the reflector in $\mathcal{D}$ provided $c_0 < g < c_1$ for some positive constants c_0, c_1, and $f \in L^p(U)$ for some $p > \frac{n+1}{2}$ [KW, Liu].

6. Singularity of solutions

In this section we prove part (c) of Theorem C. We say (i) in Theorem C(b) is violated if there is a point $q \in \Sigma$ such that the straight line $\overline{pq}$ intersects transversally with Σ at two points (see Remark 6.1

below); (ii) is violated if there exists two vectors γ_1, γ_2 such that the intersection $\mathcal{C}_{p,\gamma_1,\gamma_2} \cap \Sigma$ contains two disconnected components, where $\mathcal{C}_{p,\gamma_1,\gamma_2}$ is the reflection cone introduced in §4.4; and (iii) is violated if the least eigenvalue of the matrix in (1.14) is negative. In Corollary 6.1 we show that if there is no open subset of Σ which lies in a plane passing through the origin, then for any point $p \in \mathcal{C}_U - \overline{D}$, at least one condition of (i)–(iii) is violated.

First, we show that if (ii) or (iii) is violated, there exists smooth, positive distributions f, g such that the weak solution is not smooth.

Theorem 6.1. *Let p be a point in the cone $\mathcal{C}_U$ such that one of the following conditions is satisfied:*
(i) $\partial\Sigma$ is not R-convex with respect to points near p;
(ii) the least eigenvalue of the matrix in (1.14) is negative for some point $q \in \Sigma$.
Then there exist smooth, positive functions f, g, and a weak solution $\hat{\rho}_p$, which passes through the point p, and is not C^1 smooth near p.

Proof. Let Z_0, Z_1 be two points on Σ to be determined. Let $\{f_k\}$ be a sequence of smooth, positive functions in U which converges to the function $f_0 = \frac{2}{|B_r(X_p)|}\chi_{B_r(X_p)}$, where $r > 0$ small, $X_p = \frac{p}{|p|}$, $B_r(X_p) \subset S^n$ is the geodesic ball, and χ is the characteristic function. Let $\{g_k\}$ be a sequence of smooth, positive functions on Σ which converges to $g_0 = \delta_{Z_0} + \delta_{Z_1}$ weakly as measures. From §4.3, there is a weak solution $\hat{\rho}_k$ with densities f_k and g_k such that the graph $\Gamma_{\hat{\rho}_k}$ passes through the point p.

Suppose $\hat{\rho}_k$ is C^1 smooth in $B_r(X_p)$ for (a subsequence of) $k \to \infty$. Then for any given k, by Lemma 4.4 we have $T_{\hat{\rho}_k}(B_r(X_p)) \subset \overline{\Sigma}$. By the gradient estimate in §4.2, we may assume that $\hat{\rho}_k \to \hat{\rho}_0$ as $k \to \infty$. Then $\hat{\rho}_0$ is an R-convex reflector with densities f_0, g_0. Since $T_{\hat{\rho}_k}(B_r(X_p)) \subset \overline{\Sigma}$, we have

$$(6.1) \qquad\qquad T_{\hat{\rho}_0}(B_r(X_p)) \subset \overline{\Sigma}.$$

Since $\hat{\rho}_0$ is R-convex, it is twice differentiable almost everywhere. Hence the mapping $T_{\hat{\rho}_0}$ is single valued almost everywhere, and $T_{\hat{\rho}_0}(X)$ is either Z_1 or Z_2 for almost all X.

Let $B_i = \{X \in B_r(X_p) \mid T_{\hat{\rho}_0}(X) = Z_i\}$ and $\Gamma_i = \{X\hat{\rho}_0(X) \mid X \in B_i\}$, $i = 0, 1$. We claim that Γ_0 lies in one ellipsoid E_0 and Γ_1 lies in another ellipsoid E_1. Indeed, let X', X'' be two points in B_0 and let E' and E'' be, respectively, the supporting ellipsoids of $\Gamma_{\hat{\rho}_0}$ at $p' = X'\hat{\rho}_0(X')$ and $p'' = X''\hat{\rho}_0(X'')$. Since $\Gamma_{\hat{\rho}_0}$ is R-convex, we see that p' is located on or in the interior of E'', and p'' is located on or in the interior of E'. But since E' and E'' have the common foci O and Z_0, we must have $E' = E''$.

Obviously Z_0 is a focus of E_0 and Z_1 is a focus of E_1. Express the ellipsoids E_0 and E_1 as radial graphs of e_0 and e_1, as in (5.1). Then

$\hat{\rho}_0(X) = \min\{e_0(X), e_1(X)\}$ for any $X \in B_r(X_p)$. By our choice of g_0, we have $|B_0| = |B_1| = \frac{1}{2}|B_r(X_p)|$. Hence there is a point $\hat{X} \in B_{(1-\varepsilon)r}(X_p)$ such that $e_0(\hat{X}) = e_1(\hat{X}) = \hat{\rho}_0(\hat{X})$, where $\varepsilon > 0$ satisfies $(1-\varepsilon)^n > \frac{1}{2}$. Let γ_0 and γ_1 be the normal of E_0 and E_1 at $\hat{p} = \hat{X}\hat{\rho}_0(\hat{X})$. For any $t \in [0,1]$, denote

$$\gamma_t = \frac{(1-t)\gamma_0 + t\gamma_1}{|(1-t)\gamma_0 + t\gamma_1|}.$$

Let $p_{k,t}$ be the unique point on $\Gamma_{\hat{\rho}_k}$ with normal γ_t, and let $E_{k,t}$ be the unique supporting ellipsoid of $\Gamma_{\hat{\rho}_k}$ at $p_{k,t}$, where $t \in [0,1]$. The uniqueness is due to that an R-convex surface is strictly convex in the usual sense, and ρ_k is C^1 by assumption. Hence we have $p_{k,t} \to \hat{p}$ at $k \to \infty$, uniformly for $t \in [0,1]$. By the Harnack inequality and the gradient estimate in §4.2, $E_{k,t}$ converges to a supporting ellipsoid E_t of $\hat{\rho}_0$ at $\hat{p}$.

Case (i): By assumption, we may choose $Z_0, Z_1 \in \Sigma$ such that the intersection $\mathcal{C}_{p',\gamma_0,\gamma_1} \cap \Sigma$ is disconnected and Z_0 and Z_1 lie in two separate components, for all points p' near p, and also for the point $\hat{p}$ determined above. But since $E_{k,t}$ is a supporting ellipsoid of $\Gamma_{\hat{\rho}_k}$ at $p_{k,t}$, E_t is a supporting ellipsoid of $\Gamma_{\hat{\rho}}$ at $\hat{p}$, for all $t \in [0,1]$. Let Z_t be the focus of E_t on Σ. Then the set $\{Z_t \mid t \in [0,1]\}$ is a curve on Σ which connects Z_0 and Z_1. It follows that Z_0 and Z_1 lie in the same component of the intersection $\mathcal{C}_{\hat{p},\gamma_0,\gamma_1} \cap \Sigma$. We reach a contradiction. Hence when k is sufficiently large, the weak solution $\hat{\rho}_k$ is not C^1 smooth in $B_r(X_p)$.

Case (ii): Let ξ is the eigenvector corresponding to the least eigenvalue. Assume that $\xi \perp e_n$. Choose Z_0 and Z_1 near q such that the ellipsoids E_0 and E_1 satisfy (5.3). Then by Remark 5.2, E_t is not a supporting ellipsoid. We also reach a contradiction.   q.e.d.

The idea in the above proof is from [W1], page 362. See also §7.3 of [MTW]. Case (ii) is related to Loeper's counterexample to the regularity of potential functions in optimal transportation [L], in which Loeper also uses the idea of approximation to a discrete measure concentrated at two points.

Next we show that condition (i) in Theorem C is necessary. First we make a remark.

Remark 6.1. Note that if there is a point $q \in \Sigma$ such that the vector $p - q$ is tangent to Σ at q, then either there is a point p_1 near p, such that the straight line $\overline{p_1 q}$ intersects transversally with Σ at more than one point, or Σ lies in a plane. In the latter case, the plane divides the reflector Γ into two pieces. In one piece (1.14) holds and in the other one (1.14) is violated. Hence in the latter case, condition (iii) in Theorem C is also violated. In Theorem 6.1, we have already shown that (iii) is necessary. Hence it suffices to consider the former case.

Theorem 6.2. *Let p be a point in $\mathcal{C}_U$. If there is a point $q \in \Sigma$ such that the straight line $\overline{pq}$ intersects transversally with Σ at two points Z_0 and Z_1, then there exist smooth, positive distributions f and g such that the weak solution $\hat{\rho}$ is not C^1 smooth near p.*

Proof. Denote by $\mathcal{C}_{\overline{pq},r}$ the cylinder $\{p \in \mathbb{R}^{n+1} \mid \text{dist}(p, \overline{pq}) < r\}$. Then for any $C_0 > 0$, the two components of $\Sigma \cap \mathcal{C}_{\overline{pq},C_0 r}$ which contain, respectively, Z_0 and Z_1 are disconnected, provided r is sufficiently small.

Choose the same f_k, g_k as above and let $\hat{\rho}_k$ be the weak solution such that $\Gamma_{\hat{\rho}_k}$ passes through p. Suppose there is a subsequence, still denoted as $\hat{\rho}_k$, which is C^1 smooth in $B_r(X_0)$. As above, one can show that $\hat{\rho}_k$ sub-converges to $\hat{\rho}_0$ and $\hat{\rho}_0 = \min\{e_0, e_1\}$. Choose a point $\hat{p} \in \Gamma_{\hat{\rho}_0}$ at which $e_0 = e_1$. Then $\hat{\rho}_0$ cannot be C^1 smooth at $\hat{p}$, as the ellipsoids E_0 and E_1 have different foci Z_0 and Z_1. Denote $B_i = \{X \in B_r(X_0) \mid T_{\hat{\rho}_0}(X) = Z_i\}$, $i = 0, 1$. Then $|B_0| = |B_1| = \frac{1}{2}|B_r(X_0)|$. Choose $X_0 \in B_0$ and $X_1 \in B_1$ near $\hat{X} = \frac{\hat{p}}{|\hat{p}|}$ such that $\hat{X}$ lies on shortest path (denoted as ℓ) in S^n connecting X_0 and X_1.

Since $\hat{\rho}_k$ is C^1, $T_{\hat{\rho}_k}(\ell)$ is a continuous curve in Σ. For any sequence $\{X_k\} \subset \ell$, let E_k be the supporting ellipsoid of $\Gamma_{\hat{\rho}_k}$ at $X_k \hat{\rho}_k(X_k)$. Assume X_k sub-converges to a point X. Then E_k sub-converges to an ellipsoid E, and E is a supporting ellipsoid of $\hat{\rho}_0$ at $\hat{p} = X \hat{\rho}_0(X)$. Let Y be the direction of the reflected ray by E at $\hat{p}$. It is easy to check that the line segment $L = \{q = \hat{p} + tY \mid |q| < R\}$ is completely contained in the cylinder $\mathcal{C}_{\overline{pq},C_0 r}$, where we choose R large such that $\Gamma_{\hat{\rho}_0}$ and Σ are contained in the ball $B_R(0)$, and C_0 depends only on R. Therefore when k is sufficiently large, $T_{\hat{\rho}_k}(\ell)$ is completely contained in the cylinder $\mathcal{C}_{\overline{pq},C_0 r}$.

But $T_{\hat{\rho}_k}(\ell)$ is a continuous curve in Σ which connects a point near Z_0 to another point near Z_1, and the two components of $\Sigma \cap \mathcal{C}_{\overline{pq},C_0 r}$ containing Z_0 and Z_1 are disconnected. This is impossible.   q.e.d.

Next we show that for any point $p \in \mathcal{C}_U - \overline{\mathcal{D}}$, at least one condition in Theorem C(b) is violated, under a very mild condition.

Lemma 6.1. *Suppose there is no open subset of Σ which is contained in a plane passing through the origin. Then for any point $p \in \mathcal{C}_U - \overline{\mathcal{D}}$, at least one condition in Theorem C(b) is violated.*

From Lemma 6.1 we have:

Corollary 6.1. *Suppose there is no open subset of Σ which is contained in a plane passing through the origin. Then for any point $p \in \mathcal{C}_U - \overline{\mathcal{D}}$, there exist smooth, positive distributions f and g such that the weak solution $\hat{\rho}$ passing through p is not C^1 smooth near p.*

Proof of Lemma 6.1. Let $\mathcal{D}_i, \mathcal{D}_{ii}$, and $\mathcal{D}_{iii}$ be, respectively, the sets of points in $\mathcal{C}_U$ at which conditions (i), (ii), and (iii) hold. Denote

by $\overline{\mathcal{D}}_i^c, \overline{\mathcal{D}}_{ii}^c$, and $\overline{\mathcal{D}}_{iii}^c$ the complements of $\overline{\mathcal{D}}_i, \overline{\mathcal{D}}_{ii}$, and $\overline{\mathcal{D}}_{iii}$. Then (i) is violated for any point $p \in \overline{\mathcal{D}}_i^c$, and (ii) is violated for any point $p \in \overline{\mathcal{D}}_{ii}^c$. It suffices to verify that for any point $p \in \overline{\mathcal{D}}_{iii}^c$, the least eigenvalue of the matrix

$$II + \frac{\cos \beta}{2 \cos^2 \alpha} \frac{|q|}{|p| \, |p - q|} I$$

is negative at some point $q \in \Sigma$.

Consider the function $\frac{\cos \beta}{2 \cos^2 \alpha_t} \frac{|q|}{|p_t| \, |p_t - q|}$ (as a function of t on the straight line $\{p_t = q + t(p - q), \ t \in \mathbb{R}\}$). This function cannot be a constant in any interval of t, provided $\cos \beta \neq 0$ and $q \neq 0$, where $2\alpha_t$ is angle between the vectors $\overrightarrow{p_t O}$ and $\overrightarrow{p_t q}$. Recall that β is the angle between Oq and the normal of Σ at q. Therefore if there is no open subset of Σ on which $\cos \beta \equiv 0$, (iii) is violated for any point $p \in \overline{\mathcal{D}}_{iii}^c$. But if $\cos \beta \equiv 0$ in an open subset of Σ, this open subset must be a domain in a plane passing through the origin. q.e.d.

7. R-concave solutions

In previous sections we studied the existence and regularity of R-convex solutions. As with the standard Monge-Ampère equation (1.5), one can also introduce an R-concave solution. The definition is the same as that of R-convex solution, except that the inequality in (4.1) changes direction. More precisely, we say $\Gamma = \Gamma_\rho$ is R-concave if for any point $\bar{X} \in U$, there is an ellipsoid $E = \{Xe(X) \mid X \in S^n\}$, with one focus at the origin and the other one on Σ, which satisfies

(7.1)
$$\rho(\bar{X}) = e(\bar{X}),$$
$$\rho(X') \geq e(X') \quad \forall \ X' \in U.$$

For the existence of weak R-concave solutions, note that the uniform and gradient estimates in §4.2 still hold, so we have the same results as in Theorem A.

Theorem 7.1. *Suppose the distributions f and g satisfy the balance condition (1.9). Then for any point p in $\mathcal{C}_U$ with $|p| > 2 \sup\{|q| : q \in \Sigma\}$, there is an R-concave weak solution ρ such that the reflector Γ_ρ passes through the point p. If furthermore $\Sigma \subset \mathcal{C}_V$ for a domain $V \subset S^n$ and $\overline{U} \cap \overline{V} = \emptyset$, then for any point $p \in \mathcal{C}_U$, there is a weak solution ρ_p such that Γ_{ρ_p} passes through p.*

Theorem 7.1 can be proved in the same way as that of Theorem A, with just some minor modifications. We omit the details here. For the regularity, we have a counterpart of Theorem C.

Theorem 7.2. *Suppose the distributions f and g satisfy (1.9). Let ρ_p be a weak solution which passes through the point p.*

(a): *There exists a region $\hat{\mathcal{D}} \subset \mathcal{C}_U$, depending on U and Σ but independent of f, g, such that if $p \in \hat{\mathcal{D}}$ and f, g are smooth and positive, then ρ_p is smooth near p.*

(b): *A point $p \in \hat{\mathcal{D}}$ if and only if*
(i) $|(q - p) \cdot \nu| > 0 \quad \forall \ q \in \Sigma$;
(ii) $\partial\Sigma$ is R-convex with respect to points near p; and
(iii) for any $q \in \Sigma$,

$$(7.2) \qquad II + \frac{\cos\beta}{2\cos^2\alpha} \frac{|q|}{|p|\,|p - q|} I < 0,$$

where the notations are the same as those in (1.14).

(c): *If one of the above conditions is violated, there exist smooth, positive distributions f, g such that the weak solution ρ_p is not C^1 smooth near p.*

The proof is very similar to that of Theorem C. Note that conditions (i) and (ii) are the same as in Theorem C but the inequality in (iii) has changed its direction, and as in the a priori estimate (Lemma 3.1) one needs to change u to $-u$.

Theorem 7.2 applies to cases when the point p and the origin lie on different sides of the tangent planes of Σ. In such situation, one has $\cos\beta < 0$ and so (7.2) holds provided Σ is concave, i.e., the second fundamental form II along the vector $p - q$ is negative. Therefore by Theorem 7.2, the weak solution ρ_p in Theorem 7.1 is smooth near p, provided $p \in \hat{\mathcal{D}}$.

We conclude the paper with two remarks, one on the uniqueness of solutions, and the other one on the case when Σ is a planer domain.

The uniqueness. From Theorem A and Theorem 7.1, there are two solutions to the reflector problem. But in general there is another solution such that the matrix $\mathcal{W}$ in (1.10) is indefinite. Indeed, one can first choose a smooth, positive function ρ in a domain $U \subset S^n$ such that $\mathcal{W}$ is indefinite, and let Σ be the area in a large sphere illuminated by Γ_ρ. By Theorem A and Theorem 7.1, we obtain another two solutions, one R-convex and the other one R-concave.

For the uniqueness of R-convex solutions, if the reflector Γ is completely in the region $\mathcal{D}$, the uniqueness can be proved by Aleksandrov's argument for the Monge-Ampère equation, as in the far field case [W1]. But if (1.13) is violated, there may be infinitely many solutions. For example, consider the reflector problem in $\mathbb{R}^2$. Let $\Sigma = B_r(0) \cap \{x_1 = 0\}$ be a segment, and let $U \subset S^1$ be an arc near $(0, 1)$. By solving an ode, there is a solution Γ such that the light reflected by $\Gamma^- = \Gamma \cap \{x_1 < 0\}$ forms a pattern g^- on Σ, and the light reflected by $\Gamma^+ = \Gamma \cap \{x_1 > 0\}$

forms another pattern g^+ on Σ. There are infinitely many decompositions $g = g^- + g^+$ for which the reflector is smooth and R-convex. However, we expect that the uniqueness holds if (1.13) is satisfied.

The case Σ is a planer domain. Let us assume that Σ is contained in $\{x_{n+1} = 0\}$, $\partial\Sigma$ is R-convex, and the reflector does not intersect with the plane, so that condition (i) in Theorem C(b) holds. We have three cases:

(i): The reflector and the origin lie on the same side of the plane.
(ii): The reflector and the origin lie on different sides of the plane.
(iii): The plane passes through the origin.

In case (i), (1.14) is satisfied and an R-convex solution is smooth, provided f, g are positive and smooth. But an R-concave solution is in general not smooth, due to Theorem 7.2(c). In case (ii), (7.2) holds and so an R-concave solution is smooth, but an R-convex solution is in general not smooth by Theorem C(c). In case (iii), the equation becomes the standard Monge-Ampère equation, and the solution is smooth if it is strictly convex [P2, C1].

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